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ON ITERATIVE POSITIVE SOLUTIONS FOR A CLASS OF SINGULAR INFINITE-POINT P-LAPLACIAN FRACTIONAL DIFFERENTIAL EQUATION WITH SINGULAR SOURCE TERMS*

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Abstract Based on properties of Green's function, the existence of unique positive solution for singular infinite-point p-Laplacian fractional differential system is established, moreover, an iterative sequence and convergence rate are given which are important for practical application, and an example is given to demonstrate the validity of our main results.

Keywords fractional differential, iterative positive solution, sequential techniques, mixed monotone operator, singular problem.

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1. Introduction

Fractional calculus have been shown to be more accurate and realistic than integer order models and it also provides an excellent tool to describe the hereditary properties of material and processes, particularly in viscoelasticity, electrochemistry, porous media, and so on. Fractional derivatives arise in a variety of different areas such as physics, chemistry, electrical networks, economics, rheology, biology chemical, image processing, and so on. There has been a significant development in the study of fractional differential equations in recent years, for an extensive collection

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of such literature, readers can refer to [1–5, 7–9, 9–11, 15, 17–21]. In [22], Zhang and Liu investigated the following infinite-point fractional differential equation:

$$D_{0+}^{\alpha}u(t) = f(t, x(t), D_{0^+}^{\alpha-2}u(t), D_{0^+}^{\alpha-1}u(t)),\ 0 < t < 1,$$

with infinite-point boundary condition

$$u(0) = 0, D_{0^+}^{\alpha-1}u(0) = \ldots = \sum_{j=1}^{\infty}\alpha_j u(\xi_j), u^{(i)}(1) = \sum_{i=1}^{\infty}\alpha_j u(\xi_j),$$

where $2 < \alpha \leq 3$, $f \in [0,1]\times\mathbb{R}^3 \to \mathbb{R}$ is a Caratheodory function, $\xi_i, \gamma_i \in (0,1)$ and $\{\xi_i\}_{i=1}^{+\infty}, \{\gamma_i\}_{i=1}^{+\infty}$ are two monotonic sequence with $\lim_{i\to}\xi_i = a, \lim_{i\to}\gamma_i = b, a, b \in (0,1), \alpha_i, \beta_i \in \mathbb{R}$, $D_{0+}^{\alpha}u$ is the standard Riemann-Liouville derivative. The authors established the existence of at least one solution for this equation by Mawhin's continuation theorem. Lucas [16]investigated the following p-Laplacian fractional differential equation

$$\begin{cases} D_{0+}^{\alpha_1}(\varphi_{r_1}(D_{0^+}^{\beta_1}u(t))) + \lambda f(t, u(t), v(t)) = 0,\ 0 < t < 1, \\ D_{0+}^{\alpha_2}(\varphi_{r_2}(D_{0^+}^{\beta_2}u(t))) + \mu g(t, u(t), v(t)) = 0,\ 0 < t < 1, \end{cases}$$

with p-point boundary condition

$$\begin{cases} u^{(j)}(0) = 0, j = 0, 1, 2, \ldots, n-2; D_{0^+}^{\beta_1}u(0) = 0, D_{0^+}^{p_1}u(1) = \displaystyle\sum_{i=1}^{N} a_i D_{0^+}^{q_1}u(\xi_i), \\ v^{(j)}(0) = 0, j = 0, 1, 2, \ldots, m-2; D_{0^+}^{\beta_1}v(0) = 0, D_{0^+}^{p_2}v(1) = \displaystyle\sum_{i=1}^{M} b_i D_{0^+}^{q_2}v(\eta_i), \end{cases}$$

where $p_1, p_2, q_1, q_2 \in \mathbb{R}, p_1 \in [1, n-2], p_2 \in [1, m-2]$, $q_1 \in [0, p_1]$, $q_2 \in [0, p_2]$,$\xi_i, a_i \in \mathbb{R}, i = 1, 2, \ldots, N(N \in \mathbb{N})$, $0 < \xi_1 < \ldots < \xi_N \leq 1$. The existence and nonexistence of positive solutions is obtained by Guo-Krasnosel'skii theorem.Jong [13] studied the following $p-$Laplacian fractional differential equations:

$$D_{0^+}^{\beta}\left(\varphi_p\left(D_{0+}^{\alpha}u\right)\right)(t) = f(t, u(t)), 0 < t < 1,$$

with m point boundary condition

$$u(0) = 0, D_{0^+}^{\gamma}u(1) = \sum_{i=1}^{m-2}\xi_i D_{0^+}^{\gamma}u(\eta_i),$$

$$D_{0+}^{\alpha}u(0) = 0, \varphi_p(D_{0+}^{\alpha}u(1)) = \sum_{i=1}^{m-2}\zeta_i\varphi_p(D_{0+}^{\alpha}u(\eta_i)),$$

where $1 < \alpha, \beta \leq 2, 3 < \alpha + \beta \leq 4, 0 < \gamma \leq 1, \alpha - \gamma - 1 > 0$, $0 < \eta_i, \zeta_i, \xi_i < 1(i = 1, 2, \ldots, \infty)$, $\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\gamma-1} < 1, \sum_{i=1}^{m-2}\zeta_i\eta_i^{\beta-1} < 1 < 1$, p-Laplacian operator φ_p is defined as $\varphi_p(s) = |s|^{p-2}s,\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$, and $f \in C([0,1]\times(0,+\infty), [0,+\infty))$. The authors obtained the existence and uniqueness of solutions by using the fixed point theorem for mixed monotone operators.

Motivated by the excellent results above, in this paper, we will devote to considering the following infinite-point singular p-Laplacian fractional differential equation:

$$\begin{gathered} D_{0+}^{\alpha}\left(\varphi_p\left(D_{0+}^{\gamma}u\right)\right)(t)+\lambda^{\frac{1}{q-1}}f(t,u(t),D_{0+}^{\mu_1}u(t),D_{0+}^{\mu_2}u(t),\cdots,D_{0+}^{\mu_{n-2}}u(t))=0,\\ 0<t<1, \end{gathered} \tag{1.1}$$

with boundary condition

$$\begin{aligned} &u^{(j)}(0)=0, j=1,2,\ldots,n-2; D_{0+}^{r_1}u(1)=\sum_{j=1}^{\infty}\eta_j D_{0+}^{r_2}u(\xi_j),\\ &D_{0+}^{\alpha}u(0)=0; \varphi_p(D_{0+}^{\alpha}u(1))=\sum_{i=1}^{\infty}\zeta_j\varphi_p(D_{0+}^{\alpha}u(\xi_j)), \end{aligned} \tag{1.2}$$

where $1<\alpha\leq 2$, $n-1<\gamma\leq n(n\geq 3)$, $r_1,r_2\in[2,n-2]$, $r_2\leq r_1$, p-Laplacian operator φ_p is defined as $\varphi_p(s)=|s|^{p-2}s$, $p,\ q>1$, $\frac{1}{p}+\frac{1}{q}=1, i-1<\mu_i\leq i(i=1,2,\ldots,n-2)$ and $0<\eta_i,\zeta_i,\xi_i<1(i=1,2,\ldots,\infty)$, $f\in C((0,1)\times(0,+\infty)^{n-1},\ \mathbb{R}_+^1))(\mathbb{R}_+^1=[0,+\infty)$ and $f(t,x_1,x_2,\ldots,x_{n-1})$ has singularity at $x_i=0\ (i=1,2,\ldots,n-1)$ and $t=0,\ 1$, $D_{0+}^{\alpha}u$, $D_{0+}^{\gamma}u$, $D_{0+}^{\mu_i}u(i=1,2,\ldots,n-2)$, $D_{0+}^{r_i}u(i=1,2)$ are the standard Riemann-Liouville derivative.

