Journal of Applied Analysis and Computation
Volume 13, Number 5, October 2023, 2720–2738
Website:http://www.jaac-online.com
DOI:10.11948/20220555

LIMIT CYCLES FOR PIECEWISE LINEAR SYSTEMS WITH IMPROPER NODE*

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Abstract This paper is concerned with the number of limit cycles of planar piecewise linear systems for improper node-improper node and improper node-node types with a straight line of separation. We obtain some sufficient conditions for the existence and stability of limit cycles and prove that the systems have at least two nested limit cycles in some parameter regions.

Keywords Limit cycle, piecewise linear system, improper node.

MSC(2010) 34C05, 34C07, 34C25.

1. Introduction

Since the 1950s, piecewise smooth(PWS) systems are widely applied to some theoretical researches and practical applications, such as the mechanics, electrical engineering and mathematical biology and so on, see [1,3,4,21].

In particular, piecewise linear systems(PWLS) also have many applications in many real-world systems, see [14,20]. There are still some open problems for PWLS, such as the existence and number of limit cycles, which is related to Hilbert's 16th problem. Filippov [4] classified singular points for planar discontinuous systems. In 1991, Lum and Chua [15] conjectured that continuous PWS systems have at most one limit cycle. This conjecture was proved by Freire etc [5] in 1998. Giannakopoulos and Pliete [8] considered a special class of planar PWLS and proved the existence of at most two limit cycles. Freire etc [6] gave a Liénard-like canonical form with seven parameters and proved the existence of two limit cycles surrounding the sliding set for focus-focus type. Han and zhang [9] proved that PWLS have two limit cycles near a focus of either FF, FP or PP type (F and P represent the focus and parabolic, respectively). Huan and Yang [10] provided an example along with numerical simulations to illustrate the existence of 3 limit cycles for the general PWLS, and Llibre and Ponce [16] gave a rigorous computer-assisted proof of the quoted numerical result. Hou and Liu [13] considered the the number of limit cycles for a class of piecewise Hamiltonian systems, and gave upper bounds of the number of limit cycles bifurcated from a period annulus of a piecewise polynomial Hamiltonian system.

In the following, we first introduce some definitions and concepts. Assume that the plane divides into two regions by a straight line $x = 0$: the right half-plane $\mathbb{R}^2_-$

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*The authors were supported by National Natural Science Foundation of China(11661017) and Science and Technology Plan Project of Guizhou Province(ZK[2022]G118).
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and the left half-plane $\mathbb{R}^2_+$, i.e.

$$\mathbb{R}^2_- = \{(x,y) \in \mathbb{R}^2 : x < 0\},$$
$$\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x > 0\}.$$

The general planar piecewise smooth linear system can be written as

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{F}^-(\mathbf{x}) = (F_1^-(\mathbf{x}), F_2^-(\mathbf{x}))^T = A^-\mathbf{x} + \mathbf{b}^-, \ \mathbf{x} \in \mathbb{R}^2_-, \\ \mathbf{F}^+(\mathbf{x}) = (F_1^+(\mathbf{x}), F_2^+(\mathbf{x}))^T = A^+\mathbf{x} + \mathbf{b}^+, \ \mathbf{x} \in \mathbb{R}^2_+, \end{cases} \tag{1.1}$$

where $\mathbf{x} = (x,y)^T$, A^- and A^+ are 2×2 constant matrices and $\mathbf{b}^-$ and $\mathbf{b}^+$ are constant vectors. If $\mathbf{F}^-(0,y)$=$\mathbf{F}^+(0,y)$, system (1.1) is continuous. The equilibrium (x,y) is called visible(invisible) for the subsystem on $\mathbb{R}^2_-$ if $\mathbf{F}^-(x,y) = 0$ for $x < 0(x > 0)$. Similarly, the equilibrium (x,y) is called visible(invisible) for the subsystem on $\mathbb{R}^2_+$ if $\mathbf{F}^+(x,y) = 0$ for $x > 0(x < 0)$. The point $(0,y)$ is called a *crossing point* if $F_1^-(0,y)F_1^+(0,y) > 0$ and the set of the crossing points is defined as follows:

$$\Sigma^c = \{(0,y) : F_1^-(0,y)F_1^+(0,y) > 0\}.$$

The point $(0,y)$ is called a *sliding point* if $F_1^-(0,y)F_1^+(0,y) \leq 0$ and the set of the sliding points is defined as follows:

$$\Sigma^s = \{(0,y) : F_1^-(0,y)F_1^+(0,y) \leq 0\}.$$

The sliding set is attractive if $F_1^+(0,y) < 0$ and $F_1^-(0,y) > 0$, while the sliding set is repulsive if $F_1^+(0,y) > 0$ and $F_1^-(0,y) < 0$. According to the convex method of Filippov, the solutions of system (1.1) satisfy the equation

$$\dot{\mathbf{x}} = \lambda \mathbf{F}^-(\mathbf{x}) + (1-\lambda)\mathbf{F}^+(\mathbf{x}), \ \mathbf{x} \in \Sigma^s,$$

where λ is selected so that the above vector field is tangent to the sliding set, that is

$$\lambda F_1^-(\mathbf{x}) + (1-\lambda)F_1^+(\mathbf{x}) = 0, \ \mathbf{x} \in \Sigma^s.$$

Then, for $\mathbf{x} \in \Sigma^s$ and $|F_1^+(\mathbf{x})| + |F_1^-(\mathbf{x})| \neq 0$, we have

$$\lambda(y) = \frac{F_1^+(\mathbf{x})}{F_1^+(\mathbf{x}) - F_1^-(\mathbf{x})},$$

and the so-called sliding solutions are given by

$$\begin{aligned} \dot{x} &= 0, \\ \dot{y} &= g(y) \\ &= \frac{F_1^+(\mathbf{x})F_2^-(\mathbf{x}) - F_1^-(\mathbf{x})F_2^+(\mathbf{x})}{F_1^+(\mathbf{x}) - F_1^-(\mathbf{x})}, \ \mathbf{x} \in \Sigma^s. \end{aligned}$$

If $|F_1^+(0,y)|+|F_1^-(0,y)| \neq 0$ and $F_1^+(0,y) \cdot F_1^-(0,y) = 0$, the point $(0,y)$ is called a *tangency point*. We call the point $(0,y)$ is a *singular sliding point* if $F_1^+(0,y) = F_1^-(0,y) = 0$. There are three cases: (1) both vector fields are tangent to the line $x = 0$, (2) one of them is tangent while the other one vanishes, and (3) both vector fields vanish. The points of case (1) are called *double tangency points*, while the points of case (2) and (3) are called *boundary equilibrium points*.

If $(0, y') \in \Sigma^s$ with $g(y') = 0$, the point $(0, y')$ is called a *pseudoequilibrium*, see [6]. An invisible double tangency point with close orbits spiraling around it is called a *pseudofocus*. In the attractive sliding set, the pseudoequilibrium is a stable *pseudonode* if $g'(y) < 0$ and a *pseudosaddle* if $g'(y) > 0$. In the repulsive sliding set, the pseudoequilibrium is an unstable *pseudonode* if $g'(y) > 0$ and a *pseudosaddle* if $g'(y) < 0$.

Liénard canonical form [6] was given as follows:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} T^- & -1 \\ D^- & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ a^- \end{pmatrix}, \ x < 0, \\ \begin{pmatrix} T^+ & -1 \\ D^+ & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -b \\ a^+ \end{pmatrix}, \ x > 0. \end{cases} \tag{1.2}$$

System (1.2) is a linear refracting system if $b = 0$. In [18], Li and Chen proved the uniqueness of crossing limit cycles for planar piecewise linear systems with a line of discontinuity and without sliding sets. Buzzi, Medrado and Torres [2] studied the generic bifurcation of refracted systems.

According to singularity type of the subsystems, system (1.2) can be divided into 6 cases: FF, FN, FS, SS, SN, NN, where F, S and N represent the focus or center, saddle and node, respectively. Huan and Yang [11] proved that system (1.2) for SS has at least two limit cycles. Wang etc [22] showed that two limit cycles can appear in the system (1.2) for FS. In [23], the authors showed that system (1.2) for SN has two limit cycles. Freire etc [7] proved that there are at least three limit cycles for FS, FN and FF. Llibre and Zhang [17] proved that the maximum number of crossing limit cycles is two for systems (1.2) with a center. Huan and Yang [12] showed the number of limit cycles is for NN (not including the improper node N'). Zhao etc [24] studied the global dynamics of refracting system (1.2) for NN, $N'N$ and $N'N'$. Li and Chen [19] proved that there are no sliding periodic orbits for NN, $N'N$ and $N'N'$.

