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OSCILLATION OF SECOND-ORDER HALF-LINEAR NEUTRAL NONCANONICAL DYNAMIC EQUATIONS

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Abstract In this paper, we shall establish some new criteria for the oscillation of certain second-order noncanonical dynamic equations with a sublinear neutral term. This task is accomplished by reducing the involved nonlinear dynamic equation to a second-order linear dynamic inequality. We also establish some new oscillation theorems involving certain integral conditions. Three examples, illustrating our results, are presented. Our results generalize results for corresponding differential and difference equations.

Keywords Half-linear dynamic equation, delay, second-order, noncanonical, oscillation.

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1. Introduction

The usual notation and terminology for time scales as can be found in Bohner and Peterson [10] will be used throughout. Here, we are concerned with obtaining some new criteria for the oscillation of second-order half-linear dynamic equations with a sublinear neutral term of the form

$$\left(a\left(y^{\Delta}\right)^{\alpha}\right)^{\Delta}(t)+q(t)f(x(g(t)))=0, \quad t\geq t_0, \tag{E}$$

where $y(t)=x(t)+p(t)x^{\beta}(\delta(t))$. For an arbitrary time scale $\mathbb{T}$ (i.e., a nonempty closed subset of the real numbers) with $\sup\mathbb{T}=\infty$, we set $[t_0,\infty)_{\mathbb{T}}=[t_0,\infty)\cap\mathbb{T}$, and we assume throughout that

(H$_1$) $a,p,q\in \mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$;

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(H$_2$) $g, \delta \in \mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$ are nondecreasing with $\delta(t) \le t$, $g(t) \le t$, and

$$\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} g(t) = \infty;$$

(H$_3$) $\alpha \ge 1$ and $0 < \beta \le 1$ are ratios of positive odd integers;

(H$_4$) $f \in \mathrm{C}(\mathbb{R},\mathbb{R})$ and $f(x)/x^{\alpha} \ge M > 0$ for $x \ne 0$.

Our presented results will also employ the hypotheses

$$A(t) := \int_t^{\infty} \frac{\Delta s}{a^{1/\alpha}(s)} < \infty \quad \text{for all} \quad t \ge t_0; \tag{H$_5$}$$

$$\int_{t_0}^{\infty} q(s)\Delta s = \infty; \tag{H$_6$}$$

$$\lim_{t\to\infty} \frac{p(t)A^{\beta}(\delta(t))}{A(t)} = 0; \tag{H$_7$}$$

$$\lim_{t\to\infty} p(t) = 0. \tag{H$_8$}$$

Recall that a solution of (E) is a nontrivial real-valued function x satisfying (E) for sufficiently large t. Solutions vanishing identically in some neighborhood of infinity will be excluded from our consideration. A solution x of (E) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. An equation itself is said to be oscillatory if all its solutions are oscillatory.

The problem of investigating oscillation criteria for various types of dynamic equations has been a very active research area over the past two decades, see [2, 5, 11, 30]. A large number of papers and monographs has been devoted to this problem; for some recent contributions, we refer to [1, 3, 6, 13, 14, 16–18, 25–29, 31] and the references contained therein. In particular, oscillatory behavior of solutions to half-linear equations has been the subject of numerous studies; see, e.g., the papers [7–9, 20–23] for more details. We point out that analysis of qualitative behavior of half-linear equations is important not only for the further development of oscillation theory, but for practical reasons too since half-linear equations have numerous applications in the study of p-Laplace equations, and so forth; see, e.g., the papers [7,8,19,24] (we also refer to [19,24] for models from mathematical biology, where oscillation and/or delay actions may be formulated by means of cross-diffusion terms).

The purpose of this paper is to provide some new oscillation criteria for (E) in noncanonical form, i.e., satisfying (H$_5$), via a comparison with second-order linear dynamic inequalities. We also establish new theorems involving integral conditions that ensure the oscillation of (E). For related results in the case of corresponding differential equations, i.e., $\mathbb{T} = \mathbb{R}$, we refer to [12, 20, 21]. For related results in the case of corresponding difference equations, i.e., $\mathbb{T} = \mathbb{Z}$, we refer to [4]. For related results in the time scales case, we also refer to [7–9, 15, 23].

After this introduction, Section 2 gives three auxiliary results that are needed in the proofs of our five main theorems in Section 3. We conclude the paper in Section 4 with three examples, illustrating our theoretical findings.

2. Auxiliary Results

In this section, we give some auxiliary results that are used in the remainder of this paper.