In this paper, we investigate the existence of positive solutions for a singular infinite-point p-Laplacian BVP(1.1,1.2). Compared with [22], the equation in this paper is p-Laplacian fractional differential equation and the method which we used in this paper is mixed monotone operator and Sequential techniques. Compared with [13], fractional derivatives are involved in the nonlinear terms for BVP(1.1,1.2) and value at infinite points are involved in the boundary conditions of the BVP(1.1,1.2).

2. Preliminaries and lemmas

Some basic definitions and lemmas about the theory of fractional calculus which are useful for the following research, reader can refer to the recent literature such as [11, 14, 19], we omit some definitions and properities of fractional calculus here.

Lemma 2.1. *Let $y\in L^1(0,1)\cap C(0,1)$, then the equation of the BVPs*

$$-D_{0+}^{\gamma}u(t)=y(t),\ 0<t<1, \tag{2.1}$$

with boundary condition (1.2) *has integral representation*

$$u(t)=\int_0^1 G(t,s)y(s)ds, \tag{2.2}$$

where

$$G(t,s)=\frac{1}{\Delta\Gamma(\gamma)}\begin{cases}\Gamma(\gamma)t^{\gamma-1}P(s)(1-s)^{\gamma-r_1-1}-\Delta(t-s)^{\gamma-1},\ 0\leq s\leq t\leq 1,\\ \Gamma(\gamma)t^{\gamma-1}P(s)(1-s)^{\gamma-r_1-1},\ 0\leq t\leq s\leq 1,\end{cases} \tag{2.3}$$

in which

$$P(s)=\frac{1}{\Gamma(\gamma-r_1)}-\frac{1}{\Gamma(\gamma-r_2)}\sum_{s\le\xi_j}\eta_j\left(\frac{\xi_j-s}{1-s}\right)^{\gamma-r_2-1}(1-s)^{r_1-r_2},$$

$$\Delta=\frac{\Gamma(\gamma)}{\Gamma(\gamma-r_1)}-\frac{\Gamma(\gamma)}{\Gamma(\gamma-r_2)}\sum_{j=1}^{\infty}\eta_j\xi_j^{\gamma-r_2-1}\neq 0.$$

Proof. First, we prove (2.3). By means of the definition of fractional differential integral, we can reduce (2.1) to an equivalent integral equation

$$u(t)=-I_{0+}^{\gamma}y(t)+C_1t^{\gamma-1}+C_2t^{\gamma-2}+\ldots+C_nt^{\gamma-n},$$

for $C_i\in\mathbb{R}$(i=1,2,...,n). From $u^{(j)}(0)=0(j=0,1,2,\ldots,n-2)$, we have $C_i=0(i=2,3,\ldots,n)$. Consequently, we get

$$u(t)=C_1t^{\gamma-1}-I_{0+}^{\gamma}y(t).$$

By some properties of the fractional integrals and fractional derivatives, we have

$$\begin{aligned} D_{0+}^{r_1}u(t)&=C_1\frac{\Gamma(\gamma)}{\Gamma(\gamma-r_1)}t^{\gamma-r_1-1}-I_{0+}^{\gamma-r_1}y(t),\\ D_{0+}^{r_2}u(t)&=C_1\frac{\Gamma(\gamma)}{\Gamma(\gamma-r_2)}t^{\gamma-r_2-1}-I_{0+}^{\gamma-r_2}y(t). \end{aligned}\tag{2.4}$$

On the other hand, $D_{0+}^{r_1}u(1)=\sum_{j=1}^{\infty}\eta_jD_{0+}^{r_2}u(\xi_j)$ combining with (2.4), we get

$$\begin{aligned} C_1&=\int_0^1\frac{(1-s)^{\gamma-r_1-1}}{\Gamma(\gamma-r_1)\Delta}y(s)ds-\sum_{j=1}^{\infty}\eta_j\int_0^{\xi_j}\frac{(\xi_j-s)^{\gamma-r_2-1}}{\Gamma(\gamma-r_2)\Delta}y(s)ds\\ &=\int_0^1\frac{(1-s)^{\gamma-r_1-1}P(s)}{\Delta}y(s)ds, \end{aligned}$$

where

$$P(s)=\frac{1}{\Gamma(\gamma-r_1)}-\frac{1}{\Gamma(\gamma-r_2)}\sum_{s\le\xi_j}\eta_j\left(\frac{\xi_j-s}{1-s}\right)^{\gamma-r_2-1}(1-s)^{r_1-r_2},$$

$$\Delta=\frac{\Gamma(\gamma)}{\Gamma(\gamma-r_1)}-\frac{\Gamma(\gamma)}{\Gamma(\gamma-r_2)}\sum_{j=1}^{\infty}\eta_j\xi_j^{\gamma-r_2-1}.$$

Hence,

$$\begin{aligned} u(t)&=C_1t^{\gamma-1}-I_{0+}^{\gamma}y(t)\\ &=-\int_0^t\frac{\Delta(t-s)^{\gamma-1}}{\Gamma(\gamma)\Delta}y(s)ds+\int_0^1\frac{(1-s)^{\gamma-r_1-1}t^{\gamma-1}P(s)}{\Delta}y(s)ds. \end{aligned}$$

Therefore, we get (2.3).

Moreover, by (2.3), for $i=1,2,\ldots,n-2$, we have

$$\begin{aligned} &D_{0+}^{\mu_i}G(t,s)\\ =&\frac{1}{\Delta\Gamma(\gamma-\mu_i)}\begin{cases}\Gamma(\gamma)t^{\gamma-\mu_i-1}P(s)(1-s)^{\gamma-r_1-1}-\Delta(t-s)^{\gamma-\mu_i-1},\ 0\le s\le t\le 1,\\ \Gamma(\gamma)t^{\gamma-\mu_i-1}P(s)(1-s)^{\gamma-r_1-1},\ 0\le t\le s\le 1.\end{cases} \end{aligned}\tag{$*$}$$

□

Lemma 2.2. *Let $f \in C((0,1] \times (0,+\infty)^{n-1}, [0,+\infty)$, then the BVP (1.1,1.2) has a unique solution*

$$u(t) = \lambda \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau, u(\tau), D_{0^+}^{\mu_1}u(\tau), D_{0^+}^{\mu_2}u(\tau), \cdots, D_{0^+}^{\mu_{n-2}}u(\tau))d\tau\right)ds, \tag{2.5}$$

where

$$H(t,s) = H_1(t,s) + H_2(t,s), \tag{2.6}$$

in which

$$H_1(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, \ 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, \ 0 \le t \le s \le 1, \end{cases} \tag{2.7}$$

$$H_2(t,s) = \frac{t^{\alpha-1}}{\overline{\Delta}\Gamma(\alpha)}\left[\sum_{\xi_j > s} \zeta_j \left[\xi_j^{\alpha-1}(1-s)^{\alpha-1} - (\xi_j - s)^{\alpha-1}\right] + \sum_{s \ge \xi_j} \zeta_j \xi_j^{\alpha-1}(1-s)^{\alpha-1}\right], \ t,s \in [0,1], \tag{2.8}$$

in which

$$\overline{\Delta} = 1 - \sum_{i=1}^{\infty} \zeta_i \xi_i^{\alpha-1}.$$

Easily, we have

$$H_2(t,s) = \frac{1}{\overline{\Delta}} \sum_{i=1}^{\infty} \zeta_i H_1(\xi_i, s) \cdot t^{\alpha-1}. \tag{2.9}$$