In this paper, we consider the existence and number of crossing limit cycles of system (1.2) for $N'N$ and $N'N'$. The paper is organized as follows. We deal with the existence and number of limit cycles of system (1.2) for $N'N'$ and $N'N$ in section 2 and 3, respectively.

2. Case $N'N'$

In this section, we consider the existence and number of limit cycles of system (1.2) with $N'N'$. First, we analyse the Poincaré map of system (1.2) for $N'N'$.

2.1. The Poincaré map of $N'N'$

If both the left and right subsystems of system (1.2) have improper nodes, then system (1.2) can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} T^- & -1 \\ \frac{(T^-)^2}{4} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ a^- \end{pmatrix}, \ x < 0, \\ \begin{pmatrix} T^+ & -1 \\ \frac{(T^+)^2}{4} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -b \\ a^+ \end{pmatrix}, \ x > 0. \end{cases} \tag{2.1}$$

The left subsystem (2.1) has an improper node $N'_L(\frac{4a^-}{(T^-)^2}, \frac{4a^-}{T^-})$ with eigenvalues $\lambda^-_{1,2} = \frac{T^-}{2}$ and the right subsystem (2.1) has an improper node $N'_R(\frac{4a^+}{(T^+)^2}, \frac{4a^+}{T^+} + b)$ with eigenvalues $\lambda^+_{1,2} = \frac{T^+}{2}$. Moreover, the invariant manifolds of the left and right subsystems (2.1) are given by

$$l^- : \ y = \lambda^- x + \frac{a^-}{\lambda^-},$$

and

$$l^+ : \ y = \lambda^+ x + \frac{a^+}{\lambda^+} + b,$$

respectively. Assume that l^- and l^+ intersect the straight line $x = 0$ at points $(0, \ y^-_m)$ and $(0, \ b + y^+_m)$ respectively, i.e.

$$y^-_m = \frac{a^-}{\lambda^-}, \quad y^+_m = \frac{a^+}{\lambda^+}.$$

2.1.1. The left Poincaré map

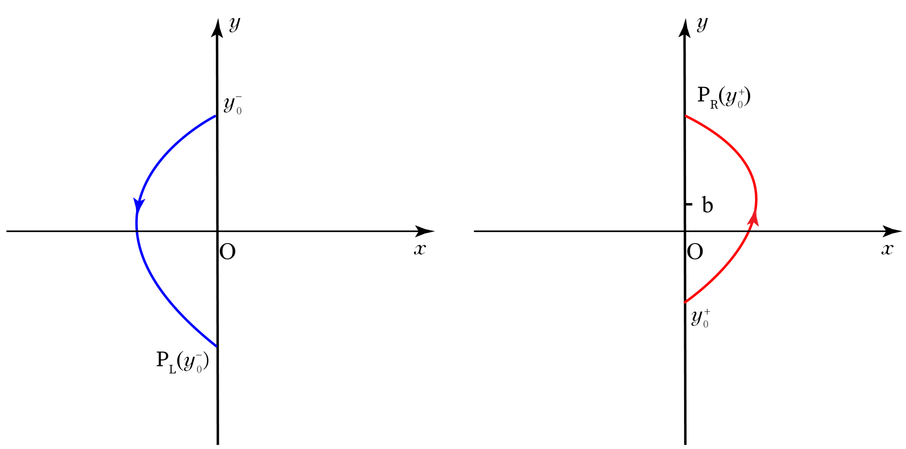

Figure 1. Illustration of $P_L(y^-_0)$ and $P_R(y^+_0)(b > 0)$.

Suppose that the orbit of the left subsystem (2.1) starting at the initial point $(0, y^-_0)$ with $y^-_0 > 0$ goes into the left zone $\mathbb{R}^2_-$ and reaches $x = 0$ again at some point $(0, y_1^-)$ with $y_1^- < 0$ after some finite time $t_0^- > 0$, see Figure 1(a). Then we can define a left Poincaré map P_L satisfying

$$P_L(0) = 0, \quad P_L(y_0^-) = y_1^-, \quad y_0^- > 0 > y_1^-.$$

The solution of the left subsystem (2.1) for the initial value $(0,\ y_0^-)$ is

$$\begin{pmatrix} x(t^-) \\ y(t^-) \end{pmatrix} = \begin{pmatrix} -t^- e^{\frac{T^-t^-}{2}} \\ (1 - \frac{T^-t^-}{2}) \cdot e^{\frac{T^-t^-}{2}} \end{pmatrix} y_0^- + \begin{pmatrix} \frac{(2T^-t^- - 4)\cdot a^- e^{\frac{T^-t^-}{2}} + 4a^-}{(T^-)^2} \\ (t^- - \frac{4}{T^-}) \cdot a^- e^{\frac{T^-t^-}{2}} + \frac{4a^-}{T^-} \end{pmatrix}. \tag{2.2}$$

Substituting $x(t^-) = 0$ into (2.2), we have that the parametric representations of the left Poincaré map with respect to t^- are

$$\begin{aligned} y_0^-(t^-) &= \frac{2a^- \cdot [(T^-t^- - 2) \cdot e^{\frac{T^-t^-}{2}} + 2]}{t^-(T^-)^2 e^{\frac{T^-t^-}{2}}}, \\ y_1^-(t^-) &= \frac{2a^- \cdot (T^-t^- - 2e^{\frac{T^-t^-}{2}} + 2)}{t^-(T^-)^2}, \ t^- > 0. \end{aligned} \tag{2.3}$$

From [24], we know that $a^- > 0$ is necessary for the existence of limit cycles of system (2.1). In what follows, we only deal with P_L for $a^- > 0$.

Lemma 2.1. *For the left subsystem* (2.1)*, the left Poincaré map P_L is well defined by* (2.3) *if and only if $a^- > 0$. Moreover, the following conditions hold.*

1. $y_0^-(t^-)$ is increasing and $y_1^-(t^-)$ is decreasing with respect to t^-.

2. When $T^- > 0$, the domain and range of P_L are $(0,\ y_m^-)$ and $(-\infty,\ 0)$, respectively.

(I) P_L is decreasing with respect to y_0^-.

(II) P_L has $y_0^- = y_m^-$ as an asymptote.

(III) $P_L''(y_0^-) < 0$.

3. When $T^- < 0$, the domain and range of P_L are $(0,\ +\infty)$ and $(y_m^-,\ 0)$, respectively.

(I) P_L is decreasing with respect to y_0^-.

(II) P_L has $y_1^- = y_m^-$ as an asymptote.

(III) $P_L''(y_0^-) > 0$.

4. We define $P_L(0) = 0$. Then P_L is continuous at $y_0^- = 0$ and the first two derivatives of P_L at $y_0^- = 0$ are

$$P_L'(0) = -1, \quad P_L''(0) = -\frac{4T^-}{3a^-}.$$

Proof. It follows from (2.3) that

$$\begin{aligned} (y_0^-(t^-))' &= -\frac{2a^- \cdot g(t^-)}{(t^-)^2(T^-)^2 e^{\frac{T^-t^-}{2}}}, \\ (y_1^-(t^-))' &= -\frac{2a^- \cdot h(t^-)}{(t^-)^2(T^-)^2}, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} g(t^-) &= T^-t^- - 2e^{\frac{T^-t^-}{2}} + 2, \\ h(t^-) &= (T^-t^- - 2) \cdot e^{\frac{T^-t^-}{2}} + 2. \end{aligned}$$

Note that $g''(t^-) < 0$ and $g'(0) = 0$. We have that $g'(t^-) < g'(0) = 0$ as $t^- > 0$ and $g(t^-) < g(0) = 0$ as $t^- > 0$. Similarly, we can know that $h(t^-) > 0$ as $t^- > 0$. Thus, $y_0^-(t^-)$ and $y_1^-(t^-)$ are both monotone with respect to t^- for $a^- > 0$. Therefore, the left Poincaré map $P_L : y_0^- \mapsto y_1^-$ is well defined by (2.3).

1. Since $g(t^-) < 0$, $h(t^-) > 0$ and $a^- > 0$, we have $(y_0^-(t^-))' > 0$ and $(y_1^-(t^-))' < 0$. Thus $y_0^-(t^-)$ is increasing and $y_1^-(t^-)$ is decreasing with respect to t^-.

