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LONG-TIME ASYMPTOTIC BEHAVIOR OF FISHER-KPP EQUATION FOR NONLOCAL DISPERSAL IN ASYMMETRIC KERNEL

De-Yu Kong[1] and Bang-Sheng Han[1,†]

Abstract In this paper, we main consider the asymptotic spreading speeds and the long-time asymptotic behavior of a nonlocal with asymmetric kernel diffusion Fisher-KPP equation

$$u_t(t,x) = k * u(t,x) - u(t,x) + f\left(u(t,x)\right),\ t>0,\ x\in\mathbb{R}.$$

On the basis of the spreading speeds $c_r^* = c(\lambda_r^*)$ and $c_l^* = c(\lambda_l^*)$, the long-time asymptotic behavior is given by constructing a suitable upper solution and lower solution and using the tool of comparison principle. In particular, the core difficulty and breakthrough point is the lower bounds part. In this regard, we improve the "forward-backward spreading" method which was first proposed by Xu et al. (J Funct Anal 280(2021)108957) to fit the corresponding lower solution so that the asymptotic behavior can be obtained for the initial values that decays within a certain range of asymptotic decay rate $\lambda_1 \in (0, \lambda^+)$ and $\lambda_2 \in (\lambda^-, 0)$.

Keywords Spreading speed, nonlocal diffusion, fisher-KPP, asymptotic, comparison principle.

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1. Introduction

This paper is concerned with the asymptotic spreading speeds (the concept proposed by Aroson and Weinberger [1]) and long-time asymptotic behavior of the following nonlocal reaction-diffusion equation

$$\begin{cases} u_t(t,x) = k * u(t,x) - u(t,x) + f\left(u(t,x)\right),\ t>0,\ x\in\mathbb{R}, \\ u(0,x) = u_0(x),\ x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $u_0(x) \in C(\mathbb{R})$, $f \in C^1\left([0,1]\right)$ and satisfies the condition of Fisher-KPP type:

(P) $f(0) = f(1) = 0,\ f(u) > 0,\ f'(0) > 0$ and $f(u) \le f'(0)u$ for $u \in (0,1)$.

The convolution integral operator is given as follows

$$k * u(t,x) - u(t,x) = \int_{\mathbb{R}} k(x-y)u(t,y)dy - u(t,x),\ t>0,\ x\in\mathbb{R}.$$

†The corresponding author.
[1]School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan, 611756, China
Email: KongDY@my.swjtu.edu.cn(D. Kong), hanbangsheng@swjtu.edu.cn(B. Han)

Here, the diffusion kernel function $k(x)$ including the asymmetric case satisfy:

(K1) $\int_{\mathbb{R}} k(x)dx = 1$ and exist some $\lambda \in \mathbb{R}$ such that $\int_{\mathbb{R}} k(x)e^{\lambda x}dx < +\infty$.

(K2) There exist $x^+ \in \mathbb{R}^+$ and $x^- \in \mathbb{R}^-$ such that $k(x^{\pm}) > 0$.

Therefore, we point out that the system (1.1) is exponentially bounded and light-tailed (it is indicated by (K1)) which ensures the spatial propagation mode can be consider the asymptotic spreading speeds. Moreover, fat-tailed kernel as well as the property that $u_0(x)$ is compactly supported on $\mathbb{R}$ will causes acceleration propagation, which means the average spreading speeds approaches infinity [7].

The important feature of (1.1) is not only the diffusion term is a convolution integral operator $k * u(t,x) - u(t,x)$, but also the kernel function including the asymmetric case. First of all, in terms of nonlocal diffusion it can better represent natural phenomena as it also covers long distances motion. This topic has indeed attracted the research and attention of many scholars. Therefore, for (1.1) and its concrete form, we refer to some papers about the traveling wave solution [3–5, 8], entire solution [6, 12, 14] and other problems including spreading speeds [2, 10, 15]. Secondly, the asymmetric kernel means the probability distribution of the population jumping from location y to location x (that is, the value of $k(x-y)$ may be different from that of $k(y-x)$). As it do increased the difficulty, so there are very few studies that we know of the contain only the article [9, 13, 16, 17]. In fact, it was not until 2008 that Coville et al. [5] began to deal with the case of asymmetric kernel function, and only proposed symbols that would affect the asymptotic spreading speed without giving specific instructions. In this regard, understanding the asymptotic spreading speeds in nonlocal diffusion system with asymmetric kernel is of great benefit to the future study of nonlocal dispersals.

