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DYNAMICS OF A TWO-PREY AND ONE PREDATOR SYSTEM WITH QUADRATIC SELF INTERACTION*

Lingling Liu[1], Ke-wei Ding[2] and Zhiheng Yu[3,†]

Abstract A two-prey and one-predator system with quadratic self-interaction is discussed on subsets of special biological sense, none of which is closed under operations of the polynomial ring. The known work studied the stability of the boundary equilibria and gave invariant algebraic surfaces up to degree two but no further discussion for bifurcations. In this paper, we investigate the finite and infinite equilibria and their qualitative properties in the first octant. Moreover, we discuss their bifurcations, such as transcritical bifurcation on boundary equilibria, and give the bifurcation diagram. Finally, simulation examples are given to illustrate the theoretical results in this paper.

Keywords Coexistence, stability, saddle-node, transcritical bifurcation.

MSC(2010) 34C23, 34C60, 37G10.

1. Introduction

Recently, many researchers [1–3, 5, 8, 10, 12–14, 16, 17] have been increasingly interested in predator and prey systems. As said in [14], the Lotka-Volterra model describes interactions between several species in an ecosystem, predators and preys. Hsu *et al.* [10] studied the Lotka-Volterra model with two predators and a single prey and obtained the parameter range of the validity of the principle of competitive exclusion, and provided a wide range of parameter values for the coexistence of two predators numerically. The integrability of a 3-dimensional Lotka-Volterra system was considered by Aziz [2] and Bountis *et al.* [3]. Attention to Hopf bifurcation is paid to this model for obtaining a stable periodic orbit which implies the coexistence of two competing predators [11, 15]. Llibre and Xiao [13] considered a class of 3-dimensional Lotka-Volterra systems with eight parameters that contain

†The corresponding author.
1School of Sciences and Institute for Artificial Intelligence, Southwest Petroleum University, 610500 Chengdu, China
2School of Computer Science and Technology, Southwest Minzu University, 610041 Chengdu, China
3School of Mathematics, Southwest Jiaotong University, 610031 Chengdu, China
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Email: a600aa@163.com(L. Liu), bluedkw@163.com(K. Ding), yuzhiheng9@163.com(Z. Yu)

two predators competing for the same food. They obtained sufficient and necessary conditions for the principle of competitive exclusion to hold and gave the global dynamical behavior of the three species.

A 3-dimensional system of Lotka-Volterra type with nine parameters(two-prey and one-predator) was proposed by [14]

$$\begin{cases} \dfrac{dx}{dt} = x(a_1 - b_1 x - c_1 y), \\ \dfrac{dy}{dt} = y(-a_2 + b_2 x + c_2 z), \\ \dfrac{dz}{dt} = z(a_3 - b_3 y - c_3 z), \end{cases} \tag{1.1}$$

where x and z are the prey A and prey B populations respectively, y is the predator population, a_1 and a_3 are the growth rates of prey A and prey B in the absence of the predator, a_2 is the death rate of the predator in the absence of the preys, c_1 and b_3 are the death rates of prey A and prey B due to predation, b_2 and c_2 are the consumption rates of the predator over prey A and prey B, b_1 and c_3 are the quadratic self interaction rates of prey A and prey B. All parameters $a_i, b_i, c_i (i = 1, 2, 3)$ are positive. In [14], they gave an observer design by using the convergence characteristics to monitor the system (1.1). In 2018, Aybar *et al.* (see [1]) discussed the stability of the equilibria and gave some numerical simulations to illustrate these analytical results. They found the invariant algebraic surfaces up to degree 2. However, the complete qualitative properties of the equilibria and their local bifurcations are still unknown.

In this paper, we first simplify the system (1.1) and investigate the conditions of the exact number of equilibria and then study the qualitative properties of the finite and infinite equilibria in Section 2. We prove that no periodic orbit bifurcates from Hopf bifurcation for system (1.1). In Section 3, we study their bifurcations such as transcritical bifurcation on boundary equilibria. In section 4, the bifurcation diagram and simulation example are given to illustrate the theoretical results.

2. Existence of the Equilibria

First of all, we make the transformation

$$x \mapsto a_1^{-1} b_1 \, x, \; y \mapsto a_1^{-1} c_1 \, y, \; z \mapsto a_1^{-1} c_2 z, \; t \mapsto a_1 \, t$$

to reduce system (1.1) to the form

$$\begin{cases} \dfrac{dx}{dt} = x(1 - x - y), \\ \dfrac{dy}{dt} = y(-k_1 + k_2 x + z), \\ \dfrac{dz}{dt} = z(k_3 - k_4 y - k_5 z), \end{cases} \tag{2.1}$$

where $k_1 = a_1^{-1} a_2$, $k_2 = b_1^{-1} b_2$, $k_3 = a_1^{-1} a_3$, $k_4 = c_1^{-1} b_3$ and $k_5 = c_2^{-1} c_3$ and the number of parameters is reduced to five.

