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NEW CLASS OF NONLINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH THEORETICAL ANALYSIS VIA FIXED POINT APPROACH: NUMERICAL AND EXACT SOLUTIONS

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Abstract The analysis of fractional Integro-differential equations is valuable for researchers in the science community. For the present work, we examine the analysis of a newly technique called the Fractional Decomposition Method (FDM) via fixed point approach applies to nonlinear fractional Volterra Integro-Differential equations. Then, we implement the method on four test problems such as; Fractional Volterra Integro-Differential Equations (FVIDE). We present exact and approximate solutions to fractional Volterra Integro-Differential equations. The Caputo fractional derivative will be considered in the current work.

Keywords Fixed-point theory, fractional calculus, Volterra integro-differential equation, Caputo fractional derivative.

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1. Introduction

Many researchers studied in recent time the subject of fractional Volterra integro-differential equations (FVIDEs) because its applicability in various fields of engineering and science [3,6,8–10,12,14,16–19]. Traian Lalescu was the first scientist to study the Volterra integral equations (VIE) in his 1908 thesis. Many applications of VIE can be found in many areas of science such as; demography, and in insurance Mathematics and Physics. When someone converts any IVP or BVP to IE, this usually result in an Integro-differential equations (IDE) which appear in various scientific models. Both integral and differential operators show up in many integro-differential equations. Because of that, we are required to search for a reliable and efficient technique to find analytical solutions of fractional differential equations.

It is well-known among the research community to convert IVP to VIE, and to convert VIE to IVP. Lately, many powerful and reliable techniques have been presented to find analytical approximate solutions for fractional VIDE, and to name

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few for example, the Chebyshev pseudo-spectral method [24], the Homotopy Analysis Method (HAM) [18], Taylor Expansion Method [16], the Fractional Power Series Method [14], the Fractional Differential Transform Method (FDTM) [3], Fractional Adomian Decomposition Method (FADM) [25], Fractional Homotopy Perturbation Method (FHPM) [9], and Fractional Laplace Decomposition Method (FLDM) [16]. In addition, M. Rawashdeh has developed new theorems that help in finding analytical approximate solutions to fractional nonlinear PDEs using the FNDM [22, 23]. The first author of the current paper was the first researcher to combine both the natural decomposition method (NDM) along with the (ADM) to solve linear and nonlinear ODEs and PDEs in a thesis authored by S. Maitama in 2014, see [25].

We implement the newly techniques (FDM) to four linear and nonlinear Volterra integro-differential equations to show our new method is valid and efficient which will prove the simplicity and the easiness of current algorithm.

First, we explore the FVIDE given by:

$$D_t^{\frac{3}{4}}\left(v(t)\right)=\frac{6\,t^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)}-\left(\frac{t^2\,e^t}{5}\right)v(t)+\int_0^t e^t\tau v(\tau)d\tau, \tag{1.1}$$

along with I.C.:

$$v(0)=0. \tag{1.2}$$

It is known that $v(t)=t^3$ is the exact solution of the above equation (1.1).

Second, we take a look at the linear FVIDE given by:

$$D_t^{\beta}\left(v\left(t\right)\right)=\frac{8t^{\frac{3}{2}}}{3\Gamma\left(0.5\right)}-t^2-\frac{t^3}{3}+v\left(t\right)+\int_0^t v\left(\tau\right)d\tau, \quad 0<t,\,\beta\leq 1, \tag{1.3}$$

along with I.C.:

$$v(0)=0. \tag{1.4}$$

Third, we examine the nonlinear FVIDE given as:

$$D_t^{\frac{6}{5}}\left(v(t)\right)=\frac{5t^{\frac{4}{5}}}{2\Gamma\left(\frac{4}{5}\right)}-\frac{t^9}{252}+\int_0^t\left(t-\tau\right)^2\left(v(\tau)\right)^3d\tau,\ 0\leq t<1, \tag{1.5}$$

along with I.C's.:

$$v'(0)=v(0)=0. \tag{1.6}$$

Note that the above equation has this exact solution $v(t)=t^2$.

Finally, we explore the nonlinear FVIDE given as:

$$D_t^{\beta}\left(v\left(t\right)\right)=1+\int_0^t v'\left(\tau\right)v\left(\tau\right)d\tau, \quad 0<\beta\leq 1, \tag{1.7}$$

with I.C.:

$$v(0)=0. \tag{1.8}$$

The rest of current research is present in this fashion: In Section 2, we present definitions and some background of fractional calculus. Section 3is devoted for some of the theories of the N-transformations along with some important properties of the natural transform. In Section 4, we examine in details the convergence analysis of the Fractional Decomposition Method (FDM) applied to the nonlinear FVIDE. In section 5, we implement the FDM to four linear and nonlinear FIDE's. Finally, we devote section 6 for our conclusion of this current work.

2. Fractional Calculus Background

In this section, we shall look at important definitions and properties which are useful whenever someone talk about the subject of fractional calculus [7,11,13,15,21].

Definition 2.1. If $h(t) \in \mathbb{R}$, where $t > 0$. Then $h(t)$ is in the space $C_\mu,\ \mu \in \mathbb{R}$ if $\exists\, p \in \mathbb{R}$ such that $h(t) = t^p g(t)$, where $g(t)$ in $C\,[0,\infty)$, and $h(t) \in C_\mu^m$ if $h^{(m)} \in C_\mu,\ m = 1, 2, \ldots.$

Definition 2.2. The Riemann-Liouville fractional integral for the function g of order $\beta \geq 0$, is defined as:

$$\left\{ \begin{aligned} & J^\beta\left(g(s)\right) = \frac{1}{\Gamma(\beta)} \int_0^s (s-t)^{\beta-1} g(t)\, dt,\ \beta > 0,\ s > 0 \\ & J^0\left(g(s)\right) = g(s) \end{aligned} \right\}. \tag{2.1}$$

Definition 2.3. The Caputo fractional derivative of g is:

$${}^cD^\beta\left(g(s)\right) = J^{k-\beta} D^k\left(g(s)\right) = \frac{1}{\Gamma(k-\beta)} \int_0^s (s-t)^{k-\beta-1} g^{(k)}(t) dt, \tag{2.2}$$

for $k-1 < \beta \leq k,\ k = 1, 2, \ldots,\ s > 0,\ g \in C_{-1}^k$.

Definition 2.4 ([20])**.** The Gamma function can be defined as:

$$\Gamma(w) = \int_0^\infty e^{-s}\, s^{w-1}\, ds,\ w > 0. \tag{2.3}$$

Definition 2.5 ([4])**.** Given a complete (Y, ρ) metric space. Then $F : Y \to Y$ is called a contraction mapping on Y if we can find $0 < C < 1$ with $\rho\left(F(x), F(y)\right) \leq C\,\rho(x, y), \forall\, x, y \in Y$.

Theorem 2.1 ([4])**.** *Given complete Y nonempty metric space ρ, and $F : Y \to Y$ is a contraction mapping, then F has a unique fixed-point, such that $F\left(y^*\right) = y^*$.*

Theorem 2.2 ([4])**.** *Given a non-empty complete metric space (Y, ρ) with a map $F : Y \to Y$ which is of a contraction type. Then the mapping has a unique fixed-point $y^* \in Y$ (i.e. $F\left(y^*\right) = y^*$). Furthermore, y^* can be obtained by starting with an element $y_0 \in Y$ and define a sequence $\{y_n\}$ by $F\left(y_{n-1}\right) = y_n$ for $n \geq 1$. Then $y_n \to y^*$.*

3. The Natural-Adomian Method

For the sake of the definition and important properties, we refer the readers to read more about the background of the general integral transform, Laplace, Sumudu and natural transform method and its related properties for any given function $\zeta(x)$, $x \in \mathbb{R}$, see for example [5].

Definition 3.1. Consider $\zeta\,(s)$ to be a piece-wise continuous function over $\mathbb{R}$ and $M, K, p, q > 0$ where $p < q$. Suppose that

$$A = \left\{\zeta\,(s) : \left|\zeta\,(s)\right| < Me^{p\,s}\chi_{(s_2,\infty)}(s) + Ke^{qs}\chi_{(-\infty,s_1)}(s)\right\}.$$

So, $|\zeta(s)| \leq M e^{p\,s}$ for $s \longrightarrow \infty$ i.e. $s > s_2$ and $|\zeta(s)| \leq K e^{q\,s}$ for $s \longrightarrow -\infty$ i.e. $s < s_1$.