As a type of spatial propagation, the asymptotic spreading speeds c_r^* and c_l^* of the nonlocal diffusion equation are available in Lutscher et al [11]. Here, for any non-negative and compactly supported initial data u_0 and $\varepsilon > 0$,

$$\begin{cases} \lim\limits_{t\to+\infty} \sup_{x\in(-\infty,(c_l^*-\varepsilon)t]\cup[(c_r^*+\varepsilon)t,+\infty)} |u(t,x)| = 0, \\ \lim\limits_{t\to+\infty} \inf_{x\in((c_l^*+\varepsilon)t,\ (c_r^*-\varepsilon)t)} |u(t,x)-1| = 0. \end{cases}$$

The principle of construction is that the spreading speeds and traveling wave solution which is a special form of solution that satisfies $u(t,x) = q(x-ct)$, as two spatial propagation modes, have certain similarities. That is, in monotone dynamics, the minimum wave speed is numerically consistent with spreading speeds. In addition, to the best of our knowledge, the conclusion of system (1.1) is also that the signs of c_r^* and c_l^* are dependent on $f'(0)$ and the function

$$E(k) \triangleq \operatorname{sign}\left(\int_{\mathbb{R}} k(x)xdx\right)\left[1 - \inf_{\lambda\in\mathbb{R}}\left\{\int_{\mathbb{R}} k(x)e^{\lambda x}dx\right\}\right],$$

which can describe the asymmetry of diffusion kernel [17].

The purpose of this paper is to further explore the spreading speeds on the above basis and deduce the corresponding long-time asymptotic behavior under the initial value conditions of a wide range of exponential decay rates $\lambda_1 \in (0, \lambda^+)$, $\lambda_2 \in (\lambda^-, 0)$. The basis tool is upper and lower solutions method and our improved "forward-backward spreading" method which plays an important role in lower bound part. Furthermore, the key difficulty is to improve the "forward-backward spreading" method (naming follows the classical case that $c_r^* = -c_l^* > 0$) such that the desired asymptotic result is obtained by combining with the corresponding lower solutions.

2. Asymptotic spreading speeds

In this section, first denote some useful notations, definitions and lemmas about spreading speeds and lower solution in Subsection 2.1. Then the proof of long-time asymptotic behavior is presented in Subsection 2.2.

2.1. Mathematical preliminaries

Denote

$$\lambda^+ \triangleq \sup\left\{\lambda > 0 \mid \int_{\mathbb{R}} k(x)e^{\lambda x}dx < +\infty\right\},$$
$$\lambda^- \triangleq \inf\left\{\lambda < 0 \mid \int_{\mathbb{R}} k(x)e^{\lambda x}dx < +\infty\right\},$$

which implies that $\lambda \in (\lambda^-, \lambda^+)$ satisfy the second half of the requirement (K1). Actually, condition (K2) is related to the function $E(k)$, which also ensures the species did not just spread in one direction.

Asymptotic spreading speeds

To characterize the asymptotic spreading speeds, here we summarize the speeds of corresponding traveling wave solutions. For system (1.1), By defining the function $u(t,x) = q(x-ct)$ and plugging it into the equation and linearizing it, we have

$$-cq'(x) = \int_{\mathbb{R}} k(x-y)q(y)dy - q(x) - f'(0)q(x).$$

Using the eigenroot method, do the exponential transformation $e^{\lambda x}$ and simplification, finally get

$$c(\lambda) = \frac{\int_{\mathbb{R}} k(x)e^{\lambda x}dx - 1 + f'(0)}{\lambda},\ \lambda \in (-\infty, 0) \cup (0, +\infty).$$

In this regard, for $\lambda \in (\lambda^-, 0) \cup (0, \lambda^+)$, the spreading speeds can be expressed as

$$c_r^* = \inf_{\lambda \in (0,\lambda^+)} \{c(\lambda)\} = c(\lambda_r^*) = \frac{\int_{\mathbb{R}} k(x)e^{\lambda_r^* x}dx - 1 + f'(0)}{\lambda_r^*} = \int_{\mathbb{R}} k(x)e^{\lambda_r^* x}xdx,$$
$$c_l^* = \sup_{\lambda \in (\lambda^-,0)} \{c(\lambda)\} = c(\lambda_l^*) = \frac{\int_{\mathbb{R}} k(x)e^{\lambda_l^* x}dx - 1 + f'(0)}{\lambda_l^*} = \int_{\mathbb{R}} k(x)e^{\lambda_l^* x}xdx.$$

In fact, the existence of parameters λ_r^* and λ_l^* can be obtained by the property of $c(\lambda)$. Corresponding the following function can be defined

$$c_\xi(\lambda) = \frac{\int_{\mathbb{R}} k(x)e^{\lambda x}dx - 1 + f'(0) - \xi}{\lambda}$$

with the arbitrary constant $\xi \in (0, f'(0))$. Function $c_\xi(\lambda)$ is actually a little bit closer to the origin than function $c(\lambda)$, essentially for use with the lower solutions constructed later. For any small $\varepsilon > 0$, there have the arbitrary constants $\xi_1,\ \xi_2 \in (0, f'(0))$ to satisfies that

$$c_r^*(\xi_1) \triangleq \inf_{\lambda_1 \in (0,\lambda^+)} \{c_{\xi_1}(\lambda_1)\} = c_r^* - \varepsilon, \quad c_l^*(\xi_2) \triangleq \sup_{\lambda_2 \in (\lambda^-,0)} \{c_{\xi_2}(\lambda_2)\} = c_l^* + \varepsilon. \quad (2.1)$$