2.1. Finite equilibria

Finite equilibria of system (1.1) were discussed in [1]. The reduced system (2.1) containing fewer parameters helps us in the computation of center manifolds, normal forms, and determining quantities. In view of the physical meaning, we only consider equilibria of system (2.1) in the closure $\bar{\mathcal{Q}}$ of first octant $\mathcal{Q} := \{(x, y, z) : x > 0, y > 0, z > 0\}$ for all possibilities of $(k_1, k_2, k_3, k_4, k_5) \in \mathbb{R}_+^5$. We will partition the parameter space $\mathbb{R}_+^5$

$$\mu := (k_1, k_2, k_3, k_4, k_5) \tag{2.2}$$

for various cases of equilibria. In order to state our results easily, we partition the parameter space $\mathbb{R}_+^5$ into the following subregions:

$$\begin{aligned}
&\mathcal{P}_7 := \{\mu \in \mathbb{R}_+^5 : k_2 > k_1, \max\{k_1k_5, k_4 - \frac{k_1k_4}{k_2}\} < k_3 < k_4 + k_1k_5\},\\
&\mathcal{P}_{61} := \{\mu \in \mathbb{R}_+^5 : k_2 > k_1, k_3 > k_4 + k_1k_5\},\\
&\mathcal{P}_{62} := \{\mu \in \mathbb{R}_+^5 : k_2 > k_1, k_1k_5 < k_3 < k_4 - \frac{k_1k_4}{k_2}\},\\
&\mathcal{P}_{63} := \{\mu \in \mathbb{R}_+^5 : k_2 < k_1, k_1k_5 < k_3 < k_4 + k_1k_5\},\\
&\mathcal{P}_{64} := \{\mu \in \mathbb{R}_+^5 : k_2 > k_1, k_4 - \frac{k_1k_4}{k_2} < k_3 < k_1k_5\},\\
&\mathcal{P}_{51} := \{\mu \in \mathbb{R}_+^5 : k_2 < k_1, k_3 > k_4 + k_1k_5\},\\
&\mathcal{P}_{52} := \{\mu \in \mathbb{R}_+^5 : k_2 > k_1, k_3 < \min\{k_1k_5, k_4 - \frac{k_1k_4}{k_2}\}\},\\
&\mathcal{P}_{53} := \{\mu \in \mathbb{R}_+^5 : k_2 < k_1, k_5(k_1 - k_2) < k_3 < k_1k_5\},\\
&\mathcal{P}_4 := \{\mu \in \mathbb{R}_+^5 : k_2 < k_1, k_3 < k_5(k_1 - k_2)\},
\end{aligned}$$

with the surfaces

$$\begin{aligned}
&\mathcal{S}_{11} := \{\mu \in \mathbb{R}_+^5 : k_1 = k_2, 0 < k_3 < k_2k_5\},\\
&\mathcal{S}_{12} := \{\mu \in \mathbb{R}_+^5 : k_1 = k_2, k_2k_5 < k_3 < k_4 + k_2k_5\},\\
&\mathcal{S}_{13} := \{\mu \in \mathbb{R}_+^5 : k_1 = k_2, k_3 > k_4 + k_2k_5\},\\
&\mathcal{S}_{21} := \{\mu \in \mathbb{R}_+^5 : k_3 = k_4 + k_1k_5, 0 < k_1 < k_2\},\\
&\mathcal{S}_{22} := \{\mu \in \mathbb{R}_+^5 : k_3 = k_4 + k_1k_5, k_1 > k_2\},\\
&\mathcal{S}_{31} := \{\mu \in \mathbb{R}_+^5 : k_3 = k_1k_5, 0 < k_1 < \frac{k_2k_4}{k_4 + k_2k_5}\},\\
&\mathcal{S}_{32} := \{\mu \in \mathbb{R}_+^5 : k_3 = k_1k_5, \frac{k_2k_4}{k_4 + k_2k_5} < k_1 < k_2\},\\
&\mathcal{S}_{33} := \{\mu \in \mathbb{R}_+^5 : k_3 = k_5k_1, k_1 > k_2\},\\
&\mathcal{S}_4 := \{\mu \in \mathbb{R}_+^5 : k_3 = k_5(k_1 - k_2), k_1 > k_2\},\\
&\mathcal{S}_{51} := \{\mu \in \mathbb{R}_+^5 : k_3 = k_4 - \frac{k_1k_4}{k_2}, 0 < k_1 < \frac{k_2k_4}{k_4 + k_2k_5}\},\\
&\mathcal{S}_{52} := \{\mu \in \mathbb{R}_+^5 : k_3 = k_4 - \frac{k_1k_4}{k_2}, \frac{k_2k_4}{k_4 + k_2k_5} < k_1 < k_2\},
\end{aligned}$$

and their boundaries

$$T_1 := \{\mu \in \mathbb{R}_+^5 : k_1 = \frac{k_2k_4}{k_4 + k_2k_5}, k_3 = \frac{k_2k_4k_5}{k_4 + k_2k_5}\},$$

$$T_2 := \{\mu \in \mathbb{R}_+^5 : k_1 = k_2, k_3 = k_2k_5\},$$
$$T_3 := \{\mu \in \mathbb{R}_+^5 : k_1 = k_2, k_3 = k_4 + k_2k_5\}.$$

Lemma 2.1. *System* (2.1) *has at most 7 isolated equilibria in the first quadrant $\mathcal{Q}$. The number of equilibria and corresponding parameter conditions are described in Table 1.*

Table 1. The number of Equilibria.

