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HYPERSTABILITY RESULTS FOR GENERALIZED QUADRATIC FUNCTIONAL EQUATIONS IN $(2,\alpha)$-BANACH SPACES

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Abstract In this article, we utilize a special version of some recent fixed point theorem to investigate the hyperstability of the following generalized quadratic functional equation with F is the unknown function from a special subset X of a (2,β)-normed space over the field $\mathbb{F}$ into a (2,α)-Banach space over the field $\mathbb{K}$:

$$F(ax_1 + bx_2) + F(cx_1 + dx_2) = rF(x_1) + sF(x_2)$$

for all $x_1, x_2 \in X$, where $a, b, c, d \in \mathbb{F}$ and $r, s \in \mathbb{K}$. In this way, we generalize several earlier outcomes.

Keywords Functional equations, hyperstability, fixed point theory.

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1. Introduction

Stability of functional equations (see e.g. [20,26]) popped up as a result of a famous question posed by Ulam in 1940 in his celebrated talk at Wisconsin University. The stability problem presented by Ulam can be rewritten as follows (see e.g. [22]):

- **Ulam's Problem in 1940:** If $(G^*, *_1)$ is a group and $(G^{**}, *_2)$ is a metric group with the metric d. Is it true that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $H : G^* \to G^{**}$ satisfies

$$d(H(x_1 *_1 x_2), H(x_1) *_2 H(x_2)) < \delta$$

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for all $x_1, x_2 \in G^*$, then a homomorphism $F : G^* \to G^{**}$ exists such that

$$d(H(x_1), F(x_1)) < \varepsilon$$

for all $x_1 \in G^*$?

Many mathematicians interacted with the question of Ulam. For instance, in 1941, Hyers introduced a partial answer to Ulam's question (see, e.g., [11] for some details and references therein). It is well-known that the following result is the most classical result concerning the Hyers-Ulam stability, which is due to Aoki, 1950 (see [2] for more details).

Theorem 1.1. *Assume that N_1 is a normed space, N_2 is a complete normed space, $n_1 \in [0, \infty)$ and q is a real number with $q \neq 1$. Let $g : N_1 \to N_2$ be an operator such that*

$$\|g(x_1 + x_2) - g(x_1) - g(x_2)\| \le n_1(\|x_1\|^q + \|x_2\|^q)$$

for all $x_1, x_2 \in N_1 \setminus \{0\}$. Then there exists a unique addtive function $h : N_1 \to N_2$ with

$$\|g(x_1) - h(x_1)\| \le \frac{n_1\|x_1\|^q}{|1 - 2^{q-1}|}$$

for all $x_1 \in N_1 \setminus \{0\}$.

It should be noted that Gajda (see [20]) proved the above theorem for $q > 1$ and Rassias proved it for $q < 0$ (see [28]). Nowadays, it is known that, for $q < 0$, every solution to the first inequality in Theorem 1.1 must be additive (see. e.g., [7]). Moreover, very recently, much more precise results than Theorem 1.1, but only for functions taking real values, have been obtained (by applying the Banach limit technique) in [4, 16].

Hyperstability and superstability are strongly related to the notion of stability mentioned above (see, e.g., [11]). In fact, they are considered as particular kinds of stability (see, e.g., [26]). It should be noted that a functional equation $\mathcal{D}$ is said to be *hyperstable* if, roughly speaking, any function f satisfying $\mathcal{D}$ approximately (in some sense) must be actually a true solution of $\mathcal{D}$ (see e.g. [11, 21]). It should be remarked that Maksa and Páles in [25] seems to be the first who used the term *hyperstability*, and the first *hyperstability* result was published in [13] and concerned ring homomorphisms.

For given two vector spaces X and Y over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, the following functional equation:

$$F(ax_1 + bx_2) + F(cx_1 + dx_2) = rF(x_1) + sF(x_2) \tag{1.1}$$

for all $x_1, x_2 \in X$, where $F : X \to Y$ is an unknow function, $a, b, c, d \in \mathbb{F}$ and $r, s \in \mathbb{K}$, is an interesting functional equation since it generalizes many well-known equations as follows:

1. For $\mathbb{F} = \mathbb{K} = \mathbb{R}$ and $a = b = c = d = 1, r = s = 2$, (1.1) becomes the most famous functional equation, which is the *Cauchy* functional equation

$$F(x_1 + x_2) = F(x_1) + F(x_2)$$

for all $x_1, x_2 \in X$.

2. For $\mathbb{F} = \mathbb{K} = \mathbb{R}$ and $a = b = c = d = 1/2, r = s = 1$, (1.1) becomes the well-known *Jensen* functional equation

$$F\left(\frac{x_1 + x_2}{2}\right) = \frac{F(x_1) + F(x_2)}{2}$$

for all $x_1, x_2 \in X$.

3. For a fixed real number p with $0 < p < 1$, $\mathbb{F} = \mathbb{K} = \mathbb{R}$ and $a = d = p, b = c = 1 - p, r = s = 1$, (1.1) becomes the well-known functional equation of the *p-Wright affine functions*

$$F(px_1 + (1-p)x_2) + F((1-p)x_1 + px_2) = F(x_1) + F(x_2)$$

for all $x_1, x_2 \in X$.

4. For $\mathbb{F} = \mathbb{K} = \mathbb{R}$ and $a = b = c = 1, d = -1, r = s = 2$, (1.1) becomes the well-known *quadratic* functional equation

$$F(x_1 + x_2) + F(x_1 - x_2) = 2F(x_1) + 2F(x_2)$$

for all $x_1, x_2 \in X$. For more details and results on the quadratic functional equation the reader is referred to [14, 23].

5. For $\mathbb{F} = \mathbb{K}$ and $b = c, d = -a, r = s = (a^2 + b^2)$, (1.1) becomes the well-known *Euler-Lagrange* functional equation

$$F(ax_1 + bx_2) + F(bx_1 - ax_2) = (a^2 + b^2)(F(x_1) + F(x_2))$$

for all $x_1, x_2 \in X$.

6. For $\mathbb{F} = \mathbb{R}$ and $c = a$ and $b = d$, (1.1) becomes some generalized linear functional equation

$$F(ax_1 + bx_2) = AF(x_1) + BF(x_2)$$

for all $x_1, x_2 \in X$, where $A = r/2$ and $B = s/2$.

Although the hyperstability results of a functional equation close to (1.1) on Banach spaces have been studied in [12], the hyperstability results of (1.1) on $(2, \alpha)$-Banach spaces is still open to the researcher.

The goal of this paper is to investigate the hyperstability results for the interesting functional equation (1.1) on $(2, \alpha)$-Banach spaces. Our results can be seen as some generalization of the exciting results obtained in [1]. In some sense, these results also generalize the hyperstability results obtained in [3, 7, 8]. In addition, in [29], the authors obtained hyperstability results of some Cauchy-Jensen functional equations in 2-Banach spaces while our results are in $(2, \alpha)-$Banach spaces. So, our results can be seen as some improvements of the results in [29] for the generalized quadratic equation. Lastly, our results also generalize the hyperstability results obtained in [15] for the general linear equation in quasi-Banach spaces.

In the rest of this current section, the organization of this paper is described. Section 2 gives basic concepts needed in this paper such as several definitions related to $(2, \alpha)$-normed spaces and the fixed point theorem that is our main tool in the proofs of main hyperstability results. In Section 3, we investigate the hyperstability of the functional equation of interest, and in Section 4, we conclude our work.