Proof. The proof is the similar to Lemma 2.2 of [13], we omit it here. □

Lemma 2.3. *Let $\Delta, \overline{\Delta} > 0$, then the Green functions defined by (2.3) satisfies:*
(1) *$G : [0,1] \times [0,1] \to \mathbb{R}_+^1$ is continuous and $G(t,s) > 0$, for all $t,s \in (0,1)$;*
(2)

$$\frac{1}{\Gamma(\gamma)} t^{\gamma-1} j(s) \le G(t,s) \le a^\star t^{\gamma-1}, \qquad t,s \in [0,1], \tag{2.10}$$

$$\frac{1}{\overline{\Delta}} \bar{j}(s) \le H(t,s) \le b^\star t^{\alpha-1}, \qquad t,s \in [0,1], \tag{2.11}$$

where

$$j(s) = (1-s)^{\gamma - r_1 - 1}[1 - (1-s)^{r_1}], \qquad \bar{j}(s) = \sum_{i=1}^{\infty} \zeta_i H_1(\eta_i, s),$$

$$b^\star = \frac{1}{\overline{\Delta}\Gamma(\alpha)}\left(1 + \sum_{i=1}^{\infty} \zeta_i \left(1 - \xi_i^{\alpha-1}\right)\right), \qquad a^\star = \frac{1}{\Delta\Gamma(\gamma - r_1)},$$

in which Δ is defined as in Lemma 2.1 and $\overline{\Delta}$ is defined as in Lemma 2.2.

Proof. Let

$$G_\star(t,s) = \frac{1}{\Gamma(\gamma)}\begin{cases} t^{\gamma-1}(1-s)^{\gamma-r_1-1} - (t-s)^{\gamma-1}, \ 0 \le s \le t \le 1, \\ t^{\gamma-1}(1-s)^{\gamma-r_1-1}, \ 0 \le t \le s \le 1. \end{cases}$$

From [12], for $r_1 \in [2, n-2]$, we have

$$0 \leq t^{\gamma-1}(1-s)^{\gamma-r_1-1}[1-(1-s)^{r_1}] \leq \Gamma(\alpha)G_\star(t,s) \leq t^{\gamma-1}(1-s)^{\gamma-r_1-1}. \quad (2.12)$$

By direct calculation, we get $P'(s) \geq 0, \ \ s \in [0,1]$, and so $P(s)$ is nondecreasing with respect to s. For $r_2 \leq r_1$, $r_1, r_2 \in [2, n-2]$, $s \in [0,1]$, we get

$$\begin{aligned}\Gamma(\gamma)P(s) = & \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_1)} - \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_2)}\sum_{s\leq\xi_j}\eta_j\left(\frac{\xi_j-s}{1-s}\right)^{\gamma-r_2-1}(1-s)^{r_1-r_2} \\ & \geq \Gamma(\gamma)P(0) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_1)} - \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_2)}\sum\eta_j\xi_j^{\gamma-r_2-1} = \Delta.\end{aligned} \quad (2.13)$$

By (2.3) and (2.13), we have

$$\Delta\Gamma(\gamma)G(t,s) \geq \begin{cases}\Delta t^{\gamma-1}(1-s)^{\gamma-r_1-1} - \Delta(t-s)^{\gamma-1}, \ 0 \leq s \leq t \leq 1, \\ \Delta t^{\gamma-1}(1-s)^{\gamma-r_1-1}, \ 0 \leq t \leq s \leq 1.\end{cases} \quad (2.14)$$

So, by (2.12) and (2.14), we have

$$\begin{aligned}\Delta\Gamma(\gamma)G(t,s) \geq & \Delta\Gamma(\gamma)G_\star(t,s) \\ \geq & \Delta t^{\gamma-1}(1-s)^{\gamma-r_1-1}[1-(1-s)^{r_1}],\end{aligned} \quad (2.15)$$

hence,

$$G(t,s) \geq \frac{1}{\Gamma(\gamma)}t^{\gamma-1}j(s).$$

On the other hand,

$$P(s) = \frac{1}{\Gamma(\gamma-r_1)} - \frac{1}{\Gamma(\gamma-r_2)}\sum_{s\leq\xi_j}\eta_j\left(\frac{\xi_j-s}{1-s}\right)^{\gamma-r_2-1}(1-s)^{r_1-r_2} \leq \frac{1}{\Gamma(\gamma-r_1)},$$

clearly,

$$\Delta\Gamma(\gamma)G(t,s) \leq \Gamma(\gamma)t^{\gamma-1}P(s)(1-s)^{\gamma-r_1-1},$$

hence,

$$G(t,s) \leq a^\star t^{\gamma-1}.$$

So the proof of (2.10) is completed, and now we will proof of 2.11.

Since $H_1(t,s) \geq 0$ for all $(t,s) \in [0,1]\times[0,1]$, we have

$$G(t,s) \geq H_2(t,s) = \frac{\overline{j}(s)}{\overline{\Delta}}t^{\alpha-1}.$$

On the other hand, we get

$$\begin{aligned}H(t,s) &= H_1(t,s) + \frac{1}{\Delta}\sum_{i=1}^{\infty}\zeta_i H_1(\xi_i,s)\cdot t^{\alpha-1} \\ &\leq \frac{1}{\Gamma(\alpha)}t^{\alpha-1} + \frac{1}{\overline{\Delta}\Gamma(\alpha)}\sum_{i=1}^{\infty}\zeta_i(1-\xi_i^{\alpha-1}) \\ &= \frac{1}{\overline{\Delta}\Gamma(\alpha)}\left(1+\sum_{i=1}^{\infty}\zeta_i(1-\xi_i^{\alpha-1})\right)\cdot t^{\alpha-1} = b^\star t^{\alpha-1}.\end{aligned}$$

□

Remark 2.1. The main idea of proof of (2.11) comes from [13].

Lemma 2.4. *For any* $(t,s)\in[0,1]\times[0,1]$*, the following inequalities hold:*

$$G(t,s)\le\frac{1}{\Delta\Gamma(\gamma-r_1)}(1-s)^{\gamma-r_1-1},$$
$$H(t,s)\le\frac{1}{\Delta\Gamma(\alpha)}(1-s)^{\alpha-1}.$$

Proof. From Lemma 2.1 and Lemma 2.2, we easily complete this proof.

Let P be a normal cone of a Banach space E, and $e\in P$, $e>\theta$, where θ is a zero element of E. Define a component of P by $Q_e=\{u\in P|$ there exists a constant $C>0$ such that $\frac{1}{C}e\preceq u\preceq Ce\}$. $A:Q_e\times Q_e\to P$ is said to be mixed monotone if $A(u,y)$ is non-decreasing in u and non-increasing in y, i.e., $u_1\preceq u_2$ $(u_1,u_2\in Q_e)$ implies $A(u_1,y)\preceq A(u_2,y)$ for any $y\in Q_e$, and $y_1\preceq y_2$ $(y_1,y_2\in Q_e)$ implies $A(u,y_1)\succeq A(u,y_2)$ for any $u\in Q_e$. The element $u^\star\in Q_e$ is called a fixed point of A if $A(u^\star,u^\star)=u^\star$. □

Lemma 2.5 ([6]). *Let* E *is a Banach space and* P *be a normal cone of a Banach space* E*. Suppose that* $A:Q_e\times Q_e\to Q_e$ *is a mixed monotone operator and there exists a constant* σ*,* $0<\sigma<1$*, such that*

$$A\left(lx,\frac{1}{l}y\right)\succeq l^\sigma A(x,y),\ \ x,\ \ y\in Q_e,\ \ 0<l<1, \tag{2.16}$$

then A *has a unique fixed point* $x^\star\in Q_e$*, and for any* $x_0\in Q_e$*, we have*

$$\lim_{k\to\infty}x_k=x^\star,$$

where

$$x_k=A(x_{k-1},x_{k-1}),k=1,2,\cdots,$$

and the convergence rate is

$$\|x_k-x^\star\|=o(1-r^{\sigma^k}),$$

where r *is a constant,* $0<r<1$*, and dependent on* x_0*.*

3. Main result

In this section, we will prove the existence of positive solutions for the BVP(1.1,1.2) by the method of sequential technique.