Conditions	Number	Equilibria
$\mathcal{P}_7$	7	$O, E_{01}, E_{02}, E_1, E_2, E_3, E_*$
$\mathcal{P}_{61} \bigcup \mathcal{P}_{62} \bigcup \mathcal{S}_{21} \bigcup \mathcal{S}_{51}$	6	$O, E_{01}, E_{02}, E_1, E_2, E_3$
$\mathcal{P}_{63} \bigcup \mathcal{S}_{12}$	6	$O, E_{01}, E_{02}, E_1, E_2, E_*$
$\mathcal{P}_{64} \bigcup \mathcal{S}_{32}$	6	$O, E_{01}, E_{02}, E_2, E_3, E_*$
$\mathcal{P}_{51} \bigcup \mathcal{S}_{13} \bigcup \mathcal{S}_{22} \bigcup T_3$	5	$O, E_{01}, E_{02}, E_1, E_2$
$\mathcal{P}_{52} \bigcup \mathcal{S}_{31} \bigcup \mathcal{S}_{52} \bigcup T_1$	5	$O, E_{01}, E_{02}, E_2, E_3$
$\mathcal{P}_{53} \bigcup \mathcal{S}_{11} \bigcup \mathcal{S}_{33} \bigcup T_2$	5	$O, E_{01}, E_{02}, E_2, E_*$
$\mathcal{P}_4 \bigcup \mathcal{S}_4$	4	O, E_{01}, E_{02}, E_2

Proof. The equilibria of system (2.1) are determined by the polynomial system

$$\begin{cases} x(1-x-y) = 0, \\ y(-k_1 + k_2x + z) = 0, \\ z(k_3 - k_4y - k_5z) = 0. \end{cases} \tag{2.3}$$

Every boundary equilibrium, i.e., an equilibrium on the boundary $\partial\bar{\mathcal{Q}}$, has at least one vanished component. Obviously, $O : (0, 0, 0)$, $E_{01} : (1, 0, 0)$, $E_{02} : (0, 0, \frac{k_3}{k_5})$ are equilibria on axes, which exist for all $\mu \in \mathbb{R}_+^5$. Known from (2.3), other boundary equilibria, lying on coordinate planes, are $E_1 : (0, \frac{k_3-k_1k_5}{k_4}, k_1)$, $E_2 : (1, 0, \frac{k_3}{k_5})$ and $E_3 : (\frac{k_1}{k_2}, 1 - \frac{k_1}{k_2}, 0)$ which exist as $k_3 > k_1k_5$, $\mu \in \mathbb{R}_+^5$ and $k_2 > k_1$ respectively. In addition, system (2.1) has a unique interior equilibrium $E_* : (x_*, y_*, z_*)$ in $\mathcal{Q}$, where

$$x_* = \frac{k_1k_5 - k_3 + k_4}{k_2k_5 + k_4}, y_* = \frac{k_3 - k_1k_5 + k_2k_5}{k_2k_5 + k_4}, z_* = \frac{k_1k_4 + k_2k_3 - k_2k_4}{k_2k_5 + k_4},$$

if and only if $\max\{k_5(k_1 - k_2), k_4 - \frac{k_1k_4}{k_2}\} < k_3 < k_4 + k_1k_5$. □

Theorem 2.1. *Equilibria of system* (2.1) *have the following qualitative properties:*
(i) *The origin O is a saddle of type 2, having two positive and one negative eigenvalues.*
(ii) *E_{01} is either a saddle of type 1 if $k_2 < k_1$ or a saddle of type 2 if $k_2 > k_1$. E_{01} is degenerate if $k_2 = k_1$; E_{02} is either a saddle of type 1 if $k_3 < k_1k_5$ or a saddle of type 2 if $k_3 > k_1k_5$. E_{02} is degenerate if $k_3 = k_1k_5$; E_2 is either a stable node if $k_3 < k_5(k_1 - k_2)$ or a saddle of type 1 if $k_3 > k_5(k_1 - k_2)$. E_2 is degenerate if $k_3 = k_5(k_1 - k_2)$.*
(iii) *For $k_3 > k_1k_5$, E_1 is either a stable node or focus if $k_3 > k_4 + k_1k_5$ or a saddle of type 1 if $k_3 < k_4 + k_1k_5$. If $k_3 = k_4 + k_1k_5$, E_1 is degenerate. For $k_2 > k_1$, E_3 is either a stable node or focus if $k_3 < k_4 - \frac{k_1k_4}{k_2}$ or a saddle of type 1 if $k_3 > k_4 - \frac{k_1k_4}{k_2}$.*

If $k_3 = k_4 - \frac{k_1 k_4}{k_2}$, E_3 *is degenerate.*
(iv) *If* $\max\{k_5(k_1-k_2), k_4 - \frac{k_1k_4}{k_2}\} < k_3 < k_4 + k_1k_5$, E_* *is a stable node or focus.*

Proof.
Compute the Jacobian matrix of the vector field (2.1)

$$A := \begin{pmatrix} 1-2x-y & -x & 0 \\ k_2 y & -k_1 + k_2 x + z & y \\ 0 & -k_4 z & k_3 - k_4 y - 2k_5 z \end{pmatrix}.$$