Note that for any $\zeta(s)$ in the class A with $r, w > 0$ we have:

$$\begin{aligned}\left|\int_{-\infty}^{\infty} e^{-r\,s}\zeta(s\,w)\,ds\right| &\leq M\int_{0}^{\infty} e^{-r\,s}e^{p|s\,w|}ds + K\int_{-\infty}^{0} e^{-r\,s}e^{q|s\,w|}ds\\ &= M\int_{s_2}^{\infty} e^{(pw-r)s}ds + K\int_{-\infty}^{s_1} e^{(q\,w-r)s}ds.\end{aligned}$$

The above is convergent if $p\,w - r < 0$ and $q\,w - r > 0$, thus $p\,w < r < q\,w$ and so $p < \frac{r}{w} < q$. Hence, $\zeta(s)$ is of exponential order.

Then, the natural transformation (N-transformation) is given as:

$$\aleph(\zeta(s)) = L(r, w) = \int_{-\infty}^{\infty} e^{-r\,s}\,\zeta(w\,s)ds,\ r, w > 0, \tag{3.1}$$

where $\aleph$ is the N-transform of $\zeta(s)$ and r, w are the N-transform parameters. Note Equation (3.1) can be written as,

$$\aleph(\zeta(s)) = \aleph^{+}(\zeta(s)) + \aleph^{-}(\zeta(s)) = L^{+}(r, w) + L^{-}(r, w),$$

where,

$$\aleph^{+}(\zeta(s)) = L^{+}(r, w) = \int_{0}^{\infty} e^{-r\,s}\,\zeta(w\,s)\,ds,\ r, w \in (0, \infty). \tag{3.2}$$

Moreover,

$$\aleph^{-1}[L(r, w)] = \zeta(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{r\,s}{w}}\,L(r, w)\,dr. \tag{3.3}$$

Thus, Eq. (3.2) is the natural transformation and Eq. (3.3) is the inverse natural transformation.

Here are some useful N-transforms properties and we shall use them throughout this paper, see [22, 23]:

1. $\aleph^{+}[M] = \dfrac{M}{r}$.
2. $\aleph^{+}\left[x^{\beta}\right] = \dfrac{\Gamma(\beta+1)\,w^{\beta}}{r^{\beta+1}},\ \ \beta > -1.$
3. $\aleph^{+}\left[e^{bx}\right] = \dfrac{1}{(r-bw)}$.
4. Suppose that $k > 0$, where $k-1 < \beta \leq k$ and $L(r, w)$ is the natural transform of the function $\zeta(x)$, then the natural transformation of the fractional derivative in the Caputo sense of the function $\zeta(x)$ of order β denoted by ${}^{c}D^{\beta}\zeta(x)$ is given by:
$$\aleph^{+}\left[{}^{c}D^{\beta}\zeta(x)\right] = \frac{r^{\beta}}{w^{\beta}}L(r, w) - \sum_{n=0}^{k-1}\frac{r^{\beta-(n+1)}}{w^{\beta-n}}\left(D^{n}\zeta(x)\right)_{x=0}.$$

4. Convergence Analysis of the FDM for nonlinear FVIDE

In this section, we first present proofs for uniqueness and convergence theorem along with error estimate using our FDM. Consider the general nonlinear non-homogeneous FVIDEs with initial conditions given by:

$$D_t^{\beta}\left(v\left(t\right)\right)=g\left(t\right)v\left(t\right)+s\left(t\right)+\delta\int_0^t\sigma\left(\tau,t\right)F\left(v\left(\tau\right)\right)d\tau, \tag{4.1}$$

where $0<\beta\leq 1$, along with I.C.:

$$v\left(0\right)=a. \tag{4.2}$$

Note that the fractional derivative of $v\left(t\right)$ is in the sense of Caputo, $F\left(v\left(\tau\right)\right)$ is a nonlinear continuous function and $s\left(t\right)$ is the non-homogeneous term and $\left|\sigma\left(\tau,t\right)\right|\leq M$. Employ the N-transformation and property 4 to Eq. (4.1) to find:

$$\aleph^{+}\left[D_t^{\beta}\left(v\left(t\right)\right)\right]=\aleph^{+}\left[s\left(t\right)\right]+\aleph^{+}\left[g\left(t\right)v\left(t\right)+\delta\int_0^t\sigma\left(\tau,t\right)F\left(v\left(\tau\right)\right)d\tau\right]. \tag{4.3}$$

$$\frac{r^{\beta}}{w^{\beta}}\aleph^{+}\left[v\left(t\right)\right]-\sum_{k=0}^{n-1}\frac{r^{\beta-(k+1)}}{w^{\beta-k}}\left[D^{(k)}v\left(t\right)\right]_{t=0}$$
$$=L\left(r,w\right)+\aleph^{+}\left[g\left(t\right)v\left(t\right)+\delta\int_0^t\sigma\left(\tau,t\right)F\left(v\left(\tau\right)\right)d\tau\right]. \tag{4.4}$$

$$\aleph^{+}\left[v\left(t\right)\right]=\frac{a\,w^{\beta}}{r^{\beta+1}}+\frac{w^{\beta}}{r^{\beta}}L\left(r,w\right)+\frac{w^{\beta}}{r^{\beta}}\aleph^{+}\left[g\left(t\right)v\left(t\right)+\delta\int_0^t\sigma\left(\tau,t\right)F\left(v\left(\tau\right)\right)d\tau\right]. \tag{4.5}$$

Take the inverse of N-transform to find:

$$v(t)=S\left(t\right)+\aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph^{+}\left[g\left(t\right)v\left(t\right)+\delta\int_0^t\sigma\left(\tau,t\right)F\left(v\left(\tau\right)\right)d\tau\right]\right]. \tag{4.6}$$

Suppose our solution of $v(t)$ is given by:

$$v(t)=\sum_{n=0}^{\infty}v_n(t). \tag{4.7}$$

Moreover, the nonlinear part is written as:

$$F\left(v\left(\tau\right)\right)=\sum_{n=0}^{\infty}A_n\left(\tau\right). \tag{4.8}$$

Note that the Adomian polynomial of $v_0,\,v_1,\,\ldots,v_n$ are represented by A_n's which can be computed by:

$$A_n=\frac{1}{n!}\frac{d^n}{d\sigma^n}\left[F\left(\sum_{i=0}^{n}\sigma^i v_i\right)\right],\quad n=0,1,2,\ldots. \tag{4.9}$$

Note that we can simplify the formula in Eq. (4.9) to be:

$$\begin{aligned} A_0 &= F(v_0) \\ A_1 &= v_1 F'(v_1) \\ A_2 &= v_2 F'(v_0) + \frac{1}{2!} v_1^2 F''(v_0). \end{aligned} \tag{4.10}$$

One can continue in this manner to get the other polynomials.

Substitute Eq. (4.7) into Eq. (4.6) to arrive at:

$$\sum_{n=0}^{\infty} v_n(t) = S(t) + \aleph^{-1}\left[\frac{w^\beta}{r^\beta}\aleph^{+}\left[g(t)\sum_{n=0}^{\infty} v_n(t) + \delta\int_0^t \sigma(\tau,t)\sum_{n=0}^{\infty} A_n(\tau)d\tau\right]\right]. \tag{4.11}$$

Eq. (4.11) gives:

$$v_0(t) = S(t). \tag{4.12}$$

In general, one can find:

$$v_{n+1}(t) = \aleph^{-1}\left[\frac{w^\beta}{r^\beta}\aleph^{+}\left[g(t)v_n(t) + \delta\int_0^t \sigma(\tau,t)A_n(\tau)d\tau\right]\right],\ n \geq 0. \tag{4.13}$$

In this case, our solution will be:

$$v(t) = \sum_{n=0}^{\infty} v_n(t). \tag{4.14}$$

Again $F(v(t)) = \sum_{i=0}^{\infty} A_i$, where A_i's are the Adomian polynomials.