The signs of spreading speeds

It follows from $\lambda_r^* > 0,\ \lambda_l^* < 0$ that the signs of spreading speeds determined by $\int_{\mathbb{R}} k(x)e^{\lambda_r^* x}dx - 1 + f'(0)$ and $1 - \int_{\mathbb{R}} k(x)e^{\lambda_l^* x}dx - f'(0)$, respectively. As a matter of fact, the diffusion kernel function $k(x)$ largely determines the signs of the spreading speeds. On the one hand, in order to better describe the influence of the kernel function $k(x)$, and on the other hand, to avoid accurate solving of corresponding parameters λ_r^* and λ_l^*, function $E(k)$ is used here to replace the original ground variable $\int_{\mathbb{R}} k(x)e^{\lambda_r^* x}dx - 1$ and $1 - \int_{\mathbb{R}} k(x)e^{\lambda_l^* x}dx$.

After a series of verification, the following conclusion is reached:

$$\begin{aligned} sign\,(c_r^*) &= sign\,(E(k) + f'(0)) = sign\left(\int_{\mathbb{R}} k(x)e^{\lambda(k)x}dx - 1 + f'(0)\right), \\ sign\,(c_l^*) &= sign\,(E(k) - f'(0)) = sign\left(1 - \int_{\mathbb{R}} k(x)e^{\lambda(k)x}dx - f'(0)\right), \end{aligned}$$

where $\lambda(k) \in (\lambda^-, \lambda^+)$ such that

$$\int_{\mathbb{R}} k(x)e^{\lambda(k)x}dx = \min_{\lambda \in (\lambda^-,\lambda^+)} \left\{\int_{\mathbb{R}} k(x)e^{\lambda x}dx\right\}.$$

That is the core of Theorem 2.4 in the literature [17] and how it works.

The next step is to prepare for the long-time asymptotic behavior, which includes the Comparison principle and the construction and property analysis of many functions.

Denote functions

$$\begin{aligned} G(c,\lambda) &= c\lambda - \int_{\mathbb{R}} k(x)e^{\lambda x}dx + 1 - f'(0), \\ G_\xi(c,\lambda) &= c\lambda - \int_{\mathbb{R}} k(x)e^{\lambda x}dx + 1 - f'(0) + \xi \quad \text{for } c \in \mathbb{R},\ \lambda \in (\lambda^-, \lambda^+). \end{aligned}$$

The definition idea is closely related to the form of the Eq. (1.1) and the expansion and contraction of the reaction term $f(u)$.

Lemma 2.1 (Comparison principle)**.** *Suppose that the bounded continuous functions $\bar{u}(t,x)$ and $\underline{u}(t,x)$ are respectively the upper-solution and lower-solution of equation* (1.1) *for $t \in (0,T]$, in the sense that*

$$\bar{u}_t - k * \bar{u} + \bar{u} - f(\bar{u}) \geqslant 0 \geqslant \underline{u}_t - k * \underline{u} + \underline{u} - f(\underline{u}) \quad \textit{for } t \in (0,T],\ x \in \mathbb{R}.$$

If $\bar{u}(0,x) \geqslant \underline{u}(0,x)$ for $x \in \mathbb{R}$, then $\bar{u}(t,x) \geqslant \underline{u}(t,x)$ for $t \in [0,T]$, $x \in \mathbb{R}$.

Lemma 2.2. *For any $c_1 \in (c(\lambda_1) - \varepsilon, c(\lambda_1))$ and $c_2 \in (c(\lambda_2), c(\lambda_2) + \varepsilon)$ with $\varepsilon > 0$ small enough and $\lambda_1 \in (0, \lambda^+)$, $\lambda_2 \in (\lambda^-, 0)$, there are four unique constants*

$\theta_1(c_1) > \lambda_r^* > \tilde{\theta}_1(c_1) > 0$ *(denoted also by* θ_1 *and* $\tilde{\theta}_1$ *for short) and* $\theta_2(c_2) < \lambda_l^* < \tilde{\theta}_2(c_2) < 0$ *(denoted also by* θ_2 *and* $\tilde{\theta}_2$ *for short) such that*

$$G_{\xi_1}(c_1,\theta_1) = G_{\xi_1}(c_1,\tilde{\theta}_1) = 0 \quad \text{and} \quad G_{\xi_1}(c_1,\rho) > 0 \quad \text{for } \rho \in (\tilde{\theta}_1, \theta_1),$$
$$G_{\xi_2}(c_2,\theta_2) = G_{\xi_2}(c_2,\tilde{\theta}_2) = 0 \quad \text{and} \quad G_{\xi_2}(c_2,\rho) > 0 \quad \text{for } \rho \in (\theta_2, \tilde{\theta}_2).$$

