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A NUMERICAL METHOD FOR TWO-DIMENSIONAL DISTRIBUTED-ORDER FRACTIONAL NONLINEAR SOBOLEV EQUATION

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Abstract This study introduces the distributed-order fractional version of the nonlinear two-dimensional Sobolev equation. The orthonormal Chebyshev cardinal polynomials are used to construct a numerical method for this equation. To this end, some derivative matrices related to these polynomials are obtained. The proposed approach turns to solve this equation into solving a nonlinear system of algebraic equations by approximating the unknown solution using the expressed polynomials and employing their derivative matrices. The applicability and validity of this method are examined by solving three examples.

Keywords Distributed-order fractional derivative, nonlinear Sobolev equation, Chebyshev cardinal polynomials, operational matrices.

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1. Introduction

More features and capabilities of fractional derivatives compared to ordinary derivatives have drawn more attention to this issue. The degree of higher freedom of order of these derivatives (which is arbitrary) along with their memory property can be mentioned as two important factors in the high use of these derivatives in modeling different problems [24]. In recent years, many problems have been formulated with the help of this type of derivatives. For instance, see [3, 26]. An important type of fractional derivatives that has received a lot of attention recently are distributed-order fractional derivatives. This type of derivative is obtained by integrating ordinary fractional derivatives with respect to their order [16, 29]. In recent years, many applications of this type of derivative have been reported in different references. For instance, some of these applications can be seen in [1, 19, 22, 23, 30]. Simultaneously with the increase in the applications of this type of derivative, many numerical methods have been presented to solve various problems modeled by these derivatives. For instance, see [2, 5, 17, 21, 25, 27, 28].

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Linear and nonlinear Sobolev equations model many applicable problems, such as the problem of humidity movement in the soil, heat flow through different materials, fluid flow through fractured rocks, propagation of long waves, etc. [7]. In recent years, different techniques have been proposed to solve the classical fractional forms of these equations. For instance, finite difference method [6], Crank-Nicolson finite volume element method [31], Crank-Nicolson finite element method [18], a hybrid technique based on the Müntz-Legender wavelets and Müntz-Legender functions [8], local discontinuous Galerkin method [32], and discrete Legendre polynomials method [9].

Because there is no study on the distributed-order fractional form of the nonlinear Sobolev equation (as far as we know), in this paper, we introduce such a fractional form of this equation and present a suitable numerical method for its solution. So, we focus on the below equation:

$$\int_0^1 \mu(\alpha)\, {}_0^C D_t^\alpha \Psi(\mathbf{x},t)d\alpha - \sigma_1 \Delta\Psi_t(\mathbf{x},t) - \sigma_2\Delta\Psi(\mathbf{x},t) + \sigma_3\nabla\left(\Psi(\mathbf{x},t)\nabla\Psi(\mathbf{x},t)\right) + \sigma_4\Psi^2(\mathbf{x},t) = g(\mathbf{x},t), \tag{1.1}$$

where $(\mathbf{x},t)\in\Omega\times[0,1]$ with $\mathbf{x}=(x,y)$ and $\Omega=[0,1]\times[0,1]$; Ψ is the unknown solution (which is assumed to be continuous), Δ is the Laplacian operator, ∇ is the gradient operator, σ $(i=1,2,3,4)$ are given constants, ${}_0^C D_t^\alpha\Psi$ is the fractional differentiation of order α with respect to temporal variable of Ψ in the Caputo form [24]. Here, $\mu:[0,1]\longrightarrow\mathbb{R}^+\cup\{0\}$ is the distribution function with the properties $0<\int_0^1\mu(\alpha)d\alpha<\infty$ [20]. Note that the above problem is assumed to have a unique continuous solution.

In recent years, basic cardinal polynomials have been used to solve many fractional problems. For instance, some of the numerical methods generated using such polynomials for fractional problems can be found in [10–13]. The two main reasons for the extensive use of these polynomials, can be found in the simplicity of calculating their fractional derivatives and the high accuracy of their approximations. The Chebyshev Cardinal polynomials (CCPs) [14] as a special family of these polynomials have attracted more attention in recent years. The reason for this can be seen in having an explicit formula for calculating the roots of Chebyshev polynomials. Because these roots are the interpolation points that are used to construct the CCPs.

In this study, we use the CCPs to solve the above problem. For this purpose, we first define two-dimensional (2D) CCPs and get their partial derivatives matrices. We also obtain a matrix for calculating the distributed-order fractional derivative of the CCPs. By expanding the solution of the problem by these polynomials and using the obtained matrices, the proposed method creates a system of algebraic equations in which the unknowns are the coefficients of the expressed expansion. By solving this system and calculating the expressed coefficients, a solution to the main problem is obtained.

This work is organized as follows: Some preparations regarding fractional derivatives are provided in Section 2. The CCPs are given in Section 3. Required matrix relationships for these polynomials are obtained in Section 4. The numerical method is explained in Section 5. In Section 6, numerical simulations are given. The conclusion of the outcomes is investigated in Section 7.

2. Preliminaries

Here, we have reviewed a few preparations that will be used in this study.

Definition 2.1 ([24]). Suppose that f is a differentiable function in its domain and $0 < \alpha \leq 1$ is a given constant. The Caputo fractional differentiation of order α of this function is defined as

$$ {}_{0}^{C}D_{t}^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-s)^{-\alpha} f'(s)ds, & 0 < \alpha < 1, \\ f'(t), & \alpha = 1. \end{cases} \tag{2.1} $$

Note that for $\alpha = 0$, we have ${}_{0}^{C}D_{t}^{0} f(t) = f(t)$.