2. When $T^- > 0$, by (2.3), we obtain that

$$\begin{aligned} \lim_{t^- \to +\infty} y_0^-(t^-) &= y_m^-, \\ \lim_{t^- \to +\infty} y_1^-(t^-) &= -\infty, \\ \lim_{t^- \to 0^+} y_0^-(t^-) &= \lim_{t^- \to 0^+} y_1^-(t^-) \\ &= 0. \end{aligned} \tag{2.5}$$

Then, the domain and range of P_L are $(0,\ y_m^-)$ and $(-\infty,\ 0)$, respectively.

(I) From (2.4), we have

$$\begin{aligned} P_L'(y_0^-) &= \frac{(y_1^-(t^-))'}{(y_0^-(t^-))'} \\ &= \frac{[(T^-t^- - 2) \cdot e^{\frac{T^-t^-}{2}} + 2] \cdot e^{\frac{T^-t^-}{2}}}{T^-t^- - 2e^{\frac{T^-t^-}{2}} + 2} \\ &< 0. \end{aligned} \tag{2.6}$$

Then P_L is decreasing with respect to y_0^-.

(II) By (2.5), we know that P_L has $y_0^- = y_m^-$ as an asymptote.

(III) Direct computation from (2.4), it yields that

$$\begin{aligned} P_L''(y_0^-) &= \frac{(y_0^-(t^-))' \cdot (y_1^-(t^-))'' - (y_1^-(t^-))' \cdot (y_0^-(t^-))''}{((y_0^-(t^-))')^3} \\ &= \frac{(T^-)^4 (t^-)^3 e^{T^-t^-} \cdot l(t^-)}{2a^- \cdot (T^-t^- - 2e^{\frac{T^-t^-}{2}} + 2)^3}, \end{aligned} \tag{2.7}$$

where

$$l(t^-) = -T^-t^- \cdot e^{\frac{T^-t^-}{2}} + e^{T^-t^-} - 1.$$

Note that $l'(t^-) = T^- e^{\frac{T^-t^-}{2}} \cdot v(t^-)$, where $v(t^-) = e^{\frac{T^-t^-}{2}} - \frac{1}{2}T^-t^- - 1$. We have that $v'(t^-) > 0$, $v(t^-) > v(0) = 0$ as $t^- > 0$ and $l'(t^-) > 0$ as $t^- > 0$, that is $l(t^-) > l(0) = 0$ as $t^- > 0$. Therefore, $P_L''(y_0^-) < 0$.

3. When $T^- < 0$, it follows from (2.3) that

$$\lim_{t^- \to 0^+} y_0^-(t^-) = \lim_{t^- \to 0^+} y_1^-(t^-) = 0, \quad \lim_{t^- \to +\infty} y_0^-(t^-) = +\infty, \quad \lim_{t^- \to +\infty} y_1^-(t^-) = y_m^-.$$

Then the domain and range of P_L are $(0,\ +\infty)$ and $(y_m^-,\ 0)$ respectively and P_L has $y_1^- = y_m^-$ as an asymptote. By (2.6) and (2.7), P_L is decreasing with respect to y_0^- and $P_L''(y_0^-) > 0$.

4. From the proofs of statements 2 and 3, we know that P_L is continuous at $y_0^- = 0$. Next, we calculate the first two derivatives of P_L at $y_0^- = 0$. By the first equation of (2.3), we have

$$[t^-(T^-)^2 \cdot y_0^- - 2a^- \cdot (t^-T^- - 2)] \cdot e^{\frac{T^-t^-}{2}} - 4a^- = 0. \tag{2.8}$$

By Implicit Function Theorem, we obtain that

$$t^- = \frac{2}{a^-} \cdot y_0^- + \frac{2T^-}{3(a^-)^2} \cdot (y_0^-)^2 + \frac{5(T^-)^2}{18(a^-)^3} \cdot (y_0^-)^3 + \frac{17(T^-)^3}{135(a^-)^4} \cdot (y_0^-)^4 + \cdots . \tag{2.9}$$

Substituting (2.9) into the second equation of (2.3), it yields that

$$P_L(y_0^-) = -y_0^- - \frac{2T^-}{3a^-} \cdot (y_0^-)^2 - \frac{4(T^-)^2}{9(a^-)^3} \cdot (y_0^-)^3 - \frac{79(T^-)^3}{270(a^-)^4} \cdot (y_0^-)^4 + \cdots . \tag{2.10}$$

Therefore, we have

$$P_L(0) = 0, \quad P_L'(0) = -1, \quad P_L''(0) = -\frac{4T^-}{3a^-}.$$

□

2.1.2. The right Poincaré map

Suppose that the orbit of the right subsystem (2.1) starting at the initial point $(0, y_0^+)$ with $y_0^+ < b$ goes into the left zone $\mathbb{R}_+^2$ and reaches $x = 0$ again at some point $(0, y_1^+)$ with $y_1^+ > b$ after some finite time $t_0^+ > 0$, see Figure 1(b). Then we can define the right Poincaré map P_R satisfying

$$P_R(b; b) = b, \quad P_R(y_0^+; b) = y_1^+, \quad y_1^+ > b > y_0^+.$$

The solution of the right subsystem (2.1) for the initial value $(0,\ y_0^+)$ is

$$\begin{pmatrix} x(t^+) \\ y(t^+) \end{pmatrix} = \begin{pmatrix} -t^+ e^{\frac{T^+t^+}{2}} \\ (1 - \frac{T^+t^+}{2}) \cdot e^{\frac{T^+t^+}{2}} \end{pmatrix} y_0^+ + \begin{pmatrix} A(t^+) \\ B(t^+) \end{pmatrix}, \tag{2.11}$$

where

$$A(t^+) = \frac{(2T^+t^+ - 4) \cdot a^+ e^{\frac{T^+t^+}{2}} + 4a^+}{(T^+)^2} + bt^+ e^{\frac{T^+t^+}{2}},$$

and

$$B(t^+) = (t^+ - \frac{4}{T^+}) \cdot a^+ e^{\frac{T^+t^+}{2}} + \frac{4a^+}{T^+} + (\frac{T^+t^+}{2} - 1) \cdot b e^{\frac{T^+t^+}{2}} + b.$$

Substituting $x(t^+) = 0$ into (2.11), we have that the parametric representations of the right Poincaré map P_R with respect to t^+ are

$$\begin{aligned} y_0^+(t^+) &= \frac{2a^+ \cdot [(T^+t^+ - 2) \cdot e^{\frac{T^+t^+}{2}} + 2]}{t^+(T^+)^2 e^{\frac{T^+t^+}{2}}} + b, \\ y_1^+(t^+) &= \frac{2a^+ \cdot (T^+t^+ - 2e^{\frac{T^+t^+}{2}} + 2)}{t^+(T^+)^2} + b,\ t^+ > 0. \end{aligned} \tag{2.12}$$

For the right Poincaré map P_R, we have the following Lemma.

Lemma 2.2. *For the right subsystem* (2.1)*, the right Poincaré map* P_R *is well defined by* (2.12) *if and only if* $a^+ < 0$*. Moreover, the following conditions hold.*

1. $y_0^+(t^+)$ *is decreasing and* $y_1^+(t^+)$ *is increasing with respect to* t^+*.*

2. When $T^+ > 0$*, the domain and range of* P_R *are* $(b + y_m^+, b)$ *and* $(b, +\infty)$*, respectively.*

(I) P_R *is decreasing with respect to* y_0^+*.*

(II) P_R *has* $y_0^+ = b + y_m^+$ *as an asymptote.*

(III) $P_R''(y_0^+; b) > 0$*.*

3. When $T^+ < 0$*, the domain and range of* P_R *are* $(-\infty, b)$ *and* $(b, b + y_m^+)$*, respectively.*

(I) P_R *is decreasing with respect to* y_0^+*.*

(II) P_R *has* $y_1^+ = b + y_m^+$ *as an asymptote.*

(III) $P_R''(y_0^+; b) < 0$*.*

4. We define $P_R(b; b) = b$*. Then* P_R *is continuous at* $y_0^+ = b$ *and the first two derivatives of* P_R *at* $y_0^+ = b$ *are*

$$P_R'(b; b) = -1, \quad P_R''(b; b) = -\frac{4T^+}{3a^+}.$$

The proof of this Lemma is similar to the proof of Lemma 2.1 and we omit the proof.

2.1.3. The full Poincaré map

Now, we define the full Poincaré map by $P = P_R \circ P_L$ for a fixed parameter b. From Lemma 2.1 and Lemma 2.2, we can directly obtain some properties of P as follows.