2. Preliminaries

Throughout the article, we will use $\mathbb{R}$ to denote the set of reals, $\mathbb{R}_+$ the set of positive reals, $\mathbb{C}$ to denote the set of complex numbers, $\mathbb{N}$ the set of integers, $\mathbb{F}, \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We first recall some basic concepts as follows (see, e.g., [6,24]).

Definition 2.1. By a linear *2-normed space*, we mean a pair $(\mathcal{Y}, \|\cdot,\cdot\|)$ such that $\mathcal{Y}$ is an at least two-dimensional real linear space and

$$\|\cdot,\cdot\| : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$$

is a function satisfying the following conditions:

(C1) $\|y, y^*\| = 0$ if and only if y and y^* are linearly dependent;

(C2) $\|y, y^*\| = \|y^*, y\|$ for all $y, y^* \in \mathcal{Y}$

(C3) $\|y, y^* + y^{**}\| \leq \|y, y^*\| + \|y, y^{**}\|$ for all $y, y^*, y^{**} \in \mathcal{Y}$

(C4) $\|qy, y^*\| = |q|\|y, y^*\|$ for all $q \in \mathbb{R}$ and $y, y^* \in \mathcal{Y}$.

More details on stability results of functional equations in linear 2-normed spaces, which are closely related to main results in this paper, are found in [5,17].

Some generalized version of the linear 2-normed space is the $(2,\chi)$-normed space defined in the following manner.

Definition 2.2. Let χ be a fixed real number with $0 < \chi \leq 1$, and let $\mathcal{V}$ be a linear space over $\mathbb{K}$ with $\dim \mathcal{V} > 1$. A function

$$\|\cdot,\cdot\|_\chi : \mathcal{V} \times \mathcal{V} \to \mathbb{R}_+$$

is called a $(2,\chi)$-norm on $\mathcal{V}$ if and only if it satisfies the following conditions:

(C1) $\|v_1, v_2\|_\chi = 0$ if and only if v_1 and v_2 are linearly dependent;

(C2) $\|v_1, v_2\|_\chi = \|v_2, v_1\|_\chi$ for all $v_1, v_2 \in \mathcal{V}$

(C3) $\|v_1, v_2 + v_3\|_\chi \leq \|v_1, v_2\|_\chi + \|v_1, v_3\|_\chi$ for all $v_i \in \mathcal{V}, i = 1, 2, 3$

(C4) $\|\lambda v_1, v_2\|_\chi = |\lambda|^\chi \|v_1, v_2\|_\chi$ for all $\lambda \in \mathbb{K}$ and $v_1, v_2 \in \mathcal{V}$.

The pair $(\mathcal{V}, \|\cdot,\cdot\|_\chi)$ is also called a $(2,\chi)$-normed space.

Definition 2.3. A sequence $(v_n)_{n\in\mathbb{N}}$ of elements of a $(2,\chi)$-normed space $\mathcal{V}$ is called a *Cauchy sequence* if there are linearly independent $v_1, v_2 \in \mathcal{V}$ such that

$$\lim_{n,m\to\infty} \|v_n - v_m, v_1\|_\chi = 0 = \lim_{n,m\to\infty} \|v_n - v_m, v_2\|_\chi,$$

whereas $(v_n)_{n\in\mathbb{N}}$ is said to be convergent if there exists an $v_1 \in \mathcal{V}$ (called a limit of this sequence and denoted by $\lim_{n\to\infty} x_n$) with

$$\lim_{n,m\to\infty} \|v_n - v_1, v\|_\chi = 0 \text{ for all } v \in \mathcal{V}.$$

A $(2,\chi)$-normed space in which every Cauchy sequence is convergent is called a $(2,\chi)$-*Banach space*.

Let us also mention that in a $(2,\chi)$-normed space, every convergent sequence has exactly one limit, and the standard properties of the limit of a sum and a scalar product are valid. Next, it is easily seen that we have the following property.

Lemma 2.1. *If $\mathcal{V}$ is a $(2,\chi)$-normed space, $v_1, v_2, v_3 \in \mathcal{V}$ with v_2, v_3 are linearly independent, and*

$$\|v_1, v_2\|_\chi = 0 = \|v_1, v_3\|_\chi,$$

then $v_1 = 0$.

Let us yet recall a lemma from [27] as follows.

Lemma 2.2 ([27]). *If $\mathcal{V}$ is a $(2,\chi)$-normed space and $(v_n)_{n\in\mathbb{N}}$ is a convergent sequence of elements of $\mathcal{V}$, then*

$$\lim_{n\to\infty} \|v_n, v_1\|_\chi = \|\lim_{n\to\infty} v_n, v_2\|_\chi \text{ for all } v_2 \in \mathcal{V}.$$

We now give a simple example of a $(2,\chi)$-normed space as follows:

Example 2.1. For $u = (u_1, u_2)$, $v = (v_1, v_2) \in \mathcal{V} = \mathbb{R}^2$, A function $\|\cdot,\cdot\|_\chi : \mathcal{V}\times\mathcal{V} \to \mathbb{R}_+$ defined by

$$\|u, v\|_\chi = |u_1 v_2 - u_2 v_1|^\chi$$

for all $u, v \in \mathcal{V}$, where χ is a fixed real number with $0 < \chi \leq 1$, is a $(2,\chi)$-norm on $\mathcal{V}$.

Next, we recall the fixed point theorem that is our main tool in the proofs. Indeed, the main tool used in this article is Theorem 2.1, which is a special version of Theorem 1 in [9]. In order to rewrite it, we need the following assumptions.

(A1) S is a nonempty set, $(Y_1, \|\cdot,\cdot\|_\alpha)$ is a $(2,\alpha)$-Banach space, Y^* is a subset of Y_1 containing two linearly independent vectors, $j \in \mathbb{N}$,

$$f_i : S \to S,\ g_i : Y^* \to Y^*, \text{ and } L_i : S \times Y^* \to \mathbb{R}_+$$

are given mappings for all $i = 1, 2, 3, \ldots, j$.

(A2) $\mathcal{T} : Y_1^S \to Y_1^S$ is an operator satisfying the following inequality for each $\xi, \mu \in Y_1^S$:

$$\|\mathcal{T}\xi(s_1) - \mathcal{T}\mu(s_1), y\|_\alpha \leq \sum_{i=1}^{j} L_i(s_1, y)\|\xi(f_i(s_1)) - \mu(f_i(s_1)), g_i(y)\|_\alpha$$

for all $s_1 \in S$ and $y \in Y^*$.

(A3) $\Lambda : {\mathbb{R}_+}^{S\times Y^*} \to {\mathbb{R}_+}^{S\times Y^*}$ is an operator defined for each $\delta \in {\mathbb{R}_+}^{S\times Y^*}$ by

$$\Lambda\delta(s_1, y) := \sum_{i=1}^{j} L_i(s_1, y)\delta(f_i(s_1), g_i(y))$$

for all $s_1 \in S$ and $y \in Y^*$.

Now, it is the position to present the above-mentioned fixed point theorem.