Corollary 2.1 ([24]). *For $k \in \mathbb{N} \cup \{0\}$, we achieve*

$$ {}_{0}^{C}D_{t}^{\alpha} t^k = \begin{cases} 0, & k = 0, \\ \dfrac{k!}{\Gamma(k-\alpha+1)} t^{k-\alpha}, & k \geq 1. \end{cases} \tag{2.2} $$

Definition 2.2 ([4, 15]). An $(M+1)$-point Legendre Gauss-Lobatto quadrature integration can be defined over $[0,1]$ as follows:

$$ \int_0^1 h(t)dt \simeq \frac{1}{2} \sum_{i=0}^{M} \bar{w}_i h\left(\frac{1}{2}(\bar{t}_i + 1)\right), \tag{2.3} $$

where $\bar{t}_0 = -1$, $\bar{t}_M = 1$ and $\bar{t}_i$ $(i = 1, 2, \ldots, M-1)$ are the zeros of L'_M (where L_M is the Mth Legendre polynomial), and

$$ \bar{w}_i = \frac{2}{M(M+1)} \frac{1}{(L_M(\bar{t}_i))^2}. \tag{2.4} $$

In this work, we consider $M = 25$ in all computations.

3. Cardinal polynomials

A set containing $(m+1)$ CCPs of degree m can be generated on $[0,1]$ according to the following formula [14]:

$$ \bar{\varphi}_{m,i}(x) = \frac{1}{\lambda_i^{(m)}} \sum_{k=0}^{m} b_{ik}^{(m)} x^{m-k}, \qquad i = 0, 1, \ldots, m, \tag{3.1} $$

where

$$ \lambda_i^{(m)} = \prod_{\substack{l=0 \\ l \neq i}}^{m} (x_i - x_l), \quad b_{ik}^{(m)} = \begin{cases} 1 & k = 0, \\ -\dfrac{1}{k} \displaystyle\sum_{l=0}^{k} a_{il}^{(m)} b_{ik-l}^{(m)}, & k \neq 0, \end{cases} \quad a_{il}^{(m)} = \sum_{\substack{r=0 \\ r \neq i}}^{m} x_r^l, \tag{3.2} $$

and $x_i = \frac{1}{2}\left(1 - \cos\left(\frac{(2i+1)\pi}{2(m+1)}\right)\right)$ for $i = 0, 1, \ldots, m$. We can approximate a continuous function $\bar{h}$ defined on $[0, 1]$ by these polynomials as

$$\bar{h}(x) \simeq \sum_{i=0}^{m} \bar{h}(x_i)\, \bar{\varphi}_{m,i}(x) \triangleq \bar{\mathbf{H}}_m^{\intercal} \Phi_m(x), \tag{3.3}$$

where

$$\bar{\mathbf{H}}_m = \left[\bar{h}(x_0)\ \ \bar{h}(x_1)\ \ \ldots\ \ \bar{h}(x_m)\right]^{\intercal},$$

and

$$\bar{\Phi}_m(x) = [\bar{\varphi}_{m,0}(x)\ \bar{\varphi}_{m,1}(x)\ \ \ldots\ \ \bar{\varphi}_{m,m}(x)]^{\intercal}. \tag{3.4}$$

Also, for $m, n \in \mathbb{Z}^+$, we can generate the two variables CCPs as

$$\varphi_{mn,ij}(\mathbf{x}) = \bar{\varphi}_{m,i}(x)\bar{\varphi}_{n,j}(y), \qquad i = 0, 1, \ldots, m,\ j = 0, 1, \ldots, n. \tag{3.5}$$

In addition, for any continuous function with two variables h, we can consider the below approximation:

$$h(\mathbf{x}) \simeq \sum_{i=0}^{m}\sum_{j=0}^{n} h_{ij}\varphi_{mn,ij}(\mathbf{x}) \triangleq \mathbf{H}_{mn}^{\intercal} \Phi_{mn}(\mathbf{x}), \tag{3.6}$$

where

$$\mathbf{H}_{mn} = [h_{00}\ h_{01}\ \ \ldots\ \ h_{0n}\ h_{10}\ h_{11}\ \ \ldots\ \ h_{1n}\ \ \ldots\ \ h_{m0}\ h_{m1}\ \ \ldots\ \ h_{mn}]^{\intercal},$$

with $h_{ij} = h(x_i, y_j)$, and

$$\begin{aligned}\Phi_{mn}(\mathbf{x}) = &[\varphi_{mn,00}(\mathbf{x})\ \varphi_{mn,01}(\mathbf{x}) \ldots \varphi_{mn,0n}(\mathbf{x})\ \varphi_{mn,10}(\mathbf{x})\ \varphi_{mn,11}(\mathbf{x}) \ldots \\ &\varphi_{mn,1n}(\mathbf{x}) \ldots \varphi_{mn,m0}(\mathbf{x})\ \varphi_{mn,m1}(\mathbf{x}) \ldots \varphi_{mn,mn}(\mathbf{x})]^{\intercal}.\end{aligned} \tag{3.7}$$

Note that we can also rewrite (3.6) in the below form (for convenience):

$$h(\mathbf{x}) \simeq \sum_{l=0}^{(m+1)(n+1)-1} \tilde{h}_l \tilde{\varphi}_{mn,l}(\mathbf{x}) \triangleq \mathbf{H}_{mn}^{\intercal} \Phi_{mn}(\mathbf{x}), \tag{3.8}$$

where $\tilde{h}_l = h_{ij}$ and $\tilde{\varphi}_{mn,l}(\mathbf{x}) = \varphi_{mn,ij}(\mathbf{x})$ with $l = (n+1)\,i + j$ for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. Similarly, the CCPs can be used for approximating any three variables continuous function $\hat{h}$ on $[0, 1]^3$ as

$$\hat{h}(\mathbf{x}, t) \simeq \sum_{l=0}^{(m+1)(n+1)-1} \sum_{r=0}^{q} \hat{h}_{lr} \tilde{\varphi}_{mn,l}(\mathbf{x}) \bar{\varphi}_{q,r}(t) \triangleq \Phi_{mn}^{\intercal}(\mathbf{x}) \widehat{\mathbf{H}}_{mnq} \bar{\Phi}_q(t), \tag{3.9}$$

