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APPROXIMATE CONTROLLABILITY OF RIEMANN-LIOUVILLE FRACTIONAL STOCHASTIC EVOLUTION SYSTEMS*

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Abstract This paper deals with the existence as well as the approximate controllability of Riemann-Liouville fractional stochastic evolution systems of Sobolev type with nonlocal initial conditions in abstract spaces. When the operator semigroup is noncompact and the nonlocal function is not Lipschitz continuous and not compact, the existence as well as the approximate controllability of the concerned problem are investigated. Finally, an application example is given.

Keywords Fractional stochastic evolution systems, Sobolev operator, approximate controllability, nonlocal function.

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1. Introduction

Fractional differential equations have got a lot of attention because they have practical background in the fields of physics, chemistry, engineering and etc. A growing number of notable research works have been gained on this topic. Particularly, the existence as well as the exact controllability results were obtained in [1, 2, 6, 20–22, 24]. But in infinite dimensional spaces, the concept of approximate controllability is more suitable. Various approaches are employed to demonstrate the approximate controllability of fractional evolution systems under the assumption that the corresponding linear systems are approximately controllable, see [12, 15, 17, 26]. In particular, Liu et al [15] utilized a different method to prove the approximate controllability of Riemann-Liouville fractional evolution systems without the assumption of the approximate controllability of the associated linear system. But in [15], the uniqueness of mild solutions is needed.

Stochastic evolution equations are important mathematical models to characterize many phenomena in natural and social sciences. Controllability results of stochastic evolution systems with fractional derivatives are reported by several researches, see [3, 7, 8, 16, 23]. Sobolev type differential equations are valuable mathematical models with a wide rang of backgrounds in physical problems. They are

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significant tools in dealing with the fluid flow through fissured rocks. Controllability theorems of Sobolev type fractional evolution equations are established in [5, 18, 25]. But to the best of author's knowledge, the research works on the approximate controllability of Sobolev type stochastic evolution systems involving Riemann-Liouville fractional derivatives are seldom.

In this article, we consider the Riemann-Liouville fractional stochastic evolution system(FSES for short) of sobolev type

$$\begin{cases} {}^L D_t^{\alpha}(Gz(t)) + Az(t) = H(t, z(t)) + \mathcal{L}c(t) + R(t, z(t))\dfrac{dW(t)}{dt}, \quad t \in (0, \eta], \\ I_{0^+}^{1-\alpha}(Gz(t))|_{t=0} + g(z) = z_0, \end{cases} \tag{1.1}$$

where $\alpha \in (\frac{1}{p}, 1)$, ${}^L D_t^{\alpha}$ represents the α-order fractional derivative of Riemann-Liouville type, $A : D(A) \subset V \to V$ and $G : D(G) \subset V \to V$ are linear operators in a Hilbert space V and A is densely defined, $c(t)$ is the control for $t \in J := [0, \eta]$ belonging to U which is another Hilbert space, $z_0 \in V$. $\{W(t)\}_{t \geq 0}$ is a standard Q-Wiener process, H, R and g are given functions.

The main innovations are listed below.

1. The assumption of approximate controllability of the deterministic or stochastic linear system corresponding to (1.1) is removed. To be more precise, we apply a method, which is cited from [15,28], to prove that the deterministic linear system corresponding to (1.1) is approximately controllable.

2. By employing the compactness of G^{-1}, we remove the assumption of compact semigroup and achieve the approximate controllability of the FSES (1.1). The complete continuity and the Lipschitz continuity of g are deleted in our work.

2. Preliminaries

Let $(\Upsilon, \mathcal{F}, \{\mathcal{F}_t \uparrow\subset \mathcal{F}, t \geq 0\}, P)$ be a complete probability space. Let $\{W(t)\}_{t\geq 0}$ be a standard Q-Wiener process with a positive nuclear operator Q satisfying $\mathrm{Tr}Q < +\infty$. Let $\{e_n\}_{n\geq 1}$ be a completely orthogonal system of V and $\{\kappa_n \in [0, \infty) : n \geq 1\}$ a bounded sequence satisfying

$$Qe_n = \kappa_n e_n, \qquad n = 1, 2, 3, \cdots,$$

and $\{\beta_n\}_{n\geq 1}$ a sequence satisfying

$$\langle W(t), z\rangle = \sum_{n=1}^{\infty} \sqrt{\kappa_n}\langle e_n, z\rangle \beta_n(t), \qquad z \in V, \quad t \geq 0.$$

Let $L_2^0 := L_2(Q^{\frac{1}{2}}V, V)$. Then L_2^0 is a separable Hilbert space with

$$\|\pi\|_{L_2^0}^2 = Tr[(\pi Q^{\frac{1}{2}})(\pi Q^{\frac{1}{2}})^*]$$

for any $\pi \in L_2^0$. For $p \geq 2$, let $L_p(\Upsilon, V)$ consist of strongly $\mathcal{F}_\eta$-measurable random variables satisfying $E\|z\|^p < +\infty$. Denote by $L_p^{\mathcal{F}}(J, U)$ the Hilbert space of U-valued $\mathcal{F}_t$-progressively measurable random processes satisfying

$$E\int_0^{\eta} \|z(t)\|^p dt < +\infty,$$

where $E\cdot$ denotes the expectation. We suppose that $c \in L_p^{\mathcal{F}}(J, U)$.

Let $C(J, L_p(\Upsilon, V)$ be the Banach space of continuous mappings satisfying

$$\sup_{t\in J} E\|z(t)\|^p < +\infty.$$

Denote by $C_{1-\alpha}(J, L_p(\Upsilon, V)) := \{z : \cdot^{1-\alpha} z(\cdot) \in C(J, L_p(\Upsilon, V))\}$, whose norm is defined by

$$\|z\|_{C_{1-\alpha}} = \sup_{t\in J} t^{1-\alpha} E\|z(t)\|^p, \quad \forall z \in C_{1-\alpha}(J, L_p(\Upsilon, V)).$$

Let $H_{1-\alpha}(J, V)$ be a closed subspace of $C_{1-\alpha}(J, L_p(\Upsilon, V))$, whose norm is defined by

$$\|z\|_{H_{1-\alpha}} = \left(\sup_{t\in J} t^{1-\alpha} E\|z(t)\|^p\right)^{\frac{1}{p}}, \quad \forall z \in H_{1-\alpha}(J, V).$$

We first recall some definitions of fractional calculus, see [10,13] for more details.

Definition 2.1. Let $u \in L^1(J)$. The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by

$$J_t^\alpha u(t) = (g_\alpha * u)(t), \quad t > 0,$$

where

$$g_\alpha(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function and $*$ means the finite convolution.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined for all $u \in L^1(J)$ satisfying $g_{m-\alpha} * u \in W^{m,1}(J)$ by

$${}^L D_t^\alpha u(t) = D_t^m (g_{m-\alpha} * u)(t), \quad t > 0,$$

where $D_t^m = \frac{d^m}{dt^m}$ and $m = \lceil \alpha \rceil$ denote the smallest integer greater than or equal to α.

Let A and G satisfy the conditions below.

