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NEW FIXED POINT RESULTS FOR GERAGHTY CONTRACTIONS AND THEIR APPLICATIONS

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Abstract In this paper, we prove existence and uniqueness of fixed point involving Geraghty contraction in a metric space endowed with a binary relation. Moreover, we give an application to periodic boundary value problems regarding to ordinary differential equations (ODE).

Keywords Binary relation, $\mathcal{R}$-completeness, d-self-closedness.

MSC(2010) 47H10, 54H25.

1. Introduction

In 1922, Stephen Banach presented the Banach Contraction principle (BCP), a graceful and devoted tool of nonlinear functional analysis. BCP remains a source of inspiration for researchers of this domain. It ensures the existence and uniqueness of the solution. This theorem basically demonstrates that a function defined on a complete metric space (ζ, d) satisfying

$$d(\mathcal{T}a, \mathcal{T}b) \leq \eta d(a, b) \quad \text{where } 0 \leq \eta < 1$$

for all $a, b \in \zeta$, has a unique fixed point. The key component of the metric fixed-point theory is the investigation of extension of contraction principles to provide novel and advantageous fixed-point theorems. Owing to the Kannan generalized another contraction principle, which encouraged the researchers to look into more extensions of the contraction principle like Boyd and Wong, Meir-Keeler and several others. The most natural and frequently discussed concept of metric space has been improved and extended the versions, i.e., Partial metric, M-metric, pseudo metric, G-metric, b-metric and m_v-metric etc.

Ran and Reurings [15] demonstrated an application to solve matrix equation of the BCP in the context of ordered metric space. Also, more refined version proved by Nieto and López [13] was utilised to solve periodic boundary value problem. The theorems of Nieto and López [13] are further extended by many authors. With an amorphous binary relation instead of partial order, Alam and Imdad devised a fundamental generalisation of the BCP in 2015. In 2017, Ahmadullah et al. [2] introduced relation theoretic principle in metric like space and derived some

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fixed point results. Soon after, various relation-theoretic results were proposed by several researchers i.e. [3, 5]. Fixed point techniques have very fruitful results in various areas and applications in integral/differential/fractional equations e.g, [6, 7, 11, 14, 18, 21, 22].

The following famous generalization of BCP is due to Geraghty [8].

Theorem 1.1 ([8]). *Let (ζ, d) be a complete metric space and $\mathcal{T} : \zeta \to \zeta$ a map. Assume that $\exists\ \gamma \in \mathfrak{H}$, where $\mathfrak{H}$ denote the class of functions $\gamma : [0, \infty) \to [0, 1)$ which satisfy*

$$\gamma(d_n) \to 1 \quad \Rightarrow \quad d_n \to 0.$$

Such that for each $a, b \in \zeta$,

$$d(\mathcal{T}a, \mathcal{T}b) \leq \gamma(d(a, b))d(a, b).$$

Then, $\mathcal{T}$ has a unique fixed point $c \in \zeta$ and $\{\mathcal{T}^n(a)\}$ converges to c, for each $a \in \zeta$.

Amini-Harandi and Emami [9] proved monotone Geraghty contraction in partially ordered metric space and gave the application in ODE. In this paper, we shall extend the fixed point theorems of Amini-Harandi and Emami [9] to a metric space with binary relation. Also, we prove an application of our newly proved results to periodic boundary value problem.

2. Preliminaries

Let's summarize few pertinent concepts and fundamental results which will be referenced in our subsequent discussion:

Definition 2.1 ([1]). Let $\mathcal{R}$ be a binary relation on a nonempty set ζ and $a, b \in \zeta$. We say that a and b are $\mathcal{R}$-comparative if either $(a, b) \in \mathcal{R}$ or $(b, a) \in \mathcal{R}$. We denote it by $[a, b] \in \mathcal{R}$.

We collect several needed results which are taken from [1–4, 10, 12, 17].