where

$$\widehat{\mathbf{H}}_{mnq} = \begin{pmatrix} \hat{h}_{00} & \hat{h}_{01} & \ldots & \hat{h}_{0q} \\ \hat{h}_{10} & \hat{h}_{11} & \ldots & \hat{h}_{1q} \\ \vdots & \vdots & \ldots & \vdots \\ \hat{h}_{(mn+m+n)0} & \hat{h}_{(mn+m+n)1} & \ldots & \hat{h}_{(mn+m+n)q} \end{pmatrix},$$

with $\hat{h}_{lr} = \hat{h}(\mathbf{x}_l, t_r)$, and

$$\bar{\Phi}_q(t) = [\bar{\varphi}_{q,0}(t)\ \bar{\varphi}_{q,1}(t)\ \ \ldots\ \ \bar{\varphi}_{q,q}(t)]^{\intercal}. \tag{3.10}$$

4. Matrix relationships

Here, we derive some relations related to the CCPs derivatives that will be needed later.

Theorem 4.1. *For the first- and second-order derivatives of* $\bar{\Phi}_m(x)$ *in* (3.4), *we have*

$$\begin{aligned} \frac{d\bar{\Phi}_m(x)}{dx} &= \mathbf{D}_m^{(1)}\bar{\Phi}_m(x), \\ \frac{d^2\bar{\Phi}_m(x)}{dx^2} &= \mathbf{D}_m^{(2)}\bar{\Phi}_m(x), \end{aligned} \tag{4.1}$$

where

$$\mathbf{D}_m^{(1)} = \begin{pmatrix} d_{00}^{(1)} & d_{01}^{(1)} & \dots & d_{0m}^{(1)} \\ d_{10}^{(1)} & d_{11}^{(1)} & \dots & d_{1m}^{(1)} \\ \vdots & \vdots & \dots & \vdots \\ d_{m0}^{(1)} & d_{m1}^{(1)} & \dots & d_{mm}^{(1)} \end{pmatrix}, \quad \mathbf{D}_m^{(2)} = \begin{pmatrix} d_{00}^{(2)} & d_{01}^{(2)} & \dots & d_{0m}^{(2)} \\ d_{10}^{(2)} & d_{11}^{(2)} & \dots & d_{1m}^{(2)} \\ \vdots & \vdots & \dots & \vdots \\ d_{m0}^{(2)} & d_{m1}^{(2)} & \dots & d_{mm}^{(2)} \end{pmatrix},$$

with

$$d_{ij}^{(1)} = \frac{1}{\lambda_i^{(m)}} \sum_{k=0}^{m-1} b_{ik}^{(m)}(m-k)x_j^{m-k-1},$$

and

$$d_{ij}^{(2)} = \frac{1}{\lambda_i^{(m)}} \sum_{k=0}^{m-2} b_{ik}^{(m)}(m-k)(m-k-1)x_j^{m-k-2}.$$

Proof. By computing the first- and second-order derivatives of the components of $\bar{\Phi}_m(x)$, considering (3.1), and expanding the obtained outcomes in terms of the CCPs, the expressed assertions are easily proved. □

Theorem 4.2. *The first- and second-order derivatives of* $\Phi_{mn}(\mathbf{x})$ *in* (3.7) *satisfy the below equalities:*

$$\begin{aligned} \frac{\partial\Phi_{mn}(\mathbf{x})}{\partial x} &= \mathbf{P}_{mn}^{(1)}\Phi_{mn}(\mathbf{x}), \\ \frac{\partial\Phi_{mn}(\mathbf{x})}{\partial y} &= \mathbf{Q}_{mn}^{(1)}\Phi_{mn}(\mathbf{x}), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \frac{\partial^2\Phi_{mn}(\mathbf{x})}{\partial x^2} &= \mathbf{P}_{mn}^{(2)}\Phi_{mn}(\mathbf{x}), \\ \frac{\partial^2\Phi_{mn}(\mathbf{x})}{\partial y^2} &= \mathbf{Q}_{mn}^{(2)}\Phi_{mn}(\mathbf{x}), \end{aligned} \tag{4.3}$$

where

$$\mathbf{P}_{mn}^{(1)} = \mathbf{D}_m^{(1)} \otimes \mathbf{I}_n = \begin{pmatrix} d_{00}^{(1)}\mathbf{I}_n & d_{01}^{(1)}\mathbf{I}_n & \dots & d_{0m}^{(1)}\mathbf{I}_n \\ d_{10}^{(1)}\mathbf{I}_n & d_{11}^{(1)}\mathbf{I}_n & \dots & d_{1m}^{(1)}\mathbf{I}_n \\ \vdots & \vdots & \dots & \vdots \\ d_{m0}^{(1)}\mathbf{I}_n & d_{m1}^{(1)}\mathbf{I}_n & \dots & d_{mm}^{(1)}\mathbf{I}_n \end{pmatrix},$$

$$\mathbf{Q}_{mn}^{(1)} = \begin{pmatrix} \mathbf{D}_n^{(1)} & \mathbf{O}_n & \mathbf{O}_n & \dots & \mathbf{O}_n & \mathbf{O}_n \\ \mathbf{O}_n & \mathbf{D}_n^{(1)} & \mathbf{O}_n & \dots & \mathbf{O}_n & \mathbf{O}_n \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \mathbf{O}_n & \mathbf{O}_n & \mathbf{O}_n & \dots & \mathbf{D}_n^{(1)} & \mathbf{O}_n \\ \mathbf{O}_n & \mathbf{O}_n & \mathbf{O}_n & \dots & \mathbf{O}_n & \mathbf{D}_n^{(1)} \end{pmatrix},$$