Note $S(t)$ represents the I.C's and the non homogeneous part. We shall use the new form of Adomian polynomials, see [6] to find:

$$A_n = F(s_n) - \sum_{j=0}^{n-1} A_j, \quad \text{where} \quad s_n = \sum_{i=0}^{n} v_i(t).$$

Theorem 4.1 (Uniqueness Theorem)**.** *Let* $0 < \gamma < 1$ *with* $\gamma = \frac{(C_1+\lambda MTC_2)}{\Gamma(\beta+1)}t^\beta$. *Then Eq. (4.1) defines a unique solution.*

Proof. Consider the Banach space of all functions on $J = [0,T]$ which are also continuous, say $A = (C[J], \|.\|)$ having a norm $\|.\|$. Let $G : A \to A$ be define by:

$$v_{n+1}(t) = \aleph^{-1}\left[\frac{w^\beta}{r^\beta}\aleph^{+}\left[g(t)v_n(t) + \delta\int_0^t \sigma(\tau,t)A_n(\tau)d\tau\right]\right].$$

Assume $L[g(t)v(t)] = v(t)$ and $F[v(t)] = \sum_{n=0}^{\infty} A_n$. Moreover, suppose $|L(v) - L(v^*)| < C_1|v - v^*|$ and $|F(v) - F(v^*)| < C_2|v - v^*|$, where C_1, C_2 are the Lipschitz constants and v, v^* are two different solutions of Eq. (4.1).

$$\|G(v) - G(v^*)\|$$

$$
\begin{aligned}
&= \max_{t\in J}\left| \begin{array}{l} \aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[L(v)+\delta\int_0^t \sigma(\tau,t)\,F(v)d\tau\right]\right] \\ -\aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[L(v^*)+\delta\int_0^t \sigma(\tau,t)\,F(v^*)d\tau\right]\right] \end{array}\right| \\
&= \max_{t\in J}\left| \begin{array}{l} \aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[L(v)-L(v^*)\right]\right] \\ +\aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[\delta\int_0^t \sigma(\tau,t)\left(F(v)-F(v^*)\right)d\tau\right]\right] \end{array}\right| \\
&\le \max_{t\in J}\left[C_1\aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[|v-v^*|\right]\right]+\left(\delta MC_2T\right)\aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[|v-v^*|\right]\right]\right] \\
&\le \max_{t\in J}\left(C_1+\delta MTC_2\right)\left[\aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[|v-v^*|\right]\right]\right] \\
&\le \left(C_1+\delta MTC_2\right)\left[\aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[\|v(t)-v^*(t)\|\right]\right]\right] \\
&= \|v(t)-v^*(t)\|\,\frac{\left(C_1+\delta MTC_2\right)}{\Gamma\left(\beta+1\right)}t^{\beta}.
\end{aligned}
$$

Since $0<\gamma<1$, then G is contraction mapping and it follows by Theorem (2.2), there exists a unique solution to Eq. (4.1). □

Theorem 4.2 (Convergence Theorem)**.** *The series solution of Eq. (4.14) for Eq. (4.1) involving the FDM will converge provided that* $0<\gamma<1$ *where* $|v_1|<\infty$*.*

Proof. Given s_k to be the m^{th} partial sum, i.e. $s_k=\sum_{i=0}^{k}v_i(t)$. We shall prove $\{s_k\}$ is a Cauchy sequence in the Banach space A. Consider the Adomian polynomials in [26]: $F\left(s_k\right)=A_k^*+\sum_{i=0}^{k-1}A_i^*$. Let s_n and s_k be any two partial sums with $k\ge n$. Then,

$$
\begin{aligned}
\|s_k-s_n\| &= \max_{t\in J}|s_k-s_n| \\
&= \max_{t\in J}\left|\sum_{i=n+1}^{k}v_i^*(t)\right|,\ k=1,2,\ldots \\
&\le \max_{t\in J}\left|\aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[\sum_{i=n+1}^{k}L\left(v_{i-1}(t)\right)\right]\right]\right. \\
&\qquad \left.+\aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[\sum_{i=n+1}^{k}\delta\int_0^t\sigma\left(\tau,t\right)F(v)d\tau\right]\right]\right| \\
&= \max_{t\in J}\left|\aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[\sum_{i=n}^{k-1}L\left(v_i(t)\right)\right]\right]\right. \\
&\qquad \left.+\aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[\delta\int_0^t\sigma\left(\tau,t\right)\sum_{i=n}^{k-1}A_i(t)d\tau\right]\right]\right| \\
&\le \max_{t\in J}\left|\aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[L(s_{k-1})-L(s_{n-1})\right]\right]\right.
\end{aligned}
$$

$$+\aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[\delta\int_0^t \sigma(\tau,t)\,d\tau\,(F(p_{k-1})-F(p_{n-1}))\right]\right]\Bigg|$$

$$\begin{aligned}&\le C_1\max_{t\in J}\aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[\|s_{k-1}-s_{n-1}\|\right]\right]\\ &\quad+(\delta MTC_2)\max_{t\in J}\aleph^{-1}\left[\left(\frac{w}{r}\right)^{\beta}\aleph^{+}\left[\|s_{k-1}-s_{n-1}\|\right]\right]\\ &=\frac{(C_1+\delta MTC_2)\,t^{\beta}}{\Gamma(\beta+1)}\|s_{m-1}-s_{n-1}\|.\end{aligned}$$

Thus, $\|r_m-r_n\|\le\gamma\|s_{k-1}-s_{n-1}\|$. Choose $k=n+1$, then

$$\|s_{n+1}-s_n\|\le\gamma\|s_n-s_{n-1}\|\le\gamma^2\|s_{n-1}-s_{n-2}\|\le...\le\gamma^n\|s_1-s_0\|,$$

where $\gamma=\frac{(C_1+\delta MTC_2)t^{\beta}}{\Gamma(\beta+1)}$.

Similarly, using the triangle inequality

$$\begin{aligned}\|s_k-s_n\|&\le\|s_{n+1}-s_n\|+\|s_{n+2}-s_{n+1}\|+...+\|s_k-s_{k-1}\|\\ &\le\left[\gamma^n+\gamma^{n+1}+...+\gamma^{k-1}\right]\|s_1-s_0\|\\ &\le\gamma^n\left[\frac{1-\gamma^{k-n}}{1-\gamma}\right]\|v_1\|.\end{aligned}$$

But, $0<\gamma<1$, then $1-\gamma^{k-n}<1$. Thus,

$$\|s_k-s_n\|\le\frac{\gamma^n}{1-\gamma}\max_{t\in J}|v_1|. \tag{4.15}$$

Since $v(t)$ is bounded, then $\|v_1\|<\infty$. So, as $n\to\infty$, then $\|s_k-s_n\|\to 0$. So, $\{s_k\}$ is a Cauchy in A. Concluding, $v(t)=\sum_{n=0}^{\infty}v_n(t)$ converges. □

Theorem 4.3 (Error Estimates)**.** *The maximum absolute truncation error of the series solution in Equation (4.14) to Equation (4.1) is estimated to be*

$$\max_{t\in J}\left|v(t)-\sum_{k=0}^{n}v_k(t)\right|\le\frac{\gamma^n}{1-\gamma}\max_{t\in J}|v_1|.$$

Proof. From Equation (4.15) in Theorem 4.2 we conclude that $\|s_k-s_n\|\le\frac{\gamma^n}{1-\gamma}\max_{t\in J}|v_1|$. So as $k\to\infty$, we have $s_k\to v(t)$. Then $\|v(t)-s_n\|\le\frac{\gamma^n}{1-\gamma}\max_{t\in J}|v_1(t)|$. Concluding, the maximum truncation absolute error in J is

$$\max_{t\in J}\left|v(t)-\sum_{k=0}^{n}v_k(t)\right|\le\max_{t\in J}\frac{\gamma^n}{1-\gamma}|v_1(t)|=\frac{\gamma^n}{1-\gamma}\|v_1(t)\|.$$

□

5. Numerical Results and Applications

Now we employ the FDM to four examples and then we go by comparing the solutions with existing exact solutions. First, we present the methodology of the FDM:

Consider the general nonlinear FVIDEs with initial conditions given by:

$$D_t^{\beta}\left(v\left(t\right)\right)=g\left(t\right)v\left(t\right)+h\left(t\right)+\delta\int_a^t\sigma\left(\tau,t\right)F\left(v\left(\tau\right)\right)d\tau,$$
$$\text{where } k-1<\beta\leq k,\ n=1,2,3,\ldots, \tag{5.1}$$

with I.C's:

$$v^{(i)}\left(0\right)=a_i,\ i=0,1,2,3,\ldots. \tag{5.2}$$

Note that the fractional derivative is in the Caputo sense of $v\left(t\right)$, $F\left(v\left(\tau\right)\right)$ is a nonlinear continuous function and $f\left(t\right)$ is the non-homogeneous term.