Theorem 2.1. *Let hypotheses (A1)-(A3) hold and functions*

$$\varepsilon : S \times Y^* \to \mathbb{R}_+ \text{ and } \varphi : S \to Y_1$$

fulfill the following two conditions:

$$\|\mathcal{T}\varphi(s_1) - \varphi(s_1), y\|_\alpha \leq \varepsilon(s_1, y)$$

and

$$\varepsilon^*(s_1, y) := \sum_{i=1}^{\infty} (\Lambda^i \varepsilon)(s_1, y) < \infty$$

for all $s_1 \in S$ *and* $y \in Y^*$. *Then there exists a unique fixed point* ψ *of* $\mathcal{T}$ *for which*

$$\|\varphi(s_1) - \psi(s_1), y\|_\alpha \le \varepsilon^*(s_1, y)$$

for all $s_1 \in S$ *and* $y \in Y^*$. *Moreover,*

$$\psi(s_1) = \lim_{l \to \infty} (\mathcal{T}^l \varphi)(s_1)$$

for all $s_1 \in S$.

We skip the proof as it can be easily proved analogously as in [6].

3. Hyperstability Results

In this section, we introduce some hyperstability results (see, e.g., [10, 18, 19]) for the functional equation (1.1) in $(2, \alpha)$-Banach spaces. The main tool in the analysis is a fixed point theorem which is illustrated in the previous section.

In the rest of the paper, we will use $(E, \|\cdot, \cdot\|_\beta)$ to denote a $(2,\beta)$-normed space, $\mathcal{X} \subset E \setminus \{0\}$ is a nonempty set, $(Y, \|\cdot, \cdot\|_\alpha)$ to denote a $(2, \alpha)$-Banach space, Y_0 is a subset of Y containing two linearly independent vectors. The next theorem is the main hyperstability result in this paper and concerns the φ_1-hyperstability of the equation (1.1). We show under some additional assumptions that equation (1.1) is φ_i-hyperstable in class of functions $g : \mathcal{X} \to Y$ for $i \in \{1, 2, 3\}$, with the following control functions $\varphi_1, \varphi_2, \varphi_3 : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ defined by:

1. $\varphi_1(x_1, x_2, y) := A\|x_1, y\|_\beta^p \|x_2, y\|_\beta^q$ for all $x_1, x_2, y \in \mathcal{X}$, where $A \ge 0$, $p, q \in \mathbb{R}$ with $p + q \ne 0$,
2. $\varphi_2(x_1, x_2, y) := A\|x_1, y\|_\beta^p + B\|x_2, y\|_\beta^q$ for all $x_1, x_2, y \in \mathcal{X}$, where $A, B \ge 0$, $p, q \in \mathbb{R}$ with $p, q < 0$,
3. $\varphi_3(x_1, x_2, y) := A\|x_1, y\|_\beta^p \|x_2, y\|_\beta^q + B\|x_1, y\|_\beta^p + C\|x_2, y\|_\beta^q$ for all $x_1, x_2, y \in \mathcal{X}$, where $A, B, C \ge 0$, $p, q \in \mathbb{R}$ with $p, q < 0$.

For some related results, see, e.g., [3, 7].

Theorem 3.1. *Let* $A \ge 0$, $0 < \alpha, \beta \le 1$, $a, b, c, d \in \mathbb{F} \setminus \{0\}$, $r, s \in \mathbb{K} \setminus \{0\}$ *and* $p, q \in \mathbb{R}$ *with* $p + q < 0$. *Assume that there exists a positive integer* m_0 *with*

$$nx, (a + bn)x, \ (c + dn)x \in \mathcal{X} \tag{3.1}$$

for all $x \in \mathcal{X}$, $n \in \mathbb{N}$ *with* $n \ge m_0$ *and* $g : \mathcal{X} \to Y_0$ *is a surjective mapping. If an operator* $h : \mathcal{X} \to Y$ *satisfies*

$$\|h(ax_1 + bx_2) + h(cx_1 + dx_2) - rh(x_1) - sh(x_2), g(y)\|_\alpha \le A\|x_1, y\|_\beta^p \|x_2, y\|_\beta^q \tag{3.2}$$

for all $x_1, x_2, y \in \mathcal{X}$ *with* $ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$, *then* h *satisfies* (1.1) *on* $\mathcal{X}$, *i.e.*,

$$h(ax_1 + bx_2) + h(cx_1 + dx_2) = rh(x_1) + sh(x_2)$$

for all $x_1, x_2, y \in \mathcal{X}$ *with* $ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$.

Proof. We see that there exists $n_0 \in \mathbb{N}$ such that

$$U_m := \frac{1}{|r|^\alpha}|a+bm|^{(p+q)\beta} + \frac{1}{|r|^\alpha}|c+dm|^{(p+q)\beta} + |\frac{s}{r}|^\alpha m^{(p+q)\beta} < 1$$

for all $m \in \mathbb{N}$ with $m \geq n_0$. Fix $m \geq \max\{m_0, n_0\}$. Note that (3.2) gives

$$\|h((a+bm)x) + h((c+dm)x) - rh(x) - sh(mx), g(y)\|_\alpha \leq Am^{q\beta}\|x,y\|_\beta^{p+q} \quad (3.3)$$

for all $x, y \in \mathcal{X}$ with $x, (a+bm)x, (c+dm)x, y \in \mathcal{X}$. Define an operator $\mathcal{T} : Y^{\mathcal{X}} \to Y^{\mathcal{X}}$ for each $\xi \in Y^{\mathcal{X}}$ by

$$(\mathcal{T}\xi)(x) := \frac{1}{r}\xi((a+bm)x) + \frac{1}{r}\xi((c+dm)x) - \frac{s}{r}\xi(mx)$$

for all $x \in \mathcal{X}$ and let

$$\epsilon(x,z) := \frac{A}{|r|^\alpha}m^{q\beta}\|x,z\|_\beta^{p+q}$$

for all $x \in \mathcal{X}$ and $z := g(y) \in Y_0$ for all $y \in \mathcal{X}$. Then (3.3) takes the form

$$\|(\mathcal{T}h)(x) - h(x), g(y)\|_\alpha \leq \epsilon(x,z)$$

for all $x \in \mathcal{X}$ and $z := g(y) \in Y_0$ for all $y \in \mathcal{X}$.

Define the operator $\Lambda : {\mathbb{R}_+}^{\mathcal{X}\times Y_0} \to {\mathbb{R}_+}^{\mathcal{X}\times Y_0}$ for each $\eta \in {\mathbb{R}_+}^{\mathcal{X}\times Y_0}$ in such a way that

$$\Lambda\eta(x,z) := \frac{1}{|r|^\alpha}\eta((a+bm)x,z) + \frac{1}{|r|^\alpha}\eta((c+dm)x,z) + \left|\frac{s}{r}\right|^\alpha\eta(mx,z)$$

for all $x \in \mathcal{X}$ and $z := g(y) \in Y_0$ for all $y \in \mathcal{X}$. Then it is easily seen that Λ has the form described in (A3) with $j = 3$ and

$$f_1(x) = (a+bm)x,\ f_2(x) = (c+dm)x,\ f_3(x) = mx,$$

and

$$L_1(x,z) = L_2(x,z) = \frac{1}{|r|^\alpha},\ L_3(x,z) = \left|\frac{s}{r}\right|^\alpha$$

for all $x \in \mathcal{X}$ and $z := g(y) \in Y_0$ for all $y \in \mathcal{X}$. Moreover, for every $\xi, \mu \in Y^{\mathcal{X}}$ and $x, y \in \mathcal{X}$, we have