System (2.1) has eigenvalues 1, k_3 and $-k_1$ at O. Then the origin O is a saddle of type 2 (see the definition of type from page 106 of [9]). At E_{01}, system (2.1) has eigenvalues k_3, -1 and $k_2 - k_1$. It is easy to see that E_{01} is either a saddle of type 1 if $k_2 < k_1$ or a saddle of type 2 if $k_2 > k_1$. When $k_2 = k_1$, E_{01} is degenerate. We leave this case to Section 3. At E_{02}, system (2.1) has eigenvalues 1, $-k_3$ and $(k_3 - k_1k_5)/k_5$. Thus, E_{02} is either a saddle of type 1 if $k_3 < k_1k_5$ or a saddle of type 2 if $k_3 > k_1k_5$. When $k_3 = k_1k_5$, E_{02} is degenerate. We leave this case to Section 3. At E_2, system (2.1) has eigenvalues -1, $-k_3$ and $-k_1 + k_2 + \frac{k_3}{k_5}$. We obtain that E_2 is either a stable node if $k_3 < k_5(k_1 - k_2)$ or a saddle of type 1 if $k_3 > k_5(k_1 - k_2)$. When $k_3 = k_5(k_1 - k_2)$, E_2 is degenerate. We leave this case to Section 3.

For $k_3 > k_1k_5$, system (2.1) at E_1 has eigenvalues $(k_4 + k_1k_5 - k_3)/k_4$ and $-(k_1k_5/2) \pm \sqrt{k_1^2k_5^2 + 4k_1^2k_5 - 4k_1k_3}$. We can see that E_1 is either a stable node or focus if $k_3 > k_4 + k_1k_5$ or a saddle of type 1 if $k_3 < k_4 + k_1k_5$. When $k_3 = k_4 + k_1k_5$, E_1 is degenerate. We leave this case to Section 3. For $k_2 > k_1$, system (2.1) at E_3 has eigenvalues $k_3 - k_4 - \frac{k_1k_4}{k_2}$ and $(-k_1 \pm \sqrt{k_1^2 + 4k_1^2k_2 - 4k_1k_2^2})/2k_2$. We obtain that E_3 is either a stable node or focus if $k_3 < k_4 - \frac{k_1k_4}{k_2}$ or a saddle of type 1 if $k_3 > k_4 - \frac{k_1k_4}{k_2}$. When $k_3 = k_4 - \frac{k_1k_4}{k_2}$, E_3 is degenerate. We leave this case to Section 3.

We further discuss the qualitative properties of equilibrium E_* when $\max\{k_5(k_1 - k_2), k_4 - \frac{k_1k_4}{k_2}\} < k_3 < k_4 + k_1k_5$. At E_*, the characteristic polynomial of the Jacobian matrix A is

$$\Phi(\lambda) := \lambda^3 + f_2\lambda^2 + f_1\lambda + f_0,$$

where

$$\begin{aligned}
f_2 &= \frac{k_1k_4k_5 + k_2k_3k_5 - k_2k_4k_5 + k_1k_5 - k_3 + k_4}{k_2k_5 + k_4} = x_* + k_5 z_*,\\
f_1 &= \frac{1}{(k_4 + k_2k_5)^2}\{-k_1^2k_2k_5^2 - k_1^2k_4^2k_5 + k_1^2k_4k_5^2 + k_1k_2^2k_5^2 - k_1k_2k_3k_4k_5 + k_1k_2k_3k_5^2\\
&\quad + 2k_1k_2k_4^2k_5 - k_1k_2k_4k_5^2 + k_2^2k_3k_4k_5 - k_2^2k_4^2k_5 + 2k_1k_2k_3k_5 - k_1k_2k_4k_5 + k_1k_3k_4^2\\
&\quad - k_1k_3k_4k_5 + k_1k_4^2k_5 - k_2^2k_3k_5 + k_2^2k_4k_5 + k_2k_3^2k_4 - k_2k_3^2k_5 - k_2k_3k_4^2\\
&\quad + 2k_2k_3k_4k_5 - k_2k_4^2k_5 - k_2k_3^2 + k_2k_3k_4\} = k_2x_*y_* + k_5x_*z_* + k_4y_*z_*,\\
f_0 &= \frac{(k_1k_4 + k_2k_3 - k_2k_4)(k_3 - k_1k_5 + k_2k_5)(k_1k_5 - k_3 + k_4)}{(k_2k_5 + k_4)^2} = (k_4 + k_2k_5)x_*y_*z_*.
\end{aligned}$$

According to the Routh-Hurwitz criterion [7], all eigenvalues have negative real parts if and only if $f_2, f_1, f_0 > 0$ and $f_1f_2 > f_0$, implying that E_* is a stable node or focus in the case that $f_2, f_1, f_0 > 0$ and $f_1 f_2 > f_0$. It is easy to see that $f_2, f_1, f_0 > 0$. We can compute

$$f_1 f_2 - f_0 = k_2 x_*^2 y_* + k_5 x_*^2 z_* + k_5^2 x_* z_*^2 + k_4 k_5 y_* z_*^2 > 0.$$

Thus, E_* is a stable node or focus. □

Remark 2.1. Lemma 2.1 gives the parameter condition for the coexistence of exactly 4, 5, 6, and 7 equilibria. Although Proposition 1 in [1] gives the condition under which the interior equilibrium E_* is stable, we prove that E_* is always locally asymptotically stable in Theorem 2.1.