Let

$$E=\left\{u|u\in C[0,1],D_{0^+}^{\mu_i}u\in C[0,1],i=1,2,\dots,n-2\right\} \tag{2.23}$$

is a Banach space with the norm

$$\|u\|=\max\left\{\max_{t\in[0,1]}|u(t)|,\max_{t\in[0,1]}|D_{0^+}^{\mu_i}u(t)|,i=1,2,\dots,n-2\right\}.$$

Moreover, we define a cone of E by

$$P=\left\{u\in E:u(t)\ge0,D_{0^+}^{\mu_i}u(t)\ge0,\ t\in[0,1],\ i=1,2,\dots,n-2\right\},$$

clearly, P is a normal cone, and E is endowed with an order relation $u \preceq v$ if and only if $u(t) \leq v(t), D_{0^+}^{\mu_i} u(t) \leq D_{0^+}^{\mu_i} v(t), (i = 1, 2, \dots, n-2), t \in [0, 1]$. Let $e(t) = t^{\gamma-1}$ for $t \in [0, 1]$, also define a component of P by

$$Q_e = \left\{ u \in P: \text{ there exists } M \geq 1, \ \frac{1}{M} e(t) \leq u(t) \leq M e(t), \ t \in [0, 1] \right\}.$$

In order to establish the existence of positive solution for system (1.1,1.2), we shall consider the following problem:

$$\begin{cases} D_{0^+}^{\alpha} \left(\varphi_p \left(D_{0+}^{\gamma} u \right) \right)(t) + \lambda^{\frac{1}{q-1}} f(t, u(t) + \frac{1}{k}, D_{0^+}^{\mu_1} u(t) + \frac{1}{k}, \\ D_{0^+}^{\mu_2} u(t) + \frac{1}{k}, \cdots, D_{0^+}^{\mu_{n-2}} u(t) + \frac{1}{k}) = 0, 0 < t < 1, \\ u^{(j)}(0) = 0, j = 1, 2, \dots, n-2; D_{0^+}^{r_1} u(1) = \displaystyle\sum_{j=1}^{\infty} \eta_j D_{0^+}^{r_2} u(\xi_j), \\ D_{0^+}^{\alpha} u(0) = 0; \varphi_p(D_{0^+}^{\gamma} u(1)) = \displaystyle\sum_{i=1}^{\infty} \zeta_i \varphi_p(D_{0^+}^{\gamma} u(\eta_i)), \end{cases} \tag{3.1}$$

where $t \in (0, 1), k \in \{2, 3, \cdots\}$. Assume that $f : [0, 1] \times (\mathbb{R}^1 \setminus \{0\})^n \to \mathbb{R}_+^1$ is continuous.

Lemma 3.1. *u is a solution of system (3.1) if and only if $u \in C[0, 1]$ is a solution of the following nonlinear integral equation system (3.2):*

$$\begin{aligned} u(t) = \lambda \int_0^1 G(t, s) \varphi_q \left(\int_0^1 H(s, \tau) f \left(\tau, u(\tau) + \frac{1}{k}, D_{0^+}^{\mu_1} u(\tau) + \frac{1}{k}, \right. \right. \\ \left. \left. D_{0^+}^{\mu_2} u(\tau) + \frac{1}{k}, \cdots, D_{0^+}^{\mu_{n-2}} u(\tau)) d\tau + \frac{1}{k} \right) d\tau \right) ds. \end{aligned} \tag{3.2}$$

Throughout this paper, we always assume the following conditions hold.

(S_1) $f(t, x_1, x_2, \dots, x_{n-1}) = \phi(t, x_1, x_2, \dots, x_{n-1}) + \psi(t, x_1, x_2, \dots, x_{n-1})$, where $\phi : (0, 1) \times (0, +\infty)^n \to \mathbb{R}_+^1$ is continuous, $\phi(t, x_1, x_2, \dots, x_{n-1})$ may be singular at $t = 0, 1$, and is nondecreasing on $x_i > 0$ $(i = 1, 2, \dots, n)$. $\psi : (0, 1) \times (0, +\infty)^{n-1} \to \mathbb{R}_+^1$ is continuous, $\psi(t, x_1, x_2, \dots, x_{n-1})$ may be singular at $t = 0, 1$, $x_i = 0$ $(i = 1, 2, \dots, n-1)$ and is nonincreasing on $x_i > 0$ $(i = 1, 2, \dots, n-1)$.

(S_2) There exists $0 < \sigma < 1$ such that, for all $x_i > 0$ $(i = 1, 2, \dots, n-1)$, and $t, l \in (0, 1)$,

$$\begin{aligned} \phi(t, lx_1, lx_2, \dots, lx_{n-1}) &\geq l^{\sigma^{\frac{1}{q-1}}} \phi(t, x_1, x_2, \dots, x_{n-1}), \\ \psi(t, l^{-1}x_1, l^{-1}x_2, \dots, l^{-1}x_{n-1}) &\geq l^{\sigma^{\frac{1}{q-1}}} \psi(t, x_1, x_2, \dots, x_{n-1}), \end{aligned}$$

where q is defined by (1.1).

(S_3)

$$\begin{aligned} 0 &< \int_0^1 \phi(\tau, 1, 1, \dots, 1) d\tau < +\infty, \\ 0 &< \int_0^1 \tau^{-(\gamma-1)\sigma^{\frac{1}{q-1}}} \psi(\tau, 1, 1, \dots, 1) d\tau < +\infty. \end{aligned}$$

Remark 1.1. *According to* (S_2) *and* (S_3)*, for all* $x_i > 0$ $(i = 1, 2, \dots, n-1)$*,* $\sigma, t \in (0,1)$*, and* $l \geq 1$*, we have*

$$\phi(t, lx_1, lx_2, \dots, lx_{n-1}) \leq l^{\sigma^{\frac{1}{q-1}}} \phi(t, x_1, x_2, \dots, x_{n-1}),$$
$$\psi(t, l^{-1}x_1, l^{-1}x_2, \dots, l^{-1}x_{n-1}) \leq l^{\sigma^{\frac{1}{q-1}}} \psi(t, x_1, x_2, \dots, x_{n-1}),$$

where $q_i (i = 1, 2)$ *is defined by* (1.1)*.*

Now we give the following Theorem.