Lemma 2.3. *For system* (2.1)*, the full Poincaré map* P *is well defined if and only if* $a^- > 0 > a^+$*. Moreover,*

$$P'(0) = 1, \quad P''(0) = \frac{4}{3} \cdot (\frac{T^-}{a^-} - \frac{T^+}{a^+}).$$

2.2. The existence and number of limit cycles of $N'N'$

In this section, we consider the existence and number for system (1.2) with $N'N'$. System (1.2) is invariant under the change of variables $(x,\ y,\ t) \to (x,\ -y,\ -t)$, and the change of parameters

$$(D^+, D^-, T^+, T^-, a^+, a^-, b) \to (D^+, D^-, -T^+, -T^-, a^+, a^-, -b). \tag{2.13}$$

Moreover, the authors [24] studied the global dynamics of system (1.2) with $b = 0$ for all node cases. Then, we only study the limit cycles of system (1.2) for $b > 0$ in the following.

The Poincaré return map can be defined as the composition of the right and left Poincaré map, i.e., $P(y_0) = P_R(P_L(y_0))$. In order to compute the fixed points of the return map $P(y_0)$, we study the zeros of the following map

$$\mathbf{D}_1(y_0; b) = P_L(y_0) - (P_R^{-1}|_{b>0})(y_0; b),\ y_0 \geq b > 0,$$

see Figure 2. For any $y_0 \in [b, +\infty)$, assume that the straight line $l\colon y = x + P_R^{-1}(y_0; b) - y_0$. Then l intersects the left Poincaré map P_L at the point $(y_0^l(y_0; b), P_L(y_0^l(y_0; b)))$. We introduce a new function

$$\mathbf{D}_2(y_0; b) = y_0^l(y_0; b) - y_0,\ y_0 \geq b > 0.$$

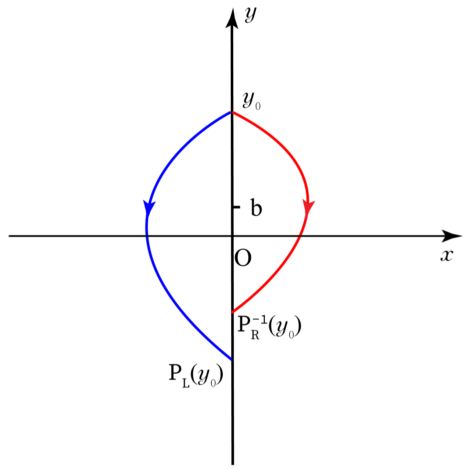

Figure 2. Illustration of $\mathbf{D}_1(y_0; b)$.

Setting $Graph(f)$ represents the set of all points which are on the graph of the function f. From (2.12) we obtain that

$$(y_0,\ y_1) \in Graph(P_R^{-1}|_{b=0}) \Leftrightarrow (y_0 + b,\ y_1 + b) \in Graph(P_R^{-1}|_{b>0}). \tag{2.14}$$

Moreover, it yields that

$$P_R^{-1}|_{b>0}(y_0; b) = P_R^{-1}|_{b=0}(y_0 - b; b) + b,\ y_0 > b. \tag{2.15}$$

For $y_0 \geq b > 0$, we have the following lemma.

Lemma 2.4. *For $a^- > 0 > a^+$ and $y_0 \geq b > 0$, the following statements hold.*

1. $Sgn(\boldsymbol{D}_1(y_0; b))$=$sgn(\boldsymbol{D}_2(y_0; b))$.

2. When $T^+ < 0$ and $T^- < 0$: for $y_0 \in (b,\ b + y_m^+)$, $\boldsymbol{D}_1(y_0; b)$ is increasing with respect to y_0 if $(T^\pm)^2 = 4D^\pm > 0$, and for $y_0 \in (b,\ b + y_{m_1}^+)$, $\boldsymbol{D}_1(y_0; b)$ is also increasing with respect to y_0 if $(T^+)^2 = 4D^+ > 0$ and $(T^-)^2 > 4D^- > 0$.

3. $\boldsymbol{D}_2(y_0; b) = \boldsymbol{D}_2(y_0 - b; 0) - b$.

Proof. 1. If $\mathbf{D}_1(y_0) > 0$ and $\mathbf{D}_2(y_0) \leq 0$, then $P_L(y_0) > (P_R^{-1}|_{b>0})(y_0)$ and $y_0^l(y_0; b) \leq y_0$. In addition, by $P_L(y_0^l(y_0; b)) - (P_R^{-1}|_{b>0})(y_0) = y_0^l(y_0; b) - y_0$, we have

$$P_L(y_0^l(y_0; b)) \leq (P_R^{-1}|_{b>0})(y_0; b) < P_L(y_0),\ y_0 \geq b > 0.$$

However, from Lemma 2.1, we know that P_L is decreasing with respect to y_0 which implies that $P_L(y_0^l(y_0; b)) \geq P_L(y_0)$. Then there is a contradiction.

2. We only consider the case $(T^\pm)^2 = 4D^\pm > 0$. The proof of case $(T^+)^2 = 4D^+ > 0$ and $(T^-)^2 > 4D^- > 0$ is similar to the former case. By Lemma 2.1, when $T^- < 0$, $P_L(y_0)$ is decreasing and $(P_L)''(y_0) > 0$ with the domain and range of $(0,\ +\infty)$ and $(y_m^-,\ 0)$. Then we have for $y_0 \in (0,\ +\infty)$

$$(P_L)''(y_0) > 0 \Rightarrow (P_L)'(y_0) > (P_L)'(0) = -1. \tag{2.16}$$

Similarly, by Lemma 2.2, we have for $y_0 \in (b,\ b + y_m^+)$,

$$(P_R^{-1})''(y_0; b) < 0 \Rightarrow ((P_R^{-1}|_{b>0}))'(y_0; b) < ((P_R^{-1}|_{b>0}))'(0; b) = -1. \tag{2.17}$$

By (2.16) and (2.17), we have

$$\mathbf{D}_1'(y_0;b) = P_L'(y_0) - ((P_R^{-1}|_{b>0}))'(y_0;b) > 0,\ y_0 \in (b,\ b+y_m^+).$$

Therefore, $\mathbf{D}_1(y_0;b)$ is increasing with respect to y_0 for $y_0 \in (b,\ b+y_m^+)$.

3. From $P_L(y_0^l(y_0;b)) - (P_R^{-1}|_{b>0})(y_0;b) = y_0^l(y_0;b) - y_0$ and (2.15), we have for any $y_0 \geq b$,

$$\begin{aligned} y_0^l(y_0;b) &= P_L(y_0^l(y_0;b)) - (P_R^{-1}|_{b>0})(y_0;b) + y_0 \\ &= P_L(y_0^l(y_0;b)) - (P_R^{-1}|_{b=0})(y_0-b;b) + (y_0-b) \\ &= y_0^l(y_0-b;0). \end{aligned}$$

Then, for any $y_0 \geq b$ we obtain that

$$\begin{aligned} \mathbf{D}_2(y_0;b) &= y_0^l(y_0;b) - y_0 \\ &= y_0^l(y_0-b;0) - (y_0-b) - b \\ &= \mathbf{D}_2(y_0-b;0) - b. \end{aligned}$$

□

By the above lemma, we have that the zeros of the map $\mathbf{D}_1(y_0;b)$ are equivalent to the zeros of the map $\mathbf{D}_2(y_0;b)$.