Those degenerate cases mentioned in Theorem 2.1 need further discussion for their qualitative properties and bifurcations and will be discussed in Section 3.

2.2. Equilibria at infinity

In this section, we discuss the equilibria at infinity of system (2.1). Note system (2.1) defines a polynomial vector field in $\mathbb{R}^3$. We use the Poincaré compactification of $\mathbb{R}^3_+$ for this study. We have the following theorem.

Theorem 2.2. *System* (2.1) *restricted to the compactification of* $\mathbb{R}^3_+$ *has two isolated equilibria* $I_i (i = 1, 2)$ *at infinity which are the endpoints of the invariant positive x-axis, and one isolated equilibrium* O_I *at infinity which is the endpoint of the invariant plane* $x = 0$ *intersection with the compactification of* $\mathbb{R}^3_+$. *Furthermore,* I_1 *is an unstable node,* I_2 *is a saddle with a 2-dimensional unstable manifold and a 1-dimensional stable manifold, and* O_I *is degenerate which has a 2-dimensional stable manifold and a 1-dimensional unstable manifold.*

Proof. Using the Poincaré compactification in $\mathbb{R}^3$ (see [6, 13] for more details), we consider the unit sphere $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \sum_{i=1}^4 x_i^2 = 1\}$ in $\mathbb{R}^4$ and its equator $S^2 = \{(x_1, x_2, x_3, x_4) \in S^3 : x_4 = 0\}$. Denote $U_1 := \{x \in S^3 : x_1 > 0\}$ and $U_2 := \{x \in S^3 : x_2 > 0\}$ and the diffeomorphisms $F_i : U_i \to \mathbb{R}^3 (i = 1, 2)$ are the inverses of the central projections from the origin to the tangent planes at the points $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ respectively.

In the local chart U_1, with the change of variables

$$x = \frac{1}{y_3}, \quad y = \frac{y_1}{y_3}, \quad z = \frac{y_2}{y_3},$$

and time-rescaling $d\tau = dt/y_3$, system (2.1) is transformed into

$$\begin{cases} \dfrac{dy_1}{d\tau} = y_1(1 + k_2 + y_1 + y_2 - y_3 - k_1 y_3), \\ \dfrac{dy_2}{d\tau} = y_2(1 + y_1 - k_4 y_1 - k_5 y_2 - y_3 + k_3 y_3), \\ \dfrac{dy_3}{d\tau} = y_3(1 + y_1 - y_3). \end{cases} \tag{2.4}$$

Note that the infinity of $\mathbb{R}^3_+$ in the local chart U_1 corresponds to the invariant plane $y_3 = 0$ with $y_1 \geq 0$ and $y_2 \geq 0$. System (2.4) in the local chart U_1 has two equilibria

$$I_1 : (0, 0, 0), \quad I_2 : (0, \frac{1}{k_5}, 0)$$

at infinity which has eigenvalues 1(with multiplicity 2), k_2+1 and 1, $k_2+1+\frac{k_2}{k_5}$, -1 respectively. Thus, I_1 is an unstable node and I_2 is a saddle with a 2-dimensional unstable manifold and a 1-dimensional stable manifold.

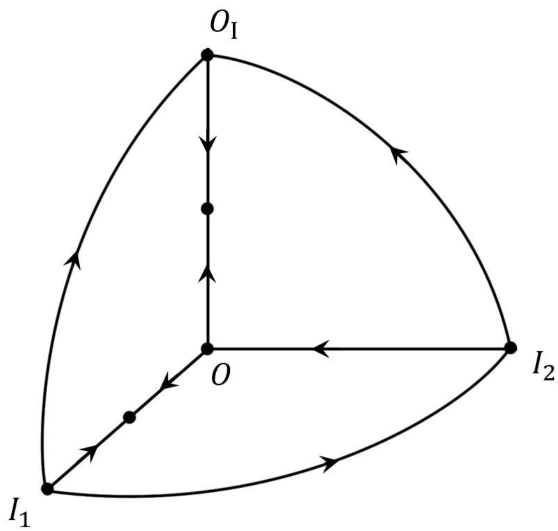

Figure 1. The phase portraits of system (2.1) near all equilibria at infinity on S^2.

Since the local chart U_1 does not contain the plane $x = 0$ at infinity, we further consider system (2.1) in the local chart U_2. Using the transformation

$$x = \frac{z_1}{z_3}, \quad y = \frac{1}{z_3}, \quad z = \frac{z_2}{z_3},$$

and time-rescaling $d\tau = dt/z_3$, system (2.1) is changed into

$$\begin{cases} \dfrac{dz_1}{d\tau} = z_1(-1 - k_2 z_1 - z_1 - z_2 + z_3 + k_1 z_3), \\ \dfrac{dz_2}{d\tau} = z_2(-k_4 - k_2 z_1 - k_5 z_2 - z_2 + k_1 z_3 + k_3 z_3), \\ \dfrac{dz_3}{d\tau} = z_3(-k_2 z_1 - z_2 + k_1 z_3). \end{cases} \tag{2.5}$$