Definition 2.2. Let ζ be a nonempty set equipped with a binary relation $\mathcal{R}$ and $\mathcal{T}$ a self-mapping on ζ.

(1) The inverse or transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by $\mathcal{R}^{-1} = \{(a, b) \in \zeta^2 : (b, a) \in \mathcal{R}\}$.

(2) The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (*i.e.*, $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$). Indeed, $\mathcal{R}^s$ is the smallest symmetric relation on ζ containing $\mathcal{R}$.

(3) $(a, b) \in \mathcal{R}^s \iff [a, b] \in \mathcal{R}$.

(4) A sequence $\{a_n\} \subset \zeta$ is called $\mathcal{R}$-preserving if

$$(a_n, a_{n+1}) \in \mathcal{R}\ \ \forall\ n \in \mathbb{N}_0.$$

(5) A binary relation $\mathcal{R}$ defined on ζ is called $\mathcal{T}$-closed if for any $a, b \in \zeta$,

$$(a, b) \in \mathcal{R} \Rightarrow (\mathcal{T}a, \mathcal{T}b) \in \mathcal{R}.$$

(6) $\mathcal{R}$ is $\mathcal{T}$-closed, then $\mathcal{R}^s$ is also $\mathcal{T}$-closed.

(7) If $\mathcal{R}$ is $\mathcal{T}$-closed, then for all $n \in \mathbb{N}_0$, $\mathcal{R}$ is also $\mathcal{T}^n$-closed, where $\mathcal{T}^n$ denotes nth iterate of $\mathcal{T}$.

(8) A subset E of ζ is called $\mathcal{R}$-directed if for each $a, b \in E$, there exists $c \in \zeta$ such that $(a, c) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$.

(9) For $a, b \in \zeta$, a path of length k (where k is a natural number) in $\mathcal{R}$ from a to b is a finite sequence $\{c_0, c_1, c_2, ..., c_k\} \subset \zeta$ satisfying the following:
(i) $c_0 = a$ and $c_k = b$,
(ii) $(c_i, c_{i+1}) \in \mathcal{R}$ for each i $(0 \leq i \leq k-1)$.
Notice that a path of length k involves $k+1$ elements of ζ, although they are not necessarily distinct.

(10) A subset E of ζ is called $\mathcal{R}$-connected if for each pair $a, b \in E$, there exists a path in $\mathcal{R}$ from a to b.
Inspired by Roldán-López-de-Hierro et al. [16], Alam and Imdad introduced the following: (i.e., a notion originated from $\mathcal{T}$-transitive subset of ζ^2 is essentially due to [16]).

(11) A binary relation $\mathcal{R}$ defined on ζ is called $\mathcal{T}$-transitive if for any $a, b, c \in \zeta$,

$$(\mathcal{T}a, \mathcal{T}b), (\mathcal{T}b, \mathcal{T}c) \in \mathcal{R} \Rightarrow (\mathcal{T}a, \mathcal{T}c) \in \mathcal{R}.$$

(12) A binary relation $\mathcal{R}$ defined on ζ is called locally transitive, if for each (effectively) $\mathcal{R}$ -preserving sequence $\{a_n\} \subset \zeta$ (with range $E = \{a_n : n \in \mathbb{N}\}$), such that $R|_E$ is transitive. Inspired by Turnici [19,20], Alam and Imdad [4] introduced these notions by localising the transitivity conditions.

(13) A binary relation $\mathcal{R}$ defined on ζ is called locally $\mathcal{T}$-transitive if for each (effectively) $\mathcal{R}$-preserving sequence $\{a_n\} \subset T(\zeta)$ (with range $E = \{a_n : n \in \mathbb{N}\}$), such that $R|E$ is transitive.