and

$$\mathbf{P}_{mn}^{(2)} = \mathbf{D}_m^{(2)} \otimes \mathbf{I}_n = \begin{pmatrix} d_{00}^{(2)}\mathbf{I}_n & d_{01}^{(2)}\mathbf{I}_n & \dots & d_{0m}^{(2)}\mathbf{I}_n \\ d_{10}^{(2)}\mathbf{I}_n & d_{11}^{(2)}\mathbf{I}_n & \dots & d_{1m}^{(2)}\mathbf{I}_n \\ \vdots & \vdots & \dots & \vdots \\ d_{m0}^{(2)}\mathbf{I}_n & d_{m1}^{(2)}\mathbf{I}_n & \dots & d_{mm}^{(2)}\mathbf{I}_n \end{pmatrix},$$

$$\mathbf{Q}_{mn}^{(2)} = \begin{pmatrix} \mathbf{D}_n^{(2)} & \mathbf{O}_n & \mathbf{O}_n & \dots & \mathbf{O}_n & \mathbf{O}_n \\ \mathbf{O}_n & \mathbf{D}_n^{(2)} & \mathbf{O}_n & \dots & \mathbf{O}_n & \mathbf{O}_n \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \mathbf{O}_n & \mathbf{O}_n & \mathbf{O}_n & \dots & \mathbf{D}_n^{(2)} & \mathbf{O}_n \\ \mathbf{O}_n & \mathbf{O}_n & \mathbf{O}_n & \dots & \mathbf{O}_n & \mathbf{D}_n^{(2)} \end{pmatrix},$$

in which $\mathbf{P}_{mn}^{(l)}$ *and* $\mathbf{Q}_{mn}^{(l)}$ *for* $l = 1, 2$ *are* $(m+1)(n+1)$*-order square matrices,* $\mathbf{D}_m^{(l)}$ *and* $\mathbf{D}_n^{(l)}$ *for* $l = 1, 2$ *are the matrices derived in Theorem 4.1,* $\otimes$ *denotes the Kronecker product,* $\mathbf{O}_n$ *is an* $(n+1)$*-order zero matrix and* $\mathbf{I}_n$ *is an* $(n+1)$*-order identity matrix.*

Proof. The proof is straightforward by considering the previous Theorem. So, we leave it to the reader. □

Theorem 4.3. *The fractional derivative of order* $0 \leq \alpha \leq 1$ *of the vector* $\bar{\Phi}_q(t)$ *in*

(3.10) *can be represented as*

$$ {}_{0}^{C}D_t^{\alpha}\bar{\Phi}_q(t) \simeq \mathbf{\Theta}_q^{(\alpha)}\bar{\Phi}_q(t), \tag{4.4} $$

where

$$ \mathbf{\Theta}_q^{(\alpha)} = \begin{pmatrix} \theta_{00}^{(\alpha)} & \theta_{01}^{(\alpha)} & \dots & \theta_{0q}^{(\alpha)} \\ \theta_{10}^{(\alpha)} & \theta_{11}^{(\alpha)} & \dots & \theta_{1q}^{(\alpha)} \\ \vdots & \vdots & \dots & \vdots \\ \theta_{q0}^{(\alpha)} & \theta_{q1}^{(\alpha)} & \dots & \theta_{qq}^{(\alpha)} \end{pmatrix}, $$

and

$$ \theta_{ij}^{(\alpha)} = \begin{cases} \dfrac{1}{\lambda_i^{(q)}} \displaystyle\sum_{k=0}^{q} b_{ik}^{(q)} t_j^{q-k}, & \alpha = 0, \\ \dfrac{1}{\lambda_i^{(q)}} \displaystyle\sum_{k=0}^{q-1} \dfrac{b_{ik}^{(q)}(q-k)!}{\Gamma(q-k-\alpha+1)} t_j^{q-k-\alpha}, & 0 < \alpha \leq 1. \end{cases} $$

Proof. We have

$$ {}_{0}^{C}D_t^{\alpha}\bar{\Phi}_q(t) = \begin{pmatrix} {}_{0}^{C}D_t^{\alpha}\bar{\varphi}_{q,0}(t) \\ {}_{0}^{C}D_t^{\alpha}\bar{\varphi}_{q,1}(t) \\ \vdots \\ {}_{0}^{C}D_t^{\alpha}\bar{\varphi}_{q,q}(t) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\lambda_0^{(q)}} \displaystyle\sum_{k=0}^{q} b_{0k}^{(q)}\, {}_{0}^{C}D_t^{\alpha} t^{q-k} \\ \dfrac{1}{\lambda_1^{(q)}} \displaystyle\sum_{k=0}^{q} b_{1k}^{(q)}\, {}_{0}^{C}D_t^{\alpha} t^{q-k} \\ \vdots \\ \dfrac{1}{\lambda_q^{(q)}} \displaystyle\sum_{k=0}^{q} b_{qk}^{(q)}\, {}_{0}^{C}D_t^{\alpha} t^{q-k} \end{pmatrix}. \tag{4.5} $$

Hence, for $\alpha = 0$, we get

$$ {}_{0}^{C}D_t^{\alpha}\bar{\Phi}_q(t) = \begin{pmatrix} \dfrac{1}{\lambda_0^{(q)}} \displaystyle\sum_{k=0}^{q} b_{0k}^{(q)} t^{q-k} \\ \dfrac{1}{\lambda_1^{(q)}} \displaystyle\sum_{k=0}^{q} b_{1k}^{(q)} t^{q-k} \\ \vdots \\ \dfrac{1}{\lambda_q^{(q)}} \displaystyle\sum_{k=0}^{q} b_{qk}^{(q)} t^{q-k} \end{pmatrix}, \tag{4.6} $$

and for $0 < \alpha \leq 1$, from the property given in (2.1), we obtain

$$ {}_0^C D_t^\alpha \bar{\Phi}_q(t) = \begin{pmatrix} \frac{1}{\lambda_0^{(q)}} \sum_{k=0}^{q-1} \frac{b_{0k}^{(q)}(q-k)!}{\Gamma(q-k-\alpha+1)} t^{q-k-\alpha} \\ \frac{1}{\lambda_1^{(q)}} \sum_{k=0}^{q-1} \frac{b_{1k}^{(q)}(q-k)!}{\Gamma(q-k-\alpha+1)} t^{q-k-\alpha} \\ \vdots \\ \frac{1}{\lambda_q^{(q)}} \sum_{k=0}^{q-1} \frac{b_{qk}^{(q)}(q-k)!}{\Gamma(q-k-\alpha+1)} t^{q-k-\alpha} \end{pmatrix}. \tag{4.7} $$