$$\begin{aligned}
\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), g(y)\|_\alpha =& \left\|\frac{1}{r}\xi((a+bm)x) + \frac{1}{r}\xi((c+dm)x) - \frac{s}{r}\xi(mx)\right.\\
&\left. - \frac{1}{r}\mu((a+bm)x) - \frac{1}{r}\mu((c+dm)x) + \frac{s}{r}\mu(mx), g(y)\right\|_\alpha\\
\leq& \frac{1}{|r|^\alpha}\|(\xi-\mu)(f_1(x)), g(y)\|_\alpha\\
&+ \frac{1}{|r|^\alpha}\|(\xi-\mu)(f_2(x)), g(y)\|_\alpha\\
&+ \left|\frac{s}{r}\right|^\alpha\|(\xi-\mu)(f_3(x)), g(y)\|_\alpha\\
=& \sum_{k=1}^{3} L_k(x, g(y))\|(\xi-\mu)(f_k(x)), g(y)\|_\alpha
\end{aligned}$$

$$=\sum_{k=1}^{3} L_k(x,g(y))\|\xi(f_k(x))-\mu(f_k(x)),g(y)\|_\alpha.$$

Therefore, the assumption (A2) is valid.

Note yet that we have

$$\varepsilon^*(x,z) := \sum_{n=0}^{\infty}\Lambda^n\varepsilon(x,y)=\varepsilon(x,z)\sum_{n=0}^{\infty}U_m^n=\frac{\varepsilon(x,z)}{1-U_m}<+\infty$$

for all $x\in\mathcal{X}$ and $z:=g(y)\in Y_0$ for all $y\in\mathcal{X}$. Consequently, in view of Theorem 2.1, there exists a unique fixed point $Q_m:\mathcal{X}\to Y$ of an operator $\mathcal{T}$, i.e.,

$$Q_m(x) := \frac{1}{r}Q_m((a+bm)x)+\frac{1}{r}Q_m((c+dm)x)-\frac{s}{r}Q_m(mx)$$

for all $x\in\mathcal{X}$ with

$$\|h(x)-Q_m(x),g(y)\|_\alpha \leq \frac{\varepsilon(x,z)}{1-U_m}$$

for all $x\in\mathcal{X}$ and $z:=g(y)\in Y_0$ for all $y\in\mathcal{X}$. Moreover, Q_m is given by the formula

$$Q_m(x):=\lim_{n\to\infty}(\mathcal{T}^n h)(x)$$

for all $x\in\mathcal{X}$.

Now, we show that for every $n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}, x_1,x_2,y\in\mathcal{X}$ with $ax_1+bx_2,\ cx_1+dx_2\in\mathcal{X}$, we have

$$\begin{aligned}&\|\mathcal{T}^n h(ax_1+bx_2)+\mathcal{T}^n h(cx_1+dx_2)-r\mathcal{T}^n h(x_1)-s\mathcal{T}^n h(x_2),g(y)\|_\alpha\\ &\leq A\times(U_m)^n\times\|x_1,y\|_\beta^p\|x_2,y\|_\beta^q.\end{aligned}\tag{3.4}$$

Clearly, if $n=0$, then (3.4) is simply (3.2). So, take $k\in\mathbb{N}_0$ and suppose that (3.4) holds for $n=k$ and every $x_1,x_2,y\in\mathcal{X}$ with $ax_1+bx_2,\ cx_1+dx_2\in\mathcal{X}$. Then,

$$\begin{aligned}&\Big\|\mathcal{T}^{k+1}h(ax_1+bx_2)+\mathcal{T}^{k+1}h(cx_1+dx_2)-r\mathcal{T}^{k+1}h(x_1)-s\mathcal{T}^{k+1}h(x_2),g(y)\|_\alpha\\
=&\Big\|\frac{1}{r}\mathcal{T}^k h((a+bm)(ax_1+bx_2))+\frac{1}{r}\mathcal{T}^k h((c+dm)(ax_1+bx_2))\\
&-\frac{s}{r}\mathcal{T}^k h(m(ax_1+bx_2))+\frac{1}{r}\mathcal{T}^k h((a+bm)(cx_1+dx_2))\\
&+\frac{1}{r}\mathcal{T}^k h((c+dm)(cx_1+dx_2))-\frac{s}{r}\mathcal{T}^k h(m(cx_1+dx_2))\\
&-r\Big(\frac{1}{r}\mathcal{T}^k h((a+bm)x_1)+\frac{1}{r}\mathcal{T}^k h((c+dm)x_1)-\frac{s}{r}\mathcal{T}^k h(mx_1)\Big)\\
&-s\Big(\frac{1}{r}\mathcal{T}^k h((a+bm)(x_2))+\frac{1}{r}\mathcal{T}^k h((c+dm)x_2)-\frac{s}{r}\mathcal{T}^k h(mx_2)\Big),g(y)\Big\|_\alpha\\
\leq&AU_m^k\Big(\frac{1}{|r|^\alpha}\|(a+bm)x_1,y\|_\beta^p\|(a+bm)x_2,y\|_\beta^q\\
&+\frac{1}{|r|^\alpha}\|(c+dm)x_1,y\|_\beta^p\|(c+dm)x_2,y\|_\beta^q+\Big|\frac{s}{r}\Big|^\alpha\|mx_1,y\|_\beta^p\|mx_2,y\|_\beta^q\Big)\\
=&AU_m^k\Big(\frac{1}{|r|^\alpha}|a+bm|^{(p+q)\beta}+\frac{1}{|r|^\alpha}|c+dm|^{(p+q)\beta}+\Big|\frac{s}{r}\Big|^\alpha m^{(p+q)\beta}\Big)\end{aligned}$$

$$\begin{aligned}&\times \|x_1, y\|_\beta^p \|x_2, y\|_\beta^q\\ =&AU_m^{k+1}\|x_1, y\|_\beta^p \|x_2, y\|_\beta^q\end{aligned}$$

for every $x_1, x_2, ax_1 + bx_2, cx_1 + dx_2, y \in \mathcal{X}$.

Thus, by induction we have shown that (3.4) holds for every $x_1, x_2 \in \mathcal{X}$ with $ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$, and $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (3.4), we obtain that Q_m satisfies

$$Q_m(ax_1 + bx_2) + Q_m(cx_1 + dx_2) = rQ_m(x_1) + sQ_m(x_2),$$

for every $x_1, x_2, ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$.