(14) The following result establish the dominance of locally $\mathcal{T}$-transitivity over other variants of transitivity:
- (i) $\mathcal{R}$ is $\mathcal{T}$ -transitive $\Leftrightarrow \mathcal{R}|_{\mathcal{T}(\zeta)}$ is transitive,
- (ii) $\mathcal{R}$ is locally $\mathcal{T}$-transitive $\Leftrightarrow \mathcal{R}|_{\mathcal{T}(\zeta)}$ is locally transitive,
- (iii) $\mathcal{R}$ is transitive $\Rightarrow \mathcal{R}$ is locally transitive $\Rightarrow R$ is locally $\mathcal{T}$-transitive,
- (iv) $\mathcal{R}$ is transitive $\Rightarrow \mathcal{R}$ is $\mathcal{T}$ -transitive $\Rightarrow \mathcal{R}$ is locally $\mathcal{T}$-transitive.

Definition 2.3 ([1,2]). Let (ζ, d) be a metric space and $\mathcal{R}$ a binary relation on ζ. Then,

(1) (ζ, d) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in ζ converges.

(2) If $\mathcal{T}$ is called $\mathcal{R}$-continuous at a if for any $\mathcal{R}$-preserving sequence $\{a_n\}$ such that $a_n \xrightarrow{d} a$, we have $\mathcal{T}(a_n) \xrightarrow{d} \mathcal{T}(a)$. Moreover, $\mathcal{T}$ is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of ζ.

(3) A binary relation $\mathcal{R}$ defined on ζ is called d-self-closed if for any $\mathcal{R}$-preserving sequence $\{a_n\}$ such that $a_n \xrightarrow{d} a$, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ with $[a_{n_k}, a] \in \mathcal{R}$ for all $k \in \mathbb{N}_0$.

Given a binary relation $\mathcal{R}$ and a self-mapping $\mathcal{T}$ on a nonempty set ζ, we use the following notations:

(i) $F(\mathcal{T})$=the set of all fixed points of $\mathcal{T}$, (*ii*) $\zeta(\mathcal{T},\mathcal{R}) := \{a \in \zeta : (a, \mathcal{T}a) \in \mathcal{R}\}$.

The following result is a relation-theoretic version of BCP:

Theorem 2.1 ([1, 2]). *Let (ζ, d) be a metric space, $\mathcal{R}$ a binary relation on ζ and $\mathcal{T}$ a self-mapping on ζ. Suppose that the following conditions hold:*

(*I*) *(ζ, d) is $\mathcal{R}$-complete,*

(*II*) *$\mathcal{R}$ is $\mathcal{T}$-closed,*

(*III*) *either $\mathcal{T}$ is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed,*

(*IV*) *$\zeta(\mathcal{T},\mathcal{R})$ is nonempty,*

(*V*) *there exists $\eta \in [0, 1)$ such that*

$$d(\mathcal{T}a, \mathcal{T}b) \leq \eta d(a, b) \ \ \forall \ a, b \in \zeta \text{ with } (a, b) \in \mathcal{R}.$$

Then, $\mathcal{T}$ has a fixed point. Moreover, if ζ is $\mathcal{R}^s$-connected, then $\mathcal{T}$ has a unique fixed point.

Using the symmetry of d, we propose the following result:

Proposition 2.1. *If (ζ, d) is a metric space, $\mathcal{R}$ is a binary relation on ζ, $\mathcal{T}$ is a self-mapping on ζ and $\gamma \in \mathfrak{H}$, then the following contractivity conditions are equivalent:*

(*I*) *$d(\mathcal{T}a, \mathcal{T}b) \leq \gamma(d(a, b))d(a, b) \ \ \forall \ a, b \in \zeta$ with $(a, b) \in \mathcal{R}$,*

(*II*) *$d(\mathcal{T}a, \mathcal{T}b) \leq \gamma(d(a, b))d(a, b) \ \ \forall \ a, b \in \zeta$ with $[a, b] \in \mathcal{R}$.*

3. Main Results

In this section, firstly we present our first result on the existence of fixed point.