Lemma 2.1. *If* (H_1)–(H_6) *and* (H_8) *hold, then any eventually positive solution of* (E) *is eventually decreasing.*

Proof. Let x be an eventually positive solution of (E), say

$$x(t)>0, \quad x(\delta(t))>0, \quad x(g(t))>0 \quad \text{for} \quad t\in[t_1,\infty)_{\mathbb{T}} \tag{2.1}$$

for some $t_1\geq t_0$. Then (E) implies $\left(a\left(y^{\Delta}\right)^{\alpha}\right)^{\Delta}<0$ on $[t_1,\infty)_{\mathbb{T}}$, so that

$$a\left(y^{\Delta}\right)^{\alpha} \quad \text{is decreasing on} \quad [t_1,\infty)_{\mathbb{T}}. \tag{2.2}$$

By (2.2), we either have

$$a\left(y^{\Delta}\right)^{\alpha}>0, \quad \text{i.e.,} \quad y^{\Delta}>0 \quad \text{on} \quad [t_1,\infty)_{\mathbb{T}}, \tag{2.3}$$

or otherwise there exists $t_2>t_1$ such that

$$a\left(y^{\Delta}\right)^{\alpha}<0, \quad \text{i.e.,} \quad y^{\Delta}<0 \quad \text{on} \quad [t_2,\infty)_{\mathbb{T}}. \tag{2.4}$$

Assume now that (2.3) holds. By (2.2), we have

$$a(t)\left(y^{\Delta}(t)\right)^{\alpha}\leq a(t_1)\left(y^{\Delta}(t_1)\right)^{\alpha} \quad \text{for all} \quad t\geq t_1.$$

Rearranging, we have

$$y^{\Delta}(t)\leq\left(\frac{a(t_1)\left(y^{\Delta}(t_1)\right)^{\alpha}}{a(t)}\right)^{1/\alpha} \quad \text{for all} \quad t\geq t_1.$$

Integrating this inequality from t_1 to $t\geq t_1$, we get

$$\begin{aligned} y(t) &\leq y(t_1)+a^{1/\alpha}(t_1)y^{\Delta}(t_1)\int_{t_1}^{t}\frac{\Delta s}{a^{1/\alpha}(s)} \\ &\overset{(2.3)}{\leq} y(t_1)+a^{1/\alpha}(t_1)y^{\Delta}(t_1)A(t_1) \quad \text{for all} \quad t\geq t_1. \end{aligned}$$

Thus, due to (H_5), y is bounded above, and together with (2.3), y has a finite positive limit, say $\ell\in(0,\infty)$, at infinity. This implies that x is bounded and

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}y(t)=\ell.$$

Hence, there exist $\varepsilon>0$ and $t^*\geq t_1$ such that $x(g(t))>\ell-\varepsilon>0$ for all $t\geq t^*$. Therefore, (E) and (H_4) imply that

$$\left(a\left(y^{\Delta}\right)^{\alpha}\right)^{\Delta}(t)\leq-(\ell-\varepsilon)^{\alpha}Mq(t) \quad \text{for all} \quad t\geq t_*.$$

Integrating from t^* to $t\geq t_*$, we find

$$a(t)\left(y^{\Delta}(t)\right)^{\alpha}\leq a(t^*)\left(y^{\Delta}(t^*)\right)^{\alpha}-(\ell-\varepsilon)^{\alpha}M\int_{t^*}^{t}q(s)\Delta s$$

for all $t \geq t_*$. Letting $t \to \infty$ implies, in view of (H_6), that

$$\lim_{t\to\infty} a(t)\left(y^{\Delta}(t)\right)^{\alpha}) = -\infty,$$

contradicting (2.3). Hence, (2.4) holds, completing the proof. □

Lemma 2.2. *Assume* (H_1)–(H_6) *and* (H_8). *Let x be an eventually positive solution of* (E). *Then $w = a^{1/\alpha}y^{\Delta}$ satisfies eventually $w \leq 0$, $w^{\Delta} \leq 0$, and*

$$\alpha w^{\Delta}\left(w^{\sigma}\right)^{\alpha-1} \leq \left(w^{\alpha}\right)^{\Delta} \leq \alpha w^{\Delta} w^{\alpha-1}. \tag{2.5}$$

Proof. Suppose x satisfies (2.1). Lemma 2.1 yields $w \leq 0$ on $[t_2,\infty)_{\mathbb{T}}$. By Pötzsche's chain rule [10, Theorem 1.90], we have

$$0 \overset{(\mathrm{E})}{\geq} \left(w^{\alpha}\right)^{\Delta} = \alpha w^{\Delta}\int_0^1 \left((1-h)w + hw^{\sigma}\right)^{\alpha-1} \mathrm{d}h. \tag{2.6}$$