Theorem 2.1. $(b>0)$ *Suppose that* $a^- > 0 > a^+$ *and* $(T^{\pm})^2 = 4D^{\pm} > 0$ *for system* (1.2)*, we have the following results.*

1. If $T^+ \cdot T^- > 0$*, then the following subcases hold.*

(I) When $T^+b > 0$*, there don't exist limit cycles.*

(II) When $T^+b < 0$*, there exists a unique stable limit cycle.*

2. If $T^+ > 0 > T^-$*, then the following subcases hold.*

(I) When $\frac{a^-}{T^-} \leq \frac{a^+}{T^+}$*, there don't exist limit cycles.*

(II) When $\frac{a^-}{T^-} > \frac{a^+}{T^+}$*,*

(a) If $y_m^- \leq y_m^+ + b$*, there don't exist limit cycles if* $y_m^- < 0 \leq y_m^+ + b$*. If* $y_m^- \leq y_m^+ + b < 0$*, there exists* $b_m > 0$ *such that system* (1.2) *has at least two limit cycles for* $b_m > b > 0$*, and system* (1.2) *has no limit cycles for* $b > b_m$*. Moreover, when* $b \in (0,\ b_m)$ *is sufficiently small, system* (1.2) *has exactly two limit cycles in which the inter one is stable and the outer one is unstable.*

(b) If $y_m^- > y_m^+ + b$*, there exists at least a stable limit cycle.*

3. If $T^- > 0 > T^+$*, then the following subcases hold.*

(I) When $\frac{a^-}{T^-} \leq \frac{a^+}{T^+}$*, there don't exist limit cycles.*

(II) When $\frac{a^-}{T^-} > \frac{a^+}{T^+}$*,*

(a) If $y_m^- \leq y_m^+ + b$*, there don't exist limit cycles if* $y_m^- - b \leq 0 < y_m^+$*. If* $0 < y_m^- - b \leq y_m^+$*, there exists* $b_m > 0$ *such that system* (1.2) *has at least two limit cycles for* $b_m > b > 0$*, and system* (1.2) *has no limit cycles for* $b > b_m$*. Moreover, when* $b \in (0,\ b_m)$ *is sufficiently small, system* (1.2) *has exactly two limit cycles in which the inter one is stable and the outer one is unstable.*

(b) If $y_m^- > y_m^+ + b$*, there exists at least a stable limit cycle.*

Proof.

1. (I) Since $b > 0$, $T^+ \cdot T^- > 0$ and $T^+b > 0$, we have $T^+ > 0$ and $T^- > 0$. By Lemma 2.1 and Lemma 2.2, it yields that the graph of P_L is below the graph

of P_R^{-1}. Indeed, the graph of P_L is below the line $y_1 = -y_0$ and the graph of P_R^{-1} is above the line $y_1 = -y_0 + 2b$. Therefore, there don't exist limit cycles.

(II) Since $b > 0$, $T^+ \cdot T^- > 0$ and $T^+ b < 0$, we have $T^+ < 0$ and $T^- < 0$. From Lemma 2.4, $\mathbf{D}_1(y_0; b)$ is increasing with respect to y_0 for $y_0 \in (b,\ b + y_m^+)$. In addition, $\mathbf{D}_1(b; b) = P_L(b) - (P_R^{-1}|_{b>0})(b; b) = P_L(b) - b < P_L(0) - b = -b < 0$ and $\mathbf{D}_1(y_0; b) > 0$ as $y_0 \to b + y_m^+$. Then, there exists a unique value $y_0^{(1)} \in (b,\ b + y_m^+)$ such that $\mathbf{D}_1(y_0^{(1)}; b) = 0$, that is, there exists a unique limit cycle for system (2.1). Since $P_L(y_0) < (P_R^{-1}|_{b>0})(y_0; b)$ for $y_0 \in [b,\ y_0^{(1)})$ and $P_L(y_0) > (P_R^{-1}|_{b>0})(y_0; b)$ for $y_0 \in (y_0^{(1)},\ b + y_m^+)$, the limit cycle is stable.

2. Freire etc [7] changed system (1.2) with $b = 0$ for $N'N'$ into the following system to reduce the parameters, that is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} 2\gamma_L & -1 \\ \gamma_L^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a_L \end{pmatrix}, & x \le 0, \\ \begin{pmatrix} 2\gamma_R & -1 \\ \gamma_R^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a_R \end{pmatrix}, & x \ge 0. \end{cases} \tag{2.18}$$

The authors [24] proved that the origin is a center if $a_R\gamma_L - a_L\gamma_R = 0$(i.e. $\frac{a^-}{T^-} = \frac{a^+}{T^+}$ for system (2.1) with $b = 0$) and system (2.18) has no limit cycles, i.e. system (2.1) with $b = 0$ has no limit cycles.

(I) If $\frac{a^-}{T^-} = \frac{a^+}{T^+}$ and $b = 0$, then the origin is a center and

$$Graph(P_L) = Graph(P_R^{-1}|_{b=0}).$$

By (2.14), it yields that $Graph(P_L) \cap Graph(P_R^{-1}|_{b>0}) = \varnothing$ and there don't exist limit cycles for system (2.1). When $\frac{a^-}{T^-} < \frac{a^+}{T^+}$, system (2.1) with $b = 0$ has no limit cycles, that is $Graph(P_L) \cap Graph(P_R^{-1}|_{b=0}) = \{0,\ 0\}$. From Lemma 2.3, we have

$$\begin{aligned} &P_R^{-1}|_{b=0}(0) = P_L(0), \\ &P_L(y_0) < P_R^{-1}|_{b=0}(y_0; b),\ \ y_0 > 0. \end{aligned} \tag{2.19}$$

Then, by (2.19) and (2.15), we obtain that $P_R^{-1}|_{b>0}(y_0; b) = P_R^{-1}|_{b=0}(y_0 - b; b) + b > P_L(y_0 - b) + b > P_L(y_0),\ \ y_0 \ge b$. Hence, there don't exist limit cycles for system (2.1).

(II) When $\frac{a^-}{T^-} > \frac{a^+}{T^+}$, system (2.1) with $b = 0$ has no limit cycles. Then $Graph(P_L) \cap Graph(P_R^{-1}|_{b=0}) = \{(0,\ 0)\}$. In addition, we have $\mathbf{D}_1(b; b) = P_L(b) - (P_R^{-1}|_{b>0})(b; b) = P_L(b) - b < P_L(0) - b = -b < 0$ and

$$\begin{aligned} \lim_{y_0 \to +\infty} \mathbf{D}_1(y_0; b) &= \lim_{y_0 \to +\infty} P_L(y_0) - (P_R^{-1}|_{b>0})(y_0; b) \\ &= y_m^- - y_m^+ - b,\ \ y_0 \ge b. \end{aligned}$$

(a) For $y_m^- \le y_m^+ + b$, we prove this statement by considering the following two cases.

(a1) If $y_m^- < 0 \le y_m^+ + b$, by Lemma 2.1 and Lemma 2.2, we know that the range of $(P_R^{-1}|_{b>0})(y_0; b)$ is $(b + y_m^+,\ b)$ for $y_0 \ge b$ and $y_m^- < P_L(y_0) \le 0$ for $y_0 \ge 0$. Then $(P_R^{-1}|_{b>0})(y_0; b) > P_L(y_0)$ for $y_0 \ge b$ and there don't exist limit cycles.

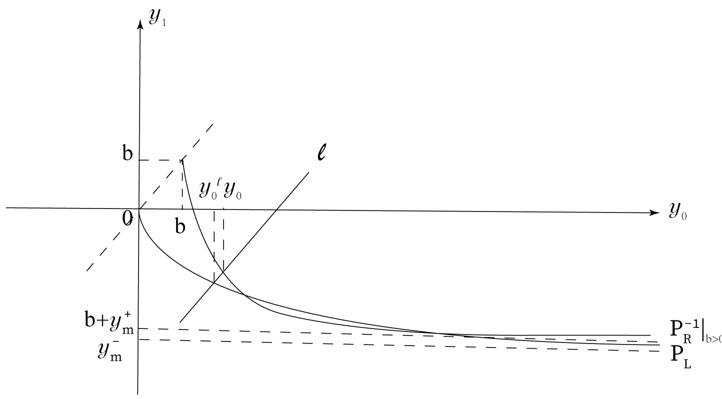

Figure 3. Graphs of P_L and $P_R^{-1}|_{b>0}$ when $y_m^- \le y_m^+ + b < 0$ and $T^+ > 0 > T^-$ for $b \in (0,\ b_m)$ is sufficiently small.

(a2) If $y_m^- \le y_m^+ + b < 0$, then

$$\lim_{y_0 \to +\infty} \mathbf{D}_1(y_0; b) \le 0.$$

By Lemma 2.4, we have $\mathbf{D}_2(y_0; b) \le 0$ as $y_0 \to +\infty$. Suppose that b_m is the maximum of $\mathbf{D}_2(y_0; 0)$ on the interval $[0,\ +\infty)$ and $\mathbf{D}_2(y_0; 0)$ reaches its maximum at $y_0 = y_0^m$, that is $\mathbf{D}_2(y_0^m; 0) = b_m$. Then $\mathbf{D}_2(0; 0) = 0$ and $b_m > \mathbf{D}_2(y_0; 0) > 0$ for $y_0 > 0$.

For $b > b_m$, $\mathbf{D}_2(y_0; b) = \mathbf{D}_2(y_0 - b; 0) - b \le b_m - b < 0$ for $y_0 \ge b$ by Lemma 2.4. Then, there don't exist limit cycles.