So, by approximating the results extracted for ${}_0^C D_t^\alpha \bar{\Phi}_q(t)$ in (4.6) and (4.7) via the CCPs, we get

$$ {}_0^C D_t^\alpha \bar{\Phi}_q(t) \simeq \begin{pmatrix} \theta_{00}^{(\alpha)} & \theta_{01}^{(\alpha)} & \dots & \theta_{0q}^{(\alpha)} \\ \theta_{10}^{(\alpha)} & \theta_{11}^{(\alpha)} & \dots & \theta_{1q}^{(\alpha)} \\ \vdots & \vdots & \dots & \vdots \\ \theta_{q0}^{(\alpha)} & \theta_{q1}^{(\alpha)} & \dots & \theta_{qq}^{(\alpha)} \end{pmatrix} \bar{\Phi}_q(t) \triangleq \mathbf{\Theta}_q^{(\alpha)} \bar{\Phi}_q(t), \tag{4.8} $$

where

$$ \theta_{ij}^{(\alpha)} = \begin{cases} \frac{1}{\lambda_i^{(q)}} \sum_{k=0}^{q} b_{ik}^{(q)} t_j^{q-k}, & \alpha = 0, \\ \frac{1}{\lambda_i^{(q)}} \sum_{k=0}^{q-1} \frac{b_{ik}^{(q)}(q-k)!}{\Gamma(q-k-\alpha+1)} t_j^{q-k-\alpha}, & 0 < \alpha \leq 1, \end{cases} $$

which completes the proof. □

Theorem 4.4. *The distributed-order fractional derivative of $\bar{\Phi}_q(t)$ in (3.10) can be approximated as*

$$ \int_0^1 \mu(\alpha)\, {}_0^C D_t^\alpha \bar{\Phi}_q(t) d\alpha \simeq \mathbf{S}_q \bar{\Phi}_q(t), \tag{4.9} $$

where

$$ \mathbf{S}_q = \begin{pmatrix} s_{00} & z_{01} & \dots & s_{0q} \\ s_{10} & s_{11} & \dots & s_{1q} \\ \vdots & \vdots & \dots & \vdots \\ s_{q0} & s_{q1} & \dots & s_{qq} \end{pmatrix}, \tag{4.10} $$

and $s_{ij} = \frac{1}{2} \sum_{r=0}^{M} \bar{w}_r \mu\left(\frac{1}{2}(\bar{t}_r + 1)\right) \theta_{ij}^{\left(\frac{1}{2}(\bar{t}_r+1)\right)}$*, in which* $\theta_{ij}^{(\alpha)}$ *is defined in Theorem 4.3.*

Proof. By employing the results obtained in Theorem 4.3, we have

$$\int_0^1 \mu(\alpha)\,{}_0^C D_t^{\alpha} \bar{\Phi}_q(t) d\alpha \simeq \left(\int_0^1 \mu(\alpha)\mathbf{\Theta}_q^{(\alpha)} d\alpha\right) \bar{\Phi}_q(t) \triangleq \mathbf{S}_q \bar{\Phi}_q(t),$$

where $\mathbf{S}_q$ is in the form of (4.10), and its elements are calculated as

$$s_{ij} = \int_0^1 \mu(\alpha)\theta_{ij}^{(\alpha)} d\alpha. \tag{4.11}$$

By evaluating the integrals in (4.11) using an $(M+1)$-point Legendre Gauss-Lobatto quadrature method, we have

$$s_{ij} = \frac{1}{2}\sum_{r=0}^{M} \bar{w}_r \mu\left(\frac{1}{2}\left(\bar{t}_r + 1\right)\right) \theta_{ij}^{\left(\frac{1}{2}(\bar{t}_r+1)\right)},$$

which ends the proof. □

5. The proposed method

In this section, we use the CCPs to solve the problem expressed in (1.1) under the below conditions:

$$\Psi(\mathbf{x}, 0) = \bar{\Psi}_0(\mathbf{x}), \tag{5.1}$$

and

$$\begin{aligned} &\Psi(0, y, t) = \bar{\Psi}_1(y, t),\ \Psi(1, y, t) = \bar{\Psi}_2(y, t), \\ &\Psi(x, 0, t) = \bar{\Psi}_3(x, t),\ \Psi(x, 1, t) = \bar{\Psi}_4(x, t), \end{aligned} \tag{5.2}$$

where $\bar{\Psi}_l$, $l = 0, 1, \ldots, 4$ are given functions (that are assumed to be continuous). To this aim, we assume that

$$\Psi(\mathbf{x}, t) \simeq \sum_{l=0}^{(m+1)(n+1)-1} \sum_{r=0}^{q} \psi_{lr} \tilde{\varphi}_{mn,l}(\mathbf{x}) \bar{\varphi}_{q,r}(t) \triangleq \Phi_{mn}^{\intercal}(\mathbf{x}) \widehat{\mathbf{\Psi}}_{mnq} \bar{\Phi}_q(t), \tag{5.3}$$

where

$$\widehat{\mathbf{\Psi}}_{mnq} = \begin{pmatrix} \psi_{00} & \psi_{01} & \cdots & \psi_{0q} \\ \psi_{10} & \psi_{11} & \cdots & \psi_{1m_3} \\ \vdots & \vdots & \ldots & \vdots \\ \psi_{(mn+m+n)0} & \psi_{(mn+m+n)1} & \cdots & \psi_{(mn+m+n)q} \end{pmatrix}.$$