Note that the integral on the right-hand side of (2.6) is nonnegative as $\alpha - 1$ is an even nonnegative integer divided by an odd positive integer. Thus, from (2.6), we obtain $w^{\Delta} \leq 0$. This implies $w^{\sigma} \leq w$ eventually. Hence,

$$w^{\sigma} \leq (1-h)w + hw^{\sigma} \leq w \quad \text{for all} \quad h \in [0,1].$$

Therefore, we obtain

$$w^{\alpha-1} \leq \left((1-h)w + hw^{\sigma}\right)^{\alpha-1} \leq \left(w^{\sigma}\right)^{\alpha-1} \quad \text{for all} \quad h \in [0,1],$$

and thus,

$$w^{\alpha-1} \leq \int_0^1 \left((1-h)w + hw^{\sigma}\right)^{\alpha-1} \mathrm{d}h \leq \left(w^{\sigma}\right)^{\alpha-1}.$$

Hence

$$\alpha w^{\Delta}\left(w^{\sigma}\right)^{\alpha-1} \leq \alpha w^{\Delta}\int_0^1 \left((1-h)w + hw^{\sigma}\right)^{\alpha-1} \mathrm{d}h \leq \alpha w^{\Delta} w^{\alpha-1},$$

proving (2.5). □

Remark 2.1. Assuming (H_1)–(H_5), (H_7) implies (H_8). To see that, in view of (H_1)–(H_5), we have $\lim_{t\to\infty} A(t) = 0$ and

$$\frac{A^{\beta}(\delta(t))}{A(t)} \geq \frac{A^{\beta}(t)}{A(t)} = \frac{1}{A^{1-\beta}(t)}.$$

Then, (H_7) implies (H_8). Thus, in this situation, we may apply Lemmas 2.1 and 2.2.

Lemma 2.3. *Assume* (H_1)–(H_7). *Let x be an eventually positive solution of* (E). *Let $w = a^{1/\alpha}y^{\Delta}$. Let $P \in (0,1)$. Then, eventually,*

$$y + wA \geq 0, \tag{2.7}$$

$$x \geq Py, \tag{2.8}$$

$$y/A \quad \textit{is nondecreasing,} \tag{2.9}$$

and there exists $\gamma > 0$ such that

$$y/A \geq \gamma. \tag{2.10}$$

Proof. Suppose x satisfies (2.1). Lemma 2.2 yields $w \le 0$ and $w^\Delta \le 0$ on $[t_2,\infty)_{\mathbb{T}}$. Hence, $w(s) \le w(t)$ for all $s \ge t \ge t_2$, so

$$y^\Delta(s) \le \frac{w(t)}{a^{1/\alpha}(s)} \quad \text{for all} \quad s \ge t \ge t_2.$$

Integrating this inequality from t to $u \ge t \ge t_2$, we find

$$-y(t) \le y(u) - y(t) \le w(t) \int_t^u \frac{\Delta s}{a^{1/\alpha}(s)} \quad \text{for all} \quad u \ge t \ge t_2,$$

and letting $u \to \infty$ yields

$$-y(t) \le w(t)A(t) \quad \text{for all} \quad t \ge t_2,$$

which proves (2.7). Next, on $[t_2,\infty)_{\mathbb{T}}$, we have

$$\left(\frac{y}{A}\right)^\Delta = \frac{y^\Delta A - yA^\Delta}{AA^\sigma} = \frac{y^\Delta A + a^{-1/\alpha} y}{AA^\sigma} = \frac{wA + y}{a^{1/\alpha}AA^\sigma} \overset{(2.7)}{\ge} 0,$$

and so (2.9) holds, while

$$\frac{y(t)}{A(t)} \overset{(2.9)}{\ge} \frac{y(t_2)}{A(t_2)} =: \gamma > 0 \quad \text{for all} \quad t \ge t_2$$

proves (2.10). Now, since $y(t) = x(t) + p(t)x^\beta(\delta(t)) \ge x(t)$, we get

$$\begin{aligned} x(t) &= y(t) - p(t)x^\beta(\delta(t)) \ge y(t) - p(t)y^\beta(\delta(t)) \\ &\overset{(2.9)}{\ge} y(t) - p(t)\left(\frac{A(\delta(t))y(t)}{A(t)}\right)^\beta \\ &= y(t)\left[1 - p(t)\frac{A^\beta(\delta(t))}{A(t)}\left(\frac{y}{A}\right)^{\beta-1}(t)\right] \\ &\overset{(2.10)}{\ge} y(t)\left[1 - \gamma^{\beta-1}p(t)\frac{A^\beta(\delta(t))}{A(t)}\right] \end{aligned}$$