Theorem 3.1. *Suppose that* (S_1)*-*(S_4) *hold. Then the PFDE* (1.1, 1.2) *has a unique positive solution* $(u^\star, v^\star)$*, for all* $t \in [0,1]$*, which satisfies*

$$\frac{1}{M} t^{\gamma-1} \leq u^\star(t) \leq M t^{\gamma-1}.$$

Moreover, for any $u_0 \in Q_e$*, constructing a successively sequence:*

$$\begin{aligned} u_{k+1}(t) =& \lambda \int_0^1 G(t,s) \varphi_q \bigg(\int_0^1 H(s,\tau) \bigg(\phi\bigg(\tau, u_k(\tau) + \frac{1}{k}, D_{0^+}^{\mu_1} u_k(\tau) + \frac{1}{k}, \cdots, \\ & D_{0^+}^{\mu_{n-2}} u_k(\tau) + \frac{1}{k} \bigg) + \psi((t, u_k(\tau) + \frac{1}{k}, D_{0^+}^{\mu_1} u_k(\tau) + \frac{1}{k}, \cdots, \\ & D_{0^+}^{\mu_{n-2}} u_k(\tau)) d\tau + \frac{1}{k} \bigg) d\tau \bigg) ds, \end{aligned} \tag{3.3}$$

and we have $\|u_k - u^\star\| \to 0$ *as* $k \to \infty$*, the convergence rate is* $\|u_k - u^\star\| = o\left(1 - r^{\sigma^k}\right)$*, where* r *is a constant,* $0 < r < 1$*, and dependent on* u_0*.*

Proof. We first consider the existence of a positive solution to problem (3.2). From the discussion in Section 2, we only need to consider the existence of a positive solution to BVP (3.2). In order to realize this purpose, define the operator $T_k : Q_e \times Q_e \to P$ by

$$\begin{aligned} T_k(u,v)(t) =& \lambda \int_0^1 G(t,s) \varphi_q \bigg(\int_0^1 H(s,\tau) \bigg(\phi\bigg(\tau, u(\tau) + \frac{1}{k}, D_{0^+}^{\mu_1} u(\tau) + \frac{1}{k}, \cdots, \\ & D_{0^+}^{\mu_{n-2}} u(\tau) + \frac{1}{k} \bigg) + \psi((\tau, v(\tau) + \frac{1}{k}, D_{0^+}^{\mu_1} v(\tau) + \frac{1}{k}, \cdots, \\ & D_{0^+}^{\mu_{n-2}} v(\tau)) d\tau + \frac{1}{k} \bigg) d\tau \bigg) ds. \end{aligned} \tag{3.4}$$

Now we prove that $T_k : Q_e \times Q_e \to P$ is well defined. For any $u, v \in Q_e$, From (S_1) and Remark 1.1, we have

$$\begin{aligned} & \phi(\tau, u(\tau), D_{0^+}^{\mu_1} u(\tau), \dots, D_{0^+}^{\mu_{n-2}} u(\tau)) \\ \leq & \phi\left(\tau, Me(\tau), D_{0^+}^{\mu_1} Me(\tau), \dots, D_{0^+}^{\mu_{n-2}} Me(\tau)\right) \\ \leq & \phi(\tau, \dots, Mb+1, Mb+1) \\ \leq & (Mb+1)^{\sigma^{\frac{1}{q-1}}} \phi(\tau, 1, \dots, 1) \\ \leq & (2bM)^{\sigma^{\frac{1}{q_1-1}}} \phi(\tau, 1, \dots, 1), \ \tau \in (0,1), \end{aligned} \tag{3.5}$$

where

$$M > \max\left\{ \left(\frac{\lambda\varphi_q(e)}{\Delta\Gamma(\gamma-r_1+1)}\right)^{\frac{1}{1-\sigma}}, \left(\frac{\lambda\varphi_q(\overline{e}}{\Gamma(\alpha)}\int_0^1 j(s)s^{(q-1)(\alpha-1)}ds)\right)^{-\frac{1}{1-\sigma}}, \right. \tag{3.6}$$
$$\left. 1, 2c, b^{-1}\right\},$$

in which

$$e = \frac{1}{\Delta\Gamma(\alpha)}(2b)^{\sigma^{\frac{1}{q-1}}}\int_0^1 \phi(\tau,1,\ldots,1)d\tau + c^{-\sigma^{\frac{1}{q-1}}}\int_0^1 \tau^{-(\gamma-1)\sigma^{\frac{1}{q-1}}}\psi(\tau,1,\ldots,1)d\tau,$$
$$\overline{e} = \int_0^1 \frac{\overline{j}(\tau)}{\overline{\Delta}}\left(c^{\sigma^{\frac{1}{q-1}}}\tau^{(\gamma-1)\sigma^{\frac{1}{q-1}}}\phi(\tau,1,\ldots,1) + (2b)^{-\sigma^{\frac{1}{q-1}}}\psi(\tau,1,\ldots,1)\right)d\tau,$$
$$b = \max\left\{\frac{\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma-\mu_{n-3})}, 1\right\}, c = \min\left\{\frac{\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma)}, 1,\right\}.$$

where $M \geq 1, b, c$ are positive constants. By (S_1) and (S_2), we also have

$$\begin{aligned}
&\psi(\tau, u(\tau), D_{0^+}^{\mu_1}u(\tau),\ldots,D_{0^+}^{\mu_{n-2}}u(\tau))\\
\leq&\psi\left(\tau, \frac{1}{M}\tau^{\gamma-1}, \frac{\Gamma(\gamma)}{M\Gamma(\gamma-\mu_1)}\tau^{\gamma-1-\mu_1},\ldots,\frac{\Gamma(\gamma)}{M\Gamma(\gamma-\mu_{n-2})}\tau^{\gamma-1-\mu_{n-2}}\right)\\
\leq&\psi\left(\tau, \frac{c}{M}\tau^{\gamma-1},\ldots,\frac{c}{M}\tau^{\gamma-1}\right)\\
\leq&\left(\frac{c}{M}\tau^{\gamma-1}\right)^{-\sigma^{\frac{1}{q-1}}}\psi(\tau,1,\ldots,1), \tau\in(0,1),
\end{aligned} \tag{3.7}$$

where c is a positive constant and is the same as above. Noting $\frac{c}{M}\tau^{\gamma-1} < 1$, and by (S_1) and (S_2), we have

$$\begin{aligned}
&\phi(\tau, u(\tau), D_{0^+}^{\mu_1}u(\tau),\ldots,D_{0^+}^{\mu_{n-2}}u(\tau))\\
\geq&\phi\left(\tau, \frac{1}{M}\tau^{\gamma-1}, \frac{\Gamma(\gamma)}{M\Gamma(\gamma-\mu_1)}\tau^{\gamma-1-\mu_1},\ldots,\frac{\Gamma(\gamma)}{M\Gamma(\gamma-\mu_{n-2})}\tau^{\gamma-1-\mu_{n-2}}\right)\\
\geq&\phi(\tau, \frac{c}{M}\tau^{\gamma-1},\ldots,\frac{c}{M}\tau^{\gamma-1})\\
\geq&(\frac{c}{M}\tau^{\gamma-1})^{\sigma^{\frac{1}{q-1}}}\phi(\tau,1,\ldots,1)\\
=&c^{\sigma^{\frac{1}{q-1}}}M^{-\sigma^{\frac{1}{q-1}}}\tau^{(\gamma-1)\sigma^{\frac{1}{q-1}}}\phi(\tau,1,\ldots,1),\ \tau\in(0,1).
\end{aligned} \tag{3.8}$$

