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A NOVEL 5D SYSTEM GENERATED INFINITELY MANY HYPERCHAOTIC ATTRACTORS WITH THREE POSITIVE LYAPUNOV EXPONENTS*

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Abstract Little seems to be known about the five-dimensional (5D) differential dynamical system with infinitely many hyperchaotic attractors, which have three positive Lyapunov exponents under no or infinitely many equilibria. This article presents a 5D dynamical system that can generate infinitely many hyperchaotic attractors. Of particular interest is the system exists not only infinitely many hyperchaotic attractors but also infinitely many periodic attractors in the following three cases: (i) no equilibria, (ii) only infinitely many non-hyperbolic equilibria, (iii) only infinitely many hyperbolic equilibria. By numerical analysis, one finds the 5D system could generate infinitely many coexisting hyperchaotic or chaotic or periodic attractors in the three kinds of equilibria cases. And one obtains the global dynamical behavior of the system, such as the Lyapunov exponential spectrum, bifurcation diagram. To study the hyperchaotic complexity of the 5D system, we rigorously show the stability of hyperbolic equilibria and some mathematical characterization for 5D Hopf bifurcation. In particular, the existence of an infinite number of isolated bifurcated periodic orbits is strictly proven. These complex dynamics studies in this paper may further contribute to a deep understanding of the hyperchaotic systems with infinitely many attractors.

Keywords Hyperchaos, multistability, equilibria and stability, coexisting attractors, Hopf bifurcation.

MSC(2010) 34C23, 34C28, 34D45.

1. Introduction

One attraction of nonlinear dynamical systems is that they can generate chaotic attractors, an essential aspect studied by nonlinear science since the 20th century. Lorenz [17] discovered the first chaotic attractor using computer numerical experiments while looking at atmospheric motion. Leonov [13–15] introduced

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the concept of hidden attractors: the attractive domain does not intersect with any neighborhood of unstable equilibria. These researches and related literature [2,4,8,10,23,25,29,30,32] make chaotic systems show more complex dynamic properties, reflecting more vital randomness and unpredictability.

Since then, chaos theory and its applications have penetrated almost all-natural and social sciences fields. Ouannas et al. [19] proposed a new secure communications approach that combines chaotic modulation and recursive encryption into one scheme. Wu et al. [34] structured a new approach for simplifying such circuit implementation, which is practical for chaos-based communications. Jahanshahi et al. [11] proposed a novel fuzzy disturbance-observer-based integral terminal sliding mode control method for the hyperchaotic financial system. Lin et al. [16] reviewed many chaotic dynamical behaviors based on memory neurons and neural networks. Fathizadeh et al. [6], using the chaotic control theory, designed a voltage generator as a control system that can reduce virus information to the environment.

Rossler [20] revealed the first four-dimensional hyperchaotic system, namely the Rossler hyperchaotic system. In contrast to chaotic phenomena, 4D hyperchaos expands in two directions. Thus the minimum dimension of an autonomous ODE system that generates hyperchaos is four and has at least one nonlinear term. In order to obtain n-dimensional ($n \geq 5$) hyperchaotic autonomous ODE systems with $n-2$ positive Lyapunov exponents, two conditions need to be satisfied: (i) the minimal dimension of the phase space of hyperchaotic attractors should be at least n; (ii) in the equations at least a nonlinear term. Therefore, studying high-dimensional hyperchaotic systems with a corresponding number of positive Lyapunov exponents is essential. The related research on hyperchaos has recently appeared in some literature [1,5,24,33,42] and its references. Based on the analytic system with hyperbolic equilibria, Shen et al. [21,22] constructed n-dimensional continuous-time autonomous systems with multiple positive Lyapunov exponents. The work in these two papers obtain exciting results, but the methodology does not apply to the case of no equilibria and non-hyperbolic equilibria. Also, they do not give a further theoretical analysis of the system except for the number of equilibria and the real part of eigenvalues. This situation motivates us further to investigate the properties of n dimensional chaos and hyperchaos system with $n-2$ positive Lyapunov exponents.

Universally known, studying equilibrium points plays a significant role in understanding hyperchaotic or chaotic attractors. It is easy to see that the number of isolated equilibrium points must be finite in high-dimensional polynomial systems. But studying the number of attractors (including equilibria) and the number of limit cycles is still challenging, which is related to the famous Hilbert's sixteenth problem in 2D planar polynomial systems. As the number of dimensions increases, it becomes more difficult to research the relationship between the number of isolated equilibria and the number of chaos (or hyperchaos) in high-dimensional dynamical systems. For systems with infinite isolated equilibria, some scholars have numerically studied the existence of infinite isolated chaotic attractors, but they lack theoretical results because the number of isolated attractors is difficult to be determined by analytical methods. For example, Zhang et al. [41] presented a possible way to construct a 3D system with an infinite number of chaotic attractors. Yang et al. [38] numerically obtained the existence of infinite isolated chaotic attractors for a 3D system. Zhao et al. [44] studied the coexistence of (infinitely) many attractors of the other 3D Chua's system with the smooth periodic nonlinear term. Chen et al. [3] explored a 4D hyperchaotic system with an infinite number of consecutive equilibria. A section of the book [29] proposed a 4D autonomous hyperchaotic system with infinitely many isolated equilibria or without equilibria. Moreover, a question is whether the number of hyperchaotic attractors in a 5D system with infinitely many isolated equilibria may be infinite. This paper gives a 5D analytic system with infinitely many hyperchaotic attractors under no or infinitely many equilibria, which helps study the relationship between equilibria and hyperchaos in high-dimensional systems.

In exploring the chaotic system, it is common to determine the dynamical properties of the given system from the characteristic polynomial of the linearized system at the equilibria. However, finding the eigenvalues from at least the quintic polynomial is generally impossible. Little research has been done on the hyperchaotic dynamics of 5D ODE systems since it needs more sophisticated techniques and rigorous theoretical analysis method, let alone the case of higher-dimensional ODE systems. The main difficulty and challenge in investigating the hyperchaotic dynamics of the 5D ODE system are to present rigorous mathematical proof of the existence of hyperchaos, not to mention infinitely many hyperchaotic attractors, even for infinitely many isolated periodic orbits. Although numerical methods can compute some examples, this is still theoretically imperfect in a certain sense. In particular, it seems poor to analyze further the dynamical characteristics of the given system, which makes the calculation results lack theoretical support and makes it more difficult to obtain general conclusions of (hyper-)chaotic dynamics.

Along with exploring hyperchaotic attractors, relevant studies focused on 5D polynomial systems with a finite number of coexisting attractors, with the most studied being 5D quadratic polynomial systems [9, 12, 18, 27, 31, 36, 37, 39, 43]. For example, Ojoniyi et al. [18] presented a 5D hyperchaotic system that shows coexisting hidden hyperchaotic, symmetric chaotic and periodic attractors. Yang et al. [36] studied a 5D hyperchaotic system with three positive Lyapunov exponents and three types of coexisting attractors. Zhang et al. [43] obtained a 5D hyperchaotic system with four center-type equilibrium points and coexisting hyperchaotic orbits. For the 5D cubic polynomial system, Yu et al. [40] introduced a novel multistable 5D memristive hyperchaotic system and its application. Wan et al. [28] proposed a novel variable-wing 5D memristive hyperchaotic system with a line equilibrium point. For the 5D quartic polynomial system, Yang et al. [35] investigated a 5D hyperchaotic system with six coexisting attractors. For the 5D seven-order polynomial system, Trikha [26] proposed a new 5D hyperchaotic system and its application in secure communication. The 5D polynomial hyperchaotic system has a finite number of attractors. However, little research has not provided an in-depth theoretical analysis of the 5D analytic system with infinite hyperchaotic attractors. Meanwhile, the 5D hyperchaotic system has rich applications in nonlinear circuits, secure communication, image encryption and engineering. Khalaf et al. [12] presented the application of a new 5D chaotic system in chaos synchronization and secure communication. Wei et al. [31] reported on the finding of hidden hyperchaos in a 5D extension to a known 3D self-exciting homopolar disc dynamo. Yu et al. [39] demonstrated that the hardware-based design of the 5D HFWMS can be applied to various chaos-based embedded system applications including cryptography and secure communication. Vaidyanathan et al. [27] presented a 5D hyperchaotic Rikitake dynamo system and confirm the feasibility of this system.

To date, designing a high-dimensional chaotic differential system with expected dynamic properties and studying its dynamic characteristics are still attractive but challenging. The natural question is whether there are infinitely many hyperchaotic attractors with three positive Lyapunov exponents in 5D autonomous systems. This paper constructs a new five-dimensional autonomous system based on Sprott's A chaotic system. Due to the periodicity of the sinusoidal function, the 5D system can generate not only infinitely many hyperchaotic attractors with three positive Lyapunov exponents, but also infinitely many hidden and non-hyperbolic hyperchaotic attractors with three positive Lyapunov exponents. According to the number of equilibrium points, the obtained 5D system can be divided into three types: no equilibria, an infinite number of hyperbolic equilibria, and an infinite number of non-hyperbolic equilibria. Through the normal form theory and the center manifold theory, we theoretically investigate the local dynamical features of the new system, such as the stability of equilibrium points, Hopf bifurcation and bifurcated periodic solution. In particular, it is strictly proven that the 5D system has an infinite number of isolated bifurcated periodic orbits. This may be useful for understanding the complex dynamics of infinitely many chaotic or hyperchaotic attractors. Meanwhile, one numerically obtains the coexisting hyperchaotic, chaotic and periodic attractors by investigating phase trajectories and Poincaré projections.

The paper is organized as follows. Section 2 introduces a new five-dimensional system and numerically gives three types of hyperchaotic attractors under different equilibria. Meanwhile, we theoretically investigate the stability of hyperbolic equilibrium points. Section 3 obtains the 5D system has infinitely many coexisting (hyper-) chaotic attractors in different cases according to equilibria and positive Lyapunov exponents. Section 4 performs a series of numerical simulations to verify the complex global dynamics of the 5D system with varying parameters. Section 5 gives the existence of Hopf bifurcation and approximate expressions for the bifurcated periodic solution. The final section is a brief conclusion and discussion.

2. A new five-dimensional hyperchaotic system

This section introduces a new five-dimensional hyperchaotic dynamical system with three different types of equilibria under different parameter conditions. Interestingly, this system has infinitely many hyperchaotic attractors with three positive Lyapunov exponents in all three different states.

2.1. Form of 5D hyperchaotic system

Consider the following five-dimensional system

$$\begin{cases} \dot{x} = F(x,y,z,u,v) = ax + by + v \\ \dot{y} = G(x,y,z,u,v) = cx + dy + e\sin z \\ \dot{z} = H(x,y,z,u,v) = f + gy \\ \dot{u} = K(x,y,z,u,v) = h_1 \sin z + h_2 u \\ \dot{v} = L(x,y,z,u,v) = l + kx \end{cases} \tag{2.1}$$

where x, y, z, u, v are the state variables, and $a, b, c, d, e, f, g, h_1, h_2, k$ are non-zero system parameters and l is real system parameter. For the parameter condition

$a + d + h_2 < 0$, one can yield the function

$$\nabla V(t) = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} + \frac{\partial K}{\partial z} + \frac{\partial L}{\partial v} = a + d + h_2 < 0,$$

where $V(t)$ denotes the volume of a region with a smooth boundary in $\mathbb{R}^5$. By Liouville's theorem, one has $V(t) = \exp(et)V(0)$, implying that any volume in $\mathbb{R}^5$ will contract exponentially to zero.

Let $T = \frac{1}{e}\left(\frac{cl}{k} + \frac{df}{g}\right)$, it is easy to see system (2.1) has three types of equilibria.

(I_1) If $|T| > 1$, system (2.1) has no equilibria.

(I_2) If $|T| = 1$, system (2.1) has an infinite number of non-hyperbolic equilibria.

1) Case $T = 1$,

$$(x, y, z, u, v) = \left(-\frac{l}{k}, -\frac{f}{g}, 2n\pi + \frac{\pi}{2}, -\frac{h_1}{h_2}, \frac{al}{k} + \frac{bf}{g}\right) = P_{0n}, \quad n = 0, \pm 1, \pm 2, ...,$$

2) Case $T = -1$,

$$(x, y, z, u, v) = \left(-\frac{l}{k}, -\frac{f}{g}, 2n\pi - \frac{\pi}{2}, \frac{h_1}{h_2}, \frac{al}{k} + \frac{bf}{g}\right) = Q_{0n}, \quad n = 0, \pm 1, \pm 2, ...$$

(I_3) If $|T| < 1$, system (2.1) has an infinite number of hyperbolic equilibria.

1) Case $0 \le T < 1$,

$$(x, y, z, u, v) = \left(-\frac{l}{k}, -\frac{f}{g}, z_{1n}, -\frac{h_1}{h_2}T, \frac{al}{k} + \frac{bf}{g}\right) = P_{1n},\ z_{1n} = 2n\pi + \arcsin T,$$

and

$$(x, y, z, u, v) = \left(-\frac{l}{k}, -\frac{f}{g}, z_{2n}, -\frac{h_1}{h_2}T, \frac{al}{k} + \frac{bf}{g}\right) = P_{2n},\ z_{2n} = (2n+1)\pi + \arcsin T,$$

2) Case $-1 < T < 0$,

$$(x, y, z, u, v) = \left(-\frac{l}{k}, -\frac{f}{g}, z_{3n}, -\frac{h_1}{h_2}T, \frac{al}{k} + \frac{bf}{g}\right) = P_{3n},\ z_{3n} = (2n+1)\pi + \arcsin T,$$

and

$$(x, y, z, u, v) = \left(-\frac{l}{k}, -\frac{f}{g}, z_{4n}, -\frac{h_1}{h_2}T, \frac{al}{k} + \frac{bf}{g}\right) = P_{4n},\ z_{4n} = 2n\pi + \arcsin T,$$

where $n = 0, \pm 1, \pm 2, ...$

With the transformation $(x, y, z, u, v) \to (x, y, z + 2\pi, u, v)$, system (2.1) is invariant which means the phase portrait of system is periodic along the z direction.