In this way, for every $m \geq \max\{n_0, m_0\}$, there exists a function Q_m such that (3.4) holds on $\mathcal{X}$ and

$$\|h(x) - Q_m(x), g(y)\|_\alpha \leq \frac{\varepsilon(x, z)}{1 - U_m}$$

for all $x \in \mathcal{X}$ and $z := g(y) \in Y_0$ for all $y \in \mathcal{X}$. From the fact that $p + q < 0$, we get that at least one of p and q must be negative and so we may assume that $q < 0$. Letting $m \to \infty$, it follows that h satisfies (1.1) on $\mathcal{X}$, i.e.,

$$h(ax_1 + bx_2) + h(cx_1 + dx_2) = rh(x_1) + sh(x_2)$$

for all $x_1, x_2, ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$. □

Theorem 3.2. *For $A, B \geq 0$, $0 < \alpha$, $\beta \leq 1$, $a, b, c, d \in \mathbb{F} \setminus \{0\}$, $r, s \in \mathbb{K} \setminus \{0\}$ and $p, q \in \mathbb{R}$ with $p, q < 0$. Assume that there exists a positive integer m_0 with*

$$nx,\ \frac{1 - an}{b}x,\ \frac{cnb + d(1 - an)}{b}x \in \mathcal{X} \tag{3.5}$$

for all $x \in \mathcal{X}, n \in \mathbb{N}$ with $n \geq m_0$ and $g : \mathcal{X} \to Y_0$ is a surjective mapping. If an operator $h : \mathcal{X} \to Y$ satisfies

$$\begin{aligned}&\|h(ax_1 + bx_2) + h(cx_1 + dx_2) - rh(x_1) - sh(x_2), g(y)\|_\alpha\\ \leq&A\|x_1, y\|_\beta^p + B\|x_2, y\|_\beta^q\end{aligned} \tag{3.6}$$

for all $x_1, x_2, ax_1 + bx_2, cx_1 + dx_2, y \in \mathcal{X}$, then h satisfies (1.1) on $\mathcal{X}$, i.e.,

$$h(ax_1 + bx_2) + h(cx_1 + dx_2) = rh(x_1) + sh(x_2) \tag{3.7}$$

for all $x_1, x_2, ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$.

Proof. Setting $x_1 = mx$, $x_2 = \frac{1-am}{b}x$ (with $m \in \mathbb{N}_{m_0} := \{m \in \mathbb{N} : m \geq m_0\}$) in (3.6) to get

$$\begin{aligned}&\Big\|rh(mx) + sh\Big(\frac{1 - am}{b}x\Big) - h\Big(\frac{cbm + d(1 - am)}{b}x\Big) - h(x), g(y)\Big\|_\alpha\\ \leq&Am^{p\beta}\|x, y\|_\beta^p + B\Big|\frac{1 - am}{b}\Big|^{q\beta}\|x, y\|_\beta^q\end{aligned} \tag{3.8}$$

for all $x, y \in \mathcal{X}$. Define an operator $\mathcal{T}_m : Y^{\mathcal{X}} \to Y^{\mathcal{X}}$ for each $\zeta \in Y^{\mathcal{X}}$ and define an operator $\Lambda_m : \mathbb{R}_+^{\mathcal{X} \times Y_0} \to \mathbb{R}_+^{\mathcal{X} \times Y_0}$ for each $\eta \in \mathbb{R}_+^{\mathcal{X} \times Y_0}$ by

$$\mathcal{T}_m\zeta(x) := r\zeta(mx) + s\zeta\Big(\frac{1 - am}{b}x\Big) - \zeta\Big(\frac{cbm + d(1 - am)}{b}x\Big)$$

for all $x \in \mathcal{X}$ and

$$\Lambda_m \eta(x,z) := |r|^\alpha \eta(mx,z) + |s|^\alpha \eta\Big(\frac{1-am}{b}x, z\Big) + \eta h\Big(\frac{cbm+d(1-am)}{b}x, z\Big)$$

for all $x \in \mathcal{X}$ and $z := g(y) \in Y_0$ for all $y \in \mathcal{X}$. In addition, let

$$\epsilon_1(x,z) := Am^{p\beta}\|x,y\|_\beta^p, \ \epsilon_2(x,z) := B\Big|\frac{1-am}{b}\Big|^{q\beta}\|x,y\|_\beta^q$$

for all $x \in \mathcal{X}$ and $z := g(y) \in Y_0$ for all $y \in \mathcal{X}$. Then, it is easily seen that Λ_m has the form described in (A3) with $j = 3$ and

$$\begin{aligned} &f_1(x) = mx, \ \ f_2(x) = \frac{1-am}{b}x, \ \ f_3(x) = \frac{cbm+d(1-am)}{b}x, \\ &L_1(x,z) = |r|^\alpha, \ \ L_2(x,z) = |s|^\alpha, \ \ L_3(x,z) = 1 \end{aligned}$$

for all $x \in \mathcal{X}$ and $z := g(y) \in Y_0$ for all $y \in \mathcal{X}$. Further, (3.8) can be written in the form

$$\|\mathcal{T}_m h(x) - h(x), g(y)\|_\alpha \le \epsilon_1(x,z) + \epsilon_2(x,z) := \epsilon(x,z)$$

for all $x \in \mathcal{X}$ and $z := g(y) \in Y_0$ for all $y \in \mathcal{X}$. Moreover, for each $\zeta, \mu \in Y^{\mathcal{X}}$, we have

$$\begin{aligned} \|\mathcal{T}_m\zeta(x) - \mathcal{T}_m\mu(x), g(y)\|_\alpha \le & |r|^\alpha \|(\zeta-\mu)(f_1(x)), g(y)\|_\alpha \\ & + |s|^\alpha \|(\zeta-\mu)(f_2(x)), g(y)\|_\alpha + \|(\zeta-\mu)(f_3(x)), g(y)\|_\alpha \\ = & \sum_{i=1}^{3} L_i(x,z)\|(\zeta-\mu)(f_i(x)), g(y)\|_\alpha \end{aligned}$$

for all $x \in \mathcal{X}$ and $z := g(y) \in Y_0$ for all $y \in \mathcal{X}$ and so the hypothesis (A2) holds.

By induction, we have

$$\Lambda_m^n \epsilon_1(x,z) = (U_m)^n \epsilon_1(x,z) \ \text{ and } \ \Lambda_m^n \epsilon_2(x,z) = (V_m)^n \epsilon_2(x,z)$$

for all $x \in \mathcal{X}$ with $z := g(y)$ for every $y \in \mathcal{X}$ and

$$\begin{cases} U_m = |r|^\alpha m^{p\beta} + |s|^\alpha |\dfrac{1-am}{b}|^{p\beta} + |\dfrac{cbm+d(1-am)}{b}|^{p\beta} \\ V_m = |r|^\alpha m^{q\beta} + |s|^\alpha |\dfrac{1-am}{b}|^{q\beta} + |\dfrac{cbm+d(1-am)}{b}|^{q\beta} \end{cases}$$

for all $m \ge m_0$. As Λ_m is a linear operator, so Λ_m^n is also linear and then

$$\begin{aligned} \Lambda_m^n \epsilon(x,z) &= \Lambda_m^n \epsilon_1(x,z) + \Lambda_m^n \epsilon_2(x,z) \\ &= (U_m)^n \epsilon_1(x,z) + (V_m)^n \epsilon_2(x,z) \end{aligned}$$

for all $x \in \mathcal{X}$ and $z := g(y) \in Y_0$ for all $y \in \mathcal{X}$.