$(C1)$ $D(G) \subset D(A)$ and G is bijective;

$(C2)$ Linear operator $G^{-1} : V \to D(G)$ is compact.

From $(C1)$ and $(C2)$, G^{-1} is a bounded operator. Then $-AG^{-1} : V \to V$ is a bounded linear operator and generates a C_0-semigroup $K(t) = e^{-AG^{-1}t}, t \ge 0$ satisfying $M := \sup_{t\in J} \|K(t)\| < +\infty$.

Let the function $M_\alpha(\theta)$, $\alpha \in (0, 1)$ is defined by

$$M_\alpha(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-\alpha n)}, \quad \theta \in \mathbf{C}.$$

Then by [19], we have

$$\int_0^\infty \theta^\varrho M_\alpha(\theta) d\theta = \frac{\Gamma(1+\varrho)}{\Gamma(1+\alpha\varrho)}, \quad \varrho \ge 0.$$

It is known that the FSES (1.1) is equivalent to the nonlocal problem

$$\begin{cases} Gz(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} I_{0^+}^{1-\alpha}(Gz(t))|_{t=0} + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}[-Az(s) + H(s,z(s)) \\ \qquad\qquad + \mathcal{L}c(s)]ds + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} R(s,z(s))dW(s), \quad t \in (0,\eta], \\ I_{0^+}^{1-\alpha}(Gz(t))|_{t=0} = z_0 - g(z). \end{cases} \tag{2.1}$$

Lemma 2.1. *If* (2.1) *holds, we have*

$$\begin{aligned} z(t) = t^{\alpha-1} S_G(t)[z_0 - g(z)] + \int_0^t (t-s)^{\alpha-1} S_G(t-s)[H(s,z(s)) + \mathcal{L}c(s)]ds \\ + \int_0^t (t-s)^{\alpha-1} S_G(t-s) R(s,z(s))dW(s), \qquad t \in (0,\eta], \end{aligned}$$

where

$$S_G(t) = G^{-1} S_I(t)$$

and

$$S_I(t) = \int_0^\infty \alpha\tau M_\alpha(\tau) K(t^\alpha \tau) d\tau.$$

Proof. For each $\zeta > 0$, by using Laplace transforms

$$\hat{z}(\zeta) = \int_0^\infty e^{-\zeta s} z(s)ds, \qquad \hat{H}(\zeta) = \int_0^\infty e^{-\zeta s}[H(s,z(s)) + \mathcal{L}c(s)]ds$$

to the equation (2.1), we can acquire that

$$\begin{aligned} G\hat{z}(\zeta) = & \ \frac{1}{\zeta^\alpha}[z_0 - g(z)] - \frac{1}{\zeta^\alpha}(AG^{-1})G\hat{z}(\zeta) \\ & + \frac{1}{\zeta^\alpha}\hat{H}(\zeta) + \frac{1}{\zeta^\alpha}\int_0^\infty e^{-\zeta s} R(s,z(s))dW(s) \\ = & \ (\zeta^\alpha I + AG^{-1})^{-1}[z_0 - g(z)] + (\zeta^\alpha I + AG^{-1})^{-1}\hat{H}(\zeta) \\ & + (\zeta^\alpha I + AG^{-1})^{-1} \int_0^\infty e^{-\zeta s} R(s,z(s))dW(s) \\ = & \ \int_0^\infty e^{-\zeta^\alpha s} K(s)[z_0 - g(z)]ds + \int_0^\infty e^{-\zeta^\alpha s} K(s)\hat{H}(\zeta)ds \\ & + \int_0^\infty e^{-\zeta^\alpha s} K(s) \int_0^\infty e^{-\zeta\tau} R(\tau, z(\tau))dW(\tau)ds. \end{aligned}$$

It infers from Lemma 3.3 of [29] that

$$\begin{aligned} Gz(t) = \alpha \int_0^\infty \delta t^{\alpha-1} M_\alpha(\delta) K(t^\alpha \delta)[z_0 - g(z)]d\delta \\ + \alpha \int_0^t \int_0^\infty \delta(t-s)^{\alpha-1} M_\alpha(\delta) K((t-s)^\alpha \delta)[H(s,z(s)) + \mathcal{L}c(s)]d\delta ds \\ + \alpha \int_0^t \int_0^\infty \delta(t-s)^{\alpha-1} M_\alpha(\delta) K((t-s)^\alpha \delta) R(s,z(s))d\delta dW(s) \end{aligned}$$

$$= t^{\alpha-1}S_I(t)[z_0 - g(z)] + \int_0^t (t-s)^{\alpha-1}S_I(t-s)[H(s,z(s)) + \mathcal{L}c(s)]ds$$
$$+ \int_0^t (t-s)^{\alpha-1}S_I(t-s)R(s,z(s))dW(s).$$

Thus, the proof of Lemma 2.1 is completed. □

Definition 2.3. The mild solution of (1.1) we mean a stochastic process $z : J \to V$ satisfying

(i) for $t \geq 0$, $z(t)$ is $\mathcal{F}_t$-adapted;

(ii) for $t \in (0, \eta]$, $z(t)$ satisfies

$$z(t) = t^{\alpha-1}S_G(t)[z_0 - g(z)] + \int_0^t (t-s)^{\alpha-1}S_G(t-s)[H(s,z(s)) + \mathcal{L}c(s)]ds$$
$$+ \int_0^t (t-s)^{\alpha-1}S_G(t-s)R(s,z(s))dW(s);$$

(iii) $I_{0^+}^{1-\alpha}(Gz(t))|_{t=0} + g(z) = z_0$.

Lemma 2.2 ([5,29]). *Let assumptions* $(C1)$ *and* $(C2)$ *hold. Then* $\{S_G(t) : t \geq 0\}$ *has properties below:*

(1) $\|S_G(t)z\| \leq \frac{M\|G^{-1}\|}{\Gamma(\alpha)}\|z\|, \quad \forall t \geq 0, \quad z \in V.$

(2) $\|S_G(t)z - S_G(s)z\| \to 0$ *as* $t - s \to 0, \quad \forall t, s \geq 0, \quad z \in V.$

(3) $S_G(t)$ *is compact for each* $t \geq 0$.

Lemma 2.3 ([9]). *Assume that* $R : J \times \Upsilon \to L_2^0$ *is strongly measurable and*

$$\int_0^\eta E\|R(\delta)\|_{L_2^0}^p d\delta < +\infty.$$

Then, for $p \geq 2$, *there is* $L_R > 0$ *satisfying*

$$E\|\int_0^t R(\delta)dW(\delta)\|^p \leq L_R \int_0^t E\|R(\delta)\|_{L_2^0}^p d\delta, \quad t \in J.$$

3. Existence of mild solutions

We first make the assumptions below.