Proof. It follows from the definition of $G_\xi(c,\lambda)$ that

$$G_{\xi_i}(c_i,0) = \xi_i - f'(0) < 0 \text{ and } \frac{\partial^2}{\partial\lambda^2}G_{\xi_i}(c_i,\lambda) < 0,\ i = 1,2.$$

After a series of calculations, whether it's $\lambda^{\pm} = \pm\infty$ or $\lambda^{\pm} < \pm\infty$, we can conclude that

$$\lim_{\lambda\to\lambda\pm} G_\xi(c,\lambda) = -\infty.$$

Moreover, noticing the fact of (2.1) and

$$G_{\xi_1}(c_1,\lambda_1) = (c_1 - c_{\xi_1}(\lambda_1))\,\lambda_1 = (c_1 - c_r^*(\xi_1))\,\lambda_1 + (c_r^*(\xi_1) - c_{\xi_1}(\lambda_1))\,\lambda_1,$$
$$G_{\xi_2}(c_2,\lambda_2) = (c_2 - c_{\xi_2}(\lambda_2))\,\lambda_2 = (c_2 - c_l^*(\xi_2))\,\lambda_2 + (c_l^*(\xi_2) - c_{\xi_2}(\lambda_2))\,\lambda_2,$$

we obtain there do have some $\bar{\lambda}_1$ and $\bar{\lambda}_2$ such that

$$G_{\xi_1}(c_1,\bar{\lambda}_1) > 0 \ \text{ and } G_{\xi_2}(c_2,\bar{\lambda}_2) > 0.$$

Therefore, according to the continuous property in mathematical analysis, there exists four unique constants as above. This completes the proof. □

For the construction of lower solution, we give the corresponding function and obtain the desired function properties through the following lemma.

Lemma 2.3 ([17, Lemma 3.2]). *For any* $\delta \in (0,1)$, *define a function*

$$L(z) = z - z^{1+\delta} - \ell z^{1-\delta} \quad \text{for} \quad z > 0.$$

We have the following conclusions:

$$L^{\max} > 0 \quad \text{for} \quad \ell \in (0,\ 1/4) \quad \text{and} \quad L^{\max} \to 0^+,\ \alpha - \beta \to 0^+ \quad \text{as} \quad \ell - 1/4 \to 0^-,$$

where

$$L^{\max} \triangleq \sup_{z>0}\{L(z)\} = L(z_0) \quad \text{for some} \quad z_0 \in (\alpha,\ \beta)$$

and $(\alpha,\ \beta) \triangleq \{z > 0 \mid L(z) > 0\}$ *for* $\ell \in (0,\ 1/4)$. *Moreover, for any* $\hbar > 0$, *there exists* $\ell(\hbar) \in (0,\ 1/4)$ *such that*

$$L^{\max} = \hbar \quad \text{and} \quad \ell(\hbar) \to \frac{1}{4} \quad \text{as} \quad \hbar \to 0^+.$$

2.2. Asymptotic behavior

In this subsection, prove the asymptotic behavior with defining some necessary parameters. Here, the proof is divided into two parts. it is worth noting that the initial value here is an exponential function which satisfies a certain range $\lambda_1 \in (0,\lambda^+)$, $\lambda_2\,(\lambda^-,0)$ of exponential decay rates. The Step 2 is the core content of the article, where utilize the improved “forward-backward spreading” method with constructing the lower solutions that can be combined with this method.

Theorem 2.1 (Long-time asymptotic behavior)**.** *Suppose that the assumptions* (P)*,* (K1) *and* (K2) *hold. If* $u_0(\cdot)$ *satisfies that* $0 \leqslant u_0(x) \leqslant 1$ *for* $x \in \mathbb{R}$*,* $u_0(x_0) > 0$ *for some* $x_0 \in \mathbb{R}$ *and*

$$u_0(x) \sim O\left(\mathrm{e}^{-\lambda_1 x}\right), \quad \text{for} \quad x \geq \tilde{x}, \quad u_0(x) \sim O\left(\mathrm{e}^{-\lambda_2 x}\right), \quad \text{for} \quad x \leq -\tilde{x},$$

where $\lambda_1 \in (0, \lambda^+)$, $\lambda_2 \in (\lambda^-, 0)$ *and* $\tilde{x} \geq 0$*. Then there is some constant* $\hbar \in (0,1)$ *such that the solution* $u(t,x)$ *of equation* (1.1) *has the following properties:*

$$\begin{cases} \lim\limits_{t \to +\infty} \sup\limits_{x - x_0 \leqslant (c(\lambda_2) - \varepsilon)t} u(t,x) = 0, & (a) \\ \inf\limits_{(c(\lambda_2)+\varepsilon)t \leqslant x - x_0 \leqslant (c(\lambda_1)-\varepsilon)t} u(t,x) \geqslant \hbar, & (b) \quad \text{for all } t > 0. \\ \lim\limits_{t \to +\infty} \sup\limits_{x - x_0 \geqslant (c(\lambda_1) + \varepsilon)t} u(t,x) = 0, & (c) \end{cases}$$