Theorem 3.1. *Let (ζ, d) be a metric space, $\mathcal{R}$ a binary relation on ζ, and $\mathcal{T}$ a self-mapping on ζ. Suppose that the following conditions hold:*

(i) *(ζ, d) is a $\mathcal{R}$-complete metric space,*

(ii) *$\zeta(\mathcal{T},\mathcal{R})$ is nonempty,*

(iii) *$\mathcal{R}$ is $\mathcal{T}$-closed and locally $\mathcal{T}$-transitive,*

(iv) *$\mathcal{T}$ is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed,*

(v) *exists $\gamma \in \mathfrak{H}$ such that for each $a, b \in M$,*

$$d(\mathcal{T}a, \mathcal{T}b) \leq \gamma(d(a, b))d(a, b) \quad \forall a, b \in \zeta \text{ with } (a, b) \in \mathcal{R}.$$

Then, $\mathcal{T}$ has a fixed point.

Proof. From the condition (*ii*), there exists $a_0 \in \zeta$ such that $(a_0, \mathcal{T}a_0) \in \mathcal{R}$. If $\mathcal{T}(a_0) = a_0$, then a_0 is fixed point. Otherwise, we can choose $a_1 \in \zeta$ such that $\mathcal{T}(a_0) = a_1$. Again we can choose $a_2 \in \zeta$ such that $\mathcal{T}(a_1) = a_2$. Continuing this process, we can construct inductive sequence $\{a_n\}$ such that

$$a_{n+1} = \mathcal{T}a_n. \tag{3.1}$$

In view of (i), we have a $a_0 \in \zeta$ such that $(a_0, \mathcal{T}a_0) \in \mathcal{R}$, i.e., $(a_0, a_1) \in \mathcal{R}$. Now $(a_0, a_1) \in \mathcal{R}$ gives in view of $\mathcal{T}$-closedness of $\mathcal{R}$

$$(\mathcal{T}a_0, \mathcal{T}a_1) \in \mathcal{R} \text{ i.e., } (a_1, a_2) \in \mathcal{R}.$$

Continuing this process, we get

$$(a_n, a_{n+1}) \in \mathcal{R} \quad \forall\, n \in \mathbb{N}_0. \tag{3.2}$$

Denote

$$\delta_n := d(a_n, a_{n+1}).$$

Now,

$$d(a_{n+1}, a_{n+2}) = d(\mathcal{T}a_n, \mathcal{T}a_{n+1}) \leq \gamma(d(a_n, a_{n+1}))d(a_n, a_{n+1}) \leq d(a_n, a_{n+1})$$

so that

$$\delta_{n+1} \leq \delta_n.$$

Then, $\{\delta_n\}$ is a decreasing sequence and bounded below, so $\lim\limits_{n\to\infty} \delta_n = r \geq 0$. If $r > 0$, then we have

$$\frac{\delta_{n+1}}{\delta_n} \leq \gamma(\delta_n), \quad n = 1, 2, \cdots$$

which yields $\lim\limits_{n\to\infty} \gamma(\delta_n) = 1$. As $\gamma \in \mathfrak{H}$, we get $r = 0$. Thus

$$\lim_{n\to\infty} \delta_n = 0.$$

Now, we show that $\{a_n\}$ is a Cauchy sequence. On the contrary, assume that

$$\lim_{m,n\to\infty} \sup d(a_n, a_m) > 0. \tag{3.3}$$

By the triangle inequality

$$d(a_n, a_m) \leq d(a_n, a_{n+1}) + d(a_{n+1}, a_{m+1}) + d(a_{m+1}, a_m). \tag{3.4}$$

As $\mathcal{R}$ is locally $\mathcal{T}$-transitive, we have $(a_m, a_n) \in \mathcal{R}$. Applying assumption (v) and using (3.1), we get

$$d(a_{n+1}, a_{m+1}) = d(\mathcal{T}a_n, \mathcal{T}a_m) \leq \gamma(d(a_n, a_m))d(a_n, a_m).$$