$(A1)$ $H : J \times V \to V$ is continuous and there is $\varphi \in L^p(J, [0,\infty))$ such that

$$E\|H(t,z)\|^p \leq \varphi(t)\Phi_H(t^{1-\alpha}E\|z\|^p), \quad \forall (t,z) \in J \times L_p(\Upsilon, V),$$

where $\Phi_H : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with

$$a_1 := \liminf_{\rho\to\infty} \frac{1}{\rho}\Phi_H(\rho) < +\infty.$$

$(A2)$ $R : J \times V \to L_2^0$ is continuous and there is $\psi \in L^1(J, [0,\infty))$ satisfying

$$\int_0^t (t-s)^{p(\alpha-1)}\psi(s)ds < +\infty$$

such that

$$E\|R(t,z)\|_{L_2^0}^p \le \psi(t)\Phi_R(t^{1-\alpha}E\|z\|^p), \quad \forall (t,z)\in J\times L_p(\Upsilon,V),$$

where $\Phi_R:[0,\infty)\to[0,\infty)$ is continuous and nondecreasing with

$$a_2 := \liminf_{\rho\to\infty}\frac{1}{\rho}\Phi_R(\rho) < +\infty.$$

$(A3)$ $g: H_{1-\alpha}(J,V)\to V$ is continuous and there is $\overline{M}>0$ satisfying

$$\|g(z)\|\le \overline{M}, \quad \forall z\in D_\rho,$$

where $D_\rho=\{z\in H_{1-\alpha}(J,V): t^{1-\alpha}E\|z(t)\|^p\le\rho, t\in J\}$ for some $\rho>0$.

$(A4)$ $\mathcal{L}: L_p^{\mathcal{F}}(J,U)\to L^p(J,V)$ is bounded, and let $M_B:=\|\mathcal{L}\|$.

$(A5)$ $2^{p-1}\big(\frac{p-1}{p\alpha-1}\big)^{p-1}\eta^{p\alpha-\frac{1}{p}}\|\varphi\|_{L^p}a_1+a_2L_R\sup\limits_{t\in J}\int_0^t(t-s)^{p(\alpha-1)}\psi(s)ds<\frac{4^{1-p}\eta^{\alpha-1}}{\big(\frac{M\|G^{-1}\|}{\Gamma(\alpha)}\big)^p}$.

We define $\mathcal{Q}: D_\rho\to H_{1-\alpha}(J,V)$ by

$$\begin{aligned}(\mathcal{Q}z)(t) = {} & t^{\alpha-1}S_G(t)\big[z_0-g(z)\big]+\int_0^t(t-s)^{\alpha-1}S_G(t-s)\big[H(s,z(s))+\mathcal{L}c(s)\big]ds\\ & +\int_0^t(t-s)^{\alpha-1}S_G(t-s)R(s,z(s))dW(s), \quad t\ge 0.\end{aligned}$$

Lemma 3.1. *If assumptions $(C1),(C2)$ and $(A1)-(A5)$ are fulfilled, $\mathcal{Q}: D_\rho\to D_\rho$ for some $\rho>0$.*

Proof. If this is not true, there would exist $z\in D_\rho$ satisfying $t^{1-\alpha}E\|(\mathcal{Q}z)(t)\|^p>\rho$. According to assumptions, we have

$$\begin{aligned}\rho < {} & t^{1-\alpha}E\|(\mathcal{Q}z)(t)\|^p\\ \le {} & 4^{p-1}E\|S_G(t)z_0\|^p+4^{p-1}E\|S_G(t)g(z)\|^p\\ & +4^{p-1}\eta^{1-\alpha}E\|\int_0^t(t-s)^{\alpha-1}S_G(t-s)\big[H(s,z(s))+\mathcal{L}c(s)\big]ds\|^p\\ & +4^{p-1}\eta^{1-\alpha}E\|\int_0^t(t-s)^{\alpha-1}S_G(t-s)R(s,z(s))dW(s)\|^p\\ \le {} & 4^{p-1}\left(\frac{M\|G^{-1}\|}{\Gamma(\alpha)}\right)^p(\|z_0\|^p+\overline{M}^p)\\ & +8^{p-1}\left(\frac{M\|G^{-1}\|}{\Gamma(\alpha)}\right)^p\left(\frac{p-1}{p\alpha-1}\right)^{p-1}\eta^{p\alpha-\alpha}(\eta^{1-\frac{1}{p}}\|\varphi\|_{L^p}\Phi_H(\rho)+M_B^p\|c\|_{L^p}^p)\\ & +4^{p-1}\eta^{1-\alpha}L_R\left(\frac{M\|G^{-1}\|}{\Gamma(\alpha)}\right)^p\sup_{t\in J}\int_0^t(t-s)^{p(\alpha-1)}\psi(s)ds\Phi_R(\rho).\end{aligned}$$

This fact combining with $(A5)$ yields that $\mathcal{Q}(D_\rho)\subset D_\rho$ for certain $\rho>0$. □

Lemma 3.2. *If assumptions $(C1),(C2)$ and $(A1)-(A4)$ hold, we can acquire the equi-continuity of the set*

$$\prod := \{\xi\in C(J,V):\xi(\cdot)=\cdot^{1-\alpha}(\mathcal{Q}z)(\cdot), z\in D_\rho\}. \tag{3.1}$$

Proof. Let $\xi \in \prod$ and $\mu_1, \mu_2 \in [0, \eta]$ with $\mu_1 < \mu_2$. If $\mu_1 \equiv 0$, we have

$$
\begin{aligned}
& E\|\xi(\mu_2) - \xi(0)\|^p \\
&\leq 3^{p-1} E\|(S_G(\mu_2) - I)\big[z_0 - g(z)\big]\|^p \\
&\quad + 6^{p-1}\mu_2^{p-1}\left(\frac{M||G^{-1}||}{\Gamma(\alpha)}\right)^p \left(\frac{p-1}{p\alpha-1}\right)^{p-1}\left(\mu_2^{1-\frac{1}{p}}\|\varphi\|_{L^p}\Phi_H(\rho) + M_B^p\|c\|_{L^p}^p\right) \\
&\quad + 3^{p-1}\mu_2^{p(1-\alpha)} L_R\left(\frac{M||G^{-1}||}{\Gamma(\alpha)}\right)^p \int_0^{\mu_2} (\mu_2 - s)^{p(\alpha-1)}\psi(s)ds\Phi_R(\rho) \\
&\to 0 \qquad (\mu_2 \to 0).
\end{aligned}
$$