Note that we focus on the equilibrium with $z_1 = z_3 = 0$ and $z_2 \geq 0$ of system (2.5) corresponding to the infinity of $\mathbb{R}^3_+$ in the plane $x = 0$ for system (2.1). System (2.5) in the local chart U_2 has a unique equilibrium $O_I : (0, 0, 0)$ at infinity which has eigenvalues $-1, -k_4$, and 0. Thus, O_I is degenerate and has a 2-dimensional stable manifold and a 1-dimensional center manifold. Since the z-axis is an invariant of system (2.1) and tangent to the center space, the z-axis is the center manifold at O_I. It is easy to see that restricted on the center manifold, system (2.1) becomes $\frac{dz}{dt} = -k_5 z^2$ and the origin is a saddle-node. The solution of this system on the z-axis, except the equilibrium O_I at infinity, will approach the origin in forward time. Hence, O_I has a 1-dimensional unstable manifold on its center manifold. Thus, the phase portraits of system (2.1) near all equilibria at infinity on S^2 are shown in Figure 1. $\square$

3. Local bifurcations

Theorem 2.1 shows that system (2.1) has degenerate equilibria E_{01}, E_{02}, E_1, E_2 and E_3 if $k_2 = k_1$, $k_3 = k_1 k_5$, $k_3 = k_4 + k_1 k_5$, $k_3 = k_5(k_1 - k_2)$ and $k_3 = k_4 - \frac{k_1 k_4}{k_2}$ respectively. In this section, we discuss the local bifurcations near E_{01}, E_{02}, E_1, E_2 and E_3 for $k_2 = k_1$, $k_3 = k_1 k_5$, $k_3 = k_4 + k_1 k_5$, $k_3 = k_5(k_1 - k_2)$ and $k_3 = k_4 - \frac{k_1 k_4}{k_2}$, respectively. Let $\mu_{01} := k_2 - k_1$, $\mu_{02} := k_3 - k_1 k_5$, $\mu_1 := k_3 - (k_4 + k_1 k_5)$, $\mu_2 := k_3 - k_5(k_1 - k_2)$ and $\mu_3 := k_3 - k_4 + \frac{k_1 k_4}{k_2}$.

Theorem 3.1. **(i)** *For $k_2 = k_1$, E_{01} is a saddle-node of system (2.1). Moreover, as k_2 crosses k_1, a transcritical bifurcation happens at E_{01} such that a saddle E_{01} of type 1 changes into a saddle E_{01} of type 2 and stable node or focus E_3.*
(ii) *For $k_3 = k_1 k_5$, E_{02} is a saddle-node of system (2.1). Moreover, as k_3 crosses $k_1 k_5$, a transcritical bifurcation happens at E_{02} such that a saddle E_{02} of type 1 changed into a saddle E_{02} of type 2 and a saddle E_1 of type 1.*
(iii) *For $k_3 = k_1 k_5 + k_4$, E_1 is a saddle-node of system (2.1). Moreover, as k_3 crosses $k_4 + k_1 k_5$, a transcritical bifurcation happens at E_1 such that a saddle E_1 of type 1 and a stable node or focus E_* changed into a stable node or focus E_1.*
(iv) *For $k_1 > k_2$, $k_3 = k_5(k_1 - k_2)$, E_2 is a saddle-node of system (2.1). Moreover, as k_3 crosses $k_5(k_1 - k_2)$, a transcritical bifurcation happens at E_2 such that a stable node or focus E_2 changes into a saddle E_2 with type 1 and a stable node or focus E_*.*
(v) *For $k_2 > k_1$ and $k_3 = k_4 - \frac{k_1 k_4}{k_2}$, E_3 is a saddle-node of system (2.1). Moreover, as k_3 crosses $k_4 - \frac{k_1 k_4}{k_2}$, a transcritical bifurcation happens at E_3 such that a stable node or focus E_3 changes into a saddle E_3 of type 1 and a stable node or focus E_*.*

Proof.

When $\mu_{01} = 0$, i.e., $k_2 = k_1$, system (2.1) at E_{01} has eigenvalues k_3, -1 and 0. For sufficiently small $|\mu_{01}|$, we can transform system (2.1) into the form

$$\begin{cases} \dfrac{dx}{dt} = -x - x^2 - (k_1 + 1)xz - yz - k_1 z^2, \\ \dfrac{dy}{dt} = k_3 y - k_5 y^2 + k_4(1 + \mu_{01})yz, \\ \dfrac{dz}{dt} = \mu_{01} z + (k_1 + \mu_{01})xz + yz + (k_1 + \mu_{01})z^2, \end{cases} \tag{3.1}$$

by translating E_{01} to the origin O and diagonalizing the linear part of system (3.1) in the case that $\mu_{01} = 0$. Suspended with the parameter μ_{01}, it can be regarded as a 4-dimensional one. The center manifold theory ([4]) shows that the suspended system has a smooth 2-dimensional center manifold

$$\begin{aligned} \mathcal{W}^c_{\mu_{01}} = \{(x, y, z, \mu_{01}) \,|\, x = h_1(z, \mu_{01}), y = h_2(z, \mu_{01}),\ h_1(0,0) = h_2(0,0) = 0, \\ Dh_1(0,0) = Dh_2(0,0) = 0\} \end{aligned}$$