Hence, (3.4) becomes

$$d(a_n, a_m) \leq (1 - \gamma(d(a_n, a_m)))^{-1}[d(a_n, a_{n+1}) + d(a_{m+1}, a_m)].$$

Since

$$\limsup_{m,n\to\infty} d(a_n, a_m) > 0$$

and

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = 0$$

then,

$$\limsup_{m,n\to\infty} (1 - \gamma(d(a_n, a_m)))^{-1} = +\infty$$

which gives
$$\limsup_{m,n\to\infty} \gamma(d(a_n, a_m) = 1.$$
But as $\gamma \in \mathfrak{H}$, we have
$$\limsup_{m,n\to\infty} d(a_n, a_m) = 0$$
which contradicts (v) hence $\{a_n\}$ is a Cauchy sequence in ζ. Since (ζ, d) is a $\mathcal{R}$-complete metric space, then there exists a $c \in \zeta$ such that
$$\lim_{n\to\infty} a_n = c.$$
To prove that c is a fixed point of $\mathcal{T}$, we use assumption (iv).

Firstly, we assume that $\mathcal{T}$ is continuous. Then, we have
$$c = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \mathcal{T}^n(a_0) = \lim_{n\to\infty} \mathcal{T}^{n+1}(a_0) = \mathcal{T}(\lim_{n\to\infty} \mathcal{T}^n(a_0)) = \mathcal{T}(c),$$
hence $\mathcal{T}(a) = a$.

Now, suppose that $\mathcal{R}$ is d-self-closed. As $\{a_n\}$ is a $\mathcal{R}$-preserving sequence and $a_n \xrightarrow{d} c$, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ with $[a_{n_k}, a] \in \mathcal{R} \quad \forall\ k \in \mathbb{N}_0$. On using triangular inequality, (e), Proposition 2.1, $[a_{n_k}, c] \in \mathcal{R}$ and $a_{n_k} \xrightarrow{d} c$, we obtain
$$\begin{aligned} d(\mathcal{T}c, c) &\leq d(\mathcal{T}a_{n_k}, \mathcal{T}c) + d(\mathcal{T}a_n, c) \\ &\leq \gamma(d(a_{n_k}, c))d(a_{n_k}, c) + d(a_{n_k+1}, c) \\ &\leq d(c, a_n) + d(a_{n_k+1}, c). \end{aligned}$$
Since $d(c, a_{n_k}) \to 0$, then we get $\mathcal{T}(c) = c$. Hence, c is a fixed point of $\mathcal{T}$. □

Now, we prove a corresponding uniqueness result.

Theorem 3.2. *In addition of Theorem (3.1), if $\mathcal{T}(\zeta)$ is $\mathcal{R}^s$-connected, then $\mathcal{T}$ has a unique fixed point.*

Proof. Theorem (3.1) guarantees the existence of one fixed point of $\mathcal{T}$. If a and b are two fixed points of $\mathcal{T}$, then
$$\mathcal{T}^n(a) = a \text{ and } \mathcal{T}^n(b) = b \quad \forall\ n \in \mathbb{N}_0.$$
Clearly $a, b \in \mathcal{T}(\zeta)$. By assumption (vi), we have a path $\{c_0, c_1, c_2, ..., c_k\}$ of finite length k in $\mathcal{R}^s$ from a to b so that
$$c_0 = a, c_k = b \text{ and } [c_i, c_{i+1}] \in \mathcal{R} \text{ for any } i\ (0 \leq i \leq k-1). \tag{3.5}$$
As $\mathcal{R}$ is $\mathcal{T}$-closed, we have
$$[\mathcal{T}^n c_i, \mathcal{T}^n c_{i+1}] \in \mathcal{R} \text{ for any } i\ (0 \leq i \leq k-1) \text{ and } n \in \mathbb{N}_0. \tag{3.6}$$
Now, for all $n \in \mathbb{N}_0$ and for any i $(0 \leq i \leq k-1)$, define
$$t_n^i = d(\mathcal{T}^n c_i, \mathcal{T}^n c_{i+1}).$$
We show that
$$\lim_{n\to\infty} t_n^i = 0. \tag{3.7}$$