If $\mu_1 > 0$, we can achieve that

$$
\begin{aligned}
& E\|\xi(\mu_2) - \xi(\mu_1)\|^p \\
&\leq 9^{p-1} E\|[S_G(\mu_2) - S_G(\mu_1)][z_0 - g(z)]\|^p \\
&\quad + 9^{p-1}|\mu_2^{1-\alpha} - \mu_1^{1-\alpha}|^p E\| \int_0^{\mu_2} (\mu_2 - s)^{\alpha-1} S_G(\mu_2 - s)\big[H(s, z(s)) + \mathcal{L}c(s)\big]ds\|^p \\
&\quad + 9^{p-1}\mu_1^{p(1-\alpha)} E\| \int_0^{\mu_1} \big[(\mu_2 - s)^{\alpha-1} S_G(\mu_2 - s) - (\mu_1 - s)^{\alpha-1} S_G(\mu_2 - s)\big] \\
&\quad \times \big[H(s, z(s)) + \mathcal{L}c(s)\big]ds\|^p \\
&\quad + 9^{p-1}\mu_1^{p(1-\alpha)} E\| \int_0^{\mu_1} \big[(\mu_1 - s)^{\alpha-1} S_G(\mu_2 - s) - (\mu_1 - s)^{\alpha-1} S_G(\mu_1 - s)\big] \\
&\quad \times \big[H(s, z(s)) + \mathcal{L}c(s)\big]ds\|^p \\
&\quad + 9^{p-1}\mu_1^{p(1-\alpha)} E\| \int_{\mu_1}^{\mu_2} (\mu_2 - s)^{\alpha-1} S_G(\mu_2 - s)\big[H(s, z(s)) + \mathcal{L}c(s)\big]ds\|^p \\
&\quad + 9^{p-1}|\mu_2^{1-\alpha} - \mu_1^{1-\alpha}|^p E\| \int_0^{\mu_2} (\mu_2 - s)^{\alpha-1} S_G(\mu_2 - s) R(s, z(s))dW(s)]\|^p \\
&\quad + 9^{p-1}\mu_1^{p(1-\alpha)} E\| \int_0^{\mu_1} \big[(\mu_2 - s)^{\alpha-1} S_G(\mu_2 - s) - (\mu_1 - s)^{\alpha-1} S_G(\mu_2 - s)\big] \\
&\quad \times R(s, z(s))dW(s)\|^p \\
&\quad + 9^{p-1}\mu_1^{p(1-\alpha)} E\| \int_0^{\mu_1} \big[(\mu_1 - s)^{\alpha-1} S_G(\mu_2 - s) - (\mu_1 - s)^{\alpha-1} S_G(\mu_1 - s)\big] \\
&\quad \times R(s, z(s))dW(s)\|^p \\
&\quad + 9^{p-1}\mu_1^{p(1-\alpha)} E\| \int_{\mu_1}^{\mu_2} (\mu_2 - s)^{\alpha-1} S_G(\mu_2 - s) R(s, x(s))dW(s)]\|^p \\
&:= 9^{p-1}\sum_{\ell=1}^{9} I_\ell.
\end{aligned}
$$

By Lemma 2.2(2), we get

$$
I_1 = E\|[S_G(\mu_2) - S_G(\mu_1)][z_0 - g(z)]\|^p \to 0 \qquad (\mu_2 - \mu_1 \to 0).
$$

The equi-continuity of $\{S_G(t) : t \geq 0\}$ yields

$$
I_4 = \mu_1^{p(1-\alpha)} E\| \int_0^{\mu_1} \big[(\mu_1 - s)^{\alpha-1} S_G(\mu_2 - s) - (\mu_1 - s)^{\alpha-1} S_G(\mu_1 - s)\big]
$$

$$
\begin{aligned}
&\quad \times \big[H(s,z(s))+\mathcal{L}c(s)\big]ds\|^p \\
&\leq \mu_1^{p(1-\alpha)} \sup_{s\in[0,\mu_1]} \|S_G(\mu_2-s)-S_G(\mu_1-s)\|^p E\| \\
&\quad \times \int_0^{\mu_1} (\mu_1-s)^{\alpha-1}\big[H(s,z(s))+\mathcal{L}c(s)\big]ds\|^p \\
&\leq 2^{p-1}\mu_1^{p-1}\left(\frac{p-1}{p\alpha-1}\right)^{p-1} \sup_{s\in[0,\mu_1]} \|S_G(\mu_2-s)-S_G(\mu_1-s)\|^p \\
&\quad \times (\mu_1^{1-\frac{1}{p}}\|\varphi\|_{L^p}\Phi_H(\rho)+M_B^p\|c\|_{L^p}^p) \\
&\to 0
\end{aligned}
$$

and

$$
\begin{aligned}
I_8 &= \mu_1^{p(1-\alpha)}E\|\int_0^{\mu_1}\big[(\mu_1-s)^{\alpha-1}S_G(\mu_2-s)-(\mu_1-s)^{\alpha-1}S_G(\mu_1-s)\big] \\
&\quad \times R(s,z(s))dW(s)\|^p \\
&= \mu_1^{p(1-\alpha)}E\|\int_0^{\mu_1}(\mu_1-s)^{\alpha-1}\big[S_G(\mu_2-s)-S_G(\mu_1-s)\big]R(s,z(s))dW(s)\|^p \\
&\leq L_R\Phi_R(\rho)\mu_1^{p(1-\alpha)} \sup_{s\in[0,\mu_1]}\|S_G(\mu_2-s)-S_G(\mu_1-s)\|^p \\
&\quad \times \int_0^{\mu_1}(\mu_1-s)^{p(\alpha-1)}\psi(s)ds \\
&\to 0
\end{aligned}
$$

as $\mu_2-\mu_1\to 0$. A direct calculation shows that

$$
\begin{aligned}
I_2 &= |\mu_2^{1-\alpha}-\mu_1^{1-\alpha}|^p E\|\int_0^{\mu_2}(\mu_2-s)^{\alpha-1}S_G(\mu_2-s)\big[H(s,z(s))+\mathcal{L}c(s)\big]ds\|^p \\
&\leq 2^{p-1}|\mu_2^{1-\alpha}-\mu_1^{1-\alpha}|^p\left(\frac{M||G^{-1}||}{\Gamma(\alpha)}\right)^p\left(\frac{p-1}{p\alpha-1}\right)^{p-1}\mu_2^{p\alpha-1} \\
&\quad \times (\mu_2^{1-\frac{1}{p}}\|\varphi\|_{L^p}\Phi_H(\rho)+M_B^p\|c\|_{L^p}^p) \\
&\to 0, \\
I_3 &= \mu_1^{p(1-\alpha)}E\|\int_0^{\mu_1}\big[(\mu_2-s)^{\alpha-1}S_G(\mu_2-s)-(\mu_1-s)^{\alpha-1}S_G(\mu_2-s)\big] \\
&\quad \times \big[H(s,z(s))+\mathcal{L}c(s)\big]ds\|^p \\
&= \mu_1^{p(1-\alpha)}E\|\int_0^{\mu_1}\big[(\mu_2-s)^{\alpha-1}-(\mu_1-s)^{\alpha-1}\big]S_G(\mu_2-s)\big[H(s,z(s))+\mathcal{L}c(s)\big]ds\|^p \\
&\leq \mu_1^{p(1-\alpha)}\left(\frac{M||G^{-1}||}{\Gamma(\alpha)}\right)^p E\|\int_0^{\mu_1}\big[(\mu_2-s)^{\alpha-1}-(\mu_1-s)^{\alpha-1}\big]\big[H(s,z(s))+\mathcal{L}c(s)\big]ds\|^p \\
&\leq 2^{p-1}\mu_1^{p(1-\alpha)}\left(\frac{M||G^{-1}||}{\Gamma(\alpha)}\right)^p\Big(\int_0^{\mu_1}|(\mu_2-s)^{\alpha-1}-(\mu_1-s)^{\alpha-1}|^{\frac{p}{p-1}}ds\Big)^{p-1} \\
&\quad \times (\mu_1^{1-\frac{1}{p}}\|\varphi\|_{L^p}\Phi_H(\rho)+M_B^p\|c\|_{L^p}^p) \\
&\to 0, \\
I_5 &= \mu_1^{p(1-\alpha)}E\|\int_{\mu_1}^{\mu_2}(\mu_2-s)^{\alpha-1}S_G(\mu_2-s)\big[H(s,z(s))+\mathcal{L}c(s)\big]ds]\|^p
\end{aligned}
$$