Next, we can find $m_1 \in \mathbb{N}_{m_0}$ such that

$$U_m < 1 \ \text{ and } \ V_m < 1, \ \forall m \ge m_1.$$

Therefore, we obtain

$$\epsilon^*(x,z) := \sum_{n=0}^{\infty} \Lambda_m^n \epsilon(x,z) = \frac{\epsilon_1(x,z)}{1-U_m} + \frac{\epsilon_2(x,z)}{1-V_m} \tag{3.9}$$

for all $m \geq m_1$, $x \in \mathcal{X}$ and $z := g(y) \in Y_0$ for all $y \in \mathcal{X}$. Thus, according to Theorem 2.1, there exists a unique fixed point $Q_m : \mathcal{X} \to Y$ of an operator $\mathcal{T}_m$, i.e., $\mathcal{T}_m(Q_m) = Q_m$ for all $m \geq m_1$ such that

$$\|h(x) - Q_m(x), g(y)\|_\alpha \leq \epsilon^*(x, z)$$

for all $x \in \mathcal{X}$ and $z := g(y) \in Y_0$ for all $y \in \mathcal{X}$. Moreover, Q_m is given by the formula

$$Q_m(x) := \lim_{n\to\infty} (\mathcal{T}_m^n h)(x)$$

for all $x \in \mathcal{X}$.

To prove that Q_m satisfies (3.7), we show that

$$\begin{aligned}&\|\mathcal{T}_m^n h(ax_1 + bx_2) + \mathcal{T}_m^n h(cx_1 + dx_2) - r\mathcal{T}_m^n h(x_1) - s\mathcal{T}_m^n h(x_2), g(y)\|_\alpha \\ &\leq A \times (U_m)^n \|x_1, y\|_\beta^p + B \times (V_m)^n \|x_2, y\|_\beta^q \end{aligned} \tag{3.10}$$

for all $n \in \mathbb{N}$ and $x_1, x_2, y \in \mathcal{X}$ with $ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$. Indeed, if $n = 0$, then (3.10) is simply (3.6). So, fix $n \in \mathbb{N}$ and suppose that (3.10) holds for n and every $x_1, x_2, y \in \mathcal{X}$ with $ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$. Then we have

$$\begin{aligned}
&\Big\|\mathcal{T}_m^{n+1}\, h(ax_1 + bx_2) + \mathcal{T}_m^{n+1} h(cx_1 + dx_2) - r\mathcal{T}_m^{n+1} h(x_1) - s\mathcal{T}_m^{n+1} h(x_2), g(y)\|_\alpha \\
\leq& |r|^\alpha \|\mathcal{T}_m^n h(m(ax_1 + bx_2)) + \mathcal{T}_m^n h(m(cx_1 + dx_2)) - r\mathcal{T}_m^n h(mx_1) \\
&- s\mathcal{T}_m^n h(mx_2), g(y)\|_\alpha + |s|^\alpha \Big\|\mathcal{T}_m^n h\Big(\frac{1-am}{b}(ax_1 + bx_2)\Big) \\
&+ \mathcal{T}_m^n h\Big(\frac{1-am}{b}(cx_1 + dx_2)\Big) - r\mathcal{T}_m^n h\Big(\frac{1-am}{b}x_1\Big) - s\mathcal{T}_m^n h\Big(\frac{1-am}{b}x_2\Big), g(y)\Big\|_\alpha \\
&+ \Big\|\mathcal{T}_m^n h\Big(\frac{cmb + d(1-am)}{b}(ax_1 + bx_2)\Big) + \mathcal{T}_m^n h\Big(\frac{cmb + d(1-am)}{b}(cx_1 + dx_2)\Big) \\
&- r\mathcal{T}_m^n h\Big(\frac{cmb + d(1-am)}{b}x_1\Big) - s\mathcal{T}_m^n h\Big(\frac{cmb + d(1-am)}{b}x_2\Big), g(y)\Big\|_\alpha \\
\leq& |r|^\alpha \Big(AU_m^n \|mx_1, y\|_\beta^p + BV_m^n \|mx_2, y\|_\beta^q\Big) \\
&+ |s|^\alpha \Big(AU_m^n \Big\|\frac{1-am}{b}x_1, y\Big\|_\beta^p + BV_m^n \Big\|\frac{1-am}{b}x_2, y\Big\|_\beta^q\Big) \\
&+ AU_m^n \Big\|\frac{cmb + d(1-am)}{b}x_1, y\Big\|_\beta^p + BV_m^n \Big\|\frac{cmb + d(1-am)}{b}x_2, y\Big\|_\beta^q \\
=& AU_m^n \|x_1, y\|_\beta^p \Big(|r|^\alpha m^{p\beta} + |s|^\alpha \Big|\frac{1-am}{b}\Big|^{p\beta} + \Big|\frac{cmb + d(1-am)}{b}\Big|^{p\beta}\Big) \\
&+ BV_m^n \|x_2, y\|_\beta^q \Big(|r|^\alpha m^{q\beta} + |s|^\alpha \Big|\frac{1-am}{b}\Big|^{q\beta} + \Big|\frac{cmb + d(1-am)}{b}\Big|^{q\beta}\Big) \\
=& AU_m^{n+1} \|x_1, y\|_\beta^p + BV_m^{n+1} \|x_2, y\|_\beta^q
\end{aligned}$$

for every $x_1, x_2, y \in \mathcal{X}$ with $ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$. Thus, by induction, we have shown that (3.10) holds for every $x_1, x_2 \in \mathcal{X}$ with $ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$, $n \in \mathbb{N}$, and $m \geq m_1$. Letting $n \to \infty$ in (3.10), we obtain

$$Q_m(ax_1 + bx_2) + Q_m(cx_1 + dx_2) = rQ_m(x_1) + sQ_m(x_2) \tag{3.11}$$

for every $x_1, x_2 \in \mathcal{X}$ with $ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$.

In this way, for every $m \in \mathbb{N}_{m_1}$, we obtain a function Q_m such that (3.11) holds and

$$\|h(x) - Q_m(x), g(y)\|_\alpha \leq \frac{\epsilon_1(x,z)}{1-U_m} + \frac{\epsilon_2(x,z)}{1-V_m}$$

for all $x, y \in \mathcal{X}, m \geq m_1$ with $z := g(y)$ for all $y \in \mathcal{X}$. Since

$$\lim_{m\to\infty} \epsilon_1(x,z) = \lim_{m\to\infty} \epsilon_2(x,z) = 0$$

for all $x \in \mathcal{X}$ and $z := g(y) \in Y_0$ for all $y \in \mathcal{X}$. It follows with $m \to \infty$ that h satisfies (3.7). This completes the proof. $\square$

Remark 3.1. 1. It is clear that if $p = q$ in Theorem 3.2, the result of this theorem remains true.

2. Similar to Theorem 3.2, we can prove that equation (1.1) is φ_3-hyperstable on $\mathcal{X}$ if (with $A, B, C \geq 0$)

$$\varphi_3(x_1, x_2, y) = A\|x_1, y\|_\beta^p \times \|x_2, y\|_\beta^q + B\|x_1, y\|_\beta^p + C\|x_2, y\|_\beta^q$$

for all $x_1, x_2, y \in \mathcal{X}$ with $p, q < 0$.

Next, we derive two corollaries from Theorems 3.1 and 3.2.