Relation (5.3) and Theorems 4.1 and 4.2 yield

$$\nabla\Psi(\mathbf{x}, t) \simeq \Phi_{mn}^{\intercal}(\mathbf{x}) \left[\left(\mathbf{P}_{mn}^{(1)}\right)^{\intercal} + \left(\mathbf{Q}_{mn}^{(1)}\right)^{\intercal}\right] \widehat{\mathbf{\Psi}}_{mnq} \bar{\Phi}_q(t), \tag{5.4}$$

and

$$\Delta\Psi(\mathbf{x},t) \simeq \mathit{\Phi}^{\intercal}_{mn}(\mathbf{x})\left[\left(\mathbf{P}^{(2)}_{mn}\right)^{\intercal} + \left(\mathbf{Q}^{(2)}_{mn}\right)^{\intercal}\right]\widehat{\mathbf{\Psi}}_{mnq}\bar{\mathit{\Phi}}_q(t),$$
$$\Delta\Psi_t(\mathbf{x},t) \simeq \mathit{\Phi}^{\intercal}_{mn}(\mathbf{x})\left[\left(\mathbf{P}^{(2)}_{mn}\right)^{\intercal} + \left(\mathbf{Q}^{(2)}_{mn}\right)^{\intercal}\right]\widehat{\mathbf{\Psi}}_{mnq}\mathbf{D}^{(1)}_q\bar{\mathit{\Phi}}_q(t). \tag{5.5}$$

Also, using the cardinal property of the CCPs and (5.3) and (5.4), we obtain

$$\Psi^2(\mathbf{x},t) \simeq \mathit{\Phi}^{\intercal}_{mn}(\mathbf{x})\mathbf{U}_{mnq}\bar{\mathit{\Phi}}_q(t), \tag{5.6}$$

where

$$\mathbf{U}_{mnq} = \begin{pmatrix} \psi^2_{00} & \psi^2_{01} & \cdots & \psi^2_{0q} \\ \psi^2_{10} & \psi^2_{11} & \cdots & \psi^2_{1m_3} \\ \vdots & \vdots & \cdots & \vdots \\ \psi^2_{(mn+m+n)0} & \psi^2_{(mn+m+n)1} & \cdots & \psi^2_{(mn+m+n)q} \end{pmatrix},$$

and

$$\Psi(\mathbf{x},t)\nabla\Psi(\mathbf{x},t) \simeq \mathit{\Phi}^{\intercal}_{mn}(\mathbf{x})\mathbf{V}_{mnq}\bar{\mathit{\Phi}}_q(t), \tag{5.7}$$

where

$$\mathbf{V}_{mnq} = \begin{pmatrix} v_{00} & v_{01} & \cdots & v_{0q} \\ v_{10} & v_{11} & \cdots & v_{1m_3} \\ \vdots & \vdots & \cdots & \vdots \\ v_{(mn+m+n)0} & v_{(mn+m+n)1} & \cdots & v_{(mn+m+n)q} \end{pmatrix},$$

and

$$v_{ij} = \left[\widehat{\mathbf{\Psi}}_{mnq}\right]_{ij}\left[\left(\left(\mathbf{P}^{(1)}_{mn}\right)^{\intercal} + \left(\mathbf{Q}^{(1)}_{mn}\right)^{\intercal}\right)\widehat{\mathbf{\Psi}}_{mnq}\right]_{ij}.$$

Theorem 4.2 and (5.7) result in

$$\nabla\left(\Psi(\mathbf{x},t)\nabla\Psi(\mathbf{x},t)\right) \simeq \mathit{\Phi}^{\intercal}_{mn}(\mathbf{x})\left[\left(\mathbf{P}^{(1)}_{mn}\right)^{\intercal} + \left(\mathbf{Q}^{(1)}_{mn}\right)^{\intercal}\right]\mathbf{V}_{mnq}\bar{\mathit{\Phi}}_q(t). \tag{5.8}$$

Moreover, from (5.3) and Theorem 4.4, we have

$$\int_0^1 \mu(\alpha)\,{}^C_0D^{\alpha}_t\Psi(\mathbf{x},t)d\alpha \simeq \mathit{\Phi}^{\intercal}_{mn}(\mathbf{x})\widehat{\mathbf{\Psi}}_{mnq}\mathbf{S}_q\bar{\mathit{\Phi}}_q(t). \tag{5.9}$$

Substituting (5.5), (5.6), (5.8), and (5.9) into (1.1) gives

$$\begin{aligned}&\mathit{\Phi}^{\intercal}_{mn}(\mathbf{x})\Big\{\widehat{\mathbf{\Psi}}_{mnq}\mathbf{S}_q - \sigma_1\left[\left(\mathbf{P}^{(2)}_{mn}\right)^{\intercal} + \left(\mathbf{Q}^{(2)}_{mn}\right)^{\intercal}\right]\widehat{\mathbf{\Psi}}_{mnq}\mathbf{D}^{(1)}_q \\ &-\sigma_2\left[\left(\mathbf{P}^{(2)}_{mn}\right)^{\intercal} + \left(\mathbf{Q}^{(2)}_{mn}\right)^{\intercal}\right] + \sigma_3\left[\left(\mathbf{P}^{(1)}_{mn}\right)^{\intercal} + \left(\mathbf{Q}^{(1)}_{mn}\right)^{\intercal}\right]\mathbf{V}_{mnq} \\ &+\sigma_4\mathbf{U}_{mnq}\Big\}\,\bar{\mathit{\Phi}}_q(t) - g(\mathbf{x},t) \triangleq R(\mathbf{x},t) \simeq 0.\end{aligned} \tag{5.10}$$