$$\le 2^{p-1}\mu_1^{p(1-\alpha)}\bigg(\frac{M||G^{-1}||}{\Gamma(\alpha)}\bigg)^p\bigg(\frac{p-1}{p\alpha-1}\bigg)^{p-1}(\mu_2-\mu_1)^{p\alpha-1}$$
$$\times\int_{\mu_1}^{\mu_2}(\varphi(s)\Phi_H(\rho)+M_B^p\|c(s)\|^p)ds$$
$$\to 0,$$

$$I_6=|\mu_2^{1-\alpha}-\mu_1^{1-\alpha}|^pE\|\int_0^{\mu_2}(\mu_2-s)^{\alpha-1}S_G(\mu_2-s)R(s,z(s))dW(s)]\|^p$$
$$\le L_R\Phi_R(\rho)\bigg(\frac{M||G^{-1}||}{\Gamma(\alpha)}\bigg)^p|\mu_2^{1-\alpha}-\mu_1^{1-\alpha}|^p\int_0^{\mu_2}(\mu_2-s)^{p(\alpha-1)}\psi(s)ds$$
$$\to 0,$$

$$I_7=\mu_1^{p(1-\alpha)}E\|\int_0^{\mu_1}\big[(\mu_2-s)^{\alpha-1}S_G(\mu_2-s)-(\mu_1-s)^{\alpha-1}S_G(\mu_2-s)\big]$$
$$\times R(s,z(s))dW(s)\|^p$$
$$=\mu_1^{p(1-\alpha)}E\|\int_0^{\mu_1}\big[(\mu_2-s)^{\alpha-1}-(\mu_1-s)^{\alpha-1}\big]S_G(\mu_2-s)R(s,z(s))dW(s)\|^p$$
$$\le\mu_1^{p(1-\alpha)}\bigg(\frac{M||G^{-1}||}{\Gamma(\alpha)}\bigg)^pL_R\Phi_R(\rho)\int_0^{\mu_1}|(\mu_2-s)^{\alpha-1}-(\mu_1-s)^{\alpha-1}|^p\psi(s)ds$$
$$\to 0$$

and

$$\begin{aligned} I_9 &= \mu_1^{p(1-\alpha)}E\|\int_{\mu_1}^{\mu_2}(\mu_2-s)^{\alpha-1}S_G(\mu_2-s)R(s,z(s))dW(s)\|^p\\ &\le L_R\Phi_R(\rho)\mu_1^{p(1-\alpha)}\bigg(\frac{M||G^{-1}||}{\Gamma(\alpha)}\bigg)^p\int_{\mu_1}^{\mu_2}(\mu_2-s)^{p(\alpha-1)}\psi(s)ds\\ &\to 0\end{aligned}$$

as $\mu_2-\mu_1\to 0$. Therefore, the set $\prod$ is equi-continuous. □

Lemma 3.3. *Let conditions* $(C1),(C2)$ *and* $(A1)-(A5)$ *hold. Then* $\prod(t)=\{\xi(t):\xi\in\prod\}$ *is relatively compact for every* $t\in J$.

Proof. By $(A3)$ we can achieve that the set $\{z_0-g(z):z\in D_\rho\}$ is bounded. Since $\{S_I(t):t\ge 0\}$ is linear and bounded, the set

$$\{S_I(t)[z_0-g(z)]:z\in D_\rho,t\ge 0\}$$

is bounded. Then we can infer the relative compactness of

$$\{S_G(t)[z_0-g(z)]:z\in D_\rho,t\ge 0\}$$

because G^{-1} is compact. That is, the set

$$\{t^{1-\alpha}S_G(t)[z_0-g(z)]:z\in D_\rho,t\ge 0\}$$

is relatively compact. We denote by

$$(\mathcal{Q}_1z)(t)=\int_0^t(t-s)^{\alpha-1}S_I(t-s)\big[H(s,z(s))+\mathcal{L}c(s)\big]ds$$

$$+\int_0^t (t-s)^{\alpha-1} S_I(t-s) R(s,z(s)) dW(s).$$

According to Lemma 3.1, $\{(\mathcal{Q}_1 z)(t) : z \in D_\rho, t \geq 0\}$ is bounded. Hence we can derive that

$$\{G^{-1}(\mathcal{Q}_1 z)(t) : z \in D_\rho, t \geq 0\}$$

is relatively compact in view of the compactness of G^{-1}. Therefore, we conclude that $\{(\mathcal{Q}z)(t) : z \in D_\rho, t \geq 0\}$ is relatively compact. Consequently, the relative compactness of $\prod(t) = \{\xi(t) : \xi \in \prod\}$ is achieved. □

Theorem 3.1. *Let assumptions* $(C1), (C2)$ *and* $(A1)-(A5)$ *be fulfilled. Then the FSES* (1.1) *has a mild solution.*

Proof. Owing to Lemma 3.1, we get $\mathcal{Q}(D_\rho) \subset D_\rho$ for certain $\rho > 0$. We next prove the continuity of $\mathcal{Q}$ on D_ρ. Let $\{z_n\}$ be a sequence of D_ρ with $z_n \to z$ as $n \to \infty$. According to the continuity of H, R and g, we have

$$\begin{aligned} &H(t, z_n(t)) \to H(t, z(t)), \quad t \geq 0, \\ &R(t, z_n(t)) \to R(t, z(t)), \quad t \geq 0 \end{aligned}$$

and

$$g(z_n) \to g(z)$$

as $n \to \infty$. On the other hand, since

$$E\|S_G(t-s)\big[H(s,z_n(s)) - H(s,z(s))\big]\|^p \leq 2^p\big(\frac{M\|G^{-1}\|}{\Gamma(\alpha)}\big)^p \Phi_H(\rho)\varphi(s) \in L^1(J, R^+)$$

and

$$\begin{aligned} &E\|(t-s)^{\alpha-1} S_G(t-s)\big[R(s,z_n(s)) - R(s,z(s))\big]\|^p_{L_2^0} \\ \leq &2^p\big(\frac{M||G^{-1}||}{\Gamma(\alpha)}\big)^p \Phi_R(\rho)(t-s)^{p(\alpha-1)}\psi(s) \in L^1(J, R^+), \end{aligned}$$