Further, suppose that $t^i_{n_0} = d(\mathcal{T}^{n_0} c_i, \mathcal{T}^{n_0} c_{i+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, *i.e.* $\mathcal{T}^{n_0}(c_i) = \mathcal{T}^{n_0}(c_{i+1})$, which implies that $\mathcal{T}^{n_0+1}(c_i) = \mathcal{T}^{n_0+1}(c_{i+1})$. Consequently, we get $t^i_{n_0+1} = d(\mathcal{T}^{n_0+1} c_i, \mathcal{T}^{n_0+1} c_{i+1}) = 0$. Thus by induction, we get $t^i_n = 0 \ \forall \ n \geq n_0$, yielding thereby $\lim\limits_{n\to\infty} t^i_n = 0$. Now, suppose that $t^i_{n_0} > 0 \ \forall \ n \in \mathbb{N}_0$, then on using (3.1), assumption (v) and Proposition 2.1, we obtain

$$\begin{aligned} t^i_{n+1} = d(\mathcal{T}^{n+1} c_i, \mathcal{T}^{n+1} c_{i+1}) &\leq \gamma(d(\mathcal{T}^n c_i, \mathcal{T}^n c_{i+1})) d(\mathcal{T}^n c_i, \mathcal{T}^n c_{i+1}) \\ &= \gamma(t^i_n) t^i_n \leq t^i_n \end{aligned}$$

so that

$$t^i_{n+1} \leq t^i_n.$$

Hence, we obtain $\lim\limits_{n\to\infty} t^i_n = 0$. Thus, (3.7) is proved for each i $(0 \leq i \leq k-1)$. Now, using (3.7), we get

$$\begin{aligned} d(a,b) = d(\mathcal{T}^n c_0, \mathcal{T}^n c_k) &\leq t^0_n + t^1_n .... + t^{k-1}_n \\ &\to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

which gives $a = b$. Hence, $\mathcal{T}$ has a unique fixed point. □

Corollary 3.1. *In addition of Theorem 3.1, if one of the following conditions hold:*

(i) $\mathcal{T}(\zeta)$ *is* $\mathcal{R}^s$*-directed, or*

(ii) $\mathcal{R}|_{\mathcal{T}(\zeta)}$ *is complete.*

Then, $\mathcal{T}$ *has a unique fixed point.*

Proof. If condition (i) holds, then for each $a, b \in \mathcal{T}(\zeta)$ we have $c \in \zeta$ such that $[a, c] \in \mathcal{R}$ and $[b, c] \in \mathcal{R}$ hence $\{a, c, b\}$ is a path of length 2 in $\mathcal{R}^s$ from a to b. Hence, $\mathcal{T}(\zeta)$ is $\mathcal{R}^s$-connected and via Theorem 3.2, we are done.

Now if condition (ii) holds, then for each $a, b \in \mathcal{T}(\zeta)$, $[a, b] \in \mathcal{R}$, which yields that $\{a, b\}$ is a path of length 1 in $\mathcal{R}^s$ from a to b so that $\mathcal{T}(\zeta)$ is $\mathcal{R}^s$-connected. In view of Theorem 3.2, the conclusion follows. □

Remark 3.1. Notice that under the universal relation $\mathcal{R} = \zeta^2$, Theorem 3.2 reduces to the Theorem 1.1.

Remark 3.2. Under the relation, $\mathcal{R} = \preceq$, the partial order, we obtain the fixed point theorem of Amini-Harandi and Emami [9].