Theorem 2.1. *Let $|T| < 1$, then system (2.1) has infinite many hyperbolic equilibria $P_{in}(i = 1, 2, 3, 4; n = 0, \pm 1, \pm 2, ...)$. Furthermore, the following conclusion hold:*

(I_1) Let $a_1 = dk + aeg\sqrt{1 - T^2}$ and $a_2 = ad - bc - eg\sqrt{1 - T^2} - k$, the hyperbolic equilibria $P_{in}(i = 1, 4; n = 0, \pm 1, \pm 2, ...)$ is asymptotically stable if and only if $(a, b, c, d, e, f, g, h_1, h_2, l, k) \in \Omega_1$ where

$$\Omega_1 = \Big\{(a, b, c, d, e, f, g, h_1, h_2, l, k) | h_2 < 0, a + d < 0, aeg > 0, a_1 + (a + d)a_2 < 0,$$

$$aeg(a+d)\sqrt{1-T^2}+a_1a_2(a+d)+a_1^2<0\big\}.$$

Otherwise, the hyperbolic equilibria $P_{in}(i=1,4)$ *is unstable.*

(I_2) *Let* $\bar{a}_1=dk-aeg\sqrt{1-T^2}$ *and* $\bar{a}_2=ad-bc+eg\sqrt{1-T^2}-k$, *the hyperbolic equilibria* $P_{in}(i=2,3;n=0,\pm1,\pm2,\ldots)$ *is asymptotically stable if and only if* $(a,b,c,d,e,f,g,h_1,h_2,l,k)\in\Omega_2$ *where*

$$\begin{aligned}\Omega_2=&\big\{(a,b,c,d,e,f,g,h_1,h_2,l,k)|h_2<0,a+d<0,aeg<0,\bar{a}_1+(a+d)\bar{a}_2<0,\\&-aeg(a+d)\sqrt{1-T^2}+\bar{a}_1\bar{a}_2(a+d)+\bar{a}_1^2<0\big\}.\end{aligned}$$

Otherwise, the hyperbolic equilibria $P_{in}(i=2,3)$ *is unstable.*

Proof. (I_1) Clearly, the Jacobian matrix of system (2.1) at equilibria P_{in} is

$$J(P_{in})=\begin{pmatrix}a & b & 0 & 0 & 1\\ c & d & e\cos z_{in} & 0 & 0\\ 0 & g & 0 & 0 & 0\\ 0 & 0 & h_1\cos z_{in} & h_2 & 0\\ k & 0 & 0 & 0 & 0\end{pmatrix}.$$

The characteristic equation of the Jacobian matrix at the equilibria $P_{in}(i=1,4)$ is

$$P(\lambda)=|\lambda E-J(P_{in})|=\begin{vmatrix}\lambda-a & -b & 0 & 0 & -1\\ -c & \lambda-d & -e\cos z_{in} & 0 & 0\\ 0 & -g & \lambda & 0 & 0\\ 0 & 0 & -h_1\cos z_{in} & \lambda-h_2 & 0\\ -k & 0 & 0 & 0 & \lambda\end{vmatrix}=(\lambda-h_2)\Delta(\lambda)=0, \tag{2.2}$$

where

$$\begin{aligned}\Delta(\lambda)=&\lambda^4-(a+d)\lambda^3+(ad-bc-eg\cos z_{in}-k)\lambda^2+(dk+aeg\cos z_{in})\lambda\\&+aeg\cos z_{in}=0.\end{aligned}$$

When $(a,b,c,d,e,f,g,h_1,h_2,l,k)\in\Omega_1$, $P_{in}(i=1,4)$ is the hyperbolic equilibrium point and the characteristic values satisfy $\lambda_1=h_2,\Delta(\lambda)=0$. Let

$a_4=1,\ a_3=-a-d,\ a_2=ad-bc-eg\cos z_{in}-k,\ a_1=dk+aeg\cos z_{in},\ a_0=aeg\cos z_{in}$,

where $cosz_{in}=\sqrt{1-T^2}$ for any $n\in\mathbb{Z}$. According to the Routh-Hurwitz theorem and the condition of Theorem 2.1

$$\Delta_1=a_3=-a-d>0,\quad \Delta_2=\begin{vmatrix}a_3 & 1\\ a_1 & a_2\end{vmatrix}=a_2a_3-a_1>0,$$

$$\Delta_3=\begin{vmatrix}a_3 & 1 & 0\\ a_1 & a_2 & a_3\\ 0 & a_0 & a_1\end{vmatrix}=a_1a_2a_3-a_0a_3^2-a_1^2>0,\quad \Delta_4=a_0\Delta_3>0,$$

one obtains that all the roots of the equation $\Delta_0(\lambda) = 0$ have negative real part. Therefore when $(a, b, c, d, e, f, g, h_1, h_2, l, k) \in \Omega_1$, the hyperbolic equilibria P_{in} ($i = 1, 4$; $n = 0, \pm1, \pm2, ...$) are locally asymptotically stable. Otherwise, P_{in} is unstable equilibria.

The proof of (I_2) is similar to that of (I_1) and omitted. $\square$

2.2. Three typical hyperchaotic attractors

To illustrate that system (2.1) can produce structurally different hyperchaotic attractors, we find three particular sets of parameter values:

(A_1) $(a, b, c, d, e, f, g, h_1, h_2, l, k) = (1, -1.18, 5, -3, 10, -10, 3, -2, 1, -0.2, -4.5)$;
(A_2) $(a, b, c, d, e, f, g, h_1, h_2, l, k) = (1, -1.18, 5, -3, 10, -10, 3, -2, 1, 0, -4.5)$;
(A_3) $(a, b, c, d, e, f, g, h_1, h_2, l, k) = (1, -0.97, 5, -3, 14, -3, 3, -1, 1, 1, -3.5)$.

In these three parameter sets, the equilibrium points of the system are entirely different which means the system is also distinct.

For the parameter condition (A_1), system (2.1) has no equilibrium points when $T > 1$ (i.e. $|dfk + cgl| > |egk|$) implying that the system has hidden attractors. Denote this attractor by $\mathcal{A}_0$, as shown in Fig.1(a), where the initial value is $(0.1, 1.5, 3.6, -1, 1)$. By calculation, one can see that the hidden attractor $\mathcal{A}_0$ is hyperchaotic with three positive Lyapunov exponents, which are

$$LE_1 = 1.0000,\ LE_2 = 0.2372,\ LE_3 = 0.0601,\ LE_4 = 0.0000,\ LE_5 = -2.2973,$$

and Lyapunov dimension $D_L = 4.5647$. Fig.1(b) displays the Poincaré mapping of $\mathcal{A}_0$ on the $x - v$ plane of $y = 0$.

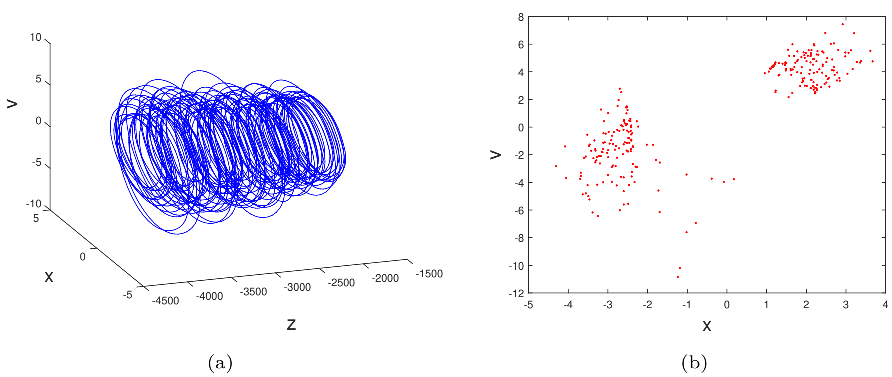
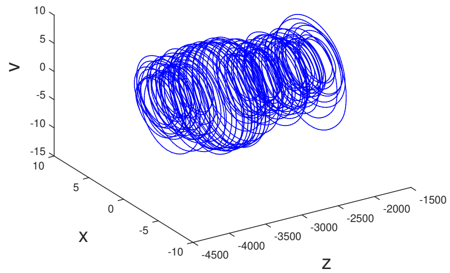


Figure 1. System (2.1) with parameter set (A_1) : (a) hidden hyperchaotic attractor $\mathcal{A}_0$; (b) Poincaré mapping on the $x - v$ plane of $y = 0$.

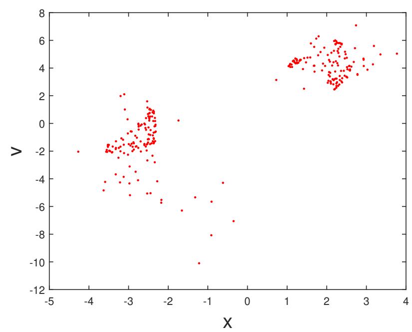
For the parameter condition (A_2), we have $T = 1$ and system (2.1) has infinitely non-hyperbolic equilibria $(x, y, z, u, v) = \left(-\frac{l}{k}, -\frac{f}{g}, 2n\pi + \frac{\pi}{2}, -\frac{h_1}{h_2}, \frac{al}{k} + \frac{bf}{g}\right) = P_{0n}$ $(n \in \mathbb{Z})$. In this case, system (2.1) has a hyperchaotic attractor, denoted by $\mathcal{B}_0$ shown in Fig.2(a), where the initial value is $(0.1, 1.5, 3.6, -1, 1)$. The corresponding Lyapunov exponents of attractor $\mathcal{B}_0$ are

$$LE_1 = 1.0000,\ LE_2 = 0.2413,\ LE_3 = 0.0652,\ LE_4 = 0.0001,\ LE_5 = -2.3065,$$

and the Lyapunov dimension is $D_L = 4.5525$. Moreover, Fig.2(b) shows the Poincaré mapping of $\mathcal{B}_0$ on the $x - v$ plane of $y = 0$, which is a different structure from Fig.1(b). For the parameter condition (A_3), it is clear that $0 \le T < 1$



(a)



(b)

Figure 2. System (2.1) with parameters set (A_2): (a) non-hyperbolic hyperchaotic attractor $\mathcal{B}_0$; (b) Poincaré mapping on the $x - v$ plane of $y = 0$.

(i.e. $|dfk + cgl| < |egk|$) and system (2.1) has infinitely many hyperbolic equilibria

$$P_{1n}\left(\frac{2}{7}, 1, z_{1n}, \frac{11}{98}, \frac{479}{700}\right), \quad P_{2n}\left(\frac{2}{7}, 1, z_{2n}, \frac{11}{98}, \frac{479}{700}\right),$$

where $z_{1n} = 2n\pi + \arcsin\left(\frac{11}{98}\right)$, $z_{2n} = (2n+1)\pi + \arcsin\left(\frac{11}{98}\right)$, $n \in \mathbb{Z}$. Under this circumstance, system (2.1) has a hyperchaotic attractor, denoted by $\mathcal{C}_0$ shown in Fig.3(a) with the same initial value $(0.1, 1.5, 3.6, -1, 1)$. The corresponding Lyapunov exponents of attractor $\mathcal{C}_0$ are

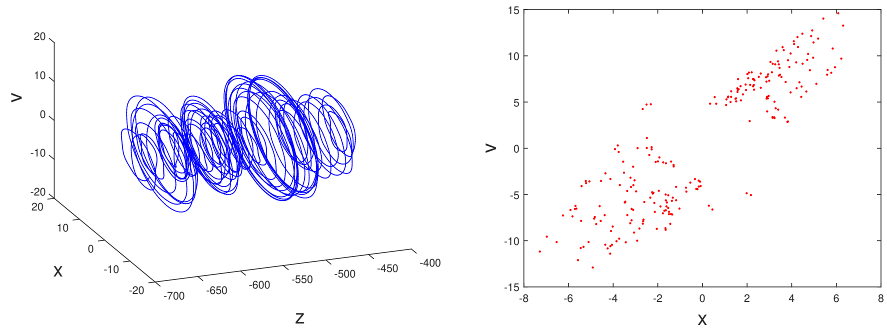
$$LE_1 = 0.9999,\ LE_2 = 0.4120,\ LE_3 = 0.1594,\ LE_4 = 0.0003,\ LE_5 = -2.5715$$

and the Lyapunov dimension is $D_L = 4.6111$. Meanwhile, Fig.3(b) shows the Poincaré mapping of $\mathcal{C}_0$ on the $x - v$ plane of $y = 0$, which is a different structure from Fig.1(b) or Fig.2(b).

Remark 2.1. When system (2.1) has three differential kind of equilibria, it can respectively generate hyperchaotic attractor with three positive Lyapunov exponents. To the best of our knowledge, this phenomenon has not be discussed in other literatures.

3. Infinite number of coexisting attractors

Generally, it is difficult to illustrate analytically the parametric regions of (hyper-) chaotic dynamics. But experiences show that Lyapunov exponent is the most convenient numerical measure for identifying (hyper-) chaotic properties. To further study the dynamics of system (2.1), numerical simulations show that the system has respectively infinitely many coexisting hyperchaotic attractors (three or two positive LEs), chaotic attractors (one positive LE) and periodic attractors (zero positive LE) in the following three cases: (i) no equilibria, (ii) only an infinite number of non-hyperbolic equilibria, (iii) only an infinite number of hyperbolic equilibria.



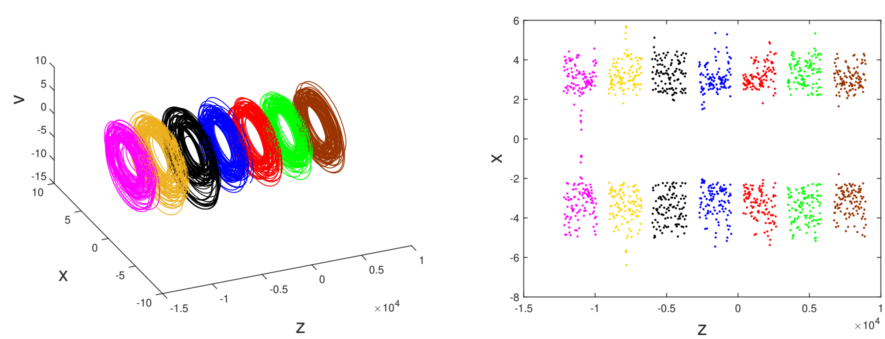
(a) (b)

Figure 3. System (2.1) with parameter set (A_3): (a) hyperchaotic attractor $\mathcal{C}_0$; (b) Poincaré mapping on the $x - v$ plane of $y = 0$.