By (S_1) and Remark 1.1, we also get

$$\begin{aligned}
&\psi(\tau, u(\tau), D_{0^+}^{\mu_1}u(\tau),\ldots,D_{0^+}^{\mu_{n-2}}u(\tau))\\
\geq&\psi\left(\tau, Me(\tau), D_{0^+}^{\mu_1}Me(\tau),\ldots,D_{0^+}^{\mu_{n-2}}Me(\tau)\right)\\
\geq&\psi\left(\tau, Mb\tau^{\gamma-1}+1, Mb\tau^{\gamma-\mu_1-1}+1,\ldots,Mb\tau^{\gamma-\mu_{n-2}-1}+1\right)\\
\geq&\psi(\tau, Mb+1, Mb+1,\cdots,Mb+1)\\
\geq&(Mb+1)^{-\sigma^{\frac{1}{q-1}}}\psi(\tau,1,\ldots,1)\\
\geq&2^{-\sigma^{\frac{1}{q-1}}}b^{-\sigma^{\frac{1}{q-1}}}M^{-\sigma^{\frac{1}{q-1}}}\psi(\tau,1,\ldots,1),\ \tau\in(0,1).
\end{aligned} \tag{3.9}$$

For any $u, v \in Q_e$, it follows from (3.5), (3.7) that

$$\begin{aligned}
&T_k(u,v)(t)\\
=&\lambda \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)\left(\phi\left(\tau, u(\tau)+\frac{1}{k}, D_{0^+}^{\mu_1}u(\tau)+\frac{1}{k},\cdots, D_{0^+}^{\mu_{n-1}}u(\tau)\right.\right.\right.\\
&\left.\left.\left.+\frac{1}{k}\right)+\psi((\tau, v(\tau)+\frac{1}{k}, D_{0^+}^{\mu_1}v(\tau)+\frac{1}{k},\cdots, D_{0^+}^{\mu_{n-1}}v(\tau))d\tau+\frac{1}{k}\right)d\tau\right)ds\\
\le&\lambda \int_0^1 \left(\frac{1}{\Delta\Gamma(\gamma-r_1)}(1-s)^{\gamma-r_1-1}\varphi_q\left(\int_0^1\left(\frac{1}{\overline{\Delta}\Gamma(\alpha)}(2bM)^{\sigma^{\frac{1}{q-1}}}\phi(\tau,1,\ldots,1)\right.\right.\right.\\
&\left.\left.\left.+\left(\frac{M}{c}\right)^{\sigma^{\frac{1}{q-1}}}\tau^{-(\gamma-1)\sigma^{\frac{1}{q-1}}}\psi(\tau,1,\ldots,1)\right)d\tau\right)ds\right.\\
=&\lambda \int_0^1 \left(\frac{1}{\Delta\Gamma(\gamma-r_1)}(1-s)^{\gamma-r_1-1}\varphi_q\left(\frac{1}{\overline{\Delta}\Gamma(\alpha)}(2bM)^{\sigma^{\frac{1}{q-1}}}\int_0^1\phi(\tau,1,\ldots,1)d\tau\right.\right.\\
&\left.\left.+\left(\frac{M}{c}\right)^{\sigma^{\frac{1}{q-1}}}\int_0^1\tau^{-(\gamma-1)\sigma^{\frac{1}{q-1}}}\psi(\tau,1,\ldots,1)d\tau\right)ds\right.\\
<&+\infty,\ t\in[0,1].
\end{aligned} \tag{3.10}$$

By (S_4), (3.10), we have that $T_k : Q_e \times Q_e \to P$ is well defined. Next, we will prove $T_k : Q_e \times Q_e \to Q_e$. The formula (3.10) implies that

$$T_k(x,z)(t) \le Mt^{\gamma-1} = Me(t),\ t\in[0,1]. \tag{3.11}$$

At the same time, by (3.8) and (3.9), we have

$$\begin{aligned}
&T_k(u,v)(t)\\
=&\lambda \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)\left(\phi\left(\tau, u(\tau)+\frac{1}{k}, D_{0^+}^{\mu_1}u(\tau)+\frac{1}{k},\cdots, D_{0^+}^{\mu_{n-1}}u(\tau)\right.\right.\right.\\
&\left.\left.\left.+\frac{1}{k}\right)+\psi((\tau, v(\tau)+\frac{1}{k}, D_{0^+}^{\mu_1}v(\tau)+\frac{1}{k},\cdots, D_{0^+}^{\mu_{n-1}}v(\tau))d\tau+\frac{1}{k}\right)d\tau\right)ds\\
\ge&\lambda \int_0^1 \frac{1}{\Gamma(\gamma)}t^{\gamma-1}j(s)\varphi_q\left(\int_0^1\frac{\overline{j}(\tau)}{\overline{\Delta}}s^{\alpha-1}\left(\left(\frac{c}{M}\right)^{\sigma^{\frac{1}{q-1}}}\tau^{(\gamma-1)\sigma^{\frac{1}{q-1}}}\phi(\tau,1,\ldots,1)\right.\right.\\
&\left.\left.+(2bM)^{-\sigma^{\frac{1}{q-1}}}\psi(\tau,1,\ldots,1)\right)d\tau\right)ds\\
=&t^{\gamma-1}\cdot\frac{\lambda}{\Gamma(\gamma)}\int_0^1 j(s)s^{(q-1)(\alpha-1)}ds\varphi_q\left(\int_0^1\frac{\overline{j}(\tau)}{\overline{\Delta}}s^{\alpha-1}\left(\left(\frac{c}{M}\right)^{\sigma^{\frac{1}{q-1}}}\tau^{(\gamma-1)\sigma^{\frac{1}{q-1}}}\right.\right.\\
&\left.\left.\times\phi(\tau,1,\ldots,1)+(2bM)^{-\sigma^{\frac{1}{q-1}}}\psi(\tau,1,\ldots,1)\right)ds.\right.
\end{aligned} \tag{3.12}$$

The formula (3.12) implies that

$$T_k(u,v)(t) \ge \frac{1}{M}t^{\gamma-1} = \frac{1}{M}e(t),\ t\in[0,1]. \tag{3.13}$$

Hence, $T_k : Q_e \times Q_e \to Q_e$, moreover, by (S_1), T_k is non-decreasing in u and non-increasing in v, hence, $T_k : Q_e \times Q_e \to Q_e$ is a mixed monotone operator.

Finally, we show that the operator T_k satisfies (2.16). For any $u, v \in Q_e$ and $l \in (0,1)$, by (S_2) and Remark 1.1, for all $t \in [0,1]$, we have

$$\begin{aligned}
&\lambda \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)\left(\phi\left(\tau, lu(\tau)+\frac{1}{k}, D_{0^+}^{\mu_1} lu(\tau)+\frac{1}{k}, \cdots, D_{0^+}^{\mu_{n-2}} lu(\tau)\right.\right.\right.\\
&\left.\left.\left.+\frac{1}{k}\right) + \psi((\tau, \frac{1}{l}v(\tau)+\frac{1}{k}, D_{0^+}^{\mu_1}\frac{1}{l}v(\tau)+\frac{1}{k}, \cdots, D_{0^+}^{\mu_{n-1}}\frac{1}{l}v(\tau))d\tau + \frac{1}{k}\right)d\tau\right)ds,\\
\geq&\lambda \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) l^{\sigma^{\frac{1}{q-1}}}\left(\phi\left(\tau, u(\tau)+\frac{1}{k}, D_{0^+}^{\mu_1} u(\tau)+\frac{1}{k}, \cdots, D_{0^+}^{\mu_{n-2}} u(\tau)\right.\right.\right.\\
&\left.\left.\left.+\frac{1}{k}\right) + \psi((\tau, v(\tau)+\frac{1}{k}, D_{0^+}^{\mu_1} v(\tau)+\frac{1}{k}, \cdots, D_{0^+}^{\mu_{n-2}} v(\tau))d\tau + \frac{1}{k}\right)d\tau\right)ds,\\
\geq&l^{\sigma} \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)\left(\phi\left(\tau, u(\tau)+\frac{1}{k}, D_{0^+}^{\mu_1} u(\tau)+\frac{1}{k}, \cdots, D_{0^+}^{\mu_{n-2}} u(\tau)\right.\right.\right.\\
&\left.\left.\left.+\frac{1}{k}\right) + \psi((\tau, v(\tau)+\frac{1}{k}, D_{0^+}^{\mu_1} v(\tau)+\frac{1}{k}, \cdots, D_{0^+}^{\mu_{n-2}} v(\tau))d\tau + \frac{1}{k}\right)d\tau\right)ds.
\end{aligned} \tag{3.14}$$

The formula (3.14) implies that

$$T_k\left(lu, \frac{1}{l}v\right) \geq l^{\sigma} T_k(u,v),\ u, v \in Q_e. \tag{3.15}$$

Hence, the Lemma 2.5 assume that there exists a unique positive solution $u_k^\star \in Q_e$ such that $T_k(u_k^\star, u_k^\star) = u_k^\star$. Consequently, $u_k^\star$ is a unique positive solution of (2.5) for every $k \in \{2, 3, \ldots\}$.

