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SEVERAL NEW INTEGRAL INEQUALITIES OF THE SIMPSON TYPE FOR (α, s, m)-CONVEX FUNCTIONS

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Dedicated to retired Professor Bo-Yan Xi at Inner Mongolia Minzu University

Abstract In the paper, the authors present some integral inequalities of the Simpson type for functions whose derivatives are (α, s, m)-convex.

Keywords Simpson type, (α, s, m)-convex function, convex function, integral inequality, derivative.

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1. A brief review

First let us review some well-known results.

In the paper [4], Dragomir and his coauthors proved the following inequalities of the Simpson type, in which the remainders were expressed in terms of derivatives lower than the fourth order.

Theorem 1.1 ([4]). *Let $f : [a, b] \to \mathbb{R}$ be a continuously four-time differentiable mapping on (a, b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty.$$

Theorem 1.2 ([4]). *Suppose $f : [a, b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on $[a, b]$ and $f' \in L_1([a, b])$. Then*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{3}\|f'\|_1, \tag{1.1}$$

where $\|f'\|_1 = \int_a^b |f(x)|\,\mathrm{d}\,x$.

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The upper bound in (1.1) for L-Lipschitzian mapping was given by $\frac{5}{36}L(b-a)$ in the paper [4].

Theorem 1.3 ([4]). *Suppose that $f:[a,b]\to\mathbb{R}$ is an absolutely continuous mapping on $[a,b]$, whose derivative belongs to $L_p([a,b])$. Then*

$$\begin{aligned}&\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\&\le\frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{1/q}(b-a)^{1/q}\|f'\|_p,\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

For more generalizations and new inequalities of the Simpson type, please refer to the papers [1,3,11,15,18,20,23,29,32,33].

In [10], H. Hudzik and L. Maligranda considered, among others, the class of functions which are called s-convex in the second sense.

Definition 1.1 ([10]). Let s be a real number $s\in(0,1]$. A function $f:\mathbb{R}_0\to\mathbb{R}_0$ is said to be s-convex (in the second sense), or say, f belongs to the class K_s^2, if

$$f(\lambda x+(1-\lambda)y)\le\lambda^s f(x)+(1-\lambda)^s f(y)$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

In [28], the concept of extended s-convex functions was introduced as follows.

Definition 1.2 ([28]). For some $s\in[-1,1]$, a function $f:I\subseteq\mathbb{R}_0\to\mathbb{R}$ is said to be extended s-convex if

$$f(\lambda x+(1-\lambda)y)\le\lambda^s f(x)+(1-\lambda)^s f(y)$$

is valid for all $x,y\in I$ and $\lambda\in(0,1)$.

In the paper [19], Sarikaya and his two coauthors obtained some inequalities for differentiable convex mappings which are connected with integral inequalities of the Simpson type. We recite several of them as follows.

Theorem 1.4 ([19, Theorem 7]). *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L_1([a,b])$, where $a,b\in I^\circ$ with $a<b$. If $|f'|$ is s-convex on $[a,b]$ for some fixed $s\in(0,1]$, then*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\&\le(b-a)\frac{(s-4)6^{s+1}+2\times5^{s+2}-2\times3^{s+2}+2}{6^{s+2}(s+1)(s+2)}[|f'(a)|+|f'(b)|].\end{aligned}\tag{1.2}$$

Theorem 1.5 ([19, Theorem 8]). *Let $f:I\subseteq[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L_1([a,b])$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $s\in(0,1]$ and $q>1$, then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|$$