4. Application to ordinary differential equations

Consider the periodic boundary value problem

$$\begin{cases} v'(p) = g(p, v(p)) & \text{if} \ \ p \in I = [0, T], \\ v(0) = v(p), \end{cases} \tag{4.1}$$

where $T > 0$ and $g : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Now, we prove the existence of solution for the problem (4.1) in presence of a lower solution.

Let $\mathfrak{A}$ denote the class of those $\rho : [0, \infty) \to [0, \infty)$ which satisfies the following conditions:

(i) ρ is increasing.

(ii) for each $a > 0, \quad \rho(a) < a,$

(iii) $\gamma(a) = \frac{\rho(a)}{a} \in \mathfrak{H}.$

Example: $\rho(p) = \upsilon p$, where $0 \leq \upsilon < 1, \rho(p) = \frac{p}{1+p}$ and $\rho(p) = \ln(1 + p)$ are in $\mathfrak{A}$.

Theorem 4.1. *Consider problem (4.1) with g continuous and suppose that there exists $\lambda > 0$ such that for $a, b \in \mathbb{R}$ with $b \geq a$*

$$0 \leq g(p, b) + \lambda b - [g(p, a) + \lambda a] \leq \lambda \rho(b - a),$$

where $\rho \in \mathfrak{A}$. Then, the existence of a lower solution for(4.1) provides the existence of a unique solution of (4.1).

Proof. Problem (4.1) is equivalent to the integral equation

$$v(p) = \int_0^T \mathcal{K}(p, q)[g(q, v(q) + \lambda v(q)]dq,$$

where

$$\mathcal{K}(p, q) = \begin{cases} \dfrac{e^{\lambda(T+q-p)}}{e^{\lambda T - 1}} & 0 \leq q < p \leq T \\ \dfrac{e^{\lambda(q-p)}}{e^{\lambda T - 1}} & 0 \leq p < q \leq T. \end{cases}$$

Define $\mathcal{F} : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ by

$$(\mathcal{F}v)(p) = \int_0^T \mathcal{K}(p, q)[g(q, v(q)) + \lambda v(q)]dq.$$

Note that if $v \in C(I, \mathbb{R})$ is a fixed point of $\mathcal{F}$ then $v \in C^1(I, \mathbb{R})$ is a solution of (4.1).

Define a binary relation as follows: $a, b \in C(I, \mathbb{R}), \quad (a, b) \in \mathcal{R}$ if and only if $a(p) \leq b(p), \quad$ for all $\; p \in I$.

Observe, $(C(I, \mathbb{R}), d)$ is a $\mathcal{R}$-complete metric space with respect to

$$d(a, b) = \sup_{p \in I} |a - b|, \quad a, b \in C(I, \mathbb{R}).$$

To show $\mathcal{R}$ is $\mathcal{F}$-closed, take $(v, w) \in \mathcal{R}$

$$g(p, v) + \lambda v \geq g(p, w) + \lambda w$$

which implies for $p \in I$, using that $\mathcal{K}(p, q) > 0$ for $(p, q) \in I \times I$, that

$$\begin{aligned} (\mathcal{F}v)(p) &= \int_0^T \mathcal{K}(p, q)[g(q, v(q)) + \lambda v(q)]dq \\ &\geq \int_0^T \mathcal{K}(p, q)[g(q, w(q)) + \lambda w(q)]dq = (\mathcal{F}w)(p). \end{aligned}$$

Hence, $(\mathcal{F}v, \mathcal{F}w) \in \mathcal{R}$.