Since $u_k^\star \in Q_e$, so $u_k^\star$ has uniform lower and upper bounds. Thus, in order to pass the solution $u_k^\star$ of (3.2) to that of (2.5), we need that the fact that $\{u_k^\star\}_{k\geq 2}$ is an equicontinuous family on $[0,1]$. In fact, by (3.5), (3.7), $u_k^\star \in Q_e$, we have

$$\begin{aligned}
&f\left(s, u_k^\star(s)+\frac{1}{k}, D_{0^+}^{\mu_1} u_k^\star(s)+\frac{1}{k}, D_{0^+}^{\mu_2} u_k^\star(s)+\frac{1}{k}, \cdots, D_{0^+}^{\mu_{n-2}} u_k^\star(s))d\tau+\frac{1}{k}\right)\\
&\leq (2bM)^{\sigma^{\frac{1}{q-1}}}\phi(s,1,1,\cdots,1) + \left(\frac{M}{c}\right)^{\sigma^{\frac{1}{q-1}}} \tau^{(\gamma-1)\sigma^{\frac{1}{q-1}}}\psi(s,1,1,\cdots,1),\ s \in (0,1),
\end{aligned}$$

and let

$$\omega(s) = (2bM)^{\sigma^{\frac{1}{q-1}}}\phi(s,1,1,\cdots,1) + \left(\frac{M}{c}\right)^{\sigma^{\frac{1}{q-1}}} \tau^{(\gamma-1)\sigma^{\frac{1}{q-1}}}\psi(s,1,1,\cdots,1),\ s \in (0,1), \tag{3.16}$$

by (S_4), we easily get that $\varphi(s) \in L^1[0,1]$. Hence, for $0 \leq t_1 \leq t_2 \leq 1$, we have

$$\begin{aligned}
&|(u_k^\star)(t_2) - (u_k^\star)(t_1)| = |T_k(u_k^\star, u_k^\star)(t_2) - T_k(u_k^\star, u_k^\star)(t_1)|\\
\leq&\lambda \int_0^1 |G(t_2,s) - G_1(t,s)|\varphi_q\left(\int_0^1 H(s,\tau)\left(\phi\left(\tau, u(\tau)+\frac{1}{k}, D_{0^+}^{\mu_1} u(\tau)+\frac{1}{k}, \cdots,\right.\right.\right.\\
&\left.\left.\left.D_{0^+}^{\mu_{n-2}} u(\tau)+\frac{1}{k}\right) + \psi((\tau, v(\tau)+\frac{1}{k}, D_{0^+}^{\mu_1} v(\tau)+\frac{1}{k}, \cdots, D_{0^+}^{\mu_{n-2}} v(\tau))d\tau + \frac{1}{k}\right)d\tau\right)ds
\end{aligned}$$

$$
\begin{aligned}
&\le \lambda \int_0^1 |G(t_2,s)-G(t_1,s)|\,\varphi_q\left(\int_0^1 b^\star s^{\alpha-1}\omega(\tau)d\tau\right)ds\\
&=\lambda b^{\star(q-1)}\int_0^1 |G(t_2,s)-G(t_1,s)|\,s^{(\alpha-1)(q-1)}\varphi_q\left(\int_0^1 b^\star \omega(\tau)d\tau\right)ds\\
&=\lambda b^{\star(q-1)}\|\omega\|_L^{q-1}\int_0^1 |G(t_2,s)-G(t_1,s)|\,s^{(\alpha-1)(q-1)}ds\\
&\le \lambda b^{\star(q-1)}\|\omega\|_L^{q-1}\int_0^1 |G(t_2,s)-G(t_1,s)|\,ds\\
&=\lambda b^{\star(q-1)}\|\omega\|_L^{q-1}\left(\int_0^1 \frac{(1-s)^{\gamma-r_1-1}t_2^{\gamma-1}P(s)}{\Delta}ds-\int_0^1 \frac{(1-s)^{\gamma-r_1-1}t_1^{\gamma-1}P(s)}{\Delta}ds\right.\\
&\quad\left.\times\int_0^{t_1}\frac{(t_1-s)^{\gamma-1}}{\Gamma(\gamma)}ds-\int_0^{t_2}\frac{(t_2-s)^{\gamma-1}}{\Gamma(\gamma)}ds\right)\\
&\le \lambda b^{\star(q-1)}\|\omega\|_L^{q-1}\left(\int_0^1 \frac{(1-s)^{\gamma-r_1-1}P(s)}{\Delta}ds\right)\left|t_2^{\gamma-1}-t_1^{\gamma-1}\right|\\
&\quad+\frac{1}{\Gamma(\gamma)}\int_0^{t_1}\left|(t_2-s)^{\gamma-1}-(t_1-s)^{\gamma-1}\right|ds+\frac{1}{\Gamma(\gamma)}\int_{t_1}^{t_2}\left|(t_2-s)^{\gamma-1}\right|ds. \qquad (3.17)
\end{aligned}
$$

Since $(t-s)^{\gamma-1}$ is uniformly continuous on $[0,1]\times[0,1]$ and $t^{\gamma-1}$ is uniformly continuous on $[0,1]$, so any $\varepsilon>0$, there exists $\delta>0$ such that for $0\le t_1\le t_2\le 1,\ t_2-t_1<\delta,\ 0<s\le t_1$,

$$
\begin{aligned}
&t_2^{\gamma-1}-t_1^{\gamma-1}<\varepsilon,\\
&(t_2-s)^{\gamma-1}-(t_1-s)^{\gamma-1}<\varepsilon.
\end{aligned}
$$

Consequently, for all $x\in D,\ 0\le t_1\le t_2\le 1$ and $t_2-t_1<\min\{\delta,\sqrt[\gamma-1]{\varepsilon}\}$, the inequality

$$
|(u_k^\star)(t_2)-(u_k^\star)(t_1)|\le \lambda b^{\star(q-1)}\|\omega\|_L^{q-1}\left(\frac{1}{\Delta\Gamma(\gamma-r_1)}+\frac{2}{\Gamma(\gamma)}\right)\varepsilon \qquad (3.18)
$$

holds. Hence, by the Arzela-Ascoli Theorem we get $\{u_k^\star\}_{k\ge 2}$ is an equiconuous family on $[0,1]$. Hence, $\{u_k^\star\}_{k\ge 2}$ is relatively compact in P, then the sequence $\{u_k^\star\}$ has a subsequence converge to $u^\star\subset P$. Without loss of generality, we still assume that $\{u_k^\star\}$ itself uniformly converges to $u^\star$, that is $\lim_{k\to\infty}u_k^\star\to u^\star$, then $(u^\star)$ is the solution of (2.5) which can be easily get by the Lebesgue dominated convergence theorem.