Employ the natural transformation and property (4) to Eq. (5.1) we arrive at:

$$\aleph^{+}\left[D_t^{\beta}\left(v\left(t\right)\right)\right]=\aleph^{+}\left[h\left(t\right)\right]+\aleph^{+}\left[g\left(t\right)v\left(t\right)+\delta\int_a^t\sigma\left(\tau,t\right)F\left(v\left(\tau\right)\right)d\tau\right]. \tag{5.3}$$

$$\frac{r^{\beta}}{w^{\beta}}\aleph^{+}\left[v\left(t\right)\right]-\sum_{k=0}^{n-1}\frac{r^{\beta-(k+1)}}{w^{\beta-k}}\left[D^{(k)}v\left(t\right)\right]_{t=0}$$
$$=L\left(r,w\right)+\aleph^{+}\left[g\left(t\right)v\left(t\right)+\delta\int_a^t\sigma\left(\tau,t\right)F\left(v\left(\tau\right)\right)d\tau\right]. \tag{5.4}$$

$$\aleph^{+}\left[v\left(t\right)\right]=\frac{w^{\beta}}{r^{\beta}}\sum_{k=0}^{n-1}\frac{r^{\beta-(k+1)}}{w^{\beta-k}}\left[D^{(k)}v\left(t\right)\right]_{t=0}+\frac{w^{\beta}}{r^{\beta}}L\left(r,w\right)$$
$$+\frac{w^{\beta}}{r^{\beta}}\aleph^{+}\left[g\left(t\right)v\left(t\right)+\delta\int_a^t\sigma\left(\tau,t\right)F\left(v\left(\tau\right)\right)d\tau\right]. \tag{5.5}$$

Plug in Eq. (5.2) into Eq. (5.5) and applying the inverse to Eq. (5.5), then we arrive at:

$$\begin{aligned}v\left(t\right)=&\aleph^{-1}\left[\sum_{k=0}^{n-1}\frac{a_k\,r^{-(k+1)}}{w^{-k}}\right]+\aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}L\left(r,w\right)\right]\\&+\aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph^{+}\left[g\left(t\right)v\left(t\right)+\delta\int_a^t\sigma\left(\tau,t\right)F\left(v\left(\tau\right)\right)d\tau\right]\right]\\=&S\left(t\right)+\aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph^{+}\left[g\left(t\right)v\left(t\right)+\delta\int_a^t\sigma\left(\tau,t\right)F\left(v\left(\tau\right)\right)d\tau\right]\right].\end{aligned} \tag{5.6}$$

Here the nonhomogeneous part and the I.Cs are represented by $S\left(t\right)$. Suppose our intended solution is given by:

$$v\left(t\right)=\sum_{n=0}^{\infty}v_n\left(t\right). \tag{5.7}$$

Also, the nonlinear term is:

$$F\left(v\left(\tau\right)\right)=\sum_{n=0}^{\infty}A_n\left(\tau\right). \tag{5.8}$$

Here, A_n's are the polynomials of v_0, v_1, $\ldots$, v_n where one can be evaluate using:

$$A_n=\frac{1}{n!}\frac{d^n}{d\delta^n}\left[F\left(\sum_{i=0}^{n}\delta^i v_i\right)\right],n=0,\ 1,\ 2,\ldots. \tag{5.9}$$

Eq. (5.9) can be presented as:

$$\begin{aligned} A_0 &= F\left(v_0\right), \\ A_1 &= v_1 F'\left(v_1\right), \\ A_2 &= v_2 F'\left(v_0\right) + \frac{1}{2!} v_1^2 F''\left(v_0\right). \end{aligned} \tag{5.10}$$

We can continue in this manner to get the other polynomials. Now, we substitute Eq. (5.7) and Eq. (5.8) into Eq. (5.6) to get:

$$\sum_{n=0}^{\infty} v_n\left(t\right) = S\left(t\right) + \aleph^{-1}\left[\frac{w^\beta}{r^\beta}\aleph^+\left[g\left(t\right)\sum_{n=0}^{\infty} v_n\left(t\right) + \delta\int_a^t \sigma\left(\tau,t\right)\sum_{n=0}^{\infty} A_n\left(\tau\right)\,d\tau\right]\right]. \tag{5.11}$$

Going through Eq. (5.11) we find:

$$\begin{aligned} v_0\left(t\right) &= S\left(t\right), \\ v_1\left(t\right) &= \aleph^{-1}\left[\frac{w^\beta}{r^\beta}\aleph^+\left[g\left(t\right)v_0\left(t\right) + \delta\int_a^t \sigma\left(\tau,t\right)A_0\left(\tau\right)d\tau\right]\right], \\ v_2\left(t\right) &= \aleph^{-1}\left[\frac{w^\beta}{r^\beta}\aleph^+\left[g\left(t\right)v_1\left(t\right) + \delta\int_a^t \sigma\left(\tau,t\right)A_1\left(\tau\right)d\tau\right]\right]. \end{aligned}$$

One can form the general recursive relation as follows:

$$v_{n+1}\left(t\right) = \aleph^{-1}\left[\frac{w^\beta}{r^\beta}\aleph^+\left[g\left(t\right)v_n\left(t\right) + \delta\int_a^t \sigma\left(\tau,t\right)A_n\left(\tau\right)d\tau\right]\right],\ n \geq 0. \tag{5.12}$$

Hence, the exact solution of the form:

$$v\left(t\right) = \sum_{n=0}^{\infty} v_n\left(t\right). \tag{5.13}$$

Example 5.1. Given the linear FVIDE:

$$D^{\frac{3}{4}}v(t) = \frac{6\,t^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)} - \frac{t^2\,e^t}{5}v(t) + \int_0^t e^t\tau v(\tau)d\tau, \tag{5.14}$$

with I.C:

$$v(0) = 0. \tag{5.15}$$

Solution. Using the natural transformation of Eq. (5.14) to find:

$$\aleph^+\left[D^{\frac{3}{4}}v(t)\right] = \aleph^+\left[\frac{6\,t^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)} - \frac{t^2\,e^t}{5}v(t) + \int_0^t e^t\tau v(\tau)d\tau\right]. \tag{5.16}$$

Now using property 4 we get:

$$\frac{r^{\frac{3}{4}}}{w^{\frac{3}{4}}}\aleph^+\left[v(t)\right] - \sum_{k=0}^{n-1}\frac{r^{\frac{3}{4}-(k+1)}}{w^{\frac{3}{4}-k}}\left[D^{(k)}\ v(t)\right]_{t=0}$$

$$=\frac{6\,w^{\frac{9}{4}}}{r^{\frac{13}{4}}}+\aleph^{+}\left[-\frac{t^{2}\,e^{t}}{5}v(t)+\int_{0}^{t}e^{t}\tau v(\tau)d\tau\right]. \tag{5.17}$$

Plug in Eq. (5.15) into Eq. (5.17) to find:

$$\aleph^{+}\left[v(t)\right]=\frac{6\,w^{3}}{r^{4}}+\frac{w^{\frac{3}{4}}}{r^{\frac{3}{4}}}\aleph^{+}\left[-\frac{t^{2}\,e^{t}}{5}v(t)+\int_{0}^{t}e^{t}\tau v(\tau)d\tau\right]. \tag{5.18}$$