for all $t \ge t_2$. From (H_7), there exists $t_3 \ge t_2$ such that

$$p(t)\frac{A^\beta(\delta(t))}{A(t)} \le (1-P)\gamma^{1-\beta} \quad \text{for all} \quad t \ge t_3,$$

and thus (2.8) holds on $[t_3,\infty)_{\mathbb{T}}$. $\square$

3. Main Results

Now, we present our five main oscillation results for (E). Throughout this section, we use the notation

$$Q(t) := Mq(t), \quad t \ge t_0.$$

The first two results give new criteria by comparing with second-order inequalities.

Theorem 3.1. *Assume* (H_1)–(H_7). *Let* $P \in (0,1)$. *If*

$$\left(a^{1/\alpha} y^{\Delta}\right)^{\Delta}(t) + \frac{P^{\alpha}}{\alpha} A^{\alpha-1}(\sigma(t)) Q(t) y(g(t)) \leq 0 \tag{3.1}$$

has no eventually positive decreasing solution, then (E) *is oscillatory.*

Proof. Assume x is a nonoscillatory solution of (E), satisfying (2.1). Let $w = a^{1/\alpha} y^{\Delta}$. Observe that Lemmas 2.1, 2.2, and 2.3 hold. On $[t_3, \infty)_{\mathbb{T}}$, we thus have

$$\begin{aligned}
0 &\overset{(\mathrm{E})}{=} \left[(w^{\alpha})^{\Delta} + q(f \circ x \circ g)\right] \frac{(w^{\sigma})^{1-\alpha}}{\alpha} \\
&\overset{(2.5)}{\geq} \left[\alpha w^{\Delta} (w^{\sigma})^{\alpha-1} + q(f \circ x \circ g)\right] \frac{(w^{\sigma})^{1-\alpha}}{\alpha} \\
&= w^{\Delta} + q \frac{(w^{\sigma})^{1-\alpha}}{\alpha} (f \circ x \circ g) \\
&\overset{(\mathrm{H}_4)}{\geq} w^{\Delta} + q \frac{(w^{\sigma})^{1-\alpha}}{\alpha} M (x \circ g)^{\alpha} \\
&= w^{\Delta} + Q \frac{(w^{\sigma})^{1-\alpha}}{\alpha} (x \circ g)^{\alpha} \\
&\overset{(2.8)}{\geq} w^{\Delta} + Q \frac{(w^{\sigma})^{1-\alpha}}{\alpha} P^{\alpha} (y \circ g)^{\alpha} \\
&= w^{\Delta} + Q \left(\frac{w^{\sigma}}{y \circ g}\right)^{1-\alpha} \frac{P^{\alpha}}{\alpha} (y \circ g) \\
&\overset{(\mathrm{H}_2)}{\geq} w^{\Delta} + Q \left(\left(\frac{w}{y}\right)^{\sigma}\right)^{1-\alpha} \frac{P^{\alpha}}{\alpha} (y \circ g) \\
&\overset{(2.7)}{\geq} w^{\Delta} + Q (A^{\sigma})^{\alpha-1} \frac{P^{\alpha}}{\alpha} (y \circ g),
\end{aligned}$$

so that (3.1) indeed has an eventually positive decreasing solution. This contradiction shows that no nonoscillatory solution of (E) can exist, and thus (E) is oscillatory. □

Theorem 3.2. *Assume* (H_1)–(H_7). *Let* $P \in (0,1)$. *If, for all* $\gamma > 0$,

$$\left(a \left(y^{\Delta}\right)^{\alpha}\right)^{\Delta}(t) + \gamma^{\alpha-1} P^{\alpha} A^{\alpha-1}(g(t)) Q(t) y(g(t)) \leq 0 \tag{3.2}$$

has no eventually positive decreasing solution, then (E) *is oscillatory.*

Proof. Assume x is a nonoscillatory solution of (E), satisfying (2.1). Let $w = a^{1/\alpha} y^{\Delta}$. Observe that Lemmas 2.1, 2.2, and 2.3 hold. On $[t_3, \infty)_{\mathbb{T}}$, we thus have

$$\begin{aligned}
(w^{\alpha})^{\Delta} &\overset{(\mathrm{E})}{=} -q(f \circ x \circ g) \\
&\overset{(\mathrm{H}_4)}{\leq} -qM(x \circ g)^{\alpha} \\
&\overset{(2.8)}{\leq} -qMP^{\alpha}(y \circ g)^{\alpha} \\
&= -P^{\alpha} Q \left(\frac{y}{A} \circ g\right)^{\alpha-1} (A \circ g)^{\alpha-1} (y \circ g)
\end{aligned}$$

$$\overset{(2.10)}{\leq} -\gamma^{\alpha-1}P^{\alpha}Q(A\circ g)^{\alpha-1}(y\circ g),$$

so that (3.2) indeed has an eventually positive decreasing solution. This contradiction shows that no nonoscillatory solution of (E) can exist, and thus (E) is oscillatory. □

The next three results deal with sufficient integral conditions to ensure oscillation of (E).