3.1. Case of no equilibria

3.1.1. Hidden hyperchaotic attractors with three positive LEs

For a clear demonstration of attractors in system (2.1), one chooses the system parameter set (A_1) as $(1, -1.18, 5, -3, 10, -10, 3, -2, 1, -0.2, -4.5)$ and the Lyapunov exponents are shown in subsection 2.2. In Fig.1(a), $z \in (-4100, -1300)$ is an approximate bound along the z direction of the hyperchaotic attractor $\mathcal{A}_0$. Due to the periodicity of the system on the z direction, system (2.1) has coexisting hidden hyperchaotic attractors in the interval $z \in (-4100 + 1000m, -1300 + 1000m)$ (for any $m \in \mathbb{Z}$), denoted by $\mathcal{A}_m$. And the initial values of hidden hyperchaotic attractor $\mathcal{A}_m$ is $(0.1, 1.5, 3.6 + 1000m\pi, -1, 1)$. When $m = 0, \pm1, \pm2, \pm3$, Fig.4 demonstrates the projections of coexisting hidden hyperchaotic attractors $\mathcal{A}_m$ on $x - z - v$ plane and the Poincaré mapping on $z - x$ plane of $v = 0$.



(a) (b)

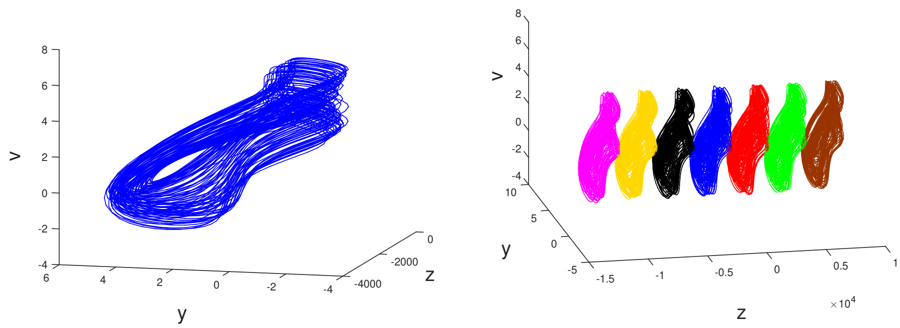
Figure 4. Coexistence of hidden hyperchaotic attractors $\mathcal{A}_m$ in ystem (2.1) with $m = 0, \pm1, \pm2, \pm3$: (a) projection on $x - z - v$ plane; (b) Poincaré mapping on $z - x$ plane cross section $v = 0$.

3.1.2. Hidden (hyper-) chaotic attractors with less than three positive LEs

Let $(a, b, c, d, e, f, g, h_1, h_2, l, k) = (1, -2.02, 5, -3, 10, -10, 3, -2, 1, -0.2, -4.5)$ and choose the initial value as $(0.1, 1.5, 3.6 + 1000m\pi, -1, 1)$ with $m = 0$. In this case, system (2.1) has hidden hyperchaotic attractor $\mathcal{D}_0^{(1)}$ which is shown in Fig.5(a). The corresponding Lyapunov exponents (two positive LEs) of $\mathcal{D}_0^{(1)}$ are

$$LE_1 = 0.9999,\ LE_2 = 0.1861,\ LE_3 = 0.0001,\ LE_4 = -0.4933,\ LE_5 = -1.6928,$$

and the Lyapunov dimension is $D_L = 4.4903$. From Fig.5(a), it is easy to see that the approximate bound along the z of $\mathcal{D}_0^{(1)}$ is $(-3200, -2000)$. Note that system (2.1) is periodic in the z direction. Thus, there exists a chaotic attractor $\mathcal{D}_m^{(1)}$ in the interval $z \in (-3200 + 1000m, -2000 + 1000m)$ for any $m \in \mathbb{Z}$. When $m = 0, \pm1, \pm2, \pm3$, Fig.5(b) shows the projections of coexisting hidden chaotic attractors $\mathcal{D}_m^{(1)}$ with initial values $(0.1, 1.5, 3.6 + 1000m\pi, -1, 1)$.



(a) (b)

Figure 5. Projection of attractor on $z - y - v$ plane: (a) hidden hyperchaotic attractor $\mathcal{D}_0^{(1)}$; (b) coexistence of hidden hyperchaotic attractors $\mathcal{D}_m^{(1)}$.

Choose parameter values as $(1, -2.18, 5, -3, 10, -10, 3, -2, 1, -0.2, -4.5)$ and the initial value $(0.1, 1.5, 3.6 + 1000m\pi, -1, 1)$ with $m = 0$. Then system (2.1) has hidden chaotic attractor $\mathcal{D}_0^{(2)}$ which is shown in Fig.6(a). And the corresponding Lyapunov exponents (one positive LE) of $\mathcal{D}_0^{(2)}$ are

$$LE_1 = 0.9998,\ LE_2 = 0.0000,\ LE_3 = -0.3019,\ LE_4 = -0.3032,\ LE_5 = -1.3948,$$

and the Lyapunov dimension is $D_L = 4.2830$. It is easy to know that system (2.1) is periodic in the z direction. Therefore let $m = 0, \pm1, \pm2, \pm3$, Fig.6(b) displays the projections of coexisting hidden hyperchaotic attractors $\mathcal{D}_m^{(2)}$ with initial values $(0.1, 1.5, 3.6 + 1000m\pi, -1, 1)$.

3.2. Case of only infinitely many non-hyperbolic equilibria

3.2.1. Hyperchaotic attractors with three positive LEs

Choose parameter set (A_2) as $(1, -1.18, 5, -3, 10, -10, 3, -2, 1, 0, -4.5)$ and the Lyapunov exponents are shown in subsection 2.2. In Fig.2(a), we note that the

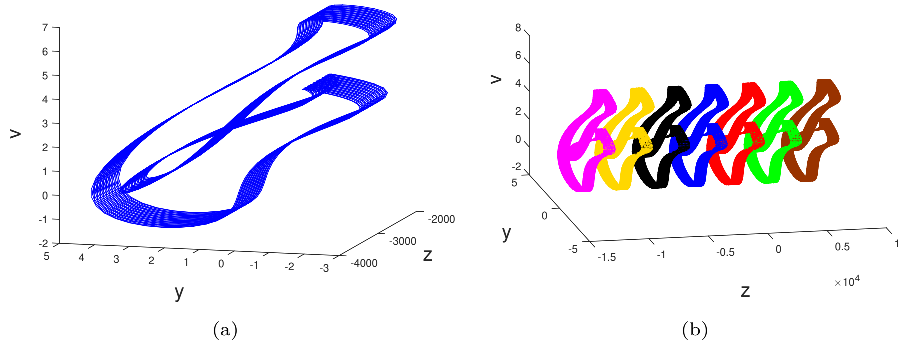


Figure 6. Projection of attractor on $z-y-v$ plane: (a) hidden chaotic attractor $\mathcal{D}_0^{(2)}$; (b) coexistence of hidden chaotic attractors $\mathcal{D}_m^{(2)}$.

hyperchaotic attractor $\mathcal{B}_0$ has an approximate bound $(-4100, -2400)$ in the direction of z. Therefore, system (2.1) has a hyperchaotic attractor in the interval $z \in (-4100 + 1000m, -2400 + 1000m)$ (for any $m \in \mathbb{Z}$), denoted by $\mathcal{B}_m$. And the initial values of the hidden attractor $\mathcal{B}_m$ is $(0.1, 1.5, 3.6 + 1000m\pi, -1, 1)$.

When $m = 0, \pm 1, \pm 2, \pm 3$, Fig.7(a) shows the projections of coexisting hyperchaotic attractors $\mathcal{B}_m$ on $x - z - v$ plane. Fig.7(b) displays the Poincaré cross section of $\mathcal{B}_m$ with $v = 0$.

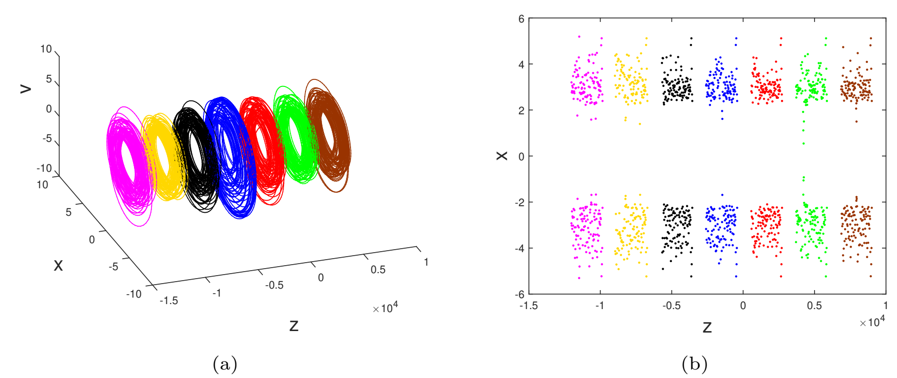


Figure 7. System (2.1) with parameter set (A_2): (a) coexistence of hyperchaotic attractors $\mathcal{B}_m$; (b) Poincaré mapping on $z - x$ plane cross section $v = 0$.

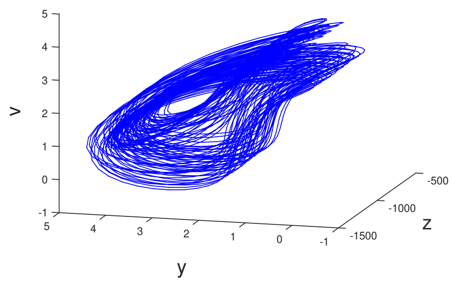
3.2.2. (Hyper-) chaotic attractors with less than three positive LEs

Let $(a, b, c, d, e, f, g, h_1, h_2, l, k) = (1, -1.12, \frac{45}{7}, -3, 5, -10, 3, -2, 1, 3.5, -4.5)$ and the initial value $(0.1, 1.5, 3.6+1000m\pi, -1, 1)$ with $m = 0$. In this case, system (2.1) has non-hyperbolic equilibria and hyperchaotic attractor $\mathcal{E}_0^{(1)}$, the projection of $\mathcal{E}_0^{(1)}$ is shown in Fig.8(a). And the corresponding Lyapunov exponents (two positive

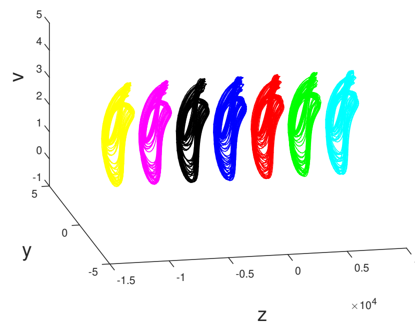
LEs) of $\mathcal{E}_0^{(1)}$ are

$$LE_1 = 0.9999,\ LE_2 = 0.1215,\ LE_3 = 0.0001,\ LE_4 = -0.5287,\ LE_5 = -1.5929,$$

and the Lyapunov dimension is $D_L = 4.3722$. From Fig.8(a), we observe that $\mathcal{E}_0^{(1)}$ has an approximate bound $(-1380, -700)$ in the z direction. System (2.1) is periodic along the z direction, thus there exists non-hyperbolic equilibria and hyperchaotic attractor $\mathcal{E}_m^{(1)}$ in the interval $z \in (-1380 + 1000m, -700 + 1000m)$ for any $m \in \mathbb{Z}$. And the initial values of chaotic attractor $\mathcal{E}_m^{(1)}$ is $(0.1, 1.5, 3.6 + 1000m\pi, -1, 1)$. When $m = 0, \pm 1, \pm 2, \pm 3$, Fig.8(b) demonstrates the projections of coexisting hyperchaotic attractors $\mathcal{E}_m^{(1)}$ on $y - z - v$ plane.



(a)



(b)

Figure 8. Projection of attractor on $z - y - v$ plane: (a) hidden chaotic attractor $\mathcal{E}_0^{(1)}$; (b) coexistence of hidden chaotic attractors $\mathcal{E}_m^{(1)}$.

We set $(a, b, c, d, e, f, g, h_1, h_2, l, k) = (1, -1.12, 5, -3, 5, -10, 3, -2, 1, 0.15, -0.15)$ and the initial value $(0.1, 1.5, 3.6 + 1000m\pi, -1, 1)$ with $m = 0$. In this case, system (2.1) has non-hyperbolic equilibria and chaotic attractor $\mathcal{E}_0^{(2)}$, which is shown in Fig.9(a). And the corresponding Lyapunov exponents (one positive LE) are

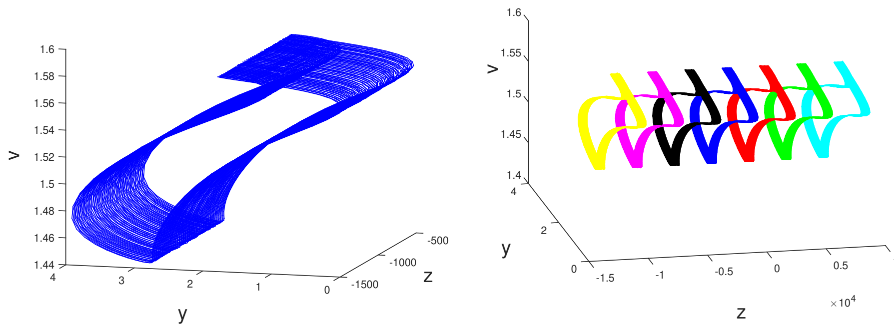
$$LE_1 = 0.9999,\ LE_2 = 0.0000,\ LE_3 = -0.1285,\ LE_4 = -0.2416,\ LE_5 = -1.6298,$$

and the Lyapunov dimension is $D_L = 4.3721$. Because of the periodicity of the system in the z direction, when $m = 0, \pm 1, \pm 2, \pm 3$, Fig.9(b) shows the projections of coexisting chaotic attractors $\mathcal{E}_m^{(2)}$.