Corollary 3.1. *Let $A \geq 0$, $0 < \alpha, \beta \leq 1$, $a, b, c, d \in \mathbb{F} \setminus \{0\}$, $r, s \in \mathbb{K} \setminus \{0\}$ and $p, q \in \mathbb{R}$ with $p + q < 0$, and let $G : \mathcal{X}^2 \to Y$ such that $G(u_0, v_0) \neq 0$ for some $u_0, v_0 \in \mathcal{X}$ with $au_0 + bv_0, cu_0 + dv_0 \in \mathcal{X}$. Assume that* (3.1) *holds with $m_0 \in \mathbb{N}$ and*

$$\|G(x_1, x_2), g(y)\|_\alpha \leq A\|x_1, y\|_\beta^p \|x_2, y\|_\beta^q \tag{3.12}$$

for all $x_1, x_2, ax_1 + bx_2, cx_1 + dx_2, y \in \mathcal{X}$, where $g : \mathcal{X} \to Y_0$ is a surjective mapping. The functional equation

$$h_0(ax_1 + bx_2) + h_0(cx_1 + dx_2) = rh_0(x_1) + sh_0(x_2) + G(x_1, x_2) \tag{3.13}$$

for all $x_1, x_2, ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$ has no solutions in the class of functions $h_0 : \mathcal{X} \to Y$.

Proof. Suppose that $h_0 : \mathcal{X} \to Y$ is a solution of (3.13). Define $f : E \to Y$ by $f(x) = h_0(x)$ for all $x \in \mathcal{X}$ and $f(x) = 0$ for $x \in E \setminus \mathcal{X}$. Then (3.2) is valid. Consequently, according to Theorem 3.2, f is a solution of (1.1) on $\mathcal{X}$, which means that $G(u_0, v_0) = 0$. This is a contradiction. $\square$

Corollary 3.2. *Let $A, B \geq 0$, $0 < \alpha, \beta \leq 1$, $a, b, c, d \in \mathbb{F} \setminus \{0\}$, $r, s \in \mathbb{K} \setminus \{0\}$ and $p, q \in \mathbb{R}$ with $p, q < 0$, and let $H : \mathcal{X}^2 \to Y$ such that $H(u_0, v_0) \neq 0$ for some $u_0, v_0 \in \mathcal{X}$ with $au_0 + bv_0, cu_0 + dv_0 \in \mathcal{X}$. Assume that* (3.5) *holds with $m_0 \in \mathbb{N}$ and*

$$\|H(x_1, x_2), g(y)\|_\alpha \leq A\|x_1, y\|_\beta^p + B\|x_2, y\|_\beta^q$$

for all $x_1, x_2, ax_1 + bx_2, cx_1 + dx_2, y \in \mathcal{X}$, where $g : \mathcal{X} \to Y_0$ is a surjective mapping. The functional equation

$$h_0(ax_1 + bx_2) + h_0(cx_1 + dx_2) = rh_0(x_1) + sh_0(x_2) + H(x_1, x_2)$$

for all $x_1, x_2, ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$ has no solutions in the class of functions $h_0 : \mathcal{X} \to Y$.

Theorem 3.3. *For $A \geq 0$, $0 < \alpha,\ \beta \leq 1$, $a, b, c, d \in \mathbb{F} \setminus \{0\}$, $r, s \in \mathbb{K} \setminus \{0\}$ and $p, q \in \mathbb{R}$ with $p + q > 0$. If there exist two sequences $\{e_m\}_{m\in\mathbb{N}}$, $\{f_m\}_{m\in\mathbb{N}}$ of the elements of $\mathbb{F}$ such that $\{e_m\}_{m\in\mathbb{N}}$ is bounded, $\lim_{m\to\infty} f_m = 0$, and there exists a positive integer m_0 with*

$$e_m x,\ f_m x,\ (ae_m + bf_m)x,\ (ce_m + df_m)x \in \mathcal{X}$$

for all $x \in \mathcal{X}, m \in \mathbb{N}$ with $m \geq m_0$ such that one of the conditions is satisfied

(C1) *$e_m \equiv 1$ and $\lim_{m\to\infty} \gamma_m^1 < 1$, where*

$$\gamma_m^1 := \frac{1}{|r|^\alpha}\Big(|a + f_m b|^{(p+q)\beta} + |c + f_m d|^{(p+q)\beta} + |s|^\alpha |f_m|^{(p+q)\beta}\Big);$$

(C2) *$ae_m + bf_m = ce_m + df_m = 1$ and $\lim_{m\to\infty} \gamma_m^2 < 1$, where*

$$\gamma_m^2 := \frac{1}{2^\alpha}(|r|^\alpha |e_m|^{(p+q)\beta} + |s|^\alpha f_m|^{(p+q)\beta});$$

(C3) *$ae_m + bf_m = 1, ce_m + df_m \neq 1$ and $\lim_{m\to\infty} \gamma_m^3 < 1$, where*

$$\gamma_m^3 := |ce_m + df_m|^{(p+q)\beta} + |r|^\alpha |e_m|^{(p+q)\beta} + |s|^\alpha |f_m|^{(p+q)\beta};$$

(C4) *$ae_m + bf_m \neq 1, ce_m + df_m = 1$ and $\lim_{m\to\infty} \gamma_m^4 < 1$, where*

$$\gamma_m^4 := |ae_m + bf_m|^{(p+q)\beta} + |r|^\alpha |e_m|^{(p+q)\beta} + |s|^\alpha |f_m|^{(p+q)\beta},$$

and $g : \mathcal{X} \to Y_0$ is a surjective mapping. Then every operator $h : \mathcal{X} \to Y$ fulfills (3.2) is a solution of (1.1) on $\mathcal{X}$, i.e.,

$$h(ax_1 + bx_2) + h(cx_1 + dx_2) = rh(x_1) + sh(x_2) \tag{3.14}$$

for all $x_1, x_2, ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$.

Proof. Replacing x_1 by $e_m x$ and x_2 by $f_m x$ in (3.2), for each

$$m \in \mathbb{N}_{m_0} := \{m \in \mathbb{N} : m \geq m_0\},$$

we get

$$\begin{aligned}&\|h((ae_m + bf_m)x) + h((ce_m + df_m)x) - rh(e_m x) - sh(f_m x), g(y)\|_\alpha \\ \leq &A|e_m|^{p\beta}|f_m|^{q\beta}\|x, y\|_\beta^{(p+q)}\end{aligned} \tag{3.15}$$

for all $x, y \in \mathcal{X}$. Let the case (C_i) holds, where $i \in \{1, 2, 3, 4\}$.

For $x, y \in \mathcal{X},\ \xi \in Y^{\mathcal{X}}$ and $\eta \in \mathbb{R}_+^{\mathcal{X}\times Y_0}$, we define:

$$\begin{aligned}
(\mathcal{T}_m)\xi(x) &:= k_1^i \xi((ae_m + bf_m)x) + k_2^i \xi((ce_m + df_m)x) - k_3^i r\xi(e_m x) - k_4^i s\xi(f_m x),\\
\epsilon(x, z) &:= k_0^i A|e_m|^{p\beta}|f_m|^{q\beta}\|x, y\|_\beta^{(p+q)},\\
\Lambda_m \eta(x, z) &:= |k_1^i|^\alpha \eta((ae_m + bf_m)x, z) + |k_2^i|^\alpha \eta((ce_m + df_m)x, z)\\
&\quad + |k_3^i r|^\alpha \eta(e_m x, z) + |k_4^i s|^\alpha \eta(f_m x, z),\ \text{with}\ z := g(y)\ \text{for all}\ y \in \mathcal{X},
\end{aligned}$$

where

$$k_1^1 = k_2^1 = k_4^1 = \frac{1}{r},\quad k_3^1 = 0,$$

$$k_1^2 = k_2^2 = 0, \quad k_3^2 = k_4^2 = -\frac{1}{2},$$
$$k_1^3 = 0, \quad k_2^3 = k_3^3 = k_4^3 = -1,$$
$$k_2^4 = 0, \quad k_1^4 = k_3^4 = k_4^4 = -1,$$
$$k_0^1 = \frac{1}{|r|^\alpha}, \quad k_0^2 = \frac{1}{2^\alpha}, \quad k_0^3 = k_0^4 = 1.$$