Theorem 3.3. *Assume* (H_1)–(H_5) *and* (H_7). *If*

$$\limsup_{t\to\infty}\left[A(t)\int_{t_1}^{t}A^{\alpha-1}(\sigma(s))Q(s)\Delta s+\frac{1}{A(g(t))}\int_{t}^{\infty}A^{\alpha}(\sigma(s))A(g(s))Q(s)\Delta s\right]>\alpha, \tag{3.3}$$

then (E) *is oscillatory.*

Proof. First note that it follows from (3.3) that there exists $P\in(0,1)$ such that

$$\limsup_{t\to\infty}\left[A(t)\int_{t_1}^{t}A^{\alpha-1}(\sigma(s))Q(s)\Delta s+\frac{1}{A(g(t))}\int_{t}^{\infty}A^{\alpha}(\sigma(s))A(g(s))Q(s)\Delta s\right]>\frac{\alpha}{P^{\alpha}}. \tag{3.4}$$

Assume x is a nonoscillatory solution of (E), satisfying (2.1). Let $w=a^{1/\alpha}y^{\Delta}$. Observe that Lemmas 2.1, 2.2, and 2.3 hold. Notice that (3.4) implies that (H_6) holds. If not, i.e., if $\int_{t_0}^{t}q(s)\Delta s$ is bounded, then

$$A(t)\int_{t_1}^{t}A^{\alpha-1}(\sigma(s))Q(s)\Delta s\leq A(t)\int_{t_1}^{t}Q(s)\Delta s\to 0\quad\text{as}\quad t\to\infty$$

and

$$\begin{aligned}\frac{1}{A(g(t))}\int_{t}^{\infty}A^{\alpha}(\sigma(s))A(g(s))Q(s)\Delta s&\leq\int_{t}^{\infty}A^{\alpha}(\sigma(s))Q(s)\Delta s\\&\leq A^{\alpha}(\sigma(t))\int_{t}^{\infty}Q(s)\Delta s\to 0\quad\text{as}\quad t\to\infty,\end{aligned}$$

where we used (H_5) and the decreasing nature of A. However, this contradicts (3.4) and hence we proved our claim. Note also that we now reach (3.1) as in the proof of Theorem 3.1. On $[t_3,\infty)_{\mathbb{T}}$, we thus have

$$(wA+y)^{\Delta}=w^{\Delta}A^{\sigma}+wA^{\Delta}+y^{\Delta}=w^{\Delta}A^{\sigma}\overset{(3.1)}{\leq}-\frac{P^{\alpha}}{\alpha}\left(A^{\sigma}\right)^{\alpha}Q(y\circ g),$$

and integrating this inequality from t to $u\geq t\geq t_3$ yields

$$\begin{aligned}-(wA+y)(t)&\overset{(2.7)}{\leq}(wA+y)(u)-(wA+y)(t)=\int_{t}^{u}(wA+y)^{\Delta}(s)\Delta s\\&\leq-\frac{P^{\alpha}}{\alpha}\int_{t}^{u}A^{\alpha}(\sigma(s))Q(s)y(g(s))\Delta s\\&=-\frac{P^{\alpha}}{\alpha}\int_{t}^{u}A^{\alpha}(\sigma(s))Q(s)\frac{y(g(s))}{A(g(s))}A(g(s))\Delta s\\&\overset{(2.9)}{\leq}-\frac{P^{\alpha}}{\alpha}\int_{t}^{u}A^{\alpha}(\sigma(s))Q(s)\frac{y(g(t))}{A(g(t))}A(g(s))\Delta s\end{aligned}$$

$$\overset{(\mathrm{H}_2)}{\leq} -\frac{P^\alpha}{\alpha}\int_t^u A^\alpha(\sigma(s))Q(s)\frac{y(t)}{A(g(t))}A(g(s))\Delta s,$$