3.3. Case of only infinitely many hyperbolic equilibria

3.3.1. Hyperchaotic attractors with three positive LEs

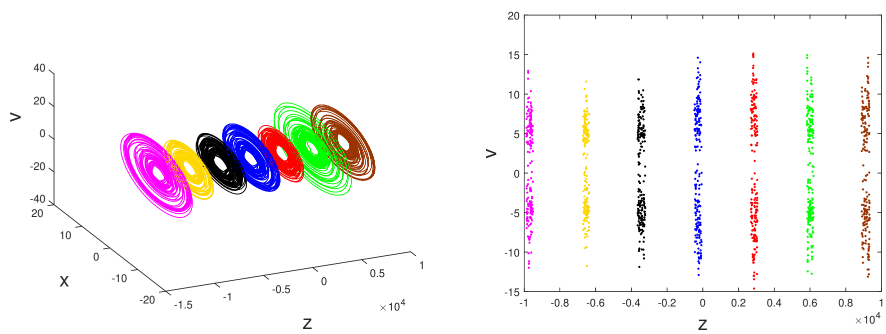
Different from the cases in 3.1 and 3.2, choose system parameter set (A_3) as $(a, b, c, d, e, f, g, h_1, h_2, l, k) = (1, -0.97, 5, -3, 14, -3, 3, -1, 1, 1, -3.5)$ and the Lyapunov exponents are shown in subsection 2.2. From Fig.3(a), hyperchaotic attractor $\mathcal{C}_0$ has an approximate bound $z \in (-560, -250)$ on the z direction. According to system (2.1) being periodic along the z direction, there exist hyperbolic equilibria and hyperchaotic attractor $\mathcal{C}_m$ (for any $m \in \mathbb{Z}$) in the interval of $z \in (-560 + 1000m, -250 + 1000m)$. And the initial values of hyperchaotic attractor



(a) (b)

Figure 9. Projection of attractor on $z-y-v$ plane: (a) hidden hyperchaotic attractor $\mathcal{E}_0^{(2)}$; (b) coexistence of hidden hyperchaotic attractors $\mathcal{E}_m^{(2)}$.

$\mathcal{C}_m$ is $(0.1, 1.5, 3.6+1000m\pi, -1, 1)$. When $m = 0, \pm 1, \pm 2, \pm 3$, Fig.10 demonstrates the projections of coexisting hyperchaotic attractors $\mathcal{C}_m$ on $y-z-v$ plane and the Poincaré mapping on $z-v$ plane of $y = 0$.



(a) (b)

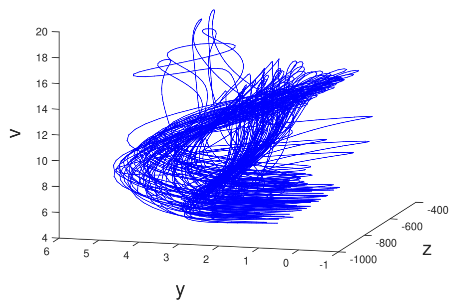
Figure 10. System (2.1) with parameters set (A_3) : (a) coexistence of hyperchaotic attractors $\mathcal{C}_m$; (b) Poincaré mapping on $z-v$ plane cross section $y = 0$.

3.3.2. (Hyper-) chaotic attractors with less than three positive LEs

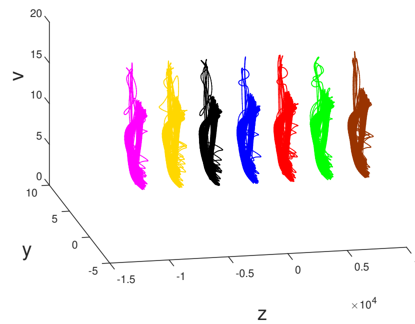
Let $(a, b, c, d, e, f, g, h_1, h_2, l, k) = (1, -4.2, 5, -3, 10, -10, 3, -2, 1, 8, -4.5)$ and the initial value $(0.1, 1.5, 3.6 + 1000m\pi, -1, 1)$ with $m = 0$. One can obtain that system (2.1) has hyperbolic equilibria and a hyperchaotic attractor $\mathcal{F}_0^{(1)}$, as shown in Fig.11(a). And the corresponding Lyapunov exponents (two positive LEs) of $\mathcal{F}_0^{(1)}$ are

$$LE_1 = 1.0000,\ LE_2 = 0.1567,\ LE_3 = 0.0000,\ LE_4 = -0.4865,\ LE_5 = -1.6698,$$

and the Lyapunov dimension is $D_L = 4.4013$. From Fig.11(a), $\mathcal{F}_0^{(1)}$ has an approximate bound $z \in (-900, -430)$ on the z direction. Since system (2.1) is periodic in the z direction, the phase portrait of system (2.1) is also periodic in this direction. Thus, there exists a hyperchaotic attractor $\mathcal{F}_m^{(1)}$ in the interval of $z \in (-900 + 1000m, -430 + 1000m)$ for any $m \in \mathbb{Z}$. When $m = 0, \pm 1, \pm 2, \pm 3$, Fig.11(b) demonstrates the projections of coexisting hyperchaotic attractors $\mathcal{F}_m^{(1)}$ on $y - z - v$ plane.



(a)



(b)

Figure 11. Projection of attractor on $z-y-v$ plane: (a) hidden chaotic attractor $\mathcal{F}_0^{(1)}$; (b) coexistence of hidden chaotic attractors $\mathcal{F}_m^{(1)}$.

Now set $(a, b, c, d, e, f, g, h_1, h_2, l, k) = (1, -5, 5, -3, 10, -10, 3, -2, 1, 8, -4.5)$ and the initial value $(0.1, 1.5, 3.6 + 1000m\pi, -1, 1)$ with $m = 0$. System (2.1) has hyperbolic equilibria and a chaotic attractor $\mathcal{F}_0^{(2)}$ in the interval of $z \in (-1150, -600)$, as shown in Fig.12(a). And the corresponding Lyapunov exponents (one positive LE) of $\mathcal{F}_0^{(2)}$ are

$$LE_1 = 0.9998,\ LE_2 = 0.0002,\ LE_3 = -0.0946,\ LE_4 = -0.2604,\ LE_5 = -1.6450,$$

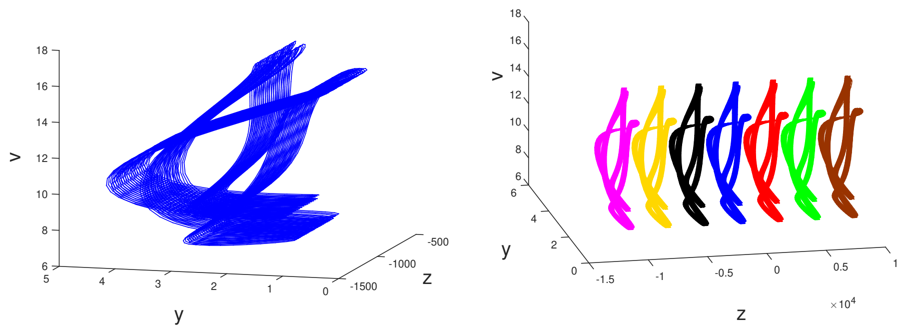
and the Lyapunov dimension is $D_L = 4.3921$. Since the phase portrait of system (2.1) is periodic in the z direction. Thus, when $m = 0, \pm 1, \pm 2, \pm 3$, Fig.12(b) shows the projections of coexisting chaotic attractor $\mathcal{F}_m^{(2)}$ in the interval of $z \in (-1150 + 1000m, -600 + 1000m)$.

3.3.3. Infinitely many periodic attractors

Let $(a, b, c, d, e, f, g, h_1, h_2, l, k) = (0.7, -0.5, 5, -2, 2, -1, 2, 1, -1, 1, -2)$ and the initial value $(0.1, 1.5, 3.6, -1, 1)$. One can find that system (2.1) has periodic attractor $\mathcal{G}_0$, which the Lyapunov exponents (zero positive LE) are

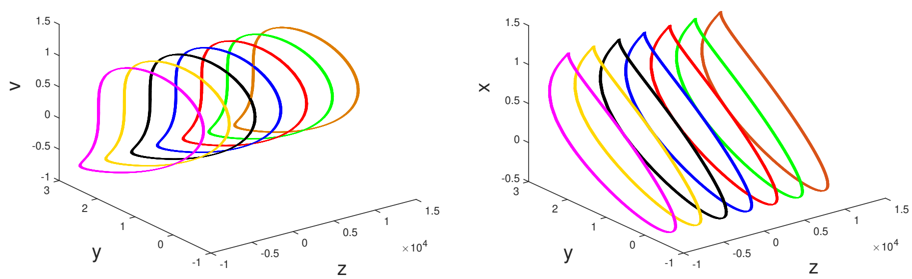
$$LE_1 = 0.0000,\ LE_2 = -0.0699,\ LE_3 = -0.2417,\ LE_4 = -0.9896,\ LE_5 = -0.9989.$$

One observes that $\mathcal{G}_0$ has an approximate bound $z \in (-0.8, 0.8)$ on the z direction. Due to the periodicity of the system (2.1) on the z direction, for any $m \in \mathbb{Z}$, there exists a periodic attractor $\mathcal{G}_m$ in the interval of $z \in (-0.8 + 1000m, 0.8 + 1000m)$. When initial points are $(0.1, 1.5, 3.6+1000m\pi, -1, 1)$ with $m = 0, \pm 1, \pm 2, \pm 3$, Fig.13 demonstrates the projections of coexisting periodic attractors $\mathcal{G}_m$ on $y - z - v$ and $z - y - x$ plane.



(a) (b)

Figure 12. Projection of attractor on $z - y - v$ plane: (a) hidden hyperchaotic attractor $\mathcal{F}_0^{(2)}$; (b) coexistence of hidden hyperchaotic attractors $\mathcal{F}_m^{(2)}$.



(a) (b)

Figure 13. Projection of periodic attractor: (a) coexistence of periodic attractor $\mathcal{G}_m$ on $z - y - v$ plane; (b) coexistence of periodic attractors $\mathcal{G}_m$ on $z - y - x$ plane.

4. Global numerical dynamic analysis

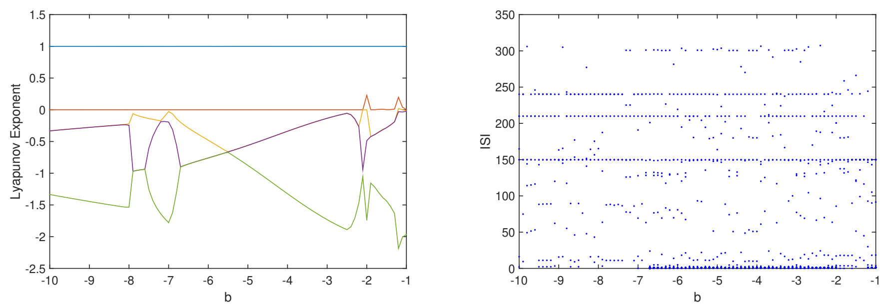
To understand the dynamics of different attractors in system (2.1), this section discusses some properties of the system with parameter changes, such as Lyapunov exponents and bifurcation diagrams. And the simulation results are further derived by using numerical methods.

Case 1. Hidden attractors: we now consider the dynamics of system (2.1) by fixing parameters $a = 1$, $c = 5$, $d = -3$, $e = 10$, $f = -10$, $g = 3$, $h_1 = -2$, $h_2 = 1$, $l = -0.2$, $k = -4.5$, and $b \in [-10, -1]$. In this case, system (2.1) has no equilibria. Moreover, Fig.14 shows system (2.1) has a large-scale hidden chaos attractor except for the small neighborhood of $b = -2, -1.2$, which has hidden hyperchaos attractors. Table 1 demonstrates a few typical chaos and hyperchaos dynamics, as well as the corresponding Lyapunov exponents of the system with parameter b varies.

Case 2. Non-hyperbolic attractors: while fixing parameters $a = 1$, $b = -1.12$, $d = -3$, $e = 5$, $f = -10$, $g = 3$, $h_1 = -2$, $h_2 = 1$, $k = -4.5$, $l \in [0.5, 5]$

Table 1. The Lyapunov exponents of system (2.1) with $(a, b, c, d, e, f, g, h_1, h_2, l, k) = (1, b, 5, -3, 10, -10, 3, -2, 1, -0.2, -4.5)$.

b	Lyapunov exponents					Dynamics
	LE_1	LE_2	LE_3	LE_4	LE_5	
-6.6	0.9995	0.0000	-0.2456	-0.8760	-0.8780	Chaotic
-2	0.9999	0.2283	0.0001	-0.4869	-1.7414	Hyperchaotic
-1.17	0.9999	0.2520	0.0650	0.0001	-2.3198	Hyperchaotic



(a) (b)

Figure 14. System (2.1) with parameters $a = 1, c = 5, d = -3, e = 10, f = -10, g = 3, h_1 = -2, h_2 = 1, l = -0.2, k = -4.5$ and $b \in [-10, -1]$: (a) LEs of hidden attractors; (b) bifurcation diagram.

and $c = \frac{1}{gl}(egk - dfk)$, system (2.1) has infinitely many isolated non-hyperbolic equilibria

$$P_{0n}\left(\frac{2}{9}l, \frac{10}{3}, 2n\pi + \frac{\pi}{2}, \frac{1}{2}, \frac{56}{15} - \frac{2}{9}l\right), \quad l \in [0.5, 5], \quad n = 0, \pm 1, \pm 2, \ldots.$$

As parameter l changes, Fig.15 shows that system (2.1) has an abundance of chaotic or hyperchaotic behaviors. It further can be observed that system (2.1) has hyperchaotic attractors in $l \in [3.3, 3.8) \cup (3.8, 3.9]$. The Lyapunov exponent spectrum and the bifurcation diagram in Fig.15 demonstrate that the system has complex behaviors as l varies. Table 2 demonstrates the Lyapunov exponents of system (2.1) with $l = 0.8$ and 3.5.