$$\leq \frac{b-a}{12}\left[\frac{1+2^{p+1}}{3(p+1)}\right]^{1/q}\left\{\left[\frac{|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right]^{1/q} + \left[\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right]^{1/q}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.6 ([19, Theorem 9]). *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1([a,b])$, where $a,b \in I^\circ$ with $a<b$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $s\in(0,1]$ and $q>1$, then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}+f(b)\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \leq \frac{b-a}{12}\left[\frac{1+2^{p+1}}{3(p+1)}\right]^{1/q}\left\{\left[\frac{(2^{s+1}-1)|f'(b)|^q+|f'(a)|^q}{2^s(s+1)}\right]^{1/q} + \left[\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^s(s+1)}\right]^{1/q}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.7 ([19, Theorem 10]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1([a,b])$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $s\in(0,1]$ and $q>1$, then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \leq \frac{b-a}{2}\left(\frac{5}{36}\right)^{1/q}\left\{\left[A(s)|f'(a)|^q+B(s)|f'(b)|^q\right]^{1/q} + \left[B(s)|f'(a)|^q+A(s)|f'(b)|^q\right]^{1/q}\right\}, \tag{1.3}$$

where

$$A(s)=\frac{(2s+1)3^{s+1}+2}{3\times 6^{(s+1)}(s+1)(s+2)}$$

and

$$B(s)=\frac{2\times 5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times 6^{(s+1)}(s+1)(s+2)}.$$

For more conclusions on this topic, please refer to [2,5–9,12,13,17,21,22,24,25,27,30] and closely related references therein.

The aim of this paper is to present several new integral inequalities of the Simpson type for functions whose derivatives are (α,s,m)-convex.

2. More definitions and a lemma

In [26], Xi, Gao, and Qi defined (α,s)-convex functions and (α,s,m)-convex functions, while they established integral inequalities of the Hermite–Hadamard type.

Definition 2.1 ([26]). For some $s \in [-1, 1]$ and $\alpha \in (0, 1]$, a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be (α, s)-convex if

$$f(tx + (1 - t)y) \le t^{\alpha s} f(x) + (1 - t^{\alpha})^s f(y)$$

for all $x, y \in I$ and $t \in (0, 1)$.

Definition 2.2 ([26]). For some $s \in [-1, 1]$ and $(\alpha, m) \in (0, 1]^2$, a function $f : [0, b] \to \mathbb{R}$ is said to be (α, s, m)-convex if

$$f(tx + m(1 - t)y) \le t^{\alpha s} f(x) + m(1 - t^{\alpha})^s f(y)$$

for all $x, y \in [0, b]$ and $t \in (0, 1)$.

Remark 2.1. By Definition 2.2 it follows that

1. if $s = 1$, then $f(x)$ is an (α, m)-convex function on $(0, b]$, see [16];
2. if $\alpha = 1$, then $f(x)$ is an extended (s, m)-convex function on $(0, b]$, see [31];
3. if $\alpha = m = 1$, then $f(x)$ is an extended s-convex function on $(0, b]$, see [28].

In the papers [1, 19], six mathematicians obtained several integral inequalities for differentiable convex mappings which are connected with integral inequalities of the Simpson type. They used the following lemma to prove these inequalities.

Lemma 2.1 ([1, Lemma 1] and [19, Lemma 1]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I° such that $f' \in L_1([a, b])$, where $a, b \in I^\circ$ with $a < b$. Then*

$$\begin{aligned} &\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \\ &= \frac{b-a}{2}\int_0^1 \left(\frac{t}{2} - \frac{1}{3}\right)\left[f'\left(tb + (1-t)\frac{a+b}{2}\right) - f'\left(ta + (1-t)\frac{a+b}{2}\right)\right]\mathrm{d}\,t. \end{aligned}$$

The main purpose of this paper is to establish some integral inequalities of the Simpson type for (α, s, m)-convex functions with the aid of Lemma 2.1.

3. Integral inequalities of the Simpson type

We now start out to present several integral inequalities of the Simpson type related to (α, s, m)-convex functions.