Employ $\aleph^{-1}$ to Eq. (5.18) to find:

$$v(t)=\frac{6\,t^{3}}{\Gamma(4)}+\aleph^{-1}\left[\frac{w^{\frac{3}{4}}}{r^{\frac{3}{4}}}\aleph^{+}\left[-\frac{t^{2}\,e^{t}}{5}v(t)+\int_{0}^{t}e^{t}\tau v(\tau)d\tau\right]\right]. \tag{5.19}$$

Suppose our intended solution given as:

$$v(t)=\sum_{n=0}^{\infty}v_{n}(t). \tag{5.20}$$

From Eq. (5.20), then Eq. (5.19) become:

$$\sum_{n=0}^{\infty}v_{n}(t)=t^{3}+\aleph^{-1}\left[\frac{w^{\frac{3}{4}}}{r^{\frac{3}{4}}}\aleph^{+}\left[-\frac{t^{2}\,e^{t}}{5}\sum_{n=0}^{\infty}v_{n}(t)+\int_{0}^{t}e^{t}\tau\sum_{n=0}^{\infty}v_{n}(\tau)d\tau\right]\right]. \tag{5.21}$$

Looking at Eq. (5.21) above, we calculate these iterations:

$$\begin{aligned}
v_{0}(t)&=t^{3},\\
v_{1}(t)&=\aleph^{-1}\left[\frac{w^{\frac{3}{4}}}{r^{\frac{3}{4}}}\aleph^{+}\left[-\frac{t^{2}\,e^{t}}{5}v_{0}(t)+\int_{0}^{t}e^{t}\tau v_{0}(\tau)d\tau\right]\right]\\
&=\aleph^{-1}\left[\frac{w^{\frac{3}{4}}}{r^{\frac{3}{4}}}\aleph^{+}\left[-\frac{t^{5}\,e^{t}}{5}+\int_{0}^{t}e^{t}\tau^{4}d\tau\right]\right]\\
&=\aleph^{-1}\left[\frac{w^{\frac{3}{4}}}{r^{\frac{3}{4}}}\aleph^{+}\left[-\frac{t^{5}\,e^{t}}{5}+\frac{t^{5}\,e^{t}}{5}\right]\right]\\
&=0,\\
v_{2}(t)&=\aleph^{-1}\left[\frac{w^{\frac{3}{4}}}{r^{\frac{3}{4}}}\aleph^{+}\left[-\frac{t^{2}\,e^{t}}{5}v_{1}(t)+\int_{0}^{t}e^{t}\tau v_{1}(\tau)d\tau\right]\right]=0.
\end{aligned}$$

Thus one can concludes that $v_1(t)=v_2(t)=\ldots=v_n(t)=0$.

Hence,

$$v(t)=\sum_{n=0}^{\infty}v_{n}(t)=v_{0}(t)+v_{1}(t)+v_{2}(t)+\ldots=t^{3}.$$

Which is in fact our intended exact solution for Eq. (5.14).

Example 5.2. Given the linear FVIDE :

$$D_{t}^{\beta}\left(v\left(t\right)\right)=\frac{8t^{\frac{3}{2}}}{3\Gamma\left(0.5\right)}-t^{2}-\frac{t^{3}}{3}+v\left(t\right)+\int_{0}^{t}v\left(\tau\right)d\tau,\quad 0\leq t\leq 1,\ 0<\beta\leq 1, \tag{5.22}$$

together with I.C:

$$v(0) = 0. \tag{5.23}$$

Solution. Taking N-transformation of Eq. (5.22) we find:

$$\aleph^{+}\left[D_t^{\beta}\left(v\left(t\right)\right)\right] = \aleph^{+}\left[\frac{8t^{\frac{3}{2}}}{3\Gamma\left(0.5\right)} - t^2 - \frac{t^3}{3} + v\left(t\right) + \int_0^t v\left(\tau\right)d\tau\right]. \tag{5.24}$$

Now using property 4 we get:

$$\begin{aligned}&\frac{r^{\beta}}{w^{\beta}}\aleph^{+}\left[v(t)\right] - \sum_{k=0}^{n-1}\frac{r^{\beta-(k+1)}}{w^{\beta-k}}\left[D^{(k)}\ v(t)\right]_{t=0}\\ =&\frac{2w^{\frac{3}{2}}}{r^{\frac{5}{2}}} - \frac{2w^2}{r^3} - \frac{2w^3}{r^4} + \aleph^{+}\left[v(t) + \int_0^t v(\tau)d\tau\right].\end{aligned} \tag{5.25}$$

Plug in Eq. (5.23) into Eq. (5.25) to find:

$$\aleph^{+}\left[v(t)\right] = \frac{2w^{\beta+\frac{3}{2}}}{r^{\beta+\frac{5}{2}}} - \frac{2w^{\beta+2}}{r^{\beta+3}} - \frac{2w^{\beta+3}}{r^{\beta+4}} + \frac{w^{\beta}}{r^{\beta}}\aleph^{+}\left[v(t) + \int_0^t v(\tau)d\tau\right]. \tag{5.26}$$

Employ $\aleph^{-1}$ of Eq. (5.26) we find:

$$v(t) = \frac{2t^{\beta+\frac{3}{2}}}{\Gamma\left(\beta+\frac{5}{2}\right)} - \frac{2t^{\beta+2}}{\Gamma\left(\beta+3\right)} - \frac{2t^{\beta+3}}{\Gamma\left(\beta+4\right)} + \aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph^{+}\left[v(t) + \int_0^t v(\tau)d\tau\right]\right]. \tag{5.27}$$

Suppose our intended solution given by:

$$v(t) = \sum_{n=0}^{\infty} v_n(t). \tag{5.28}$$

From Eq. (5.28), one can rewrite Eq. (5.27) as:

$$\begin{aligned}\sum_{n=0}^{\infty} v_n(t) =&\frac{2t^{\beta+\frac{3}{2}}}{\Gamma\left(\beta+\frac{5}{2}\right)} - \frac{2t^{\beta+2}}{\Gamma\left(\beta+3\right)} - \frac{2t^{\beta+3}}{\Gamma\left(\beta+4\right)}\\ &+ \aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph^{+}\left[\sum_{n=0}^{\infty} v_n(t) + \int_0^t \sum_{n=0}^{\infty} v_n(\tau)d\tau\right]\right].\end{aligned} \tag{5.29}$$

Then looking at Eq. (5.29) above, we calculate the iterations:

$$\begin{aligned}v_0\left(t\right) &= \frac{2\,t^{\beta+\frac{3}{2}}}{\Gamma\left(\beta+\frac{5}{2}\right)},\\ v_1\left(t\right) &= -\frac{2t^{\beta+2}}{\Gamma\left(\beta+3\right)} - \frac{2t^{\beta+3}}{\Gamma\left(\beta+4\right)} + \aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph^{+}\left[v_0\left(t\right) + \int_0^t v_0\left(\tau\right)d\tau\right]\right]\\ &= -\frac{2t^{\beta+2}}{\Gamma\left(\beta+3\right)} - \frac{2t^{\beta+3}}{\Gamma\left(\beta+4\right)} + \aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph^{+}\left[\frac{2t^{\beta+\frac{3}{2}}}{\Gamma\left(\beta+\frac{5}{2}\right)} + \int_0^t \frac{2\,\tau^{\beta+\frac{3}{2}}}{\Gamma\left(\beta+\frac{5}{2}\right)}d\tau\right]\right]\\ &= -\frac{2t^{\beta+2}}{\Gamma\left(\beta+3\right)} - \frac{2t^{\beta+3}}{\Gamma\left(\beta+4\right)} + \aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph^{+}\left[\frac{2t^{\beta+\frac{3}{2}}}{\Gamma\left(\beta+\frac{5}{2}\right)} + \frac{2t^{\beta+\frac{5}{2}}}{\Gamma\left(\beta+\frac{7}{2}\right)}\right]\right]\end{aligned}$$

$$= -\frac{2t^{\beta+2}}{\Gamma(\beta+3)} - \frac{2t^{\beta+3}}{\Gamma(\beta+4)} + \frac{2t^{2\beta+\frac{3}{2}}}{\Gamma\left(2\beta+\frac{5}{2}\right)} + \frac{2t^{2\beta+\frac{5}{2}}}{\Gamma\left(2\beta+\frac{7}{2}\right)},$$