Table 2. The Lyapunov exponents of system (2.1) with $(a, b, d, e, f, g, h_1, h_2, l, k) = (1, -1.12, -3, 5, -10, 3, -2, 1, l, -4.5)$ and $c = \frac{1}{gl}(egk - dfk)$.

l	Lyapunov exponents					Dynamics
	LE_1	LE_2	LE_3	LE_4	LE_5	
0.8	1.0000	0.0000	-0.6145	-0.6145	-0.7710	Chaotic
3.5	0.9999	0.1164	0.0001	-0.5228	-1.5937	Hyperchaotic

Case 3. Hyperbolic attractors: let us fix the parameters $b = -1.12$, $c = 5$, $d = -3$, $e = 10$, $f = -10$, $g = 3$, $h_1 = -2$, $h_2 = 1$, $l = 8, k = -4.5$ and $a \in [4.7, 13]$. In this case, system (2.1) has infinitely many isolate hyperbolic equilibria

$$P_{1n}\left(\frac{16}{9}, \frac{10}{3}, z_{1n}, \frac{1}{18}, \frac{56}{15} - \frac{16}{9}a\right), P_{2n}\left(\frac{16}{9}, \frac{10}{3}, z_{2n}, \frac{1}{18}, \frac{56}{15} - \frac{16}{9}a\right), n = 0, \pm 1, \pm 2, \ldots.$$

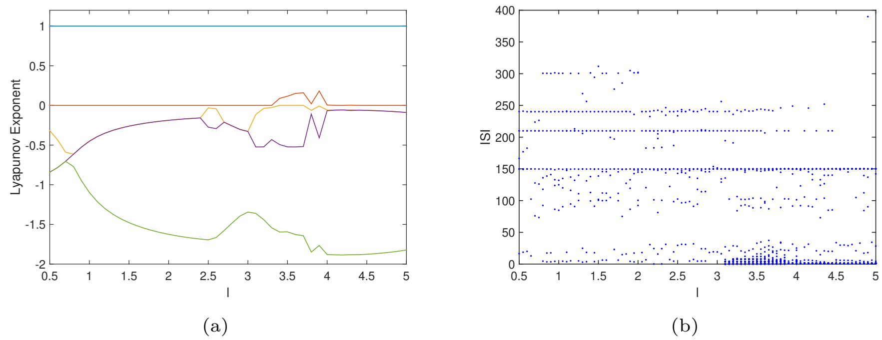


Figure 15. System (2.1) with parameters $a = 1, b = -1.12, d = -3, e = 5, f = -10, g = 3, h_1 = -2, h_2 = 1, k = -4.5, l \in [0.5, 5]$ and $c = \frac{1}{gl}(egk - dfk)$: (a) LEs of nonhyperbolic attractors; (b) bifurcation diagram.

where $a \in [4.7, 13]$, $z_{1n} = 2n\pi + \arcsin(\frac{1}{9})$ and $z_{2n} = (2n+1)\pi - \arcsin(\frac{1}{9})$. Fig.16 shows that system (2.1) is hyperchaotic with two or three positive Lyapunov exponents except for

$$a \in (7.1, 7.5) \cup (7.7, 8.2) \cup (8.4, 8.6) \cup (10.1, 10.4) \cup (10.8, 13).$$

The Lyapunov exponent spectrum and the bifurcation diagram Fig.16 can corroborate each other and exhibit the complexity of the system. Table 3 illustrates chaotic or hyperchaotic dynamic phenomenon and the corresponding Lyapunov exponents of system (2.1) with different values of parameter a.

Table 3. The Lyapunov exponents of system (2.1) with $(a, b, c, d, e, f, g, h_1, h_2, l, k) = (a, -1.12, 5, -3, 10, -10, 3, -2, 1, 8, -4.5)$.

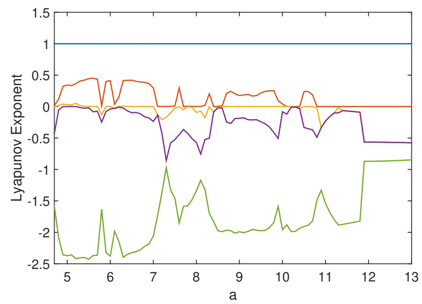
a	Lyapunov exponents					Dynamics
	LE_1	LE_2	LE_3	LE_4	LE_5	
13.0	1.0000	0.0000	-0.5760	-0.5764	-0.8476	Chaotic
6.7	0.9997	0.3923	0.0000	-0.1155	-2.2766	Hyperchaotic
5.2	0.9998	0.3650	0.0541	0.0000	-2.4190	Hyperchaotic

5. Hopf bifurcation of the 5D hyperchaotic system

This section employs the higher-dimensional Hopf bifurcation theory [7] and symbolic computations to analyze dynamical Hopf bifurcations of system (2.1).

Theorem 5.1 (Existence of Hopf Bifurcation)**.** *Suppose that* $|T| < 1$, $h_2 < 0$, $a + d < 0$, $aeg > 0$, $dk + aeg\sqrt{1 - T^2} > 0$ *and* $dk + aeg\sqrt{1 - T^2} + (a + d)(ad - bc - eg\sqrt{1 - T^2} - k) < 0$ *hold. Then, as parameter* b *varies and passes through the critical value*

$$b = b_0 = \frac{1}{c}\left(ad - eg\sqrt{1 - T^2} - k + \frac{dk + aeg\sqrt{1 - T^2}}{a + d} + \frac{aeg(a + d)\sqrt{1 - T^2}}{dk + aeg\sqrt{1 - T^2}}\right),$$



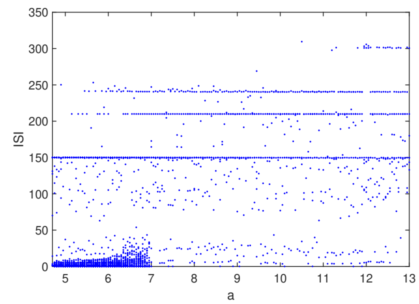


Figure 16. System (2.1) with parameters $b = -1.12, c = 5, d = -3, e = 10, f = -10, g = 3, h_1 = -2, h_2 = 1, l = 8, k = -4.5$, and $a \in [4.7, 13]$: (a) LEs of hyperbolic attractors; (b) bifurcation diagram.

system (2.1) undergoes a Hopf bifurcation at the equilibria $P_{in}(i = 1, 4; n = 0, \pm 1, ...)$, *and branches out to a periodic orbit. Furthermore, the system has a countably infinite number of periodic orbits.*

Proof. It is easy to see that linearized system of (2.1) at P_{in} $(i = 1, 4)$ yields the characteristic equation

$$\begin{aligned} P(\lambda) =& (\lambda - h_2)\left[\lambda^4 - (a+d)\lambda^3 + (ad - bc - eg\cos z_{in} - k)\lambda^2 + (dk + aeg\cos z_{in})\lambda\right] \\ &+ (\lambda - h_2)aeg\cos z_{in} \\ =& (\lambda - h_2)\left[\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0\right] = (\lambda - h_2)\Delta(\lambda) = 0, \end{aligned} \tag{5.1}$$

where $\cos z_{in} = \sqrt{1 - T^2}$.

Suppose that equation (5.1) has a pure imaginary root $\lambda = i\omega$ $(\omega \in \mathbb{R}^+)$. Substituting $\lambda = i\omega$ into (5.1) yields that

$$\Delta(\omega i) = (\omega i)^4 + a_3(\omega i)^3 + a_2(\omega i)^2 + a_1(\omega i) + a_0 = (\omega^4 - a_2\omega^2 + a_0) + i(-a_3\omega^3 + a_1\omega) = 0.$$

It follows that

$$\omega^4 - a_2\omega^2 + a_0 = 0, \quad -a_3\omega^3 + a_1\omega = 0. \tag{5.2}$$

Under the condition $(a + d)(dk + aeg\sqrt{1 - T^2}) < 0$, solving equation (5.2) gives

$$\begin{aligned} \omega &= \omega_0 = \sqrt{\frac{a_1}{a_3}} = \sqrt{-\frac{dk + aeg\sqrt{1 - T^2}}{a + d}}, \\ b &= \frac{1}{c}\left(ad - eg\sqrt{1 - T^2} - k - \omega^2 - \frac{a_0}{\omega^2}\right) \\ &= \frac{1}{c}(ad - eg\sqrt{1 - T^2} - k) + \frac{dk + aeg\sqrt{1 - T^2}}{c(a + d)} + \frac{aeg(a + d)\sqrt{1 - T^2}}{c(dk + aeg\sqrt{1 - T^2})} = b_0. \end{aligned}$$

Substituting $b = b_0$ into (5.1), one obtains

$$\lambda_1 = i\omega_0, \quad \lambda_2 = -i\omega_0, \quad \lambda_3 = \frac{1}{2}\left(-a_3 + \sqrt{-4a_2 + a_3^2 + 4\omega_0^2}\right),$$

$$\lambda_4 = h_2, \quad \lambda_5 = \frac{1}{2}\left(-a_3 - \sqrt{-4a_2 + a_3^2 + 4\omega_0^2}\right).$$

When $h_2 < 0$, $a_3 > 0$, $a_1 - a_2a_3 < 0$ and $b = b_0$, the eigenvalues $\lambda_3, \lambda_4, \lambda_5$ have negative real parts. Thus, the first condition for Hopf bifurcation [7] is satisfied.

From equation (5.1) and eigenvalues λ_3, λ_5, differentiating (5.1) concerning b, one obtains

$$P_b'(\lambda(b)) = \lambda'(b)\Delta(\lambda(b)) + (\lambda(b) - h_2)\Delta'(\lambda(b)) = 0, \tag{5.3}$$

where

$$\Delta'(\lambda(b)) = 4\lambda(b)^3\lambda'(b) + 3a_3\lambda(b)^3\lambda'(b) + 2a_2\lambda(b)^2\lambda'(b) + a_2'(b)\lambda(b)^2 + a_1\lambda'(b).$$

Substituting $\lambda = i\omega_0$ into $\Delta(i\omega_0) = 0$, it follows $\Delta'(i\omega_0) = 0$. Therefore, solving the equation $\Delta'(i\omega_0) = 0$ yields

$$\lambda'(b_0)\Big|_{\lambda=i\omega_0} = \frac{c\omega_0^2(3a_3\omega_0^2 - a_1) - 2c\omega_0^4(2\omega_0 - a_2)i}{(3a_3\omega_0^2 - a_1)^2 + 4\omega_0^2(2\omega_0 - a_2)^2}, \tag{5.4}$$

implies that

$$Re(\lambda'(b_0))\Big|_{\lambda=i\omega_0} = \frac{c\omega_0^2(3a_3\omega_0^2 - a_1)}{(3a_3\omega_0^2 - a_1)^2 + 4\omega_0^2(2\omega_0 - a_2)^2} = \frac{a_1c\omega_0^2}{2a_1^2 + 2\omega_0^2(2\omega_0 - a_2)^2} \neq 0.$$

Thus, the second condition for a Hopf bifurcation [7] is also met. Consequently, the system exists Hopf bifurcation and bifurcates a periodic orbit near equilibria $P_{in}\,(i = 1, 4)$.

Furthermore, assume that d_0 is the maximum distance from the point on the periodic orbit Γ_0 to the equilibrium $P_{ik_0}\,(k_0 \in \mathbb{Z})$, $U_0 = U_0(P_{ik_0}, d_0 + 1)$ is the region centered on P_{ik_0} and can cover periodic orbit Γ_0. Due to system (2.1) being periodic in the z direction, there must exist a periodic orbit Γ_1 which satisfies that region U_0 and $U_1 = U_1(P_{ik_1}, d_1 + 1)\,(k_0 < k_1 \in \mathbb{Z})$ do not intersect. More general, system (2.1) must exist period orbits Γ_n in the regions $U_n = U_n(P_{ik_n}, d_n + 1)\,(k_0 < k_1 < ... < k_n \in \mathbb{Z})$ along the z direction as the initial values vary. And regions $U_n = U_n(P_{ik_n}, d_n + 1)$ are independent of each other, □

Similar to the proof Theorem 5.1, one obtains the following conclusion.

Theorem 5.2 (Existence of Hopf Bifurcation)**.** *Suppose that* $|T| < 1$, $h_2 < 0$, $a + d < 0$, $aeg > 0$, $dk - aeg\sqrt{1 - T^2} > 0$ *and* $dk - aeg\sqrt{1 - T^2} + (a + d)(ad - bc + eg\sqrt{1 - T^2} - k) < 0$ *holds. Then, as* b *varies and passes through the critical value*

$$b = b_0 = \frac{1}{c}\left(ad + eg\sqrt{1 - T^2} - k + \frac{dk - aeg\sqrt{1 - T^2}}{a + d} + \frac{aeg(a + d)\sqrt{1 - T^2}}{aeg\sqrt{1 - T^2} - dk}\right),$$

system (2.1) undergoes a Hopf bifurcation at the equilibria $P_{in}(i = 2, 3; n = 0, \pm 1, ...)$, *and bifurcates out to a periodic orbit. Furthermore, the system has a countably infinite number of periodic orbits.*

Remark 5.1. When $(a + d)(dk + aeg\sqrt{1 - T^2}) \geq 0$, system (2.1) has no Hopf bifurcation at the equilibria $P_{in}(i = 1, 4; n = 0, \pm 1, \pm 2, ...)$. When $(a + d)(dk - aeg\sqrt{1 - T^2}) > 0$, system (2.1) has no Hopf bifurcation at the equilibria $P_{in}(i = 2, 3; n = 0, \pm 1, \pm 2, ...)$.

Next, one uses the canonical theory to obtain the following approximate expression of Hopf bifurcated periodic solution of system (2.1) at equilibria $P_{in}(i = 1, 4; n = 0, \pm 1, \pm 2, \ldots)$.