It is easy to check that (3.15) takes the form

$$\|\mathcal{T}h(x) - h(x), g(y)\|_\alpha \le \epsilon(x, z)$$

for all $x, y \in \mathcal{X}$, with $z := g(y)$, and Λ_m has the form described in (A3) and (A2) is valid for every $\xi, \mu \in Y^{\mathcal{X}}, x \in \mathcal{X}$. Next, we can find $n_0 \in \mathbb{N}$ such that $n_0 \ge m_0$ and $\gamma_m^i < 1$ for $m \in \mathbb{N}_{n_0}$. Therefore,

$$\varepsilon^*(x, z) := \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon(x, z) = \frac{\varepsilon(x, z)}{1 - \gamma_m^i},$$

for $m \ge n_0$, $x \in \mathcal{X}$ with $z := g(y)$ for all $y \in \mathcal{X}$. Hence, in view of Theorem 2.1, for each $m \in \mathbb{N}_{n_0}$ there exists a unique solution $G_m : \mathcal{X} \to Y$ of the equation

$$G_m(x) = k_1^i G_m((ae_m + bf_m)x) + k_2^i G_m((ce_m + df_m)x) - k_3^i r G_m(e_m x) - k_4^i s G_m(f_m x)$$

such that

$$\|h(x) - G_m(x), g(y)\|_\alpha \le \varepsilon^*(x, z) \tag{3.16}$$

for all $x, y \in \mathcal{X}$, with $z := g(y)$. Moreover,

$$G_m(ax_1 + bx_2) + G_m(cx_1 + dx_2) \ = rG_m(x_1) + sG_m(x_2)$$

for all $x_1, x_2, ax_1 + bx_2, cx_1 + dx_2 \in \mathcal{X}$. In this way, we obtain a sequence $(G_m)_{m \in \mathbb{N}_{n_0}}$ satisfying equation (3.14) such that (3.16) holds.

From the fact that $p + q > 0$, we get that at least one of p and q must be positive and so we may assume that $q < 0$. So, with $m \to \infty$, we find h satisfies (3.2), because

$$\lim_{m\to\infty} \varepsilon^*(x, z) = \|x, z\|^{(p+q)\beta} \lim_{m\to\infty} |f_m|^{q\beta} \frac{k_0^i A |e_m|^{p\beta}}{1 - \gamma_m^i} = 0$$

with $z := g(y)$ for all $y \in \mathcal{X}$. This means that using (3.16) h satisfies (1.1) on $\mathcal{X}$, which completes the proof. □

Remark 3.2. Theorem 3.3 remains valid if we replaced "$\{e_m\}$ is bounded and $\lim_{m\to\infty} f_m = 0$" by "$\{f_m\}$ is bounded and $\lim_{m\to\infty} e_m = 0$" and (C1) by

(Ć1) $f_m \equiv 1$ and $\lim_{m\to\infty} \gamma_m^1 < 1$, where

$$\gamma_m^1 := \frac{1}{|s|^\alpha}\Big(|ae_m + b|^{(p+q)\beta} + |ce_m + d|^{(p+q)\beta} + |r|^\alpha |e_m|^{(p+q)\beta}\Big).$$

Using Theorem 3.3, we deduce the following corollaries.

Corollary 3.3. *For $A \geq 0$, $0 < \alpha, \beta \leq 1$, $a, b, c, d \in \mathbb{F} \setminus \{0\}$, $r, s \in \mathbb{K} \setminus \{0\}$ and $p, q \in \mathbb{R}$ with $p + q > 0$ and $q > 0$. If*

$$|a|^{(p+q)\beta} + |c|^{(p+q)\beta} < |r|$$

and there exists a positive integer n_0 with

$$-\frac{a}{bm}x, \quad a(1 - \frac{1}{m})x, \quad (c - \frac{ad}{bm})x$$

for all $x \in \mathcal{X}, m \in \mathbb{N}_{n_0}$ and $g : \mathcal{X} \to Y_0$ is a surjective mapping. Then every operator $h : \mathcal{X} \to Y$ with (3.2) satisfies (1.1) on $\mathcal{X}$.

Proof. Putting $f_m = -\frac{a}{bm}$ and using Theorem 3.3 with condition (1), we have

$$\gamma_m^1 := \frac{1}{|r|^\alpha}\Big(|a(1 - \frac{1}{m})|^{(p+q)\beta} + |c - \frac{ad}{bm}|^{(p+q)\beta} + |s|^\alpha |\frac{a}{bm}|^{(p+q)\beta}\Big)$$

and hence

$$\lim_{m\to\infty} \gamma_m^1 := \frac{1}{|r|^\alpha}\Big(|a|^{(p+q)\beta} + |c|^{(p+q)\beta}\Big).$$

Therefore, the function h satisfies the equation (1.1) on $\mathcal{X}$. □

Corollary 3.4. *For $A \geq 0$, $0 < \alpha$, $\beta \leq 1$, $a, b, c, d \in \mathbb{F} \setminus \{0\}$, $r, s \in \mathbb{K} \setminus \{0\}$ and $p, q \in \mathbb{R}$ with $p + q > 0$ and $q > 0$. If*

$$a = c, \quad b = d, \quad \frac{|r|^\alpha}{2|a|^{(p+q)\beta}} < 1$$

and there exists a positive integer n_0 with

$$\frac{1}{a}(1 - \frac{1}{m})x, \quad \frac{1}{bm}x$$

for all $x \in \mathcal{X}, m \in \mathbb{N}_{n_0}$ and $g : \mathcal{X} \to Y_0$ is a surjective mapping. Then every operator $h : \mathcal{X} \to Y$ with (3.14) satisfies (1.1) on $\mathcal{X}$.

Proof. Setting $e_m = \frac{1}{a} - \frac{1}{am}, f_m = \frac{1}{bm}$ and using Theorem 3.3 with condition (2), we have

$$\gamma_m^2 := \frac{1}{2}\Big(|r|^\alpha |\frac{1}{a}(1 - \frac{1}{m})|^{(p+q)\beta} + |s|^\alpha |\frac{1}{bm}|^{(p+q)\beta}\Big)$$

and hence

$$\lim_{m\to\infty} \gamma_m^2 := \frac{|r|^\alpha}{2|a|^{(p+q)\beta}}.$$

Therefore, the function h satisfies the equation (1.1) on $\mathcal{X}$. □

4. Conclusion

In this article, we used a version of some recent fixed point theory to investigate the hyperstability of the generalized quadratic functional equation in (2,α)-Banach spaces. In other words, we prove that under some weak assumptions, the functions which satisfy the equation of interest approximately (in some sense) must be exact solutions of the such equation. In this way, we generalize several earlier outcomes. This work can be further extended as follows: one can obtain other hyperstability results using different control functions, and one can investigate the hyperstability of the functional equation of interest in (n,α)-Banach spaces for some $n \in \mathbb{N}$.

Conflicts of interest

The authors declare that no conflicts of interest for this manuscript

Data availability statement

Our manuscript has no associated data

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