$$v_2(t) = \aleph^{-1}\left[\frac{w^\beta}{r^\beta}\aleph^+\left[v_1(t) + \int_0^t v_1(\tau)\,d\tau\right]\right]$$

$$= -\frac{2\,t^{2\beta+2}}{\Gamma(2\beta+3)} - \frac{4\,t^{2\beta+3}}{\Gamma(2\beta+4)} + \frac{2\,t^{3\beta+\frac{3}{2}}}{\Gamma\left(3\beta+\frac{5}{2}\right)} + \frac{4\,t^{3\beta+\frac{5}{2}}}{\Gamma\left(3\beta+\frac{7}{2}\right)} - \frac{2\,t^{2\beta+4}}{\Gamma(2\beta+5)} + \frac{2\,t^{3\beta+\frac{7}{2}}}{\Gamma\left(3\beta+\frac{9}{2}\right)},$$

$$v_3(t) = \aleph^{-1}\left[\frac{w^\beta}{r^\beta}\aleph^+\left[v_2(t) + \int_0^t v_2(\tau)\,d\tau\right]\right]$$

$$= -\frac{2\,t^{3\beta+2}}{\Gamma(3\beta+3)} - \frac{6\,t^{3\beta+3}}{\Gamma(3\beta+4)} + \frac{2\,t^{4\beta+\frac{3}{2}}}{\Gamma\left(4\beta+\frac{5}{2}\right)} + \frac{6\,t^{4\beta+\frac{5}{2}}}{\Gamma\left(4\beta+\frac{7}{2}\right)} - \frac{6\,t^{3\beta+4}}{\Gamma(3\beta+5)} + \frac{6\,t^{4\beta+\frac{7}{2}}}{\Gamma\left(4\beta+\frac{9}{2}\right)} - \frac{2\,t^{3\beta+5}}{\Gamma(3\beta+6)} + \frac{2\,t^{4\beta+\frac{9}{2}}}{\Gamma\left(4\beta+\frac{11}{2}\right)}.$$

Following this direction to find out:

$$v_4(t) = \aleph^{-1}\left[\frac{w^\beta}{r^\beta}\aleph^+\left[v_3(t) + \int_0^t v_3(\tau)\,d\tau\right]\right]$$

$$= -\frac{2\,t^{4\beta+2}}{\Gamma(4\beta+3)} - \frac{8\,t^{4\beta+3}}{\Gamma(4\beta+4)} + \frac{2\,t^{5\beta+\frac{3}{2}}}{\Gamma\left(5\beta+\frac{5}{2}\right)} + \frac{8\,t^{5\beta+\frac{5}{2}}}{\Gamma\left(5\beta+\frac{7}{2}\right)} - \frac{12\,t^{4\beta+4}}{\Gamma(4\beta+5)} + \frac{12\,t^{5\beta+\frac{7}{2}}}{\Gamma\left(5\beta+\frac{9}{2}\right)} - \frac{8\,t^{4\beta+5}}{\Gamma(4\beta+6)} + \frac{8\,t^{5\alpha+\frac{9}{2}}}{\Gamma\left(5\beta+\frac{11}{2}\right)} - \frac{2\,t^{4\beta+6}}{\Gamma(4\beta+7)} + \frac{2\,t^{5\beta+\frac{11}{2}}}{\Gamma\left(5\beta+\frac{13}{2}\right)}.$$

Hence,

$$v(t) = \sum_{n=0}^{\infty} v_n(t)$$

$$= v_0(t) + v_1(t) + v_2(t) + \dots$$

$$= \frac{2\,t^{\beta+\frac{3}{2}}}{\Gamma\left(\beta+\frac{5}{2}\right)} - \frac{2t^{\beta+2}}{\Gamma(\beta+3)} - \frac{2t^{\beta+3}}{\Gamma(\beta+4)} + \frac{2t^{2\beta+\frac{3}{2}}}{\Gamma\left(2\beta+\frac{5}{2}\right)} + \frac{2t^{2\beta+\frac{5}{2}}}{\Gamma\left(2\beta+\frac{7}{2}\right)} + \dots.$$

When $\beta = \frac{1}{2}$, we get:

$$v(t) = \frac{2\,t^2}{\Gamma(3)} - \frac{2\,t^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)} - \frac{2\,t^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)} + \frac{2\,t^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)} + \frac{2\,t^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)} - \frac{2\,t^3}{\Gamma(4)} - \frac{4\,t^4}{\Gamma(5)} + \frac{2\,t^3}{\Gamma(4)} + \frac{4\,t^4}{\Gamma(5)} - \frac{2\,t^5}{\Gamma(6)} + \frac{2\,t^5}{\Gamma(6)} + \dots = t^2.$$

Which is in fact our intended solution to Eq. (5.22) for $\beta = \frac{1}{2}$.

Choosing $\beta = \{0.25, 0.5, 0.75, 1\}$ in the above equation, we find:

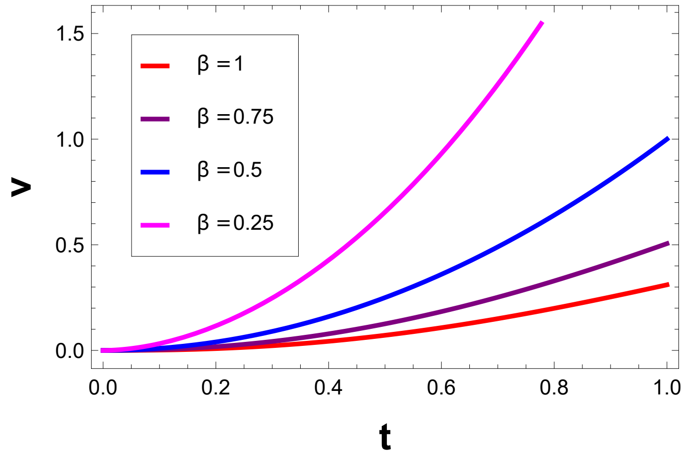

Figure 1. Numerical values for $v(t)$ of Ex. (5.2) for multiple values of β when $0 \le t \le 1$.

Table 1. The exact solutions and approximate of $v(t)$ for Ex. (5.2) for multiple values of β

t	$\beta = 0.25$	$\beta = 0.75$	$\beta = 1$	$\beta = 0.5$ Numerical	$\beta = 0.5$ Exact	$\beta = 0.5$ Absolute Error
0	0	0	0	0	0	0
0.2	0.11616208	0.01768973	0.00848809	0.04	0.04	1×10^{-17}
0.4	0.42719278	0.07855562	0.04307141	0.16	0.16	0
0.6	0.93157128	0.1837281	0.10750724	0.36	0.36	0
0.8	1.63929647	0.32885885	0.19919438	0.64	0.64	1×10^{-16}
1	2.56280703	0.50554629	0.31043634	1	1	0

Example 5.3. Given the nonlinear FVIDE:

$$D_t^{\frac{6}{5}}\left(v(t)\right) = \frac{5t^{\frac{4}{5}}}{2\Gamma\left(\frac{4}{5}\right)} - \frac{t^9}{252} + \int_0^t (t-\tau)^2 \left(v(\tau)\right)^3 d\tau, \tag{5.30}$$

together with I.C:

$$v'(0) = v(0) = 0. \tag{5.31}$$

Solution. Using natural transform of Eq. (5.30) to find:

$$\aleph^+\left[D_t^{\frac{6}{5}}\left(v(t)\right)\right] = \aleph^+\left[\frac{5t^{\frac{4}{5}}}{2\Gamma\left(\frac{4}{5}\right)} - \frac{t^9}{252} + \int_0^t (t-\tau)^2 \left(v(\tau)\right)^3 d\tau\right]. \tag{5.32}$$

Now using property 4 we get:

$$\begin{aligned} &\frac{r^{\frac{6}{5}}}{w^{\frac{6}{5}}}\aleph^+\left[v(t)\right] - \sum_{k=0}^{n-1} \frac{r^{\frac{6}{5}-(k+1)}}{w^{\frac{6}{5}-k}} \left[D^{(k)}v(t)\right]_{t=0} \\ =&\frac{2\,w^{\frac{4}{5}}}{r^{\frac{9}{5}}} - \frac{1440w^9}{r^{10}} + \aleph^+\left[\int_0^t (t-\tau)^2 \left(v(\tau)\right)^3 d\tau\right]. \end{aligned} \tag{5.33}$$