Theorem 5.3 (Periodic solution of Hopf bifurcation)**.** *Let*

$$\omega_0 = \sqrt{-\frac{dk + aeg\sqrt{1-T^2}}{a+d}}, \quad a_2 = ad - bc - eg\sqrt{1-T^2} - k,$$
$$b_0 = \frac{1}{c}\left(ad - eg\sqrt{1-T^2} - k + \frac{dk + aeg\sqrt{1-T^2}}{a+d} + \frac{aeg(a+d)\sqrt{1-T^2}}{dk + aeg\sqrt{1-T^2}}\right),$$
$$\lambda_3 = \frac{-(a+d) + \sqrt{-4a_2 + (a+d)^2 + 4\omega_0^2}}{2},$$
$$\lambda_5 = \frac{-(a+d) - \sqrt{-4a_2 + (a+d)^2 + 4\omega_0^2}}{2},$$
$$p_1 = \frac{ga^2\omega_0^2(a+d) + (a+d)(\omega_0^2 - a_2)(g+\omega_0^2)^2 + a\omega_0^2(g(a+d)^2 + a_2(\omega_0^2 - a_2)) - a_2g^2}{bg\omega_0(\omega_0^2+\lambda_3^2)(\omega_0^2+\lambda_5^2)}$$
$$- \frac{fh_1^2\sqrt{1-T^2}(a^2gh_2 + (g+\omega_0^2)(gh_2 + g(a+d) + h_2(\omega_0^2 - a_2)) + a(g^2 - \omega_0^2(\omega_0^2 - a_2) + gh_2(a+d)))}{bg(h_2+\omega_0^2)(h_2-\lambda_3)(h_2-\lambda_5)},$$
$$p_2 = -\frac{fh_1^2(g + ah_2 - h_2^2)\sqrt{1-T^2}}{(h_2^2+\omega_0^2)(h_2-\lambda_3)(h_2-\lambda_5)} - \frac{ga_2 + \omega_0^2(a_2 - a(a+d))}{\omega_0(\omega_0^2+\lambda_3^2)(\omega_0^2+\lambda_5^2)},$$
$$p_3 = \frac{h_1^2\sqrt{1-T^2}(\omega_0^2(h_2-a)(\omega_0^2-a_2) + g(h_2^3 + (h_2^2+\omega_0^2)(a+d) + 2h_2\omega_0^2 - a_2h_2))}{(h_2^2+\omega_0^2)(h_2-\lambda_3)(h_2-\lambda_5)g}$$
$$- \frac{\omega_0(\omega_0^2 - a_2)((g+\omega_0^2)(a+d) - aa_2)}{fg(\omega_0^2+\lambda_3^2)(\omega_0^2+\lambda_5^2)},$$
$$p_5 = -\frac{fh_1^2\sqrt{1-T^2}(g^3 + \omega_0^2(\omega_0^2-a_2)(a^2-ah_2) + g^2(2ah_2 + 2\omega_0^2 + h_2d - a_2) + g\omega_0^2(\omega_0^2 - a_2 + (h_2-a)(a+d)))}{bg^2(h_2^2+\omega_0^2)(h_2-\lambda_3)(h_2-\lambda_5)}$$
$$+ \frac{-a_2g^3 + g^2(a_2^2 - 2a_2\omega_0^2 + \omega_0^2(2a+d)(a+d)) - a\omega_0^2(\omega_0^2-a_2)(aa_2 + \omega_0^2(a+d))}{bg^2\omega_0(\omega_0^2+\lambda_3^2)(\omega_0^2+\lambda_5^2)}$$
$$+ \frac{g\omega_0^2(a_2^2 - a_2\omega_0^2 + aa_2(a+d) + \omega_0^2(a+d)^2)}{bg^2\omega_0(\omega_0^2+\lambda_3^2)(\omega_0^2+\lambda_5^2)},$$

and

$$N_1 = -T^2f^4\left(\frac{p_1p_3\lambda_5 + p_1p_5\lambda_3}{4\lambda_3\lambda_5} + \frac{p_1p_3\lambda_3 - 2p_2p_3\omega_0}{8(\lambda_3^2+4\omega_0^2)} + \frac{p_1p_5\lambda_5 - 2p_2p_5\omega_0}{8(\lambda_5^2+4\omega_0^2)}\right),$$
$$N_2 = -T^2f^4\left(\frac{p_2p_3\lambda_5 + p_2p_5\lambda_3}{4\lambda_3\lambda_5} + \frac{p_2p_3\lambda_3 + 2p_1p_3\omega_0}{8(\lambda_3^2+4\omega_0^2)} + \frac{p_2p_5\lambda_5 + 2p_1p_5\omega_0}{8(\lambda_5^2+4\omega_0^2)}\right).$$

Suppose that $|T| < 1$, $h_2 < 0$, $a + d < 0$, $aeg > 0$, $dk + aeg\sqrt{1-T^2} > 0$ *and* $dk + aeg\sqrt{1-T^2} + (a+d)a_2 < 0$, *then for the sufficiently small neighborhood* $|b - b_0| > 0$ *where*

$$b = b_0 = \frac{1}{c}\left(ad - eg\sqrt{1-T^2} - k + \frac{dk + aeg\sqrt{1-T^2}}{a+d} + \frac{aeg(a+d)\sqrt{1-T^2}}{dk + aeg\sqrt{1-T^2}}\right),$$

Hopf bifurcation occurs at the equilibria $P_{in}(i = 1, 4; n \in \mathbb{Z})$ *in system (2.1), and the periodic solution derived from corresponding Hopf bifurcation has the following properties:*

(i) If $\mu_2 > 0\,(resp. < 0)$ and $\beta_2 < 0\,(resp. > 0)$ where

$$\begin{aligned}
\beta_2 &= N_1 - \frac{T^2 f^4 p_1 p_2}{8\omega_0},\\
\mu_2 &= \frac{(T^2 f^4 p_1 p_2 - 8\omega_0 N_1)((dk + aeg\sqrt{1-T^2})^2 + (a_2 - 2\omega_0^2)\omega_0^2)}{8(dk + aeg\sqrt{1-T^2})c\omega_0^3},
\end{aligned}$$

then system (2.1) as a supercritical (resp. subcritical) Hopf bifurcation at equilibria $P_{in}(i = 1, 4; n \in \mathbb{Z})$, and the periodic orbit is stable (resp. unstable) when $b > b_0\,(resp. < b_0)$.

(ii) The periodic and characteristic exponents of bifurcation periodic solutions are respectively

$$\begin{aligned}
T^* &= \frac{2\pi}{\omega_0}\left(1 + \tau_2\varepsilon^2 + o\left(\varepsilon^4\right)\right),\\
\beta &= \beta_2\varepsilon^2 + o\left(\varepsilon^4\right),
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon^2 &= \frac{b - b_0}{\mu_2} + o\left((b - b_0)^2\right),\\
\tau_2 &= \frac{f^4(2p_1^2 + 5p_2^2)T^2 - 24\omega_0 N_2}{48\omega_0^2} + \frac{3\omega_0^2(a_2 - 2\omega_0)^2(f^4 p_1 p_2 T^2 - 8\omega_0 N_1)}{48(dk + aeg\sqrt{1-T^2})\omega_0^2}.
\end{aligned}$$

(iii) The approximate expression of the bifurcated periodic solution is

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} &= \begin{pmatrix} -\frac{l}{k} \\ -\frac{f}{g} \\ 2n\pi + \arcsin T \\ -\frac{h_1}{h_2}T \\ \frac{al}{k} + \frac{bf}{g} \end{pmatrix} + \varepsilon \begin{pmatrix} \frac{b_0\omega_0^2(\omega_0^2+g)}{(\omega_0^2+g)^2+a^2\omega_0^2}\cos\left(\frac{2\pi t}{T^*}\right) + \frac{ab_0\omega_0^3}{(\omega_0^2+g)^2+a^2\omega_0^2}\sin\left(\frac{2\pi t}{T^*}\right) \\ -\omega_0\sin\left(\frac{2\pi t}{T^*}\right) \\ f\cos\left(\frac{2\pi t}{T^*}\right) \\ -\frac{fh_1h_2V}{h_2^2+\omega_0^2}\cos\left(\frac{2\pi t}{T^*}\right) + \frac{fh_1V\omega_0}{h_2^2+\omega_0^2}\sin\left(\frac{2\pi t}{T^*}\right) \\ -\frac{ab_0g\omega_0^2}{(\omega_0^2+g)^2+a^2\omega_0^2}\cos\left(\frac{2\pi t}{T^*}\right) + \frac{(\omega_0^2+g)b_0gw_0}{(\omega_0^2+g)^2+a^2\omega_0^2}\sin\left(\frac{2\pi t}{T^*}\right) \end{pmatrix}\\
&+ \varepsilon^2 \begin{pmatrix} \frac{Tf^2b_0\omega_0(\omega_0^2+g)}{12(\omega_0^2+g)^2+12a^2\omega_0^2}\tilde{y}_1 + \frac{Tf^2ab_0\omega_0^3}{12(\omega_0^2+g)^2+12a^2\omega_0^2}\tilde{y}_2 + \frac{b_0\lambda_3^2}{\lambda_3^2-a\lambda_3-g}\tilde{y}_3 + \frac{b_0\lambda_5^2}{\lambda_5^2-a\lambda_5-g}\tilde{y}_5 \\ -\frac{Tf^2}{12}\tilde{y}_2 + \lambda_3\tilde{y}_3 + \lambda_5\tilde{y}_5 \\ \frac{Tf^3}{12\omega_0}\tilde{y}_1 + f\tilde{y}_3 + f\tilde{y}_5 \\ -\frac{Tf^3h_1h_2V}{12\omega_0(h_2^2+\omega_0^2)}\tilde{y}_1 + \frac{Tf^3h_1V}{12h_2^2+12\omega_0^2}\tilde{y}_2 + \frac{fh_1V}{\lambda_3-h_2}\tilde{y}_3 + \tilde{y}_4 + \frac{fh_1V}{\lambda_5-h_2}\tilde{y}_5 \\ -\frac{Tf^2ab_0g\omega_0}{12(\omega_0^2+g)^2+12a^2\omega_0^2}\tilde{y}_1 + \frac{Tf^2b_0g(\omega_0^2+g)}{12(\omega_0^2+g)^2+12a^2\omega_0^2}\tilde{y}_2 + \frac{b_0g\lambda_3}{\lambda_3^2-a\lambda_3-g}\tilde{y}_3 + \frac{b_0g\lambda_5}{\lambda_5^2-a\lambda_5-g}\tilde{y}_5 \end{pmatrix}\\
&+ o\left(\varepsilon^3\right), \qquad (5.5)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{y}_1 &= 3p_2 - p_2\cos\left(\frac{4\pi t}{T^*}\right) - 2p_1\sin\left(\frac{4\pi t}{T^*}\right),\\
\tilde{y}_2 &= -3p_1 + p_1\cos\left(\frac{4\pi t}{T^*}\right) - 2p_2\sin\left(\frac{4\pi t}{T^*}\right),
\end{aligned}$$

$$\begin{aligned}
\tilde{y}_3 =& \frac{Tf^2p_3}{4\lambda_3} + \frac{Tf^2p_3\lambda_3}{4\left(\lambda_3^2+4\omega_0^2\right)}\cos\left(\frac{4\pi t}{T^*}\right) - \frac{Tf^2p_3\omega_0}{2(\lambda_3^2+4\omega_0^2)}\sin\left(\frac{4\pi t}{T^*}\right),\\
\tilde{y}_4 =& \frac{Tf^2h_1}{4h_2} + \frac{Tf^2h_1h_2}{4\left(h_2^2+4\omega_0^2\right)}\cos\left(\frac{4\pi t}{T^*}\right) - \frac{Tf^2h_1\omega_0}{2(h_2^2+4\omega_0^2)}\sin\left(\frac{4\pi t}{T^*}\right),\\
\tilde{y}_5 =& \frac{Tf^2p_5}{4\lambda_5} + \frac{Tf^2p_5\lambda_5}{4\left(\lambda_5^2+4\omega_0^2\right)}\cos\left(\frac{4\pi t}{T^*}\right) - \frac{Tf^2p_5\omega_0}{2(\lambda_5^2+4\omega_0^2)}\sin\left(\frac{4\pi t}{T^*}\right).
\end{aligned}$$

Proof. Make a transformation

$$x \to x+\frac{l}{k},\ y \to y+\frac{f}{g},\ z \to z-(2n\pi+\arcsin T),\ u \to u+\frac{h_1}{h_2}T,\ v \to v-\left(\frac{al}{k}+\frac{bf}{g}\right),$$

then system (2.1) becomes

$$\begin{cases}
\dot{x} = ax + by + w\\
\dot{y} = cx + dy - \left(\frac{cl}{k}+\frac{df}{g}\right)(1-\cos z) + e\sqrt{1-T^2}\sin z\\
\dot{z} = gy\\
\dot{u} = h_1\sqrt{1-T^2}\sin z - h_1T(1-\cos z) + h_2u\\
\dot{v} = kx,
\end{cases} \tag{5.6}$$

which is equivalent to the following system at equilibrium $O(0,0,0,0,0)$

$$\begin{cases}
\dot{x} = ax + by + w\\
\dot{y} = cx + dy + e\sqrt{1-T^2}z - \left(\frac{cl}{2k}+\frac{df}{2g}\right)z^2 + o\left(z^3\right)\\
\dot{z} = gy\\
\dot{u} = h_1\sqrt{1-T^2}\,z - \frac{h_1T}{2}z^2 + h_2u + o\left(z^3\right)\\
\dot{v} = kx.
\end{cases} \tag{5.7}$$

Linearized system (5.7) at equilibrium O now yields the Jacobian matrix

$$J = \begin{bmatrix}
a & b & 0 & 0 & 1\\
c & d & e\sqrt{1-T^2} & 0 & 0\\
0 & g & 0 & 0 & 0\\
0 & 0 & h_1\sqrt{1-T^2} & h_2 & 0\\
k & 0 & 0 & 0 & 0
\end{bmatrix}.$$

The characteristic values of the characteristic equation give

$$\lambda_1 = i\omega_0,\quad \lambda_2 = -i\omega_0,\ \lambda_3,\ \lambda_4 = h_2,\ \lambda_5,$$

where λ_3 and λ_5 are defined as the condition in Theorem 5.3. If $|T| < 1$, $h_2 < 0$, $a + d < 0$, $aeg > 0$, $dk + aeg\sqrt{1-T^2} > 0$ and $dk + aeg\sqrt{1-T^2} + (a+d)a_2 < 0$, then there has $0 = Re(\lambda_1) \geq Re(\lambda_2) \geq Re(\lambda_3) \geq Re(\lambda_4) \geq Re(\lambda_5)$. Let

$$J\zeta_1 = \lambda_1\zeta_1,\quad J\zeta_3 = \lambda_3\zeta_3,\quad J\zeta_4 = \lambda_4\zeta_4 = h_2\zeta_4,\quad J\zeta_5 = \lambda_5\zeta_5.$$

Solving the eigenvectors $\zeta_1, \zeta_3, \zeta_4, \zeta_5$, one obtains

$$\zeta_1 = \begin{pmatrix} \frac{b_0\omega_0^2(\omega_0^2+g)-ab_0\omega_0^3 i}{(\omega_0^2+g)^2+a^2\omega_0^2} \\ \omega_0 i \\ f \\ -\frac{fh_1h_2\sqrt{1-T^2}+fh_1\sqrt{1-T^2}\omega_0 i}{h_2^2+\omega_0^2} \\ -\frac{ab_0g\omega_0^2+(\omega_0^2+g)b_0gw_0 i}{(\omega_0^2+g)^2+a^2\omega_0^2} \end{pmatrix}, \ \zeta_3 = \begin{pmatrix} \frac{b_0\lambda_3^2}{\lambda_3^2-a\lambda_3-g} \\ \lambda_3 \\ f \\ \frac{fh_1\sqrt{1-T^2}}{\lambda_3-h_2} \\ \frac{b_0g\lambda_3}{\lambda_3^2-a\lambda_3-g} \end{pmatrix},$$

$$\zeta_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \zeta_5 = \begin{pmatrix} \frac{b_0\lambda_5^2}{\lambda_5^2-a\lambda_5-g} \\ \lambda_5 \\ f \\ \frac{fh_1\sqrt{1-T^2}}{\lambda_5-h_2} \\ \frac{b_0g\lambda_5}{\lambda_5^2-a\lambda_5-g} \end{pmatrix}.$$