near the origin and the smooth functions h_1 and h_2 can be approximated as

$$h_1(z, \mu_{01}) := \phi_{21}(z, \mu_{01}) + O(\|(z, \mu_{01})\|^3)$$

and

$$h_2(z, \mu_{01}) := \phi_{22}(z, \mu_{01}) + O(\|(z, \mu_{01})\|^3)$$

respectively, where the second order approximation ϕ_{21} and ϕ_{22}, by Theorem 3 in [4], satisfies

$$(M\phi_{21})(z,\mu_{01}) := (\partial\phi_{21})/(\partial z)\big(\mu_{01}z + (k_1+\mu_{01})z^2\big) + \big(\phi_{21} + (k_1+1)\phi_{21}z + \phi_{22}z + k_1z^2\big) = O(\|(z,\mu_{01})\|^3) \tag{3.2}$$

and

$$(M\phi_{22})(z,\mu_{01}) := (\partial\phi_{22})/(\partial z)\big(\mu_{01}z + (k_1+\mu_{01})z^2\big) - \big(k_3\phi_{22} + k_4(1+\mu_{01})\phi_{22}z\big) = O(\|(z,\mu_{01})\|^3), \tag{3.3}$$

respectively. Comparing the coefficients in (3.2) and (3.3), we obtain

$$\phi_{21}(z,\mu_{01}) = -k_1z^2, \quad \phi_{22}(z,\mu_{01}) = 0.$$

Thus we obtain the restricted equation of system (3.1) to the center manifold $\mathcal{W}^c_{\mu_{01}}$, i.e.,

$$\frac{dz}{dt} = \mu_{01}z + k_1z^2 + O(|z,\mu_{01}|^3). \tag{3.4}$$

For $\mu_{01} = 0$, it shows that the origin O of system (3.4) is the unique equilibrium and that the other equilibrium arises from O as $\mu_{01} \neq 0$. Moreover, the stabilities of the two equilibria exchange as μ_{01} varies from negative to positive. Therefore, E_1 is a saddle-node at $\mu_{01} = 0$, and system (2.1) undergoes a transcritical bifurcation at E_{01} for k_2 crosses k_1.

Similar to the analysis of E_{01}, we obtain the restricted equation of system (2.1) to the center manifold $\mathcal{W}^c_{\mu_{02}}$ at E_{02}, $\mathcal{W}^c_{\mu_1}$ at E_1, $\mathcal{W}^c_{\mu_2}$ at E_2, and $\mathcal{W}^c_{\mu_3}$ at E_3, i.e.,

$$\frac{dz}{dt} = \frac{\mu_{02}}{k_5}z + z^2 + O(|z,\mu_{02}|^3), \tag{3.5}$$

$$\frac{dz}{dt} = -\frac{k_5\mu_1}{k_4}z + \frac{(k_2k_5+k_4)k_5}{k_2k_4}z^2 + O(|z,\mu_1|^3), \tag{3.6}$$

$$\frac{dz}{dt} = \mu_2z - k_5^3(k_1-k_2)(k_2k_5+k_4)z^2 + O(|z,\mu_2|^2), \tag{3.7}$$

$$\frac{dz}{dt} = \mu_3z - \frac{k_2k_5+k_4}{k_2}z^2 + O(|z,\mu_3|^3), \tag{3.8}$$

respectively. For $\mu_{02} = 0$(resp. $\mu_1 = 0$, $\mu_2 = 0$ and $\mu_3 = 0$) it shows that the origin O of system (3.5)(resp. (3.6), (3.7) and (3.8)) is the unique equilibrium and that the other equilibrium arises from O as $\mu_{02} \neq 0$ (resp. $\mu_1 \neq 0$, $\mu_2 \neq 0$ and $\mu_3 \neq 0$). Moreover, the stabilities of the two equilibria exchange as μ_{02}(resp. μ_1; μ_2 and μ_3) varies from negative to positive. Therefore, E_{02}, E_1, E_2 and E_3 are a saddle-nodes at $\mu_{02} = 0$, $\mu_1 = 0$, $\mu_2 = 0$ and $\mu_3 = 0$, respectively, and system (2.1) undergoes a transcritical bifurcation at E_{02} for $k_3 = k_1k_5$, E_1 for $k_3 = k_4 + k_1k_5$, E_2 for $k_3 = k_5(k_1-k_2)$ and E_3 for $k_3 = k_4 - \frac{k_1k_4}{k_2}$ respectively. □

Remark 3.1. Since we focus on x, y, z in the closure $\bar{\mathcal{Q}}$, there exist some different phenomena from classical transcritical bifurcations. Even we did see one more equilibrium lie in $\bar{\mathcal{Q}}$ arises from the transcritical bifurcation, the bifurcation occurs when the equilibrium loses stability.

4. Conclusions

In order to illustrate the theoretical results obtained in the previous sections, we give the bifurcation diagram and simulation for the dynamical system (2.1) in the following.