Plugin Eq. (5.31) into Eq. (5.33) to find:

$$\aleph^{+}\left[v(t)\right]=\frac{2\,w^{2}}{r^{3}}-\frac{1440w^{\frac{51}{5}}}{r^{\frac{56}{5}}}+\frac{w^{\frac{6}{5}}}{r^{\frac{6}{5}}}\aleph^{+}\left[\int_{0}^{t}\left(t-\tau\right)^{2}\left(v(\tau)\right)^{3}d\tau\right]. \tag{5.34}$$

Applying $\aleph^{-1}$ of Eq. (5.34) to find:

$$\begin{aligned} v(t)&=t^{2}-\frac{1440t^{\frac{51}{5}}}{\Gamma\left(\frac{56}{5}\right)}+\aleph^{-1}\left[\frac{w^{\frac{6}{5}}}{r^{\frac{6}{5}}}\aleph^{+}\left[\int_{0}^{t}\left(t-\tau\right)^{2}\left(v(\tau)\right)^{3}d\tau\right]\right]\\ &=t^{2}-\frac{1440t^{\frac{51}{5}}}{\Gamma\left(\frac{56}{5}\right)}+\aleph^{-1}\left[\frac{w^{\frac{6}{5}}}{r^{\frac{6}{5}}}\aleph^{+}\left[t^{2}*v^{3}(t)\right]\right]\\ &=t^{2}-\frac{1440t^{\frac{51}{5}}}{\Gamma\left(\frac{56}{5}\right)}+\aleph^{-1}\left[\frac{2w^{\frac{21}{5}}}{r^{\frac{21}{5}}}\,\aleph^{+}\left[v^{3}(t)\right]\right]. \end{aligned} \tag{5.35}$$

Suppose our intended solution is given by:

$$v(t)=\sum_{n=0}^{\infty}v_{n}(t). \tag{5.36}$$

From Eq. (5.36), one can write Eq. (5.33) as:

$$\sum_{n=0}^{\infty}v_{n}(t)=t^{2}-\frac{1440t^{\frac{51}{5}}}{\Gamma\left(\frac{56}{5}\right)}+\aleph^{-1}\left[\frac{2w^{\frac{21}{5}}}{r^{\frac{21}{5}}}\,\aleph^{+}\left[\sum_{n=0}^{\infty}A_{n}\left(t\right)\right]\right]. \tag{5.37}$$

Note that $\sum_{n=0}^{\infty}A_{n}\left(t\right)=v^{3}(t)$ is the Adomian polynomial which represents the nonlinear part. Note that:

$$\begin{aligned} &A_{0}\left(t\right)=v_{0}^{3}\left(t\right),\\ &A_{1}\left(t\right)=3v_{0}^{2}\left(t\right)v_{1}\left(t\right),\\ &A_{2}\left(t\right)=3v_{0}^{2}\left(t\right)v_{2}\left(t\right)+3v_{0}\left(t\right)v_{1}^{2}\left(t\right),\\ &A_{3}\left(t\right)=3v_{0}^{2}\left(t\right)v_{3}\left(t\right)+6v_{0}\left(t\right)v_{1}\left(t\right)v_{2}\left(t\right)+v_{1}^{3}\left(t\right),\\ &\vdots \end{aligned}$$

Looking at Eq. (5.37) above, one can calculate the iterations as:

$$\begin{aligned} v_{0}(t)&=t^{2},\\ v_{1}(t)&=-\frac{1440t^{\frac{51}{5}}}{\Gamma\left(\frac{56}{5}\right)}+\aleph^{-1}\left[\frac{2w^{\frac{21}{5}}}{r^{\frac{21}{5}}}\,\aleph^{+}\left[A_{0}\left(t\right)\right]\right]\\ &=-\frac{1440t^{\frac{51}{5}}}{\Gamma\left(\frac{56}{5}\right)}+\aleph^{-1}\left[\frac{2w^{\frac{21}{5}}}{r^{\frac{21}{5}}}\,\aleph^{+}\left[v_{0}^{3}\left(t\right)\right]\right]\\ &=-\frac{1440t^{\frac{51}{5}}}{\Gamma\left(\frac{56}{5}\right)}+\aleph^{-1}\left[\frac{2w^{\frac{21}{5}}}{r^{\frac{21}{5}}}\,\aleph^{+}\left[t^{6}\right]\right]\\ &=-\frac{1440t^{\frac{51}{5}}}{\Gamma\left(\frac{56}{5}\right)}+\frac{1440t^{\frac{51}{5}}}{\Gamma\left(\frac{56}{5}\right)}=0, \end{aligned}$$

$$\begin{aligned} v_2(t) &= \aleph^{-1}\left[\frac{2w^{\frac{21}{5}}}{r^{\frac{21}{5}}}\,\aleph^{+}\left[A_1(t)\right]\right] \\ &= \aleph^{-1}\left[\frac{2w^{\frac{21}{5}}}{r^{\frac{21}{5}}}\,\aleph^{+}\left[3v_0^2(t)\,v_1(t)\right]\right] \\ &= 0. \end{aligned}$$

Thus one can concludes that $v_1(t) = v_2(t) = ... = 0$. Hence,

$$v(t) = \sum_{n=0}^{\infty} v_n(t) = v_0(t) + v_1(t) + v_2(t) + \ldots = t^2.$$

Which is in fact our intended exact solution for Eq. (5.30).

Example 5.4. Given the nonlinear FVIDE as:

$$D_t^{\beta}\left(v(t)\right) = 1 + \int_0^t v'(\tau)v(\tau)d\tau, \;\; for\ 0 < \beta \le 1, \tag{5.38}$$

a company I.C:

$$v(0) = 0. \tag{5.39}$$

Solution. Using N-transformation of Eq. (5.38) to find:

$$\aleph^{+}\left[D_t^{\beta}\left(v(t)\right)\right] = \aleph^{+}\left[1 + \int_0^t v'(\tau)v(\tau)d\tau\right]. \tag{5.40}$$

Now using property 4 we get:

$$\frac{r^{\beta}}{w^{\beta}}\aleph^{+}\left[v(t)\right] - \sum_{k=0}^{n-1}\frac{r^{\beta-(k+1)}}{w^{\beta-k}}\left[D^{(k)}v(t)\right]_{t=0} = \frac{1}{r} + \aleph^{+}\left[\int_0^t v'(\tau)v(\tau)d\tau\right]. \tag{5.41}$$

Plugin Eq. (5.39) into Eq. (5.41) to find:

$$\aleph^{+}\left[v(t)\right] = \frac{w^{\beta}}{r^{\beta+1}} + \frac{w^{\beta}}{r^{\beta}}\aleph^{+}\left[\int_0^t v'(\tau)v(\tau)d\tau\right]. \tag{5.42}$$

Apply $\aleph^{-1}$ of Eq. (5.42) to obtain:

$$v(t) = \frac{t^{\beta}}{\Gamma(\beta+1)} + \aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph^{+}\left[\int_0^t v'(\tau)v(\tau)d\tau\right]\right]. \tag{5.43}$$

Suppose our intended solution is give by:

$$v(t) = \sum_{n=0}^{\infty} v_n(t). \tag{5.44}$$

From Eq. (5.44), one can write Eq. (5.43) as:

$$\sum_{n=0}^{\infty} v_n(t) = \frac{t^{\beta}}{\Gamma(\beta+1)} + \aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph^{+}\left[\int_0^t \sum_{n=0}^{\infty} A_n(\tau)\,d\tau\right]\right]. \tag{5.45}$$

Here $\sum_{n=0}^{\infty} A_n(\tau) = v'(\tau)v(\tau)$ is the Adomian polynomial which represents the nonlinear part.