Proof. Step 1: *Proof of the* (a) *and* (c) *in the above theorem.*

Construct a function $\bar{u}(t,x)$ satisfying

$$\bar{u}(t,x) = \min\left\{1, \Gamma_0 e^{\lambda_1(-x + c(\lambda_1)t)}, \Gamma_0 e^{\lambda_2(-x + c(\lambda_2)t)}\right\},$$

where $\Gamma_0 \geqslant 1$ is large enough to establish $\bar{u}(0,x) \geqslant u_0(x)$. Since the definition of $G(c,\lambda)$ and $\lambda_1 \in (0, \lambda^+)$, $\lambda_2 \in (\lambda^-, 0)$, we get $G(c(\lambda_1), \lambda_1) = G(c(\lambda_2), \lambda_2) = 0$. Further, a series of calculations implies that:

(1) if $x \leq c(\lambda_2)t + \lambda_2^{-1} \ln \Gamma_0$, then $\bar{u}(t,x) = \Gamma_0 e^{\lambda_2(-x + c(\lambda_2)t)}$,
(2) if $x \geq c(\lambda_1)t + \lambda_1^{-1} \ln \Gamma_0$, then $\bar{u}(t,x) = \Gamma_0 e^{\lambda_1(-x + c(\lambda_1)t)}$,
(3) if $c(\lambda_2)t + \lambda_2^{-1} \ln \Gamma_0 < x < c(\lambda_1)t + \lambda_1^{-1} \ln \Gamma_0$, then $\bar{u}(t,x) = 1$.

Moreover, take the first case, we get that

$$\begin{aligned} & \bar{u}_t(t,x) - k * \bar{u}(t,x) + \bar{u}(t,x) - f(\bar{u}(t,x)) \\ \geq & \Gamma_0 e^{\lambda_2(-x + c(\lambda_2)t)} \left[c(\lambda_2)\lambda_2 - \int_{\mathbb{R}} k(x) e^{\lambda_2 x} dx + 1 - f'(0) \right] \\ \geq & \Gamma_0 e^{\lambda_2(-x + c(\lambda_2)t)} G\left(c(\lambda_2), \lambda_2\right) \\ = & 0, \end{aligned}$$

for any $x \in \mathbb{R}$. Other cases can be the same, that is, prove that the function $\bar{u}$ is the super solution. Therefore, for $t \to +\infty$

$$\begin{aligned} \sup_{x \leqslant (c(\lambda_2) - \varepsilon)t} u(t,x) \leqslant \sup_{x \leqslant (c(\lambda_2) - \varepsilon)t} \bar{u}(t,x) \leqslant \Gamma_0 e^{\lambda_2 \varepsilon t}, \\ \sup_{x \geqslant (c(\lambda_1) + \varepsilon)t} u(t,x) \leqslant \sup_{x \geqslant (c(\lambda_1) + \varepsilon)t} \bar{u}(t,x) \leqslant \Gamma_0 e^{-\lambda_1 \varepsilon t}. \end{aligned}$$

Naturally, (a) and (c) in the above theorem be obtained.

Step 2: *Proof of the* (b) *in the above theorem.*

It follows from $u_0(x_0) > 0$ that there must have some positive constants $\hbar_1$ and d satisfying

$$u_0(x) \geqslant \hbar_1 \text{ for } x \in [-d, d] \tag{2.2}$$

by translating the x-axis. Since $f(u) \in C^1([0,1])$ and $f'(0) > 0$, there exist some constants $\hbar_2 \in (0, \hbar_1]$ such that

$$f(u) \geqslant \left(f'(0) - \frac{\xi}{2}\right) u \quad \text{for} \quad u \in [0, \hbar_2]\,,$$

where $\xi = \min\{\xi_1, \xi_2\}$. Therefore, for any $\delta \in (0,1)$, by taking $H(\delta) = \xi \hbar_2^{-\delta}/2$, we can get that

$$f(u) \geqslant (f'(0) - \xi_i)\, u + H(\delta) u^{1+\delta} \quad \text{for} \quad u \in [0, \hbar_2]\,.$$