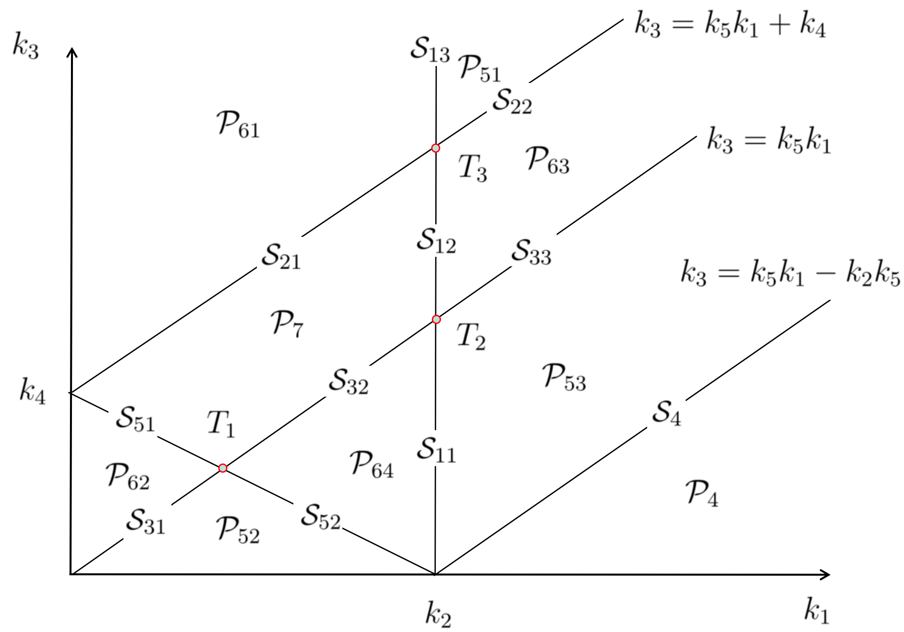

Figure 2. Bifurcation diagram on k_1k_3-plane.

We project the bifurcation surface on the k_1k_3-plane, seen in Figure 2. Make a roundtrip in Figure 2, as parameters (k_1, k_3) change from region $\mathcal{P}_4$ where there are 4 equilibria O, E_{01}, E_{02}, E_2 which always exist. Entering from region $\mathcal{P}_4$ into $\mathcal{P}_{53}$ through the component $\mathcal{S}_4$ of the transcritical curve yields one stable equilibrium E_*, and the stable equilibrium E_2 changes into an unstable saddle. Then a saddle E_1 arises from transcritical bifurcations as we cross another transcritical curve $\mathcal{S}_{33}$. When parameters cross $\mathcal{S}_{22}$ from $\mathcal{P}_{63}$ into region $\mathcal{P}_{51}$, E_* disappears and a saddle E_1 changes into a stable one. Crossing the transcritical curve $\mathcal{S}_{13}$ from region $\mathcal{P}_{51}$ into $\mathcal{P}_{61}$ implies the appearance of a stable equilibrium E_3, which survives when we enter region $\mathcal{P}_7$. Crossing the transcritical curve $\mathcal{S}_{21}$ creates an extra stable equilibrium E_*, and E_1 loses its stability. When parameters cross $\mathcal{S}_{51}$ into region $\mathcal{P}_{62}$, E_* disappears and a saddle E_2 changes into a stable one. Then entering into $\mathcal{P}_{52}$ through $\mathcal{S}_{31}$, E_1 disappears. If we continue the journey clockwise and finally cross $\mathcal{S}_{52}$ into region $\mathcal{P}_{64}$, a stable equilibrium E_* arises and E_3 loses its stability.

In order to display the transcritical bifurcation, we choose $k_1 = 0.5, k_2 = 1, k_4 = 2, k_5 = 1$ in system (2.1). Theorem 3.1 shows that the parameter value of transcritical bifurcation is $k_3 \in \mathcal{S}_{21} \bigcup \mathcal{S}_{31} \bigcup \mathcal{S}_{51}$(i.e., $k_3 = 2.5, 0.5$ and 1, respectively). We use MAPLE software to plot the first, second and third components of equilibria O, E_{01}, E_{02}, E_1, E_2, E_3 and E_* depending on k_3 in the k_3x, k_3y and k_3z-planes re-

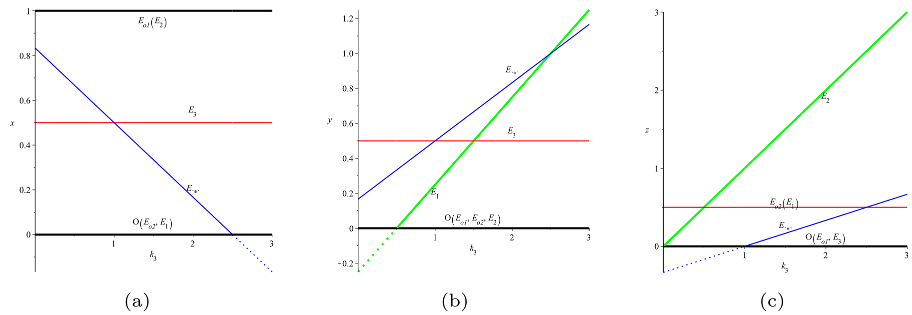

Figure 3. Bifurcation diagram of system (2.1) for $k_1 = 0.5, k_2 = 1, k_4 = 2, k_5 = 1$.

spectively. Figure 3(a) shows that E_* disappears when k_3 crosses 2.5 from $k_3 < 2.5$ to $k_3 > 2.5$. Figure 3(b) shows that E_1 arises as k_3 crosses 0.5 from $k_3 < 0.5$ to $k_3 > 0.5$. Figure 3(c) shows that E_* arises as k_3 crosses 1 from $k_3 < 1$ to $k_3 > 1$.

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