Now,

$$d(\mathcal{F}v, \mathcal{F}w) = \sup_{p \in I} |(\mathcal{F}v)(p) - (\mathcal{F}w)(p)|$$

$$\leq \sup_{p\in I}\int_0^T \mathcal{K}(p,q)[g(q,v(q))+\lambda v(q)-g(q,w(q))-\lambda w(q)]dq$$
$$\leq \sup_{p\in I}\int_0^T \mathcal{K}(p,q).\lambda\rho(v(q)-w(q))dq.$$

As the function $\rho(a)$ is increasing and $v \geq w$ then $\rho(v(q)-w(q)) \leq \rho(d(v,w))$, we obtain

$$\begin{aligned}
d(\mathcal{F}v,\mathcal{F}w) &\leq \sup_{p\in I}\int_0^T \mathcal{K}(p,q).\lambda\rho(v(q)-w(q))dq \\
&\leq \lambda.\rho(d(v,w)).\sup_{p\in I}\int_0^T \mathcal{K}(p,q)dq \\
&= \lambda.\rho(d(v,w)).\sup_{p\in I}\frac{1}{e^{\lambda p}-1}\left(\frac{1}{\lambda}e^{\lambda(T+q-p)}]_0^p+\frac{1}{\lambda}e^{\lambda(q-p)}]_p^T\right) \\
&= \lambda.\rho(d(v,w)).\frac{1}{\lambda(e^{\lambda T}-1)}(e^{\lambda T}-1)=\rho(d(v,w)) \\
&= \frac{\rho(d(v,w))}{d(v,w)}d(v,w)=\gamma(d(v,w))d(v,w).
\end{aligned}$$

Finally, let $\eta(p)$ be a lower solution for (2.1) and we will show that $\eta \leq \mathcal{F}\eta$.

Indeed,
$$\eta'(p)+\lambda\eta(p) \leq g(p,\eta(p))+\lambda\eta(p), \quad \text{for} \quad p\in I.$$

Multiplying by $e^{\lambda p}$, we get
$$(\eta(p)e^{\lambda p})' \leq [g(p,\eta(p))+\lambda\eta(p)]e^{\lambda p}, \quad \text{for} \quad p\in I$$

and this gives us
$$\eta(p)e^{\lambda p} \leq \eta(0)+\int_0^p [g(q,\eta(q))+\lambda\eta(p)]e^{\lambda q}dq, \quad \text{for} \quad p\in I \tag{4.2}$$

which implies that
$$\eta(0)e^{\lambda T} \leq \eta(T)e^{\lambda T} \leq \eta(0)+\int_0^T [g(q,\eta(q))+\lambda\eta(q)]e^{\lambda q}dq$$

and so
$$\eta(0) \leq \int_0^T \frac{e^{\lambda q}}{e^{\lambda T}-1}[g(q,\eta(q))+\lambda\eta(q)]dq.$$

From this inequality and (4.2), we obtain
$$\eta(p)e^{\lambda p} \leq \int_0^p \frac{e^{\lambda(T+q)}}{e^{\lambda T}-1}[g(q,\eta(q))+\lambda\eta(q)]dq+\int_0^T \frac{e^{\lambda q}}{e^{\lambda T}-1}[g(q,\eta(q))+\lambda\eta(q)]dq$$

and consequently
$$\eta(p) \leq \int_0^p \frac{e^{\lambda(T+q-p)}}{e^{\lambda T}-1}[g(q,\eta(q))+\lambda\eta(q)]dq+\int_0^T \frac{e^{\lambda(q-p)}}{e^{\lambda T}-1}[g(q,\eta(q))+\lambda\eta(q)]dq.$$

Hence,
$$\eta(p)=\int_0^T \mathcal{K}(p,q)[g(q,\eta(q))+\lambda\eta(q)]dq.$$

Finally, Corollary (3.1) gives that $\mathcal{F}$ has a unique fixed point. □

5. Conclusion

In this study, we established a new fixed point theorem for Geraghty contraction in a metric space equipped with a binary relation $\mathcal{R}$. In Classical contraction, we checked the condition on all elements of domain but in Relation theoretic, we checked on those elements who are comparable. Also, we provided an application to solve the periodic boundary value problem, our result generalized and improved the results of Geraghty [8], Amini-Harandi and Emami [9].

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