Furthermore, there must have some $\delta_1 > 0$, $\delta_2 > 0$ (small enough) and corresponding $\rho_1 \in (\tilde{\theta}_1, \theta_1)$, $\rho_2 \in (\theta_2, \tilde{\theta}_2)$, which not only satisfies

$$\{\rho_1(1-\delta_1),\ \rho_1(1+\delta_1) = c(\lambda_1)\} \subseteq (\tilde{\theta}_1, \theta_1),$$

$$\{\rho_2(1-\delta_2),\ \rho_2(1+\delta_2) = c(\lambda_2)\} \subseteq (\theta_2, \tilde{\theta}_2),$$

but also satisfies another important condition

$$\begin{aligned} &G_i^+ + H(\delta_i) - \sqrt{\left(G_i^+ + H(\delta_i)\right)^2 - 3H(\delta_i)(1+\delta_i)\left[G_i^0 + H(\delta_i)(1+\delta_i)\ell_i\right]} \\ &\geq 3H(\delta_i)(1+\delta_i), \end{aligned} \tag{2.3}$$

where $i = 1, 2$,

$$G_i^+ = G_{\xi_i}\left(c_i, \rho_i(1+\delta_i)\right),\ G_i^0 = G_{\xi_i}\left(c_i, \rho_i(1+\delta_i)\right),\ G_i^- = G_{\xi_i}\left(c_i, \rho_i(1+\delta_i)\right).$$

In order to better characterize the form of the equation after plugging in the following solution, denote two functions

$$g_i(x) = H(\delta_i)(1+\delta_i)x^3 - \left(G_i^+ + H(\delta_i)\right)x^2 + \left(G_i^0 + H(\delta_i)(1+\delta_i)\ell_i\right)x - \ell_i G_i^-,$$

and two important parameters $i = 1, 2$,

$$\tilde{x}_i = \frac{G_i^+ + H(\delta_i) - \sqrt{\left(G_i^+ + H(\delta_i)\right)^2 - 3H(\delta_i)(1+\delta_i)\left[G_i^0 + H(\delta_i)(1+\delta_i)\ell_i\right]}}{3H(\delta_i)(1+\delta_i)}.$$

Moreover,we declare that

$$g_i(\tilde{x}_i) = 0 \quad \text{and} \quad g_i'(\tilde{x}_i) = 0. \tag{2.4}$$

Indeed, the idea here is that the function $g_i(x)$ satisfies certain properties, that is,

$$\begin{aligned} g_i(x) =& H(\delta_i)(1+\delta_i)\left(x - \tilde{x}_i\right)^2 \left(x - x_{i3}\right) \\ =& H(\delta_i)(1+\delta_i)\left[x^3 - \left(2\tilde{x}_i + x_{i3}\right)x^2 + \left(\tilde{x}_i^2 + 2\tilde{x}_i x_{i3}\right)x - \tilde{x}_i^2 x_{i3}\right], \end{aligned}$$

which ensures that the corresponding parameters are exactly equal and here x_{i3} is another root of $g_i(x)$.

Whereafter, using the nature of diffusion, the time is divided into two segments and the lower solution functions are assigned "forward" and "backward" speeds, respectively.

Divide the time period of $[0,\tau]$ into $[0,\kappa\tau]$ and $[\kappa\tau,\tau]$, where $\tau>0$ is an arbitrary constant and

$$\kappa=\frac{X-c_2\tau}{c_1\tau-c_2\tau}\in[0,1]\,,\ \ X\in[c_2\tau,c_1\tau]\,.$$

From some simple analyzing, it is sufficient to justify that for any $c_1\in(c(\lambda_1)-\varepsilon,$ $c(\lambda_1))$ and $c_2\in(c(\lambda_2),c(\lambda_2)+\varepsilon)$, there is a constant $\hbar\in(0,1)$ such that

$$u(\tau,X)\geqslant\hbar \text{ for any given } \tau>0,\ \ X\in[c_2\tau,c_1\tau]\,. \tag{2.5}$$

In $[0,\kappa\tau]$, construct a lower solution which spread at "forward" speed $c_1\in(c(\lambda_1)-\varepsilon,c(\lambda_1))$ as follows

$$\begin{aligned}\underline{u}_1(t,x;\eta_1)=&\max\left\{0,\ L_1\left(e^{\rho_1(-x+c_1t+\eta_1)}\right)\right\}\\ =&\begin{cases}0 & \text{for } x-c_1\kappa\tau\notin\Omega_1,\\ L_1\left(e^{\rho_1(-x+c_1t+\eta_1)}\right) & \text{for } x-c_1\kappa\tau\in\Omega_1,\end{cases}\end{aligned}$$

with

$$\eta_1\in\left[-d+\rho_1^{-1}\ln\beta_1,\ d+\rho_1^{-1}\ln\alpha_1\right],\quad \Omega_1=\left(\eta_1-\rho_1^{-1}\ln\beta_1,\ \eta_1-\rho_1^{-1}\ln\alpha_1\right). \tag{2.6}$$

Therefore, from Lemma 2.3 we can choose $\ell_1\in\left(0,\frac14\right)$ close to $\frac14$ such that

$$L_1^{\max}\triangleq\max_{z>0}\{L_1(z)\}\leqslant\hbar_2\leqslant\hbar_1,\quad \rho_1^{-1}(\ln\beta_1-\ln\alpha_1)\leqslant d/2. \tag{2.7}$$