we can acquire that

$$\begin{aligned} &\quad t^{1-\alpha} E\|(\mathcal{Q}z_n)(t) - (\mathcal{Q}z)(t)\|^p \\ &\leq 3^{p-1} E\|S_G(t)[g(z_n) - g(z)]\|^p \\ &\quad + 3^{p-1} t^{1-\alpha} E\|\int_0^t (t-s)^{\alpha-1} S_G(t-s)\big[H(s,z_n(s)) - H(s,z(s))\big] ds\|^p \\ &\quad + 3^{p-1} t^{1-\alpha} E\|\int_0^t (t-s)^{\alpha-1} S_G(t-s)\big[R(s,z_n(s)) - R(s,z(s))\big] dW(s)\|^p \\ &\leq 3^{p-1}\big(\frac{M||G^{-1}||}{\Gamma(\alpha)}\big)^p E\|g(z_n) - g(z)\|^p \\ &\quad + 3^{p-1}\big(\frac{p-1}{p\alpha-1}\big)^{p-1}\eta^{p\alpha-\alpha}\int_0^t E\|S_G(t-s)\big[H(s,z_n(s)) - H(s,z(s))\big]\|^p ds \\ &\quad + 3^{p-1}\eta^{1-\alpha} L_R \int_0^t E\|(t-s)^{\alpha-1} S_G(t-s)\big[R(s,z_n(s)) - R(s,z(s))\big]\|^p_{L_2^0} ds \\ &\to 0 \quad (n \to \infty). \end{aligned}$$

This fact implies that

$$\|\mathcal{Q}z_n-\mathcal{Q}z\|^p_{H_{1-\alpha}}=\sup_{t\in J}t^{1-\alpha}E\|(\mathcal{Q}z_n)(t)-(\mathcal{Q}z)(t)\|^p\to 0$$

as $n\to\infty$. Therefore, $\mathcal{Q}:D_\rho\to D_\rho$ is continuous.

This fact together with Lemmas 3.2 and 3.3, we deduce that $\mathcal{Q}:D_\rho\to D_\rho$ is completely continuous by the Ascoli-Arzela theorem. By applying Schauder's fixed point theorem, the FSES (1.1) possesses a mild solution. □

Remark 3.1. In Theorem 3.1, we remove the compactness and Lipschitz continuity conditions of g, hence our result extends some existing conclusions of [4, 11, 27].

Remark 3.2. If $a_1=a_2\equiv 0$ in $(A1)$ and $(A2)$, the assumption $(A5)$ holds automatically.

Let conditions $(A1)$ and $(A2)$ be replaced by
$(A1)'$ $H:J\times V\to V$ is continuous and there is $N_1>0$ such that

$$E\|H(t,z)\|^p\le N_1,\qquad \forall(t,z)\in J\times L_p(\Upsilon,V).$$

$(A2)'$ $R:J\times V\to L_2^0$ is continuous and there is $N_2>0$ satisfying

$$E\|R(t,z)\|^p_{L_2^0}\le N_2,\qquad \forall(t,z)\in J\times L_p(\Upsilon,V).$$

By virtue of Theorem 3.1, we can acquire the conclusion below.

Theorem 3.2. *Assume that conditions* $(C1),(C2)$, $(A1)',(A2)',(A3)$ *and* $(A4)$ *are satisfied. Then the FSES* (1.1) *possesses a mild solution.*

Proof. Choosing $\rho>0$ large enough such that

$$\begin{aligned}\rho>\ & 4^{p-1}\big(\frac{M\|G^{-1}\|}{\Gamma(\alpha)}\big)^p(\|z_0\|^p+\overline{M}^p)\\ &+8^{p-1}\big(\frac{M\|G^{-1}\|}{\Gamma(\alpha)}\big)^p\big(\frac{p-1}{p\alpha-1}\big)^{p-1}\eta^{p\alpha-\alpha}(N_1\eta+M_B^p\|c\|^p_{L^p})\\ &+4^{p-1}\frac{\eta^{p(\alpha-1)+2-\alpha}L_R}{p(\alpha-1)+1}\big(\frac{M\|G^{-1}\|}{\Gamma(\alpha)}\big)^pN_2,\end{aligned}$$

then $\mathcal{Q}(D_\rho)\subset D_\rho$. We omit the remain proof because it is similar to the one of Theorem 3.1. □

Remark 3.3. In our Theorem 3.2, we delete the assumption $(A5)$. The uniform boundedness conditions $(A1)',(A2)'$ are strong for existence results, but they are useful in the controllability theorem.

4. Approximate controllability

Denoting by $z(t;c)$ the mild solution of (1.1) associated with c, we introduce the reachable set of (1.1) by

$$K_\eta(H):=\{z(\eta;c):c\in L_p^{\mathcal{F}}(J,U)\}.$$

Definition 4.1. Let $\overline{K_\eta(H)}$ be the closure of $K_\eta(H)$. If $\overline{K_\eta(H)} = L_p(\Upsilon, V)$, the FSES (1.1) is called approximately controllable.

We first investigate the fractional linear systems

$$\begin{cases} {}^L D_t^\alpha (Gz(t)) + Az(t) = \mathcal{L}c(t) + R(t)\dfrac{dW(t)}{dt}, \quad t \geq 0, \\ z(0) = z_0 \end{cases} \tag{4.1}$$

and

$$\begin{cases} {}^L D_t^\alpha (Gz(t)) + Az(t) = \mathcal{L}v(t), \quad t \geq 0, \\ z(0) = z_0, \end{cases} \tag{4.2}$$

where $c \in L_p^{\mathcal{F}}(J, U)$ and $v \in L^p(J, U)$.

The following conclusion is cited from [18].

Lemma 4.1. *The linear system* (4.1) *is approximately controllable iff fractional deterministic system* (4.2) *is approximately controllable on each* $[\varpi, \eta], 0 \leq \varpi \leq \eta$.

Thus, we first consider the approximate controllability of (4.2). Define a bounded linear operator $T : L^p(J, V) \to V$ by

$$Th = \int_0^\eta (\eta - s)^{\alpha-1} S_G(\eta - s) h(s) ds, \quad h \in L^p(J, V). \tag{4.3}$$

$(A6)$ For any $\epsilon > 0$ and $h \in L^p(J, V)$, there is $v_\epsilon \in L^p(J, U)$ satisfying

$$\|Th - T\mathcal{L}v_\varepsilon\| < \epsilon.$$

Lemma 4.2. *Let* $(A6)$ *hold. Then the fractional deterministic system* (4.2) *is approximately controllable.*

Proof. Since $K(t) = e^{-AG^{-1}t}, t \geq 0$, we infer that $\frac{dK(t)}{dt} = -AG^{-1}K(t), t > 0$ and for any $z_0 \in V, K(t)z_0 \in D(-AG^{-1})$. By the definition of $S_G(t)$, we can acquire that

$$\eta^{\alpha-1} S_G(\eta) z_0 \in D(-AG^{-1})$$

and

$$\frac{dS_G^2(t)z}{dt} = 2S_G(t)\frac{dS_G(t)}{dt}z, \quad \forall z \in V.$$

Since $\overline{D(-AG^{-1})} = V$, we will prove $D(-AG^{-1}) = \overline{K_\eta(0)}$. That is, we are going to find a control $v_\epsilon \in L^p(J, U)$ such that, for each $\omega \in D(-AG^{-1})$ and $\forall \epsilon > 0$,

$$\|\omega - \eta^{\alpha-1} S_G(\eta) z_0 - T\mathcal{L}v_\epsilon\| < \epsilon.$$