which upon letting $u \to \infty$ becomes

$$(wA+y)(t) \geq \frac{P^\alpha y(t)}{\alpha A(g(t))}\int_t^\infty A^\alpha(\sigma(s))Q(s)A(g(s))\Delta s \tag{3.5}$$

for all $t \geq t_3$. On the other hand, for $t \geq t_3$, we get

$$\begin{aligned} w(t) &\leq w(t) - w(t_3) = \int_{t_3}^t w^\Delta(s)\Delta s \\ &\overset{(3.1)}{\leq} -\frac{P^\alpha}{\alpha}\int_{t_3}^t A^{\alpha-1}(\sigma(s))Q(s)y(g(s))\Delta s \\ &\overset{(\mathrm{H}_2)}{\leq} -\frac{P^\alpha}{\alpha}\int_{t_3}^t A^{\alpha-1}(\sigma(s))Q(s)y(t)\Delta s, \end{aligned}$$

i.e.,

$$-(wA)(t) \geq \frac{P^\alpha}{\alpha}A(t)y(t)\int_{t_3}^t A^{\alpha-1}(\sigma(s))Q(s)\Delta s \tag{3.6}$$

for all $t \geq t_3$. Combining now (3.5) and (3.6) yields

$$\begin{aligned} y(t) =& (wA+y)(t) - (wA)(t) \\ \geq& \frac{P^\alpha y(t)}{\alpha A(g(t))}\int_t^\infty A^\alpha(\sigma(s))Q(s)A(g(s))\Delta s \\ &+ \frac{P^\alpha}{\alpha}A(t)y(t)\int_{t_3}^t A^{\alpha-1}(\sigma(s))Q(s)\Delta s, \end{aligned}$$

which upon division by $y(t)$ results in

$$\begin{aligned} \frac{\alpha}{P^\alpha} \geq& \frac{1}{A(g(t))}\int_t^\infty A^\alpha(\sigma(s))Q(s)A(g(s))\Delta s \\ &+ A(t)\int_{t_3}^t A^{\alpha-1}(\sigma(s))Q(s)\Delta s \end{aligned}$$

for all $t \geq t_3$. Taking $\limsup$ as $t \to \infty$ contradicts (3.4) and completes the proof. □

Theorem 3.4. *Assume* (H_1)–(H_5) *and* (H_7). *Let* $P \in (0,1)$. *If for any* $\ell_1 \in [t_1,\infty)_{\mathbb{T}}$, *there exists* $\ell \in [\ell_1,\infty)_{\mathbb{T}}$ *such that*

$$\limsup_{t\to\infty}\int_\ell^t \left[\frac{P^\alpha}{\alpha}A^\alpha(\sigma(s))Q(s) - \frac{a^{-1/\alpha}(s)}{4A(\sigma(s))}\right]\Delta s > 1, \tag{3.7}$$

then (E) *is oscillatory.*

Proof. Assume x is a nonoscillatory solution of (E), satisfying (2.1). Let $w = a^{1/\alpha}y^\Delta$. Observe that Lemmas 2.1, 2.2, and 2.3 hold. Notice that (3.7) implies that (H_6) holds. For if $\int_\ell^t q(s)\Delta s$ is bounded, then $\int_\ell^t A^\alpha(\sigma(s))Q(s)\Delta s$ is also bounded, which, given that ℓ_1 is arbitrarily large, contradicts (3.7). Note also that we now reach (3.1) as in the proof of Theorem 3.1. For this proof, we also introduce $v = w/y$. On $[t_3, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} v^{\Delta} &= \frac{w^{\Delta} y - w y^{\Delta}}{y y^{\sigma}} = \frac{w^{\Delta}}{y^{\sigma}} - \frac{w a^{1/\alpha} y^{\Delta} a^{-1/\alpha}}{y y^{\sigma}} \\ &= \frac{w^{\Delta}}{y^{\sigma}} - \frac{w^2 a^{-1/\alpha}}{y y^{\sigma}} = \frac{w^{\Delta}}{y^{\sigma}} - v^2 a^{-1/\alpha} \frac{y}{y^{\sigma}} \\ &\overset{(3.1)}{\leq} -\frac{P^{\alpha}}{\alpha} (A^{\sigma})^{\alpha-1} Q \frac{y \circ g}{y^{\sigma}} - v^2 a^{-1/\alpha} \frac{y}{y^{\sigma}} \\ &\overset{(\mathrm{H}_2)}{\leq} -\frac{P^{\alpha}}{\alpha} (A^{\sigma})^{\alpha-1} Q - v^2 a^{-1/\alpha}, \end{aligned}$$