Define the matrix $Q = (Re(\zeta_1), -Im(\zeta_1), \zeta_3, \zeta_4, \zeta_5)$, i.e.,

$$Q = \begin{pmatrix} \frac{b_0\omega_0^2(\omega_0^2+g)}{(\omega_0^2+g)^2+a^2\omega_0^2} & \frac{ab_0\omega_0^3}{(\omega_0^2+g)^2+a^2\omega_0^2} & \frac{b_0\lambda_3^2}{\lambda_3^2-a\lambda_3-g} & 0 & \frac{b_0\lambda_5^2}{\lambda_5^2-a\lambda_5-g} \\ 0 & -\omega_0 & \lambda_3 & 0 & \lambda_5 \\ f & 0 & f & 0 & f \\ -\frac{fh_1h_2\sqrt{1-T^2}}{h_2^2+\omega_0^2} & \frac{fh_1\omega_0}{h_2^2+\omega_0^2} & \frac{fh_1\sqrt{1-T^2}}{\lambda_3-h_2} & 1 & \frac{fh_1\sqrt{1-T^2}}{\lambda_5-h_2} \\ -\frac{ab_0g\omega_0^2}{(\omega_0^2+g)^2+a^2\omega_0^2} & \frac{(\omega_0^2+g)b_0gw_0}{(\omega_0^2+g)^2+a^2\omega_0^2} & \frac{b_0g\lambda_3}{\lambda_3^2-a\lambda_3-g} & 0 & \frac{b_0g\lambda_5}{\lambda_5^2-a\lambda_5-g} \end{pmatrix}.$$

Make a transformation

$$(x, y, z, u, v)' = Q\,(x_1, y_1, z_1, u_1, v_1)',$$

so system (5.7) can be converted to

$$\begin{cases} \dot{x_1} = -\omega_0 y_1 + P_1\,(x_1, y_1, z_1, u_1, v_1) \\ \dot{y_1} = \omega_0 x_1 + P_2\,(x_1, y_1, z_1, u_1, v_1) \\ \dot{z_1} = \lambda_4 z_1 + P_3\,(x_1, y_1, z_1, u_1, v_1) \\ \dot{u_1} = h_2 u_1 + P_4\,(x_1, y_1, z_1, u_1, v_1) \\ \dot{v_1} = \lambda_5 v_1 + P_5\,(x_1, y_1, z_1, u_1, v_1) \end{cases} \tag{5.8}$$

where

$$P_1\,(x_1, y_1, z_1, u_1, v_1) = -\frac{Tf^2}{2}(x_1 + z_1 + v_1)^2 p_1,$$
$$P_2\,(x_1, y_1, z_1, u_1, v_1) = -\frac{Tf^2}{2}(x_1 + z_1 + v_1)^2 p_2,$$

$$P_3\left(x_1, y_1, z_1, u_1, v_1\right) = -\frac{Tf^2}{2}(x_1+z_1+v_1)^2 p_3,$$
$$P_4\left(x_1, y_1, z_1, u_1, v_1\right) = -\frac{Tf^2}{2}(x_1+z_1+v_1)^2 h_1,$$
$$P_5\left(x_1, y_1, z_1, u_1, v_1\right) = -\frac{Tf^2}{2}(x_1+z_1+v_1)^2 p_5,$$

here p_1 ,p_2, p_3, p_5 as the condition in Theorem 5.3.

Through system (5.8), it can be calculated

$$\begin{aligned}
g_{02} &= \frac{1}{4}\left[\frac{\partial^2 P_1}{\partial {x_1}^2} - \frac{\partial^2 P_1}{\partial {y_1}^2} - 2\frac{\partial^2 P_2}{\partial y_1 \partial x_1} + i\left(\frac{\partial^2 P_2}{\partial {x_1}^2} - \frac{\partial^2 P_2}{\partial {y_1}^2} + 2\frac{\partial^2 P_1}{\partial y_1 \partial x_1}\right)\right] \\
&= -\frac{Tf^2}{4}\left(p_1 + ip_2\right), \\
g_{20} &= \frac{1}{4}\left[\frac{\partial^2 P_1}{\partial x_1^2} - \frac{\partial^2 P_1}{\partial y_1^2} + 2\frac{\partial^2 P_2}{\partial y_1 \partial x_1} + i\left(\frac{\partial^2 P_2}{\partial x_1^2} - \frac{\partial^2 P_2}{\partial y_1^2} - 2\frac{\partial^2 P_1}{\partial y_1 \partial x_1}\right)\right] \\
&= -\frac{Tf^2}{4}\left(p_1 + ip_2\right), \\
G_{21} &= \frac{1}{8}\left(\frac{\partial^3 P_1}{\partial x_1^3} + \frac{\partial^3 P_1}{\partial y_1^2 \partial x_1} + \frac{\partial^3 P_2}{\partial y_1 \partial x_1^2} + \frac{\partial^3 P_2}{\partial y_1^3}\right) \\
&\quad + \frac{i}{8}\left(\frac{\partial^3 P_2}{\partial x_1^3} + \frac{\partial^3 P_2}{\partial y_1^2 \partial x_1} - \frac{\partial^3 P_1}{\partial y_1 \partial x_1^2} - \frac{\partial^3 P_1}{\partial y_1^3}\right) = 0, \\
h_{11}^1 &= \frac{1}{4}\left[\frac{\partial^2 P_3}{\partial x_1^2} + \frac{\partial^2 P_3}{\partial y_1^2}\right] = -\frac{Tf^2}{4} p_3, \\
h_{11}^2 &= \frac{1}{4}\left[\frac{\partial^2 P_4}{\partial x_1^2} + \frac{\partial^2 P_4}{\partial y_1^2}\right] = -\frac{Tf^2}{4} h_1, \\
h_{11}^3 &= \frac{1}{4}\left[\frac{\partial^2 P_5}{\partial {x_1}^2} + \frac{\partial^2 P_5}{\partial {y_1}^2}\right] = -\frac{Tf^2}{4} p_5, \\
h_{20}^1 &= \frac{1}{4}\left[\frac{\partial^2 P_3}{\partial {x_1}^2} - \frac{\partial^2 P_3}{\partial {y_1}^2} - 2i\frac{\partial^2 P_3}{\partial y_1 \partial x_1}\right] = -\frac{Tf^2}{4} p_3, \\
h_{20}^2 &= \frac{1}{4}\left[\frac{\partial^2 P_4}{\partial {x_1}^2} - \frac{\partial^2 P_4}{\partial {y_1}^2} - 2i\frac{\partial^2 P_4}{\partial y_1 \partial x_1}\right] = -\frac{Tf^2}{4} h_1, \\
h_{20}^3 &= \frac{1}{4}\left[\frac{\partial^2 P_5}{\partial x_1^2} - \frac{\partial^2 P_5}{\partial {y_1}^2} - 2i\frac{\partial^2 P_5}{\partial y_1 \partial x_1}\right] = -\frac{Tf^2}{4} p_5.
\end{aligned}$$

Further, solve the following equations

$$\begin{aligned}
&D\omega_{11} = -h_{11}, \\
&\left(D - 2i\omega_0 I\right)\omega_{20} = -h_{20},
\end{aligned}$$

where

$$\mathrm{D} = \begin{pmatrix} \lambda_3 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & \lambda_5 \end{pmatrix}, \quad h_{11} = \begin{pmatrix} h_{11}^1 \\ h_{11}^2 \\ h_{11}^3 \end{pmatrix}, \quad h_{20} = \begin{pmatrix} h_{20}^1 \\ h_{20}^2 \\ h_{20}^3 \end{pmatrix}.$$

By calculation, one obtains

$$\omega_{11}=\begin{pmatrix}\omega_{11}^1\\ \omega_{11}^2\\ \omega_{11}^3\end{pmatrix}=\begin{pmatrix}\frac{Tf^2}{4\lambda_3}p_3\\ \frac{Tf^2}{4h_2}h_1\\ \frac{Tf^2}{4\lambda_5}p_5\end{pmatrix},\quad \omega_{20}=\begin{pmatrix}\omega_{20}^1\\ \omega_{20}^2\\ \omega_{20}^3\end{pmatrix}=\begin{pmatrix}\frac{Tf^2p_3\lambda_3}{4\left(\lambda_3^2+4\omega_0^2\right)}+\frac{Tf^2p_3\omega_0}{2\left(\lambda_3^2+4\omega_0^2\right)}i\\ \frac{Tf^2h_1h_2}{4\left(h_2^2+4\omega_0^2\right)}+\frac{Tf^2h_1\omega_0}{2\left(h_2^2+4\omega_0^2\right)}i\\ \frac{Tf^2p_5\lambda_5}{4\left(\lambda_5^2+4\omega_0^2\right)}+\frac{Tf^2p_5\omega_0}{2\left(\lambda_5^2+4\omega_0^2\right)}i\end{pmatrix}.$$

On the other hand,

$$\begin{aligned}
G_{110}^1=&\frac{1}{2}\left[\frac{\partial^2P_1}{\partial z_1\partial x_1}+\frac{\partial^2P_2}{\partial z_1\partial y_1}+i\left(\frac{\partial^2P_2}{\partial z_1\partial x_1}-\frac{\partial^2P_1}{\partial z_1\partial y_1}\right)\right]=-\frac{Tf^2}{4}\left(p_1+ip_2\right),\\
G_{110}^2=&\frac{1}{2}\left[\frac{\partial^2P_1}{\partial u_1\partial x_1}+\frac{\partial^2P_2}{\partial u_1\partial y_1}+i\left(\frac{\partial^2P_2}{\partial u_1\partial x_1}-\frac{\partial^2P_1}{\partial u_1\partial y_1}\right)\right]=0,\\
G_{110}^3=&\frac{1}{2}\left[\frac{\partial^2P_1}{\partial w_1\partial x_1}+\frac{\partial^2P_2}{\partial w_1\partial y_1}+i\left(\frac{\partial^2P_2}{\partial w_1\partial x_1}-\frac{\partial^2P_1}{\partial w_1\partial y_1}\right)\right]=-\frac{Tf^2}{4}\left(p_1+ip_2\right),\\
G_{101}^1=&\frac{1}{2}\left[\frac{\partial^2P_1}{\partial z_1\partial x_1}-\frac{\partial^2P_2}{\partial z_1\partial y_1}+i\left(\frac{\partial^2P_2}{\partial z_1\partial x_1}+\frac{\partial^2P_1}{\partial z_1\partial y_1}\right)\right]=-\frac{Tf^2}{4}\left(p_1+ip_2\right),\\
G_{101}^2=&\frac{1}{2}\left[\frac{\partial^2P_1}{\partial u_1\partial x_1}-\frac{\partial^2P_2}{\partial u_1\partial y_1}+i\left(\frac{\partial^2P_2}{\partial u_1\partial x_1}+\frac{\partial^2P_1}{\partial u_1\partial y_1}\right)\right]=0,\\
G_{101}^3=&\frac{1}{2}\left[\frac{\partial^2P_1}{\partial w_1\partial x_1}-\frac{\partial^2P_2}{\partial w_1\partial y_1}+i\left(\frac{\partial^2P_2}{\partial w_1\partial x_1}+\frac{\partial^2P_1}{\partial w_1\partial y_1}\right)\right]=-\frac{Tf^2}{4}\left(p_1+ip_2\right).
\end{aligned}$$

It can be calculated that

$$\begin{aligned}
g_{21}=&G_{21}+\sum_{k=1}^{3}\left(2G_{110}^k\omega_{11}^k+G_{101}^k\omega_{20}^k\right)\\
=&\left(2G_{110}^1\omega_{11}^1+G_{101}^1\omega_{20}^1\right)+\left(2G_{110}^3\omega_{11}^3+G_{101}^3\omega_{20}^3\right)\\
=&-T^2f^4\left(p_1+ip_2\right)\left(\frac{p_3\lambda_5+p_5\lambda_3}{4\lambda_3\lambda_5}+\frac{p_3\lambda_3}{8\left(\lambda_3^2+4\omega_0^2\right)}+\frac{p_5\lambda_5}{8\left(\lambda_5^2+4\omega_0^2\right)}\right)\\
&-T^2f^4\left(p_1+ip_2\right)\left(\frac{p_3\omega_0}{4\left(\lambda_3^2+4\omega_0^2\right)}+\frac{p_5\omega_0}{4\left(\lambda_5^2+4\omega_0^2\right)}\right)i\\
=&-T^2f^4\left(\frac{p_1p_3\lambda_5+p_1p_5\lambda_3}{4\lambda_3\lambda_5}+\frac{p_1p_3\lambda_3-2p_2p_3\omega_0}{8\left(\lambda_3^2+4\omega_0^2\right)}+\frac{p_1p_5\lambda_5-2p_2p_5\omega_0}{8\left(\lambda_5^2+4\omega_0^2\right)}\right)\\
&-T^2f^4\left(\frac{p_2p_3\lambda_5+p_2p_5\lambda_3}{4\lambda_3\lambda_5}+\frac{p_2p_3\lambda_3+2p_1p_3\omega_0}{8\left(\lambda_3^2+4\omega_0^2\right)}+\frac{p_2p_5\lambda_5+2p_1p_5\omega_0}{8\left(\lambda_5^2+4\omega_0^2\right)}\right)i\\
=&N_1+N_2i.
\end{aligned}$$