For $\forall \omega \in D(-AG^{-1})$, we can find a function $h \in L^p(J, V)$ satisfying

$$Th = \omega - \eta^{\alpha-1} S_G(\eta) z_0.$$

For instance, we can pick

$$h(t) = \frac{[\Gamma(\alpha)]^2 (\eta - t)^{1-\alpha} G^2}{\eta} \left[S_G(\eta - t) - 2t\frac{dS_G(\eta - t)}{dt}\right]\left[\omega - \eta^{\alpha-1} S_G(\eta) x_0\right], \ t \in (0, \eta).$$

In fact, from (4.3) we infer that

$$\begin{aligned} Th =& \int_0^\eta (\eta-s)^{\alpha-1}S_G(\eta-s)h(s)ds \\ =& \frac{[\Gamma(\alpha)]^2}{\eta}\int_0^\eta \big[S_I^2(\eta-s)-2sS_I(\eta-s)\frac{dS_I(\eta-s)}{ds}\big]ds\big[\omega-\eta^{\alpha-1}S_G(\eta)z_0\big] \\ =& \frac{[\Gamma(\alpha)]^2}{\eta}\int_0^\eta d(sS_I^2(\eta-s))\big[\omega-\eta^{\alpha-1}S_G(\eta)z_0\big] \\ =& \frac{[\Gamma(\alpha)]^2}{\eta}sS_I^2(\eta-s)|_0^\eta\big[\omega-\eta^{\alpha-1}S_G(\eta)z_0\big] \\ =& [\Gamma(\alpha)]^2S_I^2(0)\big[\omega-\eta^{\alpha-1}S_G(\eta)z_0\big] \\ =& \omega-\eta^{\alpha-1}S_G(\eta)z_0, \end{aligned}$$

where $S_I(t)=\int_0^\infty \alpha\theta M_\alpha(\theta)K(t^\alpha\theta)d\theta, t\geq 0$ and $S_G(t)=G^{-1}S_I(t),\ t\geq 0$.

For this $h\in L^p(J,V)$, the condition $(A6)$ yields that there is $v_\epsilon\in L^p(J,U)$ meeting

$$\|Th-T\mathcal{L}v_\epsilon\|<\epsilon.$$

So

$$\|\omega-\eta^{\alpha-1}S_G(\eta)z_0-T\mathcal{L}v_\epsilon\|<\epsilon,$$

and the fractional deterministic system (4.2) is approximately controllable. □

Let

$$\pi_0^\eta := \int_0^\eta (\eta-s)^{2(\alpha-1)}S_G(\eta-s)\mathcal{L}\mathcal{L}^*S_G^*(\eta-s)ds.$$

Lemma 4.3 ([17]). *The fractional deterministic system* (4.2) *is approximately controllable iff* $\|\lambda(\lambda I+\pi_0^\eta)^{-1}z\|\to 0$ *as* $\lambda\to 0^+$ *for all* $z\in V$.

Lemma 4.4 ([18]). *For every* $\Lambda\in L_p(\Upsilon,V)$, *there is* $\phi\in L_p^{\mathcal{F}}(\Upsilon;L^2(J,L_2^0))$ *satisfying*

$$\Lambda=E\Lambda+\int_0^\eta \phi(\delta)dW(\delta).$$

For any $\lambda>0$ and $h\in L_p(\Upsilon,V)$, let's now choose a control c^λ by

$$c^\lambda(t;z)=(\eta-t)^{\alpha-1}B^*S_C^*(\eta-t)(\lambda I+\pi_0^\eta)^{-1}\mathcal{P}(z),$$

where

$$\begin{aligned} \mathcal{P}(z) =&\ E\Lambda-\eta^{\alpha-1}S_G(\eta)[z_0-g(z)]-\int_0^\eta(\eta-\delta)^{\alpha-1}S_G(\eta-\delta)H(\delta,z(\delta))d\delta \\ &-\int_0^\eta[(\eta-\delta)^{\alpha-1}S_G(\eta-\delta)R(\delta,z(\delta))-\phi(\delta)]dW(\delta). \end{aligned}$$

Theorem 4.1. *Suppose that conditions* $(C1),(C2),(A1)',(A2)',(A3),(A4)$ *and* $(A6)$ *hold. Then the FSES* (1.1) *is approximately controllable.*

Proof. From Theorem 3.2 we know that the FSES (1.1) admits one mild solution z_λ in D_ρ for some $\rho>0$ associated with $c^\lambda(t;z_\lambda)$. Then $\|z_\lambda\|_{H_{1-\alpha}}\leq\rho$ and

$$z_\lambda(t)=t^{\alpha-1}S_G(t)[z_0-g(z_\lambda)]+\int_0^t(t-s)^{\alpha-1}S_G(t-s)H(s,z_\lambda(s))ds$$

$$+\int_0^t (t-s)^{\alpha-1} S_G(t-s)\mathcal{L}c^\lambda(s;z_\lambda)ds$$
$$+\int_0^t (t-s)^{\alpha-1} S_G(t-s)R(s,z_\lambda(s))dW(s).$$

Since $\lambda(\lambda I+\pi_0^\eta)^{-1}=I-\pi_0^\eta(\lambda I+\pi_0^\eta)^{-1}$, it follows that

$$z_\lambda(\eta)=\Lambda-\lambda(\lambda I+\pi_0^\eta)^{-1}\mathcal{P}(z_\lambda). \tag{4.4}$$

Assumptions $(A1)'$ and $(A2)'$ imply that

$$E\|H(s,z_\lambda(s)\|^p+E\|R(s,z_\lambda(s)\|^p_{L_2^0}\leq N_1+N_2.$$

Hence $(H(s,z_\lambda(s)),R(s,z_\lambda(s)))$ possesses a subsequence, not relabeled, weakly converging to some $(H^*(s),R^*(s))$ in $V\times L_2^0$. By virtue of the compactness of $S_G(t),t\geq 0$, we can acquire that

$$(\eta-s)^{\alpha-1}S_G(\eta-s)H(s,z_\lambda(s))\to(\eta-s)^{\alpha-1}S_G(\eta-s)H^*(s)$$

and

$$(\eta-s)^{\alpha-1}S_G(\eta-s)R(s,z_\lambda(s))\to(\eta-s)^{\alpha-1}S_G(\eta-s)R^*(s)\quad a.e.\text{ on } J\times\Upsilon.$$