Note that

$$
\begin{aligned}
&A_0(\tau) = v_0(\tau)\, v_0{}'(\tau),\\
&A_1(\tau) = v_0{}'(\tau)\, v_1(\tau) + v_0(\tau)\, v_1{}'(\tau),\\
&A_2(\tau) = v_0{}'(\tau)\, v_2(\tau) + v_1{}'(\tau)\, v_1(\tau) + v_2{}'(\tau)\, v_0(\tau),\\
&A_3(\tau) = v_0{}'(\tau)\, v_3(\tau) + v_1{}'(\tau)\, v_2(\tau) + v_2{}'(\tau)\, v_1(\tau) + v_3{}'(\tau)\, v_0(\tau),\\
&\vdots
\end{aligned}
$$

Looking at Eq. (5.45) above, one can calculate the remaining iterations as:

$$
\begin{aligned}
v_0(t) &= \frac{t^{\beta}}{\Gamma(\beta+1)},\\
v_1(t) &= \aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph + \left[\int_0^t A_0(\tau)\, d\tau\right]\right]\\
&= \aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph + \left[\int_0^t v_0(\tau)\, v_0{}'(\tau)\, d\tau\right]\right]\\
&= \aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph + \left[\int_0^t \frac{\tau^{\beta}}{\Gamma(\beta+1)} \times \frac{\beta\tau^{\beta-1}}{\Gamma(\beta+1)} d\tau\right]\right]\\
&= \aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph + \left[\frac{t^{2\beta}}{2\left(\Gamma(\beta+1)\right)^2}\right]\right]\\
&= \frac{\Gamma(2\beta+1)\, t^{3\beta}}{2\Gamma(3\beta+1)\left(\Gamma(\beta+1)\right)^2},\\
v_2(t) &= \aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph + \left[\int_0^t A_1(\tau)\, d\tau\right]\right]\\
&= \frac{\Gamma(4\beta+1)\,\Gamma(2\beta+1)\, t^{5\beta}}{2\Gamma(5\beta+1)\,\Gamma(3\beta+1)\left(\Gamma(\beta+1)\right)^3},\\
v_3(t) &= \aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph + \left[\int_0^t A_2(\tau)\, d\tau\right]\right]\\
&= \frac{4\Gamma(6\beta+1)\Gamma(4\beta+1)\Gamma(3\beta+1)\Gamma(2\beta+1)+\Gamma(6\beta+1)\Gamma(5\beta+1)(\Gamma(2\beta+1))^2}{8\Gamma(7\beta+1)\Gamma(5\beta+1)(\Gamma(3\beta+1))^2(\Gamma(\beta+1))^4} t^{7\beta}.
\end{aligned}
$$

Following this path we find:

$$
\begin{aligned}
v_4(t) &= \aleph^{-1}\left[\frac{w^{\beta}}{r^{\beta}}\aleph + \left[\int_0^t A_3(\tau)\, d\tau\right]\right]\\
&= \left(\begin{array}{l} \frac{4\,\Gamma(8\beta+1)\Gamma(6\beta+1)\Gamma(4\beta+1)\Gamma(3\beta+1)\Gamma(2\beta+1)+\Gamma(8\beta+1)\Gamma(6\beta+1)\Gamma(5\beta+1)(\Gamma(2\beta+1))^2}{8\,\Gamma(9\beta+1)\Gamma(7\beta+1)\Gamma(5\beta+1)(\Gamma(3\beta+1))^2(\Gamma(\beta+1))^5}\\ + \frac{2\,\Gamma(8\beta+1)\Gamma(7\beta+1)\Gamma(4\beta+1)(\Gamma(2\beta+1))^2}{8\,\Gamma(9\beta+1)\Gamma(7\beta+1)\Gamma(5\beta+1)(\Gamma(3\beta+1))^2(\Gamma(\beta+1))^5} \end{array}\right) t^{9\beta}.
\end{aligned}
$$

Then using Eq. (5.44), one can calculate the other iterations.

$$
v(t) = \sum_{n=0}^{\infty} v_n(t) = v_0(t) + v_1(t) + v_2(t) + \ldots
$$

$$=\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{\Gamma(2\beta+1)\,t^{3\beta}}{2\Gamma(3\beta+1)\left(\Gamma(\beta+1)\right)^{2}}$$
$$+\frac{\Gamma(4\beta+1)\,\Gamma(2\beta+1)\,t^{5\beta}}{2\Gamma(5\beta+1)\,\Gamma(3\beta+1)\left(\Gamma(\beta+1)\right)^{3}}+\ldots.$$

Choosing $\beta = 1$ the above solution becomes:

$$\begin{aligned} v(t) &= t+\frac{t^3}{6}+\frac{t^5}{30}+\frac{17t^7}{2520}+\frac{31t^9}{22680}+\ldots \\ &= \sqrt{2}\left(\frac{t}{\sqrt{2}}+\frac{t^3}{6\sqrt{2}}+\frac{t^5}{30\sqrt{2}}+\frac{17t^7}{2520\sqrt{2}}+\frac{31t^9}{22680\sqrt{2}}+\ldots\right) \\ &= \sqrt{2}\tan\left(\frac{t}{\sqrt{2}}\right). \end{aligned}$$

Which is in fact our intended solution for Eq. (5.38). Choosing $\beta=\{0.25, 0.5, 0.75, 1\}$ in Eq. (5.46), we find:

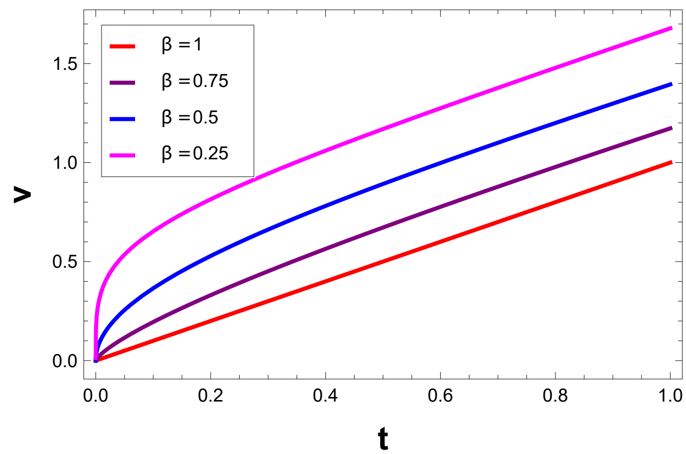

Figure 2. Numerical Solutions of $v(t)$ for Ex. (5.4) for multiple values of β when $0 \le t \le 1$.

Table 2. Numerical results of approximate and exact solutions of $v(t)$ for Ex. (5.4) for values of β

t	$\beta = 0.25$	$\beta = 0.5$	$\beta = 0.75$	$\beta = 1$ Numerical	$\beta = 1$ Exact	$\beta = 1$ Absolute Error
0	0	0	0	0	0	0
0.2	1.05181432	0.55430982	0.33396637	0.20134409	0.20134409	$5.71886982 \times 10^{-12}$
0.4	1.59435817	0.88100391	0.5909155	0.41101941	0.41101942	$1.20065037 \times 10^{-8}$
0.6	2.20948632	1.2489334	0.86164077	0.63879462	0.6387957	1.0839355×10^{-6}
0.8	2.92791531	1.72510298	1.18047557	0.8978542	0.89788154	$2.73374912 \times 10^{-5}$
1	3.76228394	2.38085999	1.59103283	1.20811287	1.20846024	$3.47367358 \times 10^{-4}$

6. Concluding Remarks

In the present work, we implement successfully the convergence analysis for the fractional decomposition method (FDM) to nonlinear FVIDE. Moreover, we found approximate and analytical solutions for both the linear and nonlinear fractional Volterra integro-differential equations. The current technique minimize the calculation difficulties of some of the well-known famous methods and the computations can be done with easy manipulations. Some famous application in FVIDEs were examined by employing the FDM and the outcomes have shown noticeable differences. Thus, the used mechanism can be implemented to various linear and nonlinear FVIDEs without doing any perturbation, discretization or linearization. In a future work, our intended aims is to employ the FDM to different linear and nonlinear FVIDEs which show up in many areas of applied science, such as Physics and Engineering.

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Declarations

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Availability of data and materials. Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.
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Consent to participate participants is aware that they can contact the Jordan University of Science and Technology Ethics Officer if they have any concerns or complaints regarding the way in which the research is or has been conducted.
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