According to the calculation and analysis, one can obtain the following critical characteristic quantities

$$\begin{aligned}
C_1(0)=&\frac{i}{2\omega_0}\left(g_{20}g_{11}-2\left|g_{11}\right|^2-\frac{1}{3}\left|g_{02}\right|^2\right)+\frac{1}{2}g_{21}\\
=&\left(\frac{1}{2}N_1-\frac{T^2f^4p_1p_2}{16\omega_0}\right)+\left(\frac{1}{2}N_2-\frac{T^2f^4}{96\omega_0}\left(4p_1^2+10p_2^2\right)\right)i,\\
\mu_2=&-\frac{\operatorname{Re}C_1(0)}{\alpha'(0)}=\frac{(T^2f^4p_1p_2-8\omega_0N_1)(a_1^2+(a_2-2\omega_0^2)\omega_0^2)}{8a_1c\omega_0^3},
\end{aligned}$$

$$\begin{aligned}\tau_2 &= -\frac{\operatorname{Im} C_1(0)+\mu_2\omega_0'(0)}{\omega_0}\\ &= \frac{a_1(f^4(2p_1^2+5p_2^2)T^2-24\omega_0N_2)+3\omega_0^2(a_2-2\omega_0)^2(f^4p_1p_2T^2-8\omega_0N_1)}{48a_1\omega_0^2},\end{aligned}$$

and

$$\beta_2 = 2\operatorname{Re} C_1(0) = N_1 - \frac{T^2f^4p_1p_2}{8\omega_0},$$

where

$$\begin{aligned}\alpha'(0) &= \operatorname{Re}\left(\lambda'\left(\mathrm{b}_0\right)\right) = \frac{a_1c\omega_0^2}{2a_1^2+2\omega_0^2(2\omega_0-a_2)^2},\\ \omega'(0) &= \operatorname{Im}\left(\lambda'\left(\mathrm{b}_0\right)\right) = \frac{-c\omega_0^4(2\omega_0-a_2)^2}{2a_1^2+2\omega_0^2(2\omega_0-a_2)^2}.\end{aligned}$$

Then the characteristic exponent of the Hopf bifurcation periodic solution can be calculated as

$$\begin{aligned}T^* &= \frac{2\pi}{\omega_0}\left(1+\tau_2\varepsilon^2+o\left(\varepsilon^4\right)\right),\\ \beta &= \beta_2\varepsilon^2+o\left(\varepsilon^4\right) = \left(N_1-\frac{T^2f^4p_1p_2}{8\omega_0}\right)\varepsilon^2+o\left(\varepsilon^4\right),\end{aligned}$$

where

$$\varepsilon^2 = \frac{b-b_0}{\mu_2}+o\left((b-b_0)^2\right).$$

If $\beta_2 < 0$ and $\mu_2 > 0$, then the Hopf bifurcation of system (5.7) at equilibrium O is supercritical, and the bifurcating periodic orbit is stable for $b > b_0$. If $\beta_2 > 0$ and $\mu_2 < 0$, then the Hopf bifurcation of system (5.7) at equilibrium O is subcritical, and the bifurcating periodic orbit is unstable for $b < b_0$.

By calculation, one has

$$\begin{aligned}q =& \varepsilon\exp\left(\frac{2\pi ti}{T^*}\right)+\frac{i\varepsilon^2}{6\omega_0}\left(g_{02}\exp\left(-\frac{4\pi ti}{T^*}\right)-3g_{20}\exp\left(\frac{4\pi ti}{T^*}\right)+6g_{11}\right)+o\left(\varepsilon^3\right)\\ =& \varepsilon\cos\left(\frac{2\pi t}{T^*}\right)+i\varepsilon\sin\left(\frac{2\pi t}{T^*}\right)+\frac{Tf^2\varepsilon^2}{12\omega_0}(p_1i-p_2)\left(\cos\left(\frac{4\pi t}{T^*}\right)+2i\sin\left(\frac{4\pi t}{T^*}\right)\right)\\ &+\frac{Tf^2\varepsilon^2}{12\omega_0}(-3p_1i+3p_2)+o\left(\varepsilon^3\right)\\ =& \varepsilon\cos\left(\frac{2\pi t}{T^*}\right)+\frac{Tf^2\varepsilon^2}{12\omega_0}\left(3p_2-p_2\cos\left(\frac{4\pi t}{T^*}\right)-2p_1\sin\left(\frac{4\pi t}{T^*}\right)\right)\\ &+\left(\varepsilon\sin\left(\frac{2\pi t}{T^*}\right)+\frac{Tf^2\varepsilon^2}{12\omega_0}\left(p_1\cos\left(\frac{4\pi t}{T^*}\right)-2p_2\sin\left(\frac{4\pi t}{T^*}\right)-3p_1\right)\right)i+o\left(\varepsilon^3\right),\\ y_1 =& \operatorname{Re}\, q = \varepsilon\cos\left(\frac{2\pi t}{T^*}\right)+\frac{Tf^2\varepsilon^2}{12\omega_0}\left(3p_2-p_2\cos\left(\frac{4\pi t}{T^*}\right)-2p_1\sin\left(\frac{4\pi t}{T^*}\right)\right)+o\left(\varepsilon^3\right),\\ y_2 =& \operatorname{Im}\, q = \varepsilon\sin\left(\frac{2\pi t}{T^*}\right)+\frac{Tf^2\varepsilon^2}{12\omega_0}\left(p_1\cos\left(\frac{4\pi t}{T^*}\right)-2p_2\sin\left(\frac{4\pi t}{T^*}\right)-3p_1\right)+o\left(\varepsilon^3\right),\\ y_3 =& \omega_{11}^1|q|^2+\operatorname{Re}\left(\omega_{20}^1q^2\right)+o\left(|q|^3\right)\\ =& \frac{Tf^2p_3}{4\lambda_3}\left(y_1^2+y_2^2\right)+\frac{Tf^2p_3\lambda_3}{4\left(\lambda_3^2+4\omega_0^2\right)}\left(y_1^2-y_2^2\right)-\frac{Tf^2p_3\omega_0}{\lambda_3^2+4\omega_0^2}(y_1y_2)+o\left(|q|^3\right),\end{aligned}$$

$$
\begin{aligned}
y_4 &= \omega_{11}^2|q|^2 + \mathrm{Re}\left(\omega_{20}^2 q^2\right) + o\left(|q|^3\right)\\
&= \frac{Tf^2h_1}{4h_2}\left(y_1^2+y_2^2\right) + \frac{Tf^2h_1h_2}{4\left(h_2^2+4\omega_0^2\right)}\left(y_1^2-y_2^2\right) - \frac{Tf^2h_1\omega_0}{h_2^2+4\omega_0^2}(y_1y_2) + o\left(|q|^3\right),\\
y_5 &= \omega_{11}^3|q|^2 + \mathrm{Re}\left(\omega_{20}^3 q^2\right) + o\left(|q|^3\right)\\
&= \frac{Tf^2p_5}{4\lambda_5}\left(y_1^2+y_2^2\right) + \frac{Tf^2p_5\lambda_5}{4\left(\lambda_5^2+4\omega_0^2\right)}\left(y_1^2-y_2^2\right) - \frac{Tf^2p_5\omega_0}{\lambda_5^2+4\omega_0^2}(y_1y_2) + o\left(|q|^3\right).
\end{aligned}
$$

Therefore, for $i = 1, 4$ and $n \in \mathbb{Z}$, the approximate expression of the bifurcated periodic solution of system (2.1) at equilibria $P_{in}\Big(-\frac{l}{k}, -\frac{f}{g}, 2n\pi+\arcsin T, -\frac{h_1}{h_2}T, \frac{al}{k}+\frac{bf}{g}\Big)$ is

$$
\begin{pmatrix} x\\ y\\ z\\ u\\ w \end{pmatrix}
= \begin{pmatrix} -\frac{l}{k}\\ -\frac{f}{g}\\ 2n\pi+\arcsin T\\ -\frac{h_1}{h_2}T\\ \frac{al}{k}+\frac{bf}{g} \end{pmatrix}
+ \begin{pmatrix}
\frac{b_0\omega_0^2(\omega_0^2+g)}{(\omega_0^2+g)^2+a^2\omega_0^2} & \frac{ab_0\omega_0^3}{(\omega_0^2+g)^2+a^2\omega_0^2} & \frac{b_0\lambda_3^2}{\lambda_3^2-a\lambda_3-g} & 0 & \frac{b_0\lambda_5^2}{\lambda_5^2-a\lambda_5-g}\\
0 & -\omega_0 & \lambda_3 & 0 & \lambda_5\\
f & 0 & f & 0 & f\\
-\frac{fh_1h_2V}{h_2^2+\omega_0^2} & \frac{fh_1V\omega_0}{h_2^2+\omega_0^2} & \frac{fh_1V}{\lambda_3-h_2} & 1 & \frac{fh_1V}{\lambda_5-h_2}\\
-\frac{ab_0g\omega_0^2}{(\omega_0^2+g)^2+a^2\omega_0^2} & \frac{(\omega_0^2+g)b_0gw_0}{(\omega_0^2+g)^2+a^2\omega_0^2} & \frac{b_0g\lambda_3}{\lambda_3^2-a\lambda_3-g} & 0 & \frac{b_0g\lambda_5}{\lambda_5^2-a\lambda_5-g}
\end{pmatrix}
\begin{pmatrix} y_1\\ y_2\\ y_3\\ y_4\\ y_5 \end{pmatrix}
$$

$$
= \begin{pmatrix} -\frac{l}{k}\\ -\frac{f}{g}\\ 2n\pi+\arcsin T\\ -\frac{h_1}{h_2}T\\ \frac{al}{k}+\frac{bf}{g} \end{pmatrix}
+ \begin{pmatrix}
\frac{b_0\omega_0^2(\omega_0^2+g)}{(\omega_0^2+g)^2+a^2\omega_0^2}y_1 + \frac{ab_0\omega_0^3}{(\omega_0^2+g)^2+a^2\omega_0^2}y_2 + \frac{b_0\lambda_3^2}{\lambda_3^2-a\lambda_3-g}y_3 + \frac{b_0\lambda_5^2}{\lambda_5^2-a\lambda_5-g}y_5\\
-\omega_0y_2 + \lambda_3y_3 + \lambda_5y_5\\
fy_1 + fy_3 + fy_5\\
-\frac{fh_1h_2V}{h_2^2+\omega_0^2}y_1 + \frac{fh_1V\omega_0}{h_2^2+\omega_0^2}y_2 + \frac{fh_1V}{\lambda_3-h_2}y_3 + y_4 + \frac{fh_1V}{\lambda_5-h_2}y_5\\
-\frac{ab_0g\omega_0^2}{(\omega_0^2+g)^2+a^2\omega_0^2}y_1 + \frac{(\omega_0^2+g)b_0gw_0}{(\omega_0^2+g)^2+a^2\omega_0^2}y_2 + \frac{b_0g\lambda_3}{\lambda_3^2-a\lambda_3-g}y_3 + \frac{b_0g\lambda_5}{\lambda_5^2-a\lambda_5-g}y_5
\end{pmatrix},
$$

which implies that (5.5) is established. Thus the proof of Theorem 5.3 is completed. □

Similar to Theorem 5, the approximate expression of the bifurcated periodic solution of equilibria $P_{in}(i = 2, 3; n = 0, \pm1, \pm2, ...)$ can be obtained, which is omitted here.

Select the parameter $(a, c, d, e, f, g, h_1, h_2, l, k) = (1, 5, -2, 2, -1, 2, 1, -1, 1, -2)$ and vary parameter b. According to Theorem 5.1, it is easy to know the bifurcation value $b_0 \approx -1.938$, implying that system (2.1) has a stable equilibrium $P_{40}(0.5, 0.5, -0.848, -0.75, 0.46)$ when $b > b_0$. Fig.17(a) indicates that the system has stable attractor P_{40} when the parameter $b = -1.92$ When parameter $b = -2 < b_0$, Fig.17(b) shows that the system has periodic attractor with the initial value is $(0.1, 1.5, 3.6, -1, 1)$. And the Lyapunov exponents of periodic attractor are

$$LE_1 = 0.0001,\ LE_2 = -0.3205,\ LE_3 = -0.3205,\ LE_4 = -0.3609,\ LE_5 = -0.9882.$$

6. Conclusion and discussion

This paper has constructed and analyzed a new five-dimensional hyperchaotic system that can generate infinitely many hyperchaotic attractors with three positive

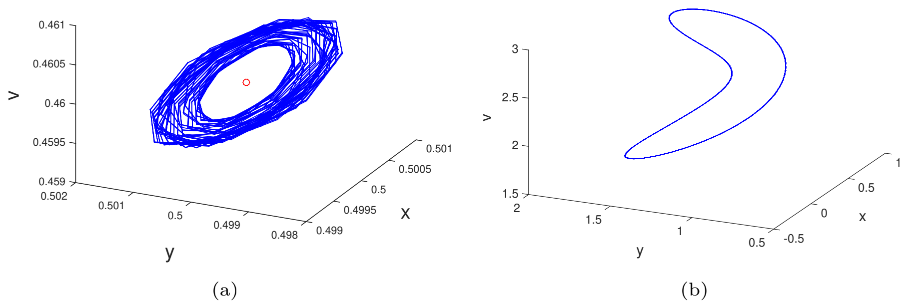


Figure 17. System (2.1) with parameters $(a, c, d, e, f, g, h_1, h_2, l, k) = (1, 5, -2, 2, -1, 2, 1, -1, 1, -2)$: (a) stable attractor for $b = -1.92$; (b) periodic attractor for $b = -2$.

Lyapunov exponents. The 5D hyperchaotic system can be classified into three categories based on the type of equilibria: no equilibria, infinitely many non-hyperbolic equilibria, and infinitely many hyperbolic equilibria. Under the three different types of equilibria, one investigates numerically and theoretically the dynamical behaviors of the hyperchaotic system with different parameters, including Lyapunov exponents, bifurcation, chaotic paths and compound structure. We obtained the stability of equilibrium points in the hyperbolic states through the center manifold theorem. Furthermore, the Hopf bifurcation and corresponding bifurcated periodic solution are derived using the normal form theory. It is strictly proved that the system has an infinite number of isolated bifurcated periodic orbits. It may be useful for understanding the complex dynamics of infinitely many (hyper-)chaotic attractors in 5D systems. Numerical results verified that this system could generate infinitely many coexisting hyperchaotic or chaotic or periodic attractors, which indicates that the system has a complex structure. These studies will contribute to the theoretical analysis and practical applications of high-dimensional hyperchaotic systems and are significant for interpreting hyperchaotic dynamics.

Although we have theoretically and numerically studied the dynamical properties of the 5D system, many complex dynamic behaviors and the problem of chaos mechanism deserve to be studied in depth. For example, finding out the relationship between the hyperchaotic behaviors and the number of equilibria of this new 5D system is still a fascinating problem. It is hoped that the theoretical and numerical results in this paper can provide insights into studying high-dimensional hyperchaotic and chaotic systems.

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