On the other hand, since

$$E\|(\eta-s)^{\alpha-1}S_G(\eta-s)\big[H(s,z_\lambda(s))-H^*(s)\big]\|^p\leq 2^pN_1\eta^{p(\alpha-1)}\left(\frac{M\|G^{-1}\|}{\Gamma(\alpha)}\right)^p$$

and

$$E\|(\eta-s)^{\alpha-1}S_G(\eta-s)\big[R(s,z_\lambda(s))-R^*(s)\big]\|^p_{L_2^0}\leq 2^pN_2\eta^{p(\alpha-1)}\left(\frac{M\|G^{-1}\|}{\Gamma(\alpha)}\right)^p,$$

we derive that

$$E\|\int_0^\eta(\eta-s)^{\alpha-1}S_G(\eta-s)[H(s,z_\lambda(s))-H^*(s)]ds\|^p\to 0$$

and

$$E\|\int_0^\eta(\eta-s)^{\alpha-1}S_G(\eta-s)[R(s,z_\lambda(s))-R^*(s)]dW(s)\|^p\to 0$$

as $\lambda\to 0^+$. Since g maps bounded subset of $H_{1-\alpha}(J,V)$ to bounded subset of V, we infer that $\eta^{\alpha-1}S_G(\eta)[z_0-g(z_\lambda)]$ tends to some g^* in V as $\lambda\to 0^+$ owing to the compactness of $S_G(t)$. Denote by

$$\begin{aligned}\vartheta:=&E\Lambda-g^*-\int_0^\eta(\eta-s)^{\alpha-1}S_G(\eta-s)H^*(s)ds\\&-\int_0^\eta(\eta-s)^{\alpha-1}S_G(\eta-s)[R^*(s)-\phi(s)]dW(s).\end{aligned}$$

Then

$$E\|\mathcal{P}(z_\lambda)-\vartheta\|^p\to 0\qquad(\lambda\to 0^+). \tag{4.5}$$

In view of (4.4), (4.5) and Lemmas 4.1–4.3, we have

$$\begin{aligned} E\|z_\lambda(\eta)-\Lambda\|^p &= E\|\lambda(\lambda I+\pi_0^\eta)^{-1}\mathcal{P}(z_\lambda)\|^p \\ &\le 2^{p-1}\|\lambda(\lambda I+\pi_0^\eta)^{-1}\|^p E\|\mathcal{P}(z_\lambda)-\vartheta\|^p \\ &\quad +2^{p-1}E\|\lambda(\lambda I+\pi_0^\eta)^{-1}\vartheta\|^p \\ &\to 0 \quad (\lambda\to 0^+). \end{aligned}$$

This fact yields the approximate controllability of the FSES (1.1). □

5. Application

Consider the Sobolev type fractional partial differential equation with stochastic term

$$\begin{cases} \quad {}^L D_t^{\frac{3}{4}}[(I-\frac{\partial^2}{\partial y^2})z(t,y)]-\frac{\partial^2}{\partial y^2}z(t,y) \\ =\dfrac{e^{-3t}\sin z(t,y)}{(30+t)(1+|z(t,y)|)}+\dfrac{e^{-5t}\cos z(t,y)}{(60+t)(1+|z(t,y)|)}\dfrac{dW(t)}{dt}+c(t),\ (t,y)\in(0,1]\times[0,\pi], \\ z(t,0)=z(t,\pi)=0, \quad t\in[0,1], \\ I_{0^+}^{1-\alpha}[(I-\frac{\partial^2}{\partial y^2})z(t,y)]|_{t=0}+\sqrt{t^{1-\alpha}|z(t,y)|+2}=z_0(y). \end{cases} \tag{5.1}$$

Let $V=U:=L^2[0,\pi]$. Denote by $D(A)=D(G):=\{z\in V: z,z'$ are absolutely continuous, $z''\in V$ and $z(0)=z(\pi)=0\}$. Define

$$Az=-\frac{\partial^2}{\partial y^2}z,\quad z\in D(A)$$

and

$$Gz=(I-\frac{\partial^2}{\partial y^2})z,\quad z\in D(G).$$

By [5,14], $A:D(A)\subset V\to V$, $G:D(G)\subset V\to V$ and

$$Az=\sum_{n=1}^{\infty}n^2\langle z,e_n\rangle e_n,\quad z\in D(A)$$

and

$$Gz=\sum_{n=1}^{\infty}(1+n^2)\langle z,e_n\rangle e_n,\quad z\in D(C),$$

where $e_n(y)=\sqrt{\frac{2}{\pi}}\sin(ny), n\in\mathbf{N}$ are eigenvectors of A corresponding to eigenvalues $\varsigma_n=n^2$. For any $z\in V$, one has

$$G^{-1}x=\sum_{n=1}^{\infty}\frac{1}{1+n^2}\langle z,e_n\rangle e_n,$$

$$-AG^{-1}x=\sum_{n=1}^{\infty}\frac{-n^2}{1+n^2}\langle z,e_n\rangle e_n$$

and

$$K(t)z=\sum_{n=1}^{\infty}e^{\frac{-n^2}{1+n^2}t}\langle z,e_n\rangle e_n==e^{-AG^{-1}t}z.$$

It is obvious that G^{-1} is compact and $\|G^{-1}\|\le 1$ and $\|K(t)\|\le 1$. Hence

$$S_G(t)=\frac{3}{4}\int_0^{\infty}G^{-1}\theta M_{\frac{3}{4}}(\theta)K(t^{\frac{3}{4}}\theta)d\theta,$$

where

$$M_{\frac{3}{4}}(\theta)=\sum_{n=1}^{\infty}\frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-\frac{3}{4}n)}.$$

Obviously,

$$\|S_G(t)z\|\le\frac{1}{\Gamma(\frac{3}{4})}\|z\|.$$

Let

$$H(t,z(t))(y)=\frac{e^{-3t}\sin z(t,y)}{(30+t)(1+|z(t,y)|)},$$
$$R(t,z(t))(y)=\frac{e^{-5t}\cos z(t,y)}{(60+t)(1+|z(t,y)|)}$$

and

$$g(z)(y)=\sqrt{t^{1-\alpha}|z(t,y)|+2}.$$

Then (5.1) can be rewritten as the abstract FSES (1.1). And conditions $(A1)'$, $(A2)'$, $(A3)$, $(A4)$ are satisfied with $N_1=\frac{1}{30^p},N_2=\frac{1}{60^p}$ where $p\ge 2$ is a fixed constant. In addition, if for $\forall\epsilon>0$ and $h\in L^p([0,1],V)$, there is $v_\epsilon\in L^p([0,1],U)$ satisfying

$$\|\int_0^1(1-s)^{-\frac{1}{4}}S_G(1-s)h(s)ds-\int_0^1(1-s)^{-\frac{1}{4}}S_G(1-s)v_\epsilon(s)ds\|\le\epsilon,$$

according to Theorem 4.1, (5.1) is approximately controllable.

Conclusion

In this manuscript, the existence as well as the approximate controllability of the Liouville-Riemann FSES (1.1) of sobolev type are considered. With the aid of assumptions $(C1)$ and $(C2)$, the compactness and boundedness of $\{S_G(t)\}_{t\ge 0}$ are achieved. By applying the compactness condition of linear operator $G^{-1}:V\to D(G)$, we easily acquire the relative compactness of the set $\Pi(t)$ for $t\in J$ without any compactness assumptions on g and the operator semigroup $K(t)$(see Lemma 3.3). Hence the achieved conclusions in our article extend and improve some existing studies.

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