so that

$$\begin{aligned} (Av)^{\Delta} &= A^{\Delta} v + A^{\sigma} v^{\Delta} = -v a^{-1/\alpha} + A^{\sigma} v^{\Delta} \\ &\leq -v a^{-1/\alpha} - \frac{P^{\alpha}}{\alpha} (A^{\sigma})^{\alpha} Q - A^{\sigma} v^2 a^{-1/\alpha} \\ &= -\frac{P^{\alpha}}{\alpha} (A^{\sigma})^{\alpha} Q - \frac{a^{-1/\alpha}}{4A^{\sigma}} (1 + 2A^{\sigma} v)^2 + \frac{a^{-1/\alpha}}{4A^{\sigma}} \\ &\leq -\frac{P^{\alpha}}{\alpha} (A^{\sigma})^{\alpha} Q + \frac{a^{-1/\alpha}}{4A^{\sigma}}. \end{aligned}$$

Integrating this last inequality from ℓ to $t \geq \ell \geq t_3$, we get

$$\begin{aligned} \int_{\ell}^{t} \left[\frac{P^{\alpha}}{\alpha} A^{\alpha}(\sigma(s)) Q(s) - \frac{a^{-1/\alpha}(s)}{4A(\sigma(s))} \right] \Delta s \leq & A(\ell) v(\ell) - A(t) v(t) \\ \leq & -A(t) v(t) \overset{(2.7)}{\leq} 1, \end{aligned}$$

which contradicts (3.7) and completes the proof. ☐

Theorem 3.5. *Assume* (H_1)–(H_5) *and* (H_7). *If*

$$\int_{t_1}^{\infty} \left[\frac{1}{a(s)} \int_{t_1}^{s} A^{\alpha}(g(u)) q(u) \Delta u \right]^{1/\alpha} \Delta s = \infty, \tag{3.8}$$

then (E) *is oscillatory.*

Proof. Assume x is a nonoscillatory solution of (E), satisfying (2.1). Let $w = a^{1/\alpha} y^{\Delta}$. Observe that Lemmas 2.1, 2.2, and 2.3 hold. Notice that (3.8) implies that (H_6) holds. Note therefore that we reach (3.2) as in the proof of Theorem 3.2. On $[t_3, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} (w^{\alpha})^{\Delta} &\overset{(3.2)}{\leq} -\gamma^{\alpha-1} P^{\alpha} (A \circ g)^{\alpha-1} Q (y \circ g) \\ &= -\gamma^{\alpha-1} P^{\alpha} (A \circ g)^{\alpha} Q \left(\frac{y}{A} \circ g \right) \\ &\overset{(2.10)}{\leq} -\gamma^{\alpha} P^{\alpha} (A \circ g)^{\alpha} Q, \end{aligned}$$

which upon integrating from t_3 to $t \geq t_3$ yields

$$w^{\alpha}(t) \leq w^{\alpha}(t) - w^{\alpha}(t_3) = \int_{t_3}^{t} (w^{\alpha})^{\Delta}(s) \Delta s$$

$$\begin{aligned}&\leq -\gamma^\alpha P^\alpha \int_{t_3}^t A^\alpha(g(s))Q(s)\Delta s\\&= -\gamma^\alpha P^\alpha M \int_{t_3}^t A^\alpha(g(s))q(s)\Delta s.\end{aligned}$$

Rearranging yields

$$y^\Delta(t) \leq -\gamma P M^{1/\alpha}\left[\frac{1}{a(t)}\int_{t_3}^t A^\alpha(g(s))q(s)\Delta s\right]^{1/\alpha}$$

for all $t \geq t_3$. Integrating this inequality again from t_3 to $t \geq t_3$ gives

$$\begin{aligned}-y(t_3) \leq& y(t) - y(t_3) = \int_{t_3}^t y^\Delta(s)\Delta s\\ \leq& -\gamma P M^{1/\alpha}\int_{t_3}^t\left[\frac{1}{a(s)}\int_{t_3}^s A^\alpha(g(u))q(u)\Delta u\right]^{1/\alpha}\Delta s,\end{aligned}$$

and so

$$\int_{t_3}^t\left[\frac{1}{a(s)}\int_{t_3}^s A^\alpha(g(u))q(u)\Delta u\right]^{1/\alpha}\Delta s \leq \frac{y(t_3)}{\gamma P M^{1/\gamma}}$$

for all $t \geq t_3$, contradicting (3.8) and completing the proof. □

4. Examples

We conclude this paper by giving three illustrating examples, one for $\mathbb{T} = \mathbb{R}$, then for $\mathbb{T} = \mathbb{Z}$, then for a general time scale.