Furthermore, using (5.1)-(5.3), we get

$$\Phi_{mn}^{\intercal}(\mathbf{x})\widehat{\mathbf{\Psi}}_{mnq}\bar{\Phi}_q(0) - \bar{\Psi}_0(\mathbf{x}) \triangleq \Pi_0(\mathbf{x}) \simeq 0, \tag{5.11}$$

and

$$\begin{aligned}
&\Phi_{mn}^{\intercal}(0,y)\widehat{\mathbf{\Psi}}_{mnq}\bar{\Phi}_q(t) - \bar{\Psi}_1(y,t) \triangleq \Pi_1(y,t) \simeq 0,\\
&\Phi_{mn}^{\intercal}(1,y)\widehat{\mathbf{\Psi}}_{mnq}\bar{\Phi}_q(t) - \bar{\Psi}_2(y,t) \triangleq \Pi_2(y,t) \simeq 0,\\
&\Phi_{mn}^{\intercal}(x,0)\widehat{\mathbf{\Psi}}_{mnq}\bar{\Phi}_q(t) - \bar{\Psi}_3(x,t) \triangleq \Pi_3(x,t) \simeq 0,\\
&\Phi_{mn}^{\intercal}(x,1)\widehat{\mathbf{\Psi}}_{mnq}\bar{\Phi}_q(t) - \bar{\Psi}_4(x,t) \triangleq \Pi_4(x,t) \simeq 0.
\end{aligned} \tag{5.12}$$

Eventually, using (5.10)-(5.12), we extract the system

$$\begin{cases}
R\left(x_i, y_j, t_l\right) = 0, & 2 \le i \le m,\ 2 \le j \le n,\ 2 \le l \le q+1,\\
\Pi_0\left(x_i, y_j\right) = 0, & 1 \le i \le m+1,\ 1 \le j \le n+1,\\
\Pi_r\left(y_j, t_l\right) = 0, & r = 1,2,\ 1 \le j \le n+1,\ 2 \le l \le q+1,\\
\Pi_r\left(x_i, t_l\right) = 0, & r = 3,4,\ 2 \le i \le m,\ 2 \le l \le q+1,
\end{cases} \tag{5.13}$$

where

$$\begin{aligned}
x_i &= \frac{1}{2}\left(1 - \cos\left(\frac{(2i-1)\pi}{2\,(m+1)}\right)\right),\\
y_j &= \frac{1}{2}\left(1 - \cos\left(\frac{(2j-1)\pi}{2\,(n+1)}\right)\right),\\
t_l &= \frac{1}{2}\left(1 - \cos\left(\frac{(2l-1)\pi}{2\,(q+1)}\right)\right).
\end{aligned}$$

By solving the above system and determining $\widehat{\mathbf{\Psi}}_{mnq}$, we get a solution for the main problem using (5.3). Note that this system is solved by "*fsolve*" command of Maple 18 (with a precision 25 decimal digits).

6. Numerical examples

In this section, we inquire about the accuracy of the expressed approach on two examples. The following formulas are applied to evaluate the accuracy of the outcomes:

$$E_2 = \left(\int_0^1 \int_0^1 \left(\Psi(\mathbf{x},1) - \tilde{\Psi}(\mathbf{x},1)\right)^2 dxdy\right)^{1/2},$$

$$E_\infty = \max_{\mathbf{x}\in\Omega}\left|\Psi(\mathbf{x},1) - \tilde{\Psi}(\mathbf{x},1)\right|,$$

in which Ψ is the true solution, and $\tilde{\Psi}$ is the numerical solution. The order of convergence (CO) of the established scheme is computed as

$$CO = \left|\log\left(\frac{\varepsilon_2}{\varepsilon_1}\right)\right| \Big/ \log\left(\frac{\mathcal{N}_2}{\mathcal{N}_1}\right),$$

where ε_1 and ε_2 are respectively the values of E_∞ obtained in the first and second implementations. Here, $\mathcal{N}_i = (m_i+1)(n_i+1)(q_i+1)$ for $i = 1, 2$ is the number of the CCPs.

Example 6.1. Consider the problem (1.1) with $\mu(\alpha) = \Gamma(4-\alpha)$, $\sigma_1 = \sigma_2 = 2, \sigma_3 = 0$, $\sigma_4 = 1$, and

$$g(\mathbf{x}, t) = \left(\frac{6t^2(t-1)}{\ln(t)} + \frac{2t\,(3t\ln(t) - 2\ln(t) - t + 1)}{\ln^2(t)} - 4t^3 - 16t^2 - 8t\right) e^{x-y} + (t^3+t^2)^2 e^{2(x-y)}.$$

The exact solution $\Psi(\mathbf{x}, t) = \left(t^3+t^2\right) e^{x-y}$ can be used to obtain other information. The outcomes derived from the presented approach are shown in Table 1. The high accuracy of the outcomes can be deduced from these results. We can also observe the high order of convergence of the results from this table. The graphical behaviors of the outcomes for $(m = n = q = 8)$ at the final time are displayed in Figure 1.

Table 1. The errors and CO regarding the outcomes of Example 6.1.

(m,n,q)	$(4,4,4)$	$(5,5,5)$	$(6,6,6)$	$(7,7,7)$	$(8,8,8)$
E_2	5.2195×10^{-05}	2.8578×10^{-06}	8.0147×10^{-08}	3.3125×10^{-09}	7.1606×10^{-11}
E_∞	1.7541×10^{-04}	7.6081×10^{-06}	2.7977×10^{-07}	8.9857×10^{-09}	2.4802×10^{-10}
CO	–	4.6874	6.0387	7.4350	8.9613

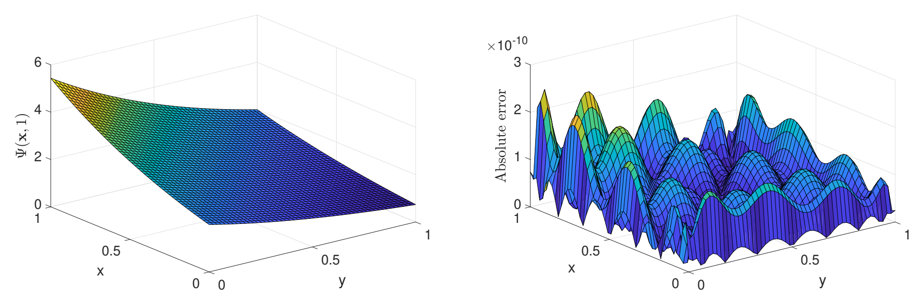

Figure 1. The outcomes obtained for $\Psi(\mathbf{x}, 1)$ (up) and related absolute error (down) with $(m = n = q = 8)$ in Example 6.1.