Next we verify that $\underline{u}_1(t,x;\eta_1)$ is a lower solution of equation (1.1). Firstly, Eq. (2.6) implies that $\Omega_1\subseteq(-d,d)$ and then combing (2.2) and (2.7), it is easy to get that

$$\underline{u}_1(0,x;\eta_1)\leqslant u_0(x)\ \ \text{for}\ \ x\in\mathbb{R}.$$

If $x-c_1t\in\Omega_1$, we have that $\underline{u}_1(t,x;\eta_1)=L_1\left(e^{\rho_1(-x+c_1t+\eta_1)}\right)$. Further, for short, denote $\tilde z_1=e^{\rho_1(-x+c_1t+\eta_1)}$, and some calculations show that

$$\begin{aligned}&\partial_t\underline{u}_1(t,x;\eta_1)-k*\underline{u}_1(t,x;\eta_1)+\underline{u}_1(t,x;\eta_1)-f(\underline{u}_1(t,x;\eta_1))\\ <&G_1^0\tilde z_1-G_1^+\tilde z_1^{1+\delta_1}-\ell_1G_1^-\tilde z_1^{1-\delta_1}-H(\delta_1)\tilde z_1^{1+\delta_1}\left[1-(1+\delta_1)\left(\tilde z_1^{\delta_1}+\ell_1\tilde z_1^{-\delta_1}\right)\right]\\ =&\tilde z_1^{1-\delta_1}\left\{H(\delta_1)(1+\delta_1)\tilde z_1^{3\delta_1}-\left(G_1^++H(\delta_1)\right)\tilde z_1^{2\delta_1}+\left[G_1^0+H(\delta_1)(1+\delta_1)\ell_1\right]\tilde z_1^{\delta_1}-\ell_1G_1^-\right\}.\end{aligned}$$

Combining the fact from (2.3) that $\tilde x_1\geq 1$ and (2.4), we assert that

$$\partial_t\underline{u}_1(t,x;\eta_1)-k*\underline{u}_1(t,x;\eta_1)+\underline{u}_1(t,x;\eta_1)-f(\underline{u}_1(t,x;\eta_1))<0. \tag{2.8}$$

If $x-c_1t\notin\bar\Omega_1$, it is easy to check that $u_1(t,x;\xi_1)=0$ and (2.8) is valid in this case. Furthermore, Lemma 2.1 implies that

$$u(t,x)\geqslant\underline{u}_1(t,x;\eta_1)\ \ \text{for } t\in[0,\kappa\tau],\ \ x\in\mathbb{R}.$$

Define $x_1(t):=c(\lambda_1)t+\eta_1-\rho_1^{-1}\ln z_1$ with $t\in[0,\kappa\tau]$ and $z_1\in(\alpha_1,\beta_1)$ satisfying $L_1(z_1)=L_1^{max}$, then it is clear that

$$u(t,x_1(t))\geqslant\underline{u}_1(t,x_1(t);\eta_1)=L_1^{\max}\ \ \text{for}\ \ t\in[0,\kappa\tau].$$

The arbitrariness of the parameter η_1 in (2.6) shows that

$$u(t,x) \geqslant L_1^{\max} \quad \text{for } t \in [0,\kappa\tau],\ x \in [c_1 t - d/2, c_1 t + d/2]\,.$$

Therefore, we get valid information about the initial value for the next period of time $[\kappa\tau,\tau]$, that is

$$u(\kappa\tau,x) \geq L_1^{\max},\ x \in [c_1\kappa\tau - d/2, c_1\kappa\tau + d/2]\,.$$

In $[\kappa\tau,\tau]$, similarly, construct a lower solution which spreads at "backward" speed $c_2 \in (c(\lambda_2), c(\lambda_2)+\varepsilon)$ as follows

$$\begin{aligned}\underline{u}_2(t,x;\eta_2) &= \max\left\{0,\ L_2\left(e^{\rho_2(-x+c_2t+\eta_2)}\right)\right\}\\ &= \begin{cases}0 & \text{for } x - c_2\kappa\tau \notin \Omega_2,\\ L_2\left(e^{\rho_2(-x+c_2t+\eta_2)}\right) & \text{for } x - c_2\kappa\tau \in \Omega_2,\end{cases}\end{aligned}$$

with

$$\eta_2 \in \left[(c_1-c_2)\,\kappa\tau + \rho_2^{-1}\ln\beta_2 - d/2,\ \ (c_1-c_2)\,\kappa\tau + \rho_2^{-1}\ln\alpha_2 + d/2\right],$$

and

$$\Omega_2 = \left(\eta_2 - \rho_2^{-1}\ln\beta_2,\ \eta_2 - \rho_2^{-1}\ln\alpha_2\right).$$