Example 4.1. Consider the second-order neutral differential equation

$$\left(t^6\left(\left[x(t) + \frac{1}{t}x^{1/3}\left(\frac{t}{2}\right)\right]'\right)^3\right)' + t^\lambda x^3\left(\frac{t}{3}\right) = 0 \tag{4.1}$$

for $t \in [3,\infty)_{\mathbb{R}}$. Here, $\alpha = 3$, $\beta = 1/3$, $p(t) = 1/t$, $\delta = t/2$, $g(t) = t/3$, $M = 1$, $Q(t) = q(t) = t^\lambda$ for $\lambda \in \mathbb{R}$. Moreover, $A(t) = 1/t$ and (H_4) is easily verified. Using Theorem 3.5, we see that (4.1) is oscillatory for all $\lambda \geq 5$.

Example 4.2. Consider the second-order neutral difference equation

$$\begin{aligned}&\Delta\left((t^2+t)^5\Delta\left(\left[x(t) + \frac{1}{t}x^\beta(t-1)\right]\right)^5\right)\\&+ (t+1)^6 x^5(t-2)(2+\sin x(t-2)) = 0, \quad t \in \mathbb{N},\end{aligned} \tag{4.2}$$

where $\beta \in (0,1]$ is a ratio of positive odd integers. Here, $\alpha = 5$, $a(t) = (t^2+t)^5$, $p(t) = 1/t$, $\delta(t) = t-1$, and $g(t) = t-2$. We also have $M = 1$, $Q(t) = (t+1)^6$, and $A(t) = \frac{1}{t}$. Therefore,

$$\lim_{t\to\infty}\frac{p(t)A^\beta(\delta(t))}{A(t)} = \lim_{t\to\infty}\frac{1}{(t-1)^\beta} = 0.$$

Thus (H_7) is satisfied. Note that

$$\frac{P^\alpha}{\alpha}A^\alpha(\sigma(s))Q(s)-\frac{a^{-1/\alpha}}{4A(\sigma(s))}=\frac{P^5}{5}(s+1)-\frac{1}{4s}.$$

Then (3.7) is clearly satisfied, and hence (4.2) is oscillatory according to Theorem 3.4.

Example 4.3. Consider the second-order neutral dynamic equation

$$\left(\sigma^3(t)t^3\left(\left[x(t)+\frac{1}{t}x^{1/3}\left(\frac{t}{2}\right)\right]^\Delta\right)^3\right)^\Delta+q_0\sigma^2(t)x^3(\lambda t)=0 \tag{4.3}$$

for $t\in[1,\infty)_{\mathbb{T}}$, where $\lambda\in(0,1]$ and $q_0>0$. Here, $\alpha=3$, $\beta=1/3$, $p(t)=1/t$, $\delta(t)=t/2$, $g(t)=\lambda t$, $M=1$, $Q(t)=q(t)=q_0\sigma^2(t)$, and

$$A(t)=\int_t^\infty\frac{1}{\sigma(s)s}\Delta s=\frac{1}{t}.$$

Condition (H_7) is clearly satisfied, since

$$\lim_{t\to\infty}\frac{p(t)A^\beta(\delta(t))}{A(t)}=\lim_{t\to\infty}\left(\frac{2}{t}\right)^{1/3}=0.$$

Now, since

$$A(t)\int_{t_1}^t A^{\alpha-1}(\sigma(s))Q(s)\Delta s=\frac{1}{t}\int_{t_1}^t\frac{q_0}{\sigma^{3-1}(s)}\sigma^2(s)\Delta s=q_0\frac{t-t_1}{t}$$

and

$$\frac{1}{A(g(t))}\int_t^\infty A^\alpha(\sigma(s))A(g(s))Q(s)\Delta s=\lambda t\int_t^\infty\frac{q_0\sigma^2(s)}{\lambda s\sigma^3(s)}\Delta s=q_0,$$

(3.3) takes the form

$$q_0>\frac{3}{2}. \tag{4.4}$$

Hence, by Theorem 3.3, (4.3) is oscillatory if (4.4) holds. Similarly, condition (3.7) takes the form

$$\limsup_{t\to\infty}\int_{t_1}^t\left[\frac{P^3q_0}{3}\frac{s}{\sigma(s)}-\frac{1}{4}\right]\frac{1}{s}\Delta s>1,$$

and so Theorem 3.4 requires, for the oscillation of (4.3), that there exist $P\in(0,1)$ and $\varepsilon>0$ such that

$$\frac{P^3q_0}{3}\frac{t}{\sigma(t)}>\frac{1}{4}+\varepsilon.$$

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