Moreover, for any $u_0(t)\in Q_e$, by Lemma 2.5, constructing a successively sequence

$$
\begin{aligned}
u_{k+1}(t)=&\lambda\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)\left(\phi\left(\tau,u_k(\tau)+\frac{1}{k},D_{0^+}^{\mu_1}u_k(\tau)+\frac{1}{k},\cdots,\right.\right.\right.\\
&\left.D_{0^+}^{\mu_{n-2}}u_k(\tau)+\frac{1}{k}\right)+\psi((t,u_k(\tau)+\frac{1}{k},D_{0^+}^{\mu_1}u_k(\tau)\\
&\left.\left.+\frac{1}{k},\cdots,D_{0^+}^{\mu_{n-2}}u_k(\tau))d\tau+\frac{1}{k}\right)d\tau\right)ds,
\end{aligned}
$$

and we have $\|u_m-u^\star\|=\to 0$ as $m\to\infty$, convergence rate

$$
\|u_m-u^\star\|=o(1-r^{\sigma^m}),
$$

r is a constant, $0 < r < 1$, and dependent on u_0. Therefore, the proof of Theorem 3.1 is completed.

4. An example

Example 4.1. Consider the following boundary value problem:

$$\begin{cases} D_{0+}^{\frac{3}{2}}\left(\varphi_3\left(D_{0+}^{\frac{5}{2}}u\right)\right)(t)+\lambda^2 f(t,u(t),D_{0+}^{\frac{1}{2}}u(t))=0,\ 0<t<1,\\ u^{(j)}(0)=0,\ j=0,1,2; D_{0+}^{r_1}u(1)=\sum_{j=1}^{\infty}\eta_j D_{0+}^{r_2}u(\xi_j),\\ D_{0+}^{\frac{5}{2}}u(0)=0;\varphi_p\left(D_{0+}^{\frac{5}{2}}u(1)\right)=\sum_{j=1}^{\infty}\zeta_i\varphi_p\left(D_{0+}^{\frac{5}{2}}u(\eta_i)\right), \end{cases} \tag{4.1}$$

where $\gamma = \frac{5}{2}, \delta = \frac{3}{2}, \alpha = \beta = \frac{3}{4},\ r_1 = r_2 = 2,\ \eta_j = \frac{1}{2j^5},\ \xi_j = \frac{1}{j^2}, \zeta_j = \frac{1}{2j^3}$, $p = 3, q = \frac{3}{2}$, and

$$\phi(t,x_1,x_2,x_3)=(t^{-\frac{1}{4}}+\text{cost})x_1^{\frac{1}{9}}+2tx_2^{\frac{1}{8}}+2x_3^{\frac{1}{16}},$$
$$\psi(t,x_1,x_2,x_3)=t^{-\frac{1}{16}}x_1^{-\frac{1}{8}}+x_2^{-\frac{1}{16}}+(2-t)x_3^{-\frac{1}{15}}.$$

Hence,

$$\begin{aligned} \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_2)}\sum_{j=1}^{\infty}\eta_j\xi_j^{\gamma-r_2-1} &= \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})}\sum_{j=1}^{\infty}\eta_j(\xi_j)^{-\frac{1}{2}}=0.4058<0.75\\ &= \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_1)}=\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})},\\ \sum_{i=1}^{\infty}\zeta_i\xi_i^{\alpha-1}=\sum_{i=1}^{\infty}\zeta_i\xi_i^{\frac{1}{2}} &= \frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{j^4}\approx 0.5411<1. \end{aligned}$$

Moreover, for any $(t, x_1, x_2, x_3) \in (0, 1) \times (0, \infty)^3$ and $0 < l < 1$, we have

$$\begin{aligned} \phi(t,lx_1,lx_2,lx_3) &= (t^{-\frac{1}{4}}+\text{cost})(lx_1)^{\frac{1}{9}}+2t(lx_2)^{\frac{1}{8}}+2(lx_3)^{\frac{1}{16}}\\ &\geq l^{\frac{1}{8}}\left((t^{-\frac{1}{4}}+\text{cost})x_1^{\frac{1}{9}}+2tx_2^{\frac{1}{8}}+2x_3^{\frac{1}{16}}\right)\\ &= l^{\frac{1}{8}}\phi(t,x_1,x_2,x_3)=l^{\sigma^{\frac{1}{q_1-1}}}\phi(t,x_1,x_2,x_3),\\ \psi(t,l^{-1}x_1,l^{-1}x_2,l^{-1}x_3) &= t^{-\frac{1}{16}}(l^{-1}x_1)^{-\frac{1}{8}}+(l^{-1}x_2)^{-\frac{1}{16}}+(2-t)(l^{-1}x_3)^{-\frac{1}{15}}\\ &\geq l^{\frac{1}{8}}\left(t^{-\frac{1}{16}}x_1^{-\frac{1}{8}}+x_2^{-\frac{1}{16}}+(2-t)x_3^{-\frac{1}{15}}\right)\\ &= l^{\frac{1}{8}}\psi(t,x_1,x_2,x_3)=l^{\sigma^{\frac{1}{q_1-1}}}\psi(t,x_1,x_2,x_3). \end{aligned}$$

Noting $\sigma = \frac{1}{2\sqrt{2}} < 1, \varsigma = \frac{2}{3},\ \psi(\tau, 1, 1, 1) = \tau^{-\frac{1}{16}} + 3 - \tau,\ \phi(\tau, 1, 1, 1) = \tau^{-\frac{1}{4}} + \cos\tau + 2\tau + 2,\ g(\tau, 1) = 3\tau + \tau^2 + \tau\sin\tau + \tau$, we have

$$0<\int_0^1\phi(\tau,1,1,1)d\tau=\int_0^1(\tau^{-\frac{1}{4}}+\cos\tau+2\tau+2)d\tau\leq 5+\frac{4}{3}<+\infty,$$

$$0 < \int_0^1 \tau^{-(\gamma-1)\sigma^{\frac{1}{q-1}}} \psi(\tau,1,1,1)d\tau = \int_0^1 \tau^{-\frac{3}{16}}(\tau^{-\frac{1}{16}} + 3 - \tau)d\tau$$
$$\le \int_0^1 \tau^{-\frac{3}{16}}(\tau^{-\frac{1}{16}} + 3)d\tau = \frac{4}{5} + \frac{48}{13} < +\infty.$$

Thus, the assumptions (S_1-S_4) of Theorem 3.1 hold. Then Theorem 3.1 implies that problem (4.1) has a unique solution.In addition, for any initial $u_0 \in Q_e$, we construct a successively sequence:

$$u_{k+1}(t) = \lambda \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)\left(\phi\left(t, u_k(t), D_{0+}^{\frac{1}{2}}u_k(t), Au_k'(t)\right)\right.\right.$$
$$\left.\left.+\psi\left(t, u_k(t), D_{0+}^{\frac{1}{2}}u_k(t), Au_k'(t)\right)\right)d\tau\right)ds, t \in [0,1],\ k = 1, 2, \cdots,$$

and we have $\|u_k - u_\lambda^\star\| \to 0$ as $k \to \infty$, the convergence rate is

$$\|u_k - u_\lambda^\star\| = o(1 - r^{\sigma^k}),$$

where r is a constant, $0 < r < 1$, and dependent on u_0.

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