For $b_m > b > 0$, it yields that $\mathbf{D}_2(y_0^m + b; b) = \mathbf{D}_2(y_0^m; 0) - b = b_m - b > 0$. Note that $y_0^m + b \in [b,\ +\infty)$. Then, $\mathbf{D}_2(y_0; b)$ has at least two zeros and system (2.1) has at least two limit cycles. When $b \in (0,\ b_m)$ is sufficiently small, $\mathbf{D}_2(y_0; b)$ has exactly two zeros by using the implicit function theorem as in Theorem 4.5 of [16], see Figure 3. We denote two zeros of $\mathbf{D}_2(y_0; b)$ by $y_0^{(2)}$ and $y_0^{(3)}$, respectively. By Lemma 2.4, we know that $P_L(y_0) < (P_R^{-1}|_{b>0})(y_0; b)$ for $y_0 \in [b,\ y_0^{(2)}) \cup (y_0^{(3)},\ +\infty)$ and $P_L(y_0) > (P_R^{-1}|_{b>0})(y_0; b)$ for $y_0 \in (y_0^{(2)},\ y_0^{(3)})$. Then, the inter limit cycle is stable and the outer one is unstable.

(b) When $y_m^- > y_m^+ + b$,

$$\lim_{y_0 \to +\infty} \mathbf{D}_1(y_0; b) > 0.$$

By Lemma 2.4, we have $\mathbf{D}_2(y_0; b) > 0$ as $y_0 \to +\infty$. Then, there is at least a value $y_0^{(4)}$ such that $\mathbf{D}_2(y_0^{(4)}; b) = 0$ on the interval $[b,\ +\infty)$ and there exists at least a limit cycle. By Lemma 2.4, we obtain that $P_L(y_0) < (P_R^{-1}|_{b>0})(y_0; b)$ for $y_0 \in [b,\ y_0^{(4)})$ and $P_L(y_0) > (P_R^{-1}|_{b>0})(y_0; b)$ for $y_0 \in (y_0^{(4)},\ +\infty)$. Then, the limit cycle is stable.

3. The proof of this statement is similar to statement 2 and is omitted here. □

The following theorem is a direct consequence of the above theorem by the change of variables $(x, y, t) \to (x, -y, -t)$.

Theorem 2.2. ($b < 0$) *Suppose that* $a^- > 0 > a^+$ *and* $(T^{\pm})^2 = 4D^{\pm} > 0$ *for system* (1.2). *Then the following conditions hold.*

1. If $T^+ \cdot T^- > 0$, *then the following subcases arise.*

(I) When $T^+ b > 0$, *there don't exist limit cycles.*

(II) When $T^+ b < 0$, *there exists a unique unstable limit cycle.*

2. If $T^+ > 0 > T^-$, *then the following subcases arise.*

(I) When $\frac{a^-}{T^-} \geq \frac{a^+}{T^+}$, *there don't exist limit cycles.*

(II) When $\frac{a^-}{T^-} < \frac{a^+}{T^+}$,

(a) If $y_m^- \geq y_m^+ + b$, *there don't exist limit cycles if* $y_m^- - b > 0 > y_m^+$, *and if* $0 > y_m^- - b \geq y_m^+$, *there is a value* $b_m > 0$ *such that system* (1.2) *has at least two limit cycles for* $b_m > -b > 0$ *and no limit cycles for* $-b > b_m$. *Moreover, when* $b \in (-b_m,\ 0)$ *is sufficiently small, system* (1.2) *has exactly two limit cycles in which the inter one is unstable and the outer one is stable.*

(b) If $y_m^- < y_m^+ + b$, *there exists at least an unstable limit cycle.*

3. If $T^- > 0 > T^+$, *then the following subcases arise.*

(I) When $\frac{a^-}{T^-} \geq \frac{a^+}{T^+}$, *there don't exist limit cycles.*

(II) When $\frac{a^-}{T^-} < \frac{a^+}{T^+}$,

(a) If $y_m^- \geq y_m^+ + b$, *there don't exist limit cycles if* $y_m^- > 0 > y_m^+ + b$, *and if* $y_m^- \geq y_m^+ + b > 0$, *there is a value* $b_m > 0$ *such that system* (1.2) *has at least two limit cycles for* $b_m > -b > 0$ *and no limit cycles for* $-b > b_m$. *Moreover, when* $b \in (-b_m,\ 0)$ *is sufficiently small, system* (1.2) *has exactly two limit cycles in which the inter one is unstable and the outer one is stable.*

(b) If $y_m^- < y_m^+ + b$, *there exists at least an unstable limit cycle.*

3. Case $N'N$

In this section, we consider the existence and number of limit cycles of system (1.2) with $N'N$. First, we deal with the Poincaré map of system (1.2) for $N'N$.

3.1. The Poincaré map of $N'N$

Assume the left subsystem of system (1.2) has an improper node and the right subsystem has a node, then system (1.2) can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} T^- & -1 \\ \frac{(T^-)^2}{4} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ a^- \end{pmatrix}, \ x < 0, \\ \begin{pmatrix} T^+ & -1 \\ D^+ & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -b \\ a^+ \end{pmatrix}, \quad x > 0, \end{cases} \tag{3.1}$$

where $(T^+)^2 > 4D^+ > 0$. The left subsystem (3.1) has an improper node $N_L'(\frac{4a^-}{(T^-)^2}, \frac{4a^-}{T^-})$ with eigenvalues $\lambda_{1,2}^- = \frac{T^-}{2}$ and the right subsystem (3.1) has a node $N_R(\frac{a^+}{D^+}, \frac{a^+T^+}{D^+} + b)$ with eigenvalues $\lambda_{1,2}^+ = \frac{T^+ \pm \sqrt{(T^+)^2 - 4D^+}}{2}$. Moreover, the invariant manifolds of the left and right subsystems (3.1) are given by

$$l^- :\ y = \lambda^- x + \frac{a^-}{\lambda^-},$$

and

$$l_1^+ :\ y = \lambda_2^+ x + \frac{a^+}{\lambda_2^+} + b,$$
$$l_2^+ :\ y = \lambda_1^+ x + \frac{a^+}{\lambda_1^+} + b,$$

respectively. Assume that l^- and $l_{1,2}^+$ intersect the straight line $x = 0$ at points $(0,\ y_m^-)$ and $(0,\ b + y_{m_{1,2}}^+)$ respectively, i.e.

$$y_m^- = \frac{a^-}{\lambda^-}, \quad y_{m_1}^+ = \frac{a^+}{\lambda_2^+}, \quad y_{m_2}^+ = \frac{a^+}{\lambda_1^+}.$$

Note that the left Poincaré map P_L of system (2.1) can be applied to system (3.1). Here we only consider the right Poincaré map P_R of system (3.1). Suppose that the orbit of the left subsystem (3.1) starting at the initial point $(0, y_0^+)$ with $y_0^+ < b$ goes into the left zone $\mathbb{R}_+^2$ and reaches $x = 0$ again at some point $(0, y_1^+)$ with $y_1^+ > b$ after some finite time $t_0^+ > 0$. Then we can define a right Poincaré map P_R satisfying

$$P_R(b) = b, \quad P_R(y_0^+; b) = y_1^+, \quad y_1^+ > b > y_0^+.$$

From [12], we obtain that the parametric representations of the right Poincaré map are

$$\begin{aligned} y_0^+(t^+) &= \frac{a^+}{D^+} \cdot \frac{\Psi_{\lambda_1,\lambda_2}(t^+)}{e^{\lambda_1 t^+} - e^{\lambda_2 t^+}} + b, \\ y_1^+(t^+) &= -\frac{a^+}{D} \cdot \frac{e^{T^+ t^+} \cdot \Psi_{\lambda_1,\lambda_2}(-t^+)}{e^{\lambda_1 t^+} - e^{\lambda_2 t^+}} + b,\ t^+ > 0, \end{aligned} \tag{3.2}$$

where $\Psi_{\lambda_1,\lambda_2}(t^+) = \lambda_1 - \lambda_2 + \lambda_2 \cdot e^{\lambda_1 t^+} + \lambda_1 \cdot e^{\lambda_2 t^+}$. Some properties of P_R are proved in the following from [12].

Lemma 3.1 (Proposition 2.4, [12]). *For the right subsystem* (3.1)*, the right Poincaré map P_R is well defined if and only if $a^+ < 0$. Moreover, the following conditions hold.*

1. $y_0^+(t^+)$ is decreasing and $y_1^+(t^+)$ is increasing with respect to t^+.