Let us choose $\ell_2 \in \left(0,\frac{1}{4}\right)$ close to $\frac{1}{4}$ such that

$$L_2^{\max} \triangleq \max_{z>0}\{L_2(z)\} \leq L_1^{\max} \leqslant \hbar_2 \leqslant \hbar_1, \quad \rho_2^{-1}(\ln\beta_2 - \ln\alpha_2) \leqslant d/2.$$

Therefore, we easily get

$$\underline{u}_2(\kappa\tau,x,\eta_2) \leq L_2^{max} \leq L_1^{max} \leq u(\kappa\tau,x),\ x \in \mathbb{R}.$$

If $x - c_2 t \in \Omega_2$, we get that $\underline{u}_2(t,x;\eta_2) = L_2\left(e^{\rho_2(-x+c_2t+\eta_2)}\right)$ and

$$\begin{aligned}&\partial_t\underline{u}_2(t,x;\eta_2) - k * \underline{u}_2(t,x;\eta_2) + \underline{u}_2(t,x;\eta_2) - f(\underline{u}_2(t,x;\eta_2))\\ \leqslant& G_{\xi_2}(c_2,\rho_2)\,e^{\rho_2(-x+c_2t+\eta_2)} - \left[G_{\xi_2}(c_2,\rho_2(1+\delta_2)) - H(\delta_2)\right]e^{\rho_2(1+\delta_2)(-x+c_2t+\eta_2)}\\ &- \ell_2 G_{\xi_2}(c_2,\rho_2(1-\delta_2))\,e^{\rho_2(1-\delta_2)(-x+c_2t+\eta_2)}.\end{aligned}$$

Similarly, noticing $g_2(\tilde{x}_2) = 0,\ g_2'(\tilde{x}_2) = 0$ and the fact from (2.3) that $\tilde{x}_2 \geq 1$, we assert that

$$\partial_t\underline{u}_2(t,x;\eta_2) - k * \underline{u}_2(t,x;\eta_2) + \underline{u}_1(t,x;\eta_2) - f(\underline{u}_2(t,x;\eta_2)) < 0. \tag{2.9}$$

If $x - c_2 t \notin \bar{\Omega}_2$, we have $u_2(t,x;\xi_1) = 0$ and (2.9). Further, Lemma 2.1 implies that

$$u(t,x) \geqslant \underline{u}_2(t,x;\eta_2) \ \text{ for } t \in [\kappa\tau,\tau],\ x \in \mathbb{R}.$$

Define $x_2(t) := c_2 t + \eta_2 - \rho_2^{-1}\ln z_2$, where $z_2 \in (\alpha_2,\beta_2)$ and satisfying $L_2^{\max} = L_2(z_2)$. We get

$$u(t,x_2(t)) \geq \underline{u}_2(t,x_2(t);\eta_2) = L_2(z_2) = L_2^{max}.$$

Then by choosing $\eta_2 := (c_1 - c_2)\kappa\tau + \rho_2^{-1} \ln z_2$, we can verify that at time $t = \tau$

$$x_2(\tau) = c_1\kappa\tau + c_2(1-\kappa)\tau = X \quad \text{and} \quad u(\tau, X) \geq L_2^{max}.$$

Noting the arbitrariness of the parameters $\kappa \in [0,1]$, we conclude that $u(t,x) \geq \hbar$ for any $t > 0$ and $x \in [c_2 t, c_1 t]$ by taking $\hbar = L_2^{max}$. Furthermore,

$$\min_{(c(\lambda_2)+\varepsilon)t \leq x \leq (c(\lambda_1)-\varepsilon)t} u(t,x) \geq \hbar \quad \text{for any} \quad t > 0,$$

which mean Eq. (2.5) holds. This completes the proof. $\square$

Remark 2.1. To be clear, when $\lambda_1 = \lambda_r^*,\ \lambda_2 = \lambda_l^*$, the conclusion becomes that there is some constant $\hbar \in (0,1)$ such that the solution $u(t,x)$ of equation (1.1) has the following properties:

$$\begin{cases} \lim\limits_{t\to+\infty} \sup\limits_{x-x_0 \leqslant (c_l^*-\varepsilon)t} u(t,x) = 0, & \\ \inf\limits_{(c_l^*+\varepsilon)t \leqslant x-x_0 \leqslant (c_r^*-\varepsilon)t} u(t,x) \geqslant \hbar, & \text{for all } t > 0. \\ \lim\limits_{t\to+\infty} \sup\limits_{x-x_0 \geqslant (c_r^*+\varepsilon)t} u(t,x) = 0, & \end{cases}$$

This corresponds exactly to the long-time asymptotic behavior in the first kind of initial value case in [17].

In fact, our conclusion extends the range of initial asymptotic decay rate, or in other words, by an improved method achieves the long-time asymptotic behavior under a wide range of initial values in the case of asymmetric diffusion kernel function.

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