Example 6.2. Consider the problem (1.1) with $\mu(\alpha) = \Gamma(4-\alpha)$, $\sigma_i = 1$ for $i = 1, 2, 3, 4$, and

$$g(\mathbf{x}, t) = \left(\frac{6t^2(t-1)}{\ln(t)} + 6t^2 + 2t^3\right)\sin(x)\cos(y) + t^9\left(\cos(x)\cos(y) - \sin(x)\sin(y)\right)^2 - t^6\sin^2(x)\cos^2(y).$$

Other information can be extracted of the true solution $\Psi(\mathbf{x}, t) = t^3\sin(x)\cos(y)$. Table 2 shows the errors and CO of the results obtained by the expressed method. These outcomes confirm the high accuracy of the scheme. For the case $(m = n = 8, q = 7)$ the extracted results are illustrated in Figure 2.

Table 2. The errors and CO regarding the outcomes of Example 6.2.

(m,n,q)	$(4,4,3)$	$(5,5,4)$	$(6,6,5)$	$(7,7,6)$	$(8,8,7)$
E_2	1.3284×10^{-05}	5.5694×10^{-07}	2.0491×10^{-08}	6.4836×10^{-10}	1.8242×10^{-11}
E_∞	3.0694×10^{-05}	1.4564×10^{-06}	5.3950×10^{-08}	1.6827×10^{-09}	5.2103×10^{-11}
CO	–	4.1528	5.6069	7.0678	8.2498

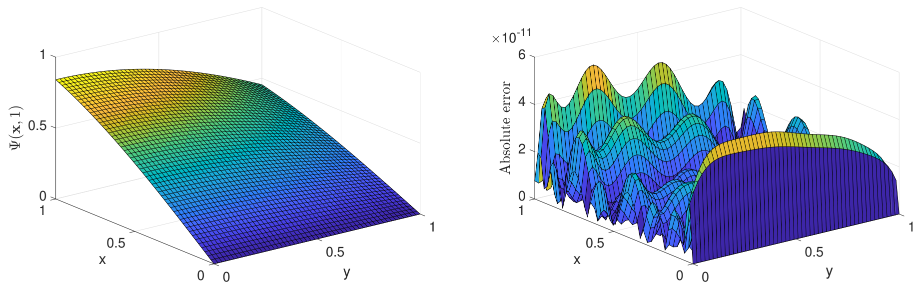

Figure 2. The outcomes obtained for $\Psi(\mathbf{x},1)$ (up) and related absolute error (down) with $(m=n=8, q=7)$ in Example 6.2.

Example 6.3. Consider the problem (1.1) with $\mu(\alpha)=\Gamma(5-\alpha)$ and $\sigma_1=\sigma_2=2$, $\sigma_3=\sigma_4=1$. The right hand function $g(\mathbf{x},t)$ and other information can be derived from the exact solution $\Psi(\mathbf{x},t)=t^4e^{x-y}\sin(\pi x)\sin(\pi y)$. Table 3 is used to provide the results obtained by the explained methodology. The high convergence of the outcomes can be observed in this table. In the case of $(m=n=9, q=8)$, the derived results are displayed in Figure 3.

Table 3. The errors and CO regarding the outcomes of Example 6.3.

(m,n,q)	$(5,5,4)$	$(6,6,5)$	$(7,7,6)$	$(8,8,7)$	$(9,9,8)$
E_2	3.2902×10^{-04}	1.0073×10^{-04}	1.1441×10^{-05}	5.1660×10^{-07}	1.1738×10^{-07}
E_∞	9.1611×10^{-04}	2.8646×10^{-04}	2.2830×10^{-05}	1.4850×10^{-06}	2.4117×10^{-07}
CO	–	0.8249	4.8049	6.4875	4.9246

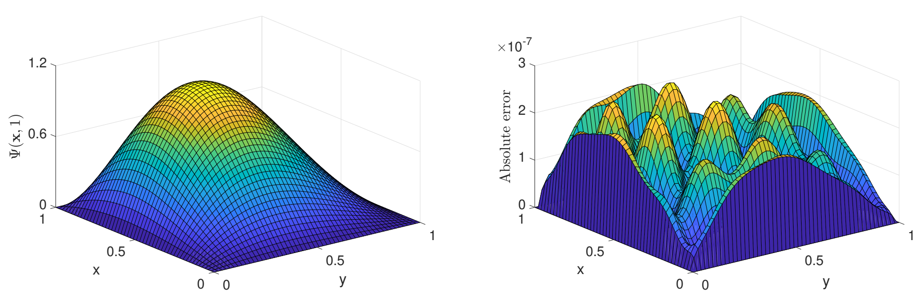

Figure 3. The outcomes obtained for $\Psi(\mathbf{x},1)$ (up) and related absolute error (down) with $(m=n=9, q=8)$ in Example 6.3.

7. Conclusion

The distributed-order fractional form of nonlinear 2D Sobolev equation was defined in this study. The orthonormal CCPs were reviewed, and some derivative matrices were obtained for them. A matrix approach was established based on these functions for this equation. In the expressed method, by approximating the unknown solution using these polynomials and employing the mentioned matrices, a nonlinear algebraic system was extracted and solved to obtain a solution for the main equation. By solving three examples, the high accuracy of the scheme was shown. Note that the expressed approach can be easily adopted for other 2D fractional linear and nonlinear fractional problems.

Data Availability

Data sharing is not applicable to this study.

Conflict of Interest

The authors declare that there is not any conflict of interest regarding this paper.

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