2. When $T^+ > 0$, the domain and range of P_R are $(b + y_{m_2}^+,\ b)$ and $(b,\ +\infty)$.

(I) P_R is decreasing with respect to y_0^+.

(II) P_R has $y_0^+ = b + y_{m_2}^+$ as an asymptote.

(III) $P_R''(y_0^+; b) > 0$.

3. When $T^+ < 0$, the domain and range of P_R are $(-\infty,\ b)$ and $(b,\ b + y_{m_1}^+)$.

(I) P_R is decreasing with respect to y_0^+.

(II) P_R has $y_1^+ = b + y_{m_1}^+$ as an asymptote.

(III) $P_R''(y_0^+; b) < 0$.

4. We define $P_R(b; b) = b$. Then P_R is continuous at $y_0^+ = b$ and the first two derivatives of P_R at $y_0^+ = b$ are

$$P_R'(b; b) = -1, \quad P_R''(b; b) = -\frac{4T^+}{3a^+}.$$

Now, we define the full Poincaré map by $P = P_R \circ P_L$ for a fixed parameter b. From Lemma 2.1 and Lemma 3.1, we can directly obtain some properties of P as follows.

Lemma 3.2. *For system* (3.1), *the full Poincaré map P is well defined if and only if $a^- > 0 > a^+$. Moreover,*

$$P'(0) = 1, \quad P''(0) = \frac{4}{3} \cdot \left(\frac{T^-}{a^-} - \frac{T^+}{a^+}\right).$$

3.2. The existence and number of limit cycles of $N'N$

In the following, we investigate the existence and number of limit cycles for system (1.2) with $N'N$. We introduce the same functions $\mathbf{D}_1(y_0; b)$ and $\mathbf{D}_2(y_0; b)$ as in the previous section.

Theorem 3.1. ($b > 0$) *Suppose that $a^- > 0 > a^+$, $(T^+)^2 > 4D^+ > 0$ and $(T^-)^2 = 4D^- > 0$ for system* (1.2).

1. If $T^+ \cdot T^- > 0$, then the following subcases arise.

(I) When $T^+b > 0$, there don't exist limit cycles.

(II) When $T^+b < 0$, there exists a unique stable limit cycle.

2. If $T^+ > 0 > T^-$, then the following subcases arise.

(I) When $\frac{a^-}{T^-} \le \frac{a^+}{T^+}$, there don't exist limit cycles.

(II) When $\frac{a^-}{T^-} > \frac{a^+}{T^+}$,

(a) If $y_m^- \le y_{m_2}^+$, there is a value $b_m > 0$ such that system (1.2) *has no limit cycles for $b > b_m$, and system* (1.2) *has at least two limit cycles for $b_m > b > 0$. Moreover, when $b \in (0,\ b_m)$ is sufficiently small, system* (1.2) *has exactly two limit cycles in which the inter one is stable and the outer one is unstable.*

(b) If $y_m^- > y_{m_2}^+$, then

(b1) When $y_m^- > y_{m_2}^+ + b$, there exists at least a stable limit cycle.

(b2) When $y_m^- \le y_{m_2}^+ + b$, there don't exist limit cycles if $y_m^- < 0 < y_{m_2}^+ + b$, and if $y_m^- \le y_{m_2}^+ + b < 0$, there is a value $b_m > 0$ such that system (1.2) *has at least two limit cycles for $b_m > b > 0$ and no limit cycles for $b > b_m$. Moreover, when $b \in (0,\ b_m)$ is sufficiently small, system* (1.2) *has exactly two limit cycles in which the inter one is stable and the outer one is unstable.*

3. If $T^- > 0 > T^+$, then the following subcases arise.

(I) When $\frac{a^-}{T^-} \le \frac{a^+}{T^+}$, there don't exist limit cycles.

(II) When $\frac{a^-}{T^-} > \frac{a^+}{T^+}$,

(a) If $y_m^- \le y_{m_1}^+$, there is a value $b_m > 0$ such that system (1.2) *has no limit cycles for $b > b_m$, and system* (1.2) *has at least two limit cycles for $b_m > b > 0$. Moreover, when $b \in (0,\ b_m)$ is sufficiently small, system* (1.2) *has exactly two limit cycles in which the inter one is stable and the outer one is unstable.*

(b) If $y_m^- > y_{m_1}^+$, then

(b1) When $y_m^- > y_{m_1}^+ + b$, there exists at least a stable limit cycle.

(b2) When $y_m^- \le y_{m_1}^+ + b$, there don't exist limit cycles if $y_m^- - b < 0 < y_{m_1}^+$, and if $0 < y_m^- - b \le y_{m_1}^+$, there is a value $b_m > 0$ such that system (1.2) *has at least two limit cycles for $b_m > b > 0$ and no limit cycles for $b > b_m$. Moreover, when $b \in (0,\ b_m)$ is sufficiently small, system* (1.2) *has exactly two limit cycles in which the inter one is stable and the outer one is unstable.*

Proof.

1. The proof of this statement is similar to the proof of the statement 1 of Theorem 2.1 and hence is omitted here.

2. Freire etc [7] changed system (1.2) with $b=0$ for $N'N$ into the following system to reduce the parameters, that is

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix}2\gamma_L & -1\\ \gamma_L^2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}0\\ a_L\end{pmatrix}, & x\le 0,\\ \begin{pmatrix}2\gamma_R & -1\\ \gamma_R^2-1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}0\\ a_R\end{pmatrix}, & x\ge 0.\end{cases} \tag{3.3}$$

The authors [24] proved that system (3.3) with $b=0$ has no limit cycles if $a_R\gamma_L-a_L\gamma_R\ge 0$ or $0>a_L\ge a_R\gamma_L-a_L\gamma_R$(i.e. $\frac{a^-}{T^-}\le\frac{a^+}{T^+}$ or $\frac{a^-}{T^-}>\frac{a^+}{T^+}$ and $y_m^-\ge y_{m_2}^+$ for system (3.1) with $b=0$) and has an unstable limit cycle if $0>a_R\gamma_L-a_L\gamma_R>a_L$(i.e. $\frac{a^-}{T^-}>\frac{a^+}{T^+}$ and $y_m^-<y_{m_2}^+$ for system (3.1) with $b=0$).

(I) If $\frac{a^-}{T^-}\le\frac{a^+}{T^+}$, then system (3.1) with $b=0$ has no limit cycles. Then, we have $Graph(P_L)\cap Graph(P_R^{-1}|_{b=0})=\{0,\ 0\}$. By Lemma 3.2, it yields that

$$\begin{aligned}&P_R^{-1}|_{b=0}(0)=P_L(0),\\ &P_L(y_0)\le P_R^{-1}|_{b=0}(y_0;b),\ y_0>0.\end{aligned} \tag{3.4}$$

From (3.4) and (2.15), we obtain that $P_R^{-1}|_{b>0}(y_0;b)=P_R^{-1}|_{b=0}(y_0-b;b)+b\ge P_L(y_0-b;b)+b>P_L(y_0),\ y_0\ge b$. Hence, there don't exist limit cycles for system (3.1).

(II) (a) We only need to consider the following two cases.

(a1) If $\frac{a^-}{T^-}>\frac{a^+}{T^+}$ and $y_m^-<y_{m_2}^+$, then system (3.1) with $b=0$ has an unstable limit cycle, that is, $Graph(P_L)\cap Graph(P_R^{-1}|_{b=0})=\{(0,\ 0),(y_0^{(5)},\ y_1^{(5)})\}$ and

$$\begin{aligned}&P_R^{-1}|_{b=0}(0)=P_L(0),\\ &P_L(y_0^{(5)})=P_R^{-1}|_{b=0}(y_0^{(5)};b),\\ &P_L(y_0)>P_R^{-1}|_{b=0}(y_0;b),\ \forall y_0\in(0,\ y_0^{(5)}),\\ &P_L(y_0)<P_R^{-1}|_{b=0}(y_0;b),\ \forall y_0\in(y_0^{(5)},\ +\infty).\end{aligned} \tag{3.5}$$

By (3.5) and (2.15), we obtain that $P_R^{-1}|_{b>0}(y_0;b)=P_R^{-1}|_{b=0}(y_0-b;b)+b>P_R^{-1}|_{b=0}(y_0;b)\ge P_L(y_0)$ for $y_0\ge y_0^{(5)}$. Then, there don't exist limit cycles for $y_0\ge y_0^{(5)}$. In the following we investigate the existence and number of limit cycles on the interval $[b,\ y_0^{(5)})$.

Let b_m be the maximum of $\mathbf{D}_2(y_0;0)$ on the interval $[0,y_0^{(5)}]$ and $\mathbf{D}_2(y_0;0)$ reaches its maximum at $y_0=y_0^m$, that is $\mathbf{D}_2(y_0^m;0)=b_m$. Then $\mathbf{D}_2(0;0)=0$ and $b_m>\mathbf{D}_2(y_0;0)>0$ for $y_0\in[b,\ y_0^{(5)})$.

For $b>b_m$, $\mathbf{D}_2(y_0;b)=\mathbf{D}_2(y_0-b;0)-b\le b_m-b<0$ for $y_0\in[b,\ y_0^{(5)})$ by Lemma 2.4. Then, there don't exist limit cycles.

For $b_m>b>0$, by (3.5) and (2.15), we have $\mathbf{D}_2(b;b)=\mathbf{D}_2(0;0)-b=-b<0$ and

$$\begin{aligned}\mathbf{D}_1(y_0^{(5)};b)&=P_L(y_0^{(5)})-(P_R^{-1}|_{b>0})(y_0^{(5)};b)\\ &=P_L(y_0^{(5)})-(P_R^{-1}|_{b=0})(y_0^{(5)};b)-b\\ &=-b<0,\ \forall y_0\in[b,\ y_0^{(5)}).\end{aligned}$$

Then, from Lemma 2.4, $\mathbf{D}_2(y_0^{(5)}; b) < 0$ for $y_0 \in [b,\ y_0^{(5)})$. In addition, $\mathbf{D}_2(y_0^m + b; b) > 0$. Note that $y_0^m + b \in [b,\ +\infty)$. Then, $\mathbf{D}_2(y_0; b)$ has at least two zeros and system (3.1) has at least two limit cycles. When $b \in (0,\ b_m)$ is sufficiently small, $\mathbf{D}_2(y_0; b)$ has exactly two zeros by using the implicit function theorem as in Theorem 4 of [16]. We denote two zeros of $\mathbf{D}_2(y_0; b)$ by $y_0^{(6)}$ and $y_0^{(7)}$, respectively. By Lemma 2.4, we know that $P_L(y_0) < (P_R^{-1}|_{b>0})(y_0; b)$ for $y_0 \in [b,\ y_0^{(6)}) \cup (y_0^{(7)},\ y_0^{(5)})$ and $P_L(y_0) > (P_R^{-1}|_{b>0})(y_0; b)$ for $y_0 \in (y_0^{(6)},\ y_0^{(6)})$. Then, the inter limit cycle is stable and the outer one is unstable.

(a2) If $\frac{a^-}{T^-} > \frac{a^+}{T^+}$ and $y_m^- = y_{m_2}^+$, then system (2.1) with $b = 0$ has no limit cycles. The proof is similar to case (b1) and is omitted here.

(b) If $\frac{a^-}{T^-} > \frac{a^+}{T^+}$ and $y_m^- > y_{m_2}^+$, then for $y_0 \geq b$ we have $\mathbf{D}_1(b; b) = P_L(b) - P_R^{-1}|_{b>0}(b) = P_L(b) - b < 0$ and

$$\begin{aligned}\lim_{y_0 \to +\infty} \mathbf{D}_1(y_0; b) &= \lim_{y_0 \to +\infty} P_L(y_0) - (P_R^{-1}|_{b>0})(y_0; b) \\ &= y_m^- - y_{m_2}^+ - b.\end{aligned}$$

In the following, we divide into two cases to analyze.

(b1) If $y_m^- > b + y_{m_2}^+$, then

$$\lim_{y_0 \to +\infty} \mathbf{D}_1(y_0; b) > 0.$$

Now, there is at least a value $y_0^{(8)}$ such that $\mathbf{D}_1(y_0^{(8)}; b) = 0$ on the interval $[b,\ +\infty)$ and there exists at least a limit cycle which is stable.

(b2) When $y_m^- < 0 < b + y_{m_2}^+$, by Lemma 2.1 and Lemma 3.1, we know that the range of $(P_R^{-1}|_{b>0})(y_0; b)$ is $(b + y_{m_2}^+,\ b)$ for $y_0 \geq b$ and $P_L(y_0) \leq 0$ for $y_0 \geq 0$. Then $(P_R^{-1}|_{b>0})(y_0; b) > P_L(y_0)$ for $y_0 \geq b$ if $b \geq -y_{m_2}^+$. Hence, there don't exist limit cycles.

When $y_m^- \leq b + y_{m_2}^+ < 0$, system (3.1) with $b = 0$ has no limit cycle, that is, $Graph(P_L) \cap Graph(P_R^{-1}|_{b=0}) = \{(0,\ 0)\}$ and

$$\lim_{y_0 \to +\infty} \mathbf{D}_1(y_0; b) \leq 0.$$

The rest of the proof is similar to the case (a1).

3. The proof of this statement is similar to statement 2 and is omitted here.

□

The following theorem is a direct consequence of the above theorem using the change of variables $(x, y, t) \to (x, -y, -t)$.

Theorem 3.2. ($b < 0$) *Suppose that* $a^- > 0 > a^+$, $(T^+)^2 > 4D^+ > 0$ *and* $(T^-)^2 = 4D^- > 0$ *for system* (1.2). *Then the following conditions hold.*

1. If $T^+ \cdot T^- > 0$, *then the following subcases arise.*

(I) When $T^+ b > 0$, *there don't exist limit cycles.*

(II) When $T^+ b < 0$, *there exists a unique unstable limit cycle.*

2. If $T^+ > 0 > T^-$, *then the following subcases arise.*

(I) When $\frac{a^-}{T^-} \geq \frac{a^+}{T^+}$, *there don't exist limit cycles.*

(II) When $\frac{a^-}{T^-} < \frac{a^+}{T^+}$,

(a) If $y_m^- \geq y_{m_2}^+$, *there is a value* $b_m > 0$ *such that system* (1.2) *has no limit cycles for* $-b > b_m$, *and for* $b_m > -b > 0$ *system* (1.2) *has at least two limit cycles.*

Moreover, when $b \in (-b_m,\ 0)$ *is sufficiently small, system* (1.2) *has exactly two limit cycles in which the inter one is unstable and the outer one is stable.*

(b) If $y_m^- < y_{m_2}^+$*, then*

(b1) When $y_m^- < y_{m_2}^+ + b$*, there exists at least an unstable limit cycle.*

(b2) When $y_m^- \geq y_{m_2}^+ + b$*, there don't exist limit cycles if* $y_m^- - b > 0 > y_{m_2}^+$*, and if* $0 > y_m^- - b \geq y_{m_2}^+$*, there is a value* $b_m > 0$ *such that system* (1.2) *has at least two limit cycles for* $b_m > -b > 0$ *and no limit cycles for* $-b > b_m$*. Moreover, when* $b \in (-b_m,\ 0)$ *is sufficiently small, system* (1.2) *has exactly two limit cycles in which the inter one is unstable and the outer one is stable.*

3. If $T^- > 0 > T^+$*, then the following subcases arise.*

(I) When $\frac{a^-}{T^-} \geq \frac{a^+}{T^+}$*, there don't exist limit cycles.*

(II) When $\frac{a^-}{T^-} < \frac{a^+}{T^+}$*,*

(a) If $y_m^- \geq y_{m_1}^+$*, there is a value* $b_m > 0$ *such that system* (1.2) *has no limit cycles for* $-b > b_m$*, and system* (1.2) *has at least two limit cycles for* $b_m > -b > 0$*. Moreover, when* $b \in (-b_m,\ 0)$ *is sufficiently small, system* (1.2) *has exactly two limit cycles in which the inter one is unstable and the outer one is stable.*

(b) If $y_m^- < y_{m_1}^+$*, then*

(b1) When $y_m^- < y_{m_1}^+ + b$*, there exists at least an unstable limit cycle.*

(b2) When $y_m^- \geq y_{m_1}^+ + b$*, there don't exist limit cycles if* $y_m^- > 0 > y_{m_1}^+ + b$*, and if* $y_m^- \geq y_{m_1}^+ + b > 0$*, there is a value* $b_m > 0$ *such that system* (1.2) *has at least two limit cycles for* $b_m > -b > 0$ *and no limit cycles for* $-b > b_m$*. Moreover, when* $b \in (-b_m,\ 0)$ *is sufficiently small, system* (1.2) *has exactly two limit cycles in which the inter one is unstable and the outer one is stable.*

Data Availability. The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interest. The authors declare that they have no conflict of interest.

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