Theorem 3.1. *For some fixed $(\alpha, m) \in (0, 1]^2$ and $s \in (-1, 1]$, let $f : \left[0, \frac{b}{m}\right] \to \mathbb{R}$ be a differentiable mapping on $\left(a, \frac{b}{m}\right)$ such that $f' \in L_1\left(\left[a, \frac{b}{m}\right]\right)$ for $b > a \ge 0$. If $|f'|^q$ is (α, s, m)-convex on $\left[0, \frac{b}{m}\right]$ for $q \ge 1$, then*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\ &\le \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-1/q}\left\{\left[S_1(\alpha, s)|f'(a)|^q + mS_2(\alpha, s)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right. \\ &\quad \left. + \left[S_1(\alpha, s)|f'(b)|^q + mS_2(\alpha, s)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right\}, \end{aligned} \tag{3.1}$$

where

$$
\begin{aligned}
S_1(\alpha,s)&=\frac{2^{\alpha s+3}-(1-\alpha s)3^{\alpha s+1}}{2\times 3^{\alpha s+2}(\alpha s+1)(\alpha s+2)},\\
S_2(\alpha,s)&=\frac{2}{3\alpha}B\biggl(\biggl(\frac{2}{3}\biggr)^{\alpha};\frac{1}{\alpha},s+1\biggr)-\frac{1}{3\alpha}B\biggl(\frac{1}{\alpha},s+1\biggr)\\
&\quad+\frac{1}{4\alpha}B\biggl(\frac{2}{\alpha},s+1\biggr)-\frac{2}{9}{}_2F_1\biggl(\frac{2}{\alpha},-s;\frac{2}{\alpha}+1;\biggl(\frac{2}{3}\biggr)^{\alpha}\biggr),\\
B(u;x,y)&=\int_0^u z^{x-1}(1-z)^{y-1}\,\mathrm{d}\,z,\quad (\textit{incomplete beta function})\\
B(x,y)&=B(1;x,y),\quad (\textit{beta function})
\end{aligned}
$$

and

$$
{}_2F_1(c,d;e;z)=\frac{\Gamma(e)}{\Gamma(d)\Gamma(e-d)}\int_0^1 t^{d-1}(1-t)^{e-d-1}(1-zt)^{-c}\,\mathrm{d}\,t
$$

(*the Gauss hypergeometric function*)

for $0<u\le 1$, $x,y>0$, $e>d>0$, $|z|<1$, *and* $c\in\mathbb{R}$.

Proof. By Lemma 2.1 and the Hölder integral inequality [14], we obtain

$$
\begin{aligned}
&\biggl|\frac{1}{6}\biggl[f(a)+4f\biggl(\frac{a+b}{2}\biggr)+f(b)\biggr]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\biggr|\\
&\le\frac{b-a}{2}\int_0^1\biggl|\frac{t}{2}-\frac{1}{3}\biggr|\biggl[\biggl|f'\biggl(tb+(1-t)\frac{a+b}{2}\biggr)\biggr|+\biggl|f'\biggl(ta+(1-t)\frac{a+b}{2}\biggr)\biggr|\biggr]\,\mathrm{d}\,t\\
&\le\frac{b-a}{2}\biggl[\int_0^1\biggl|\frac{t}{2}-\frac{1}{3}\biggr|\,\mathrm{d}\,t\biggr]^{1-1/q}\biggl\{\biggl[\int_0^1\biggl|\frac{t}{2}-\frac{1}{3}\biggr|\biggl|f'\biggl(tb+(1-t)\frac{a+b}{2}\biggr)\biggr|^q\,\mathrm{d}\,t\biggr]^{1/q}\\
&\quad+\biggl[\int_0^1\biggl|\frac{t}{2}-\frac{1}{3}\biggr|\biggl|f'\biggl(ta+(1-t)\frac{a+b}{2}\biggr)\biggr|^q\,\mathrm{d}\,t\biggr]^{1/q}\biggr\},
\end{aligned}\tag{3.2}
$$

where

$$
\int_0^1\biggl|\frac{t}{2}-\frac{1}{3}\biggr|\,\mathrm{d}\,t=\frac{5}{36}\tag{3.3}
$$

and, from the (α,s,m)-convexity of $|f'|$,

$$
\begin{aligned}
&\int_0^1\biggl|\frac{t}{2}-\frac{1}{3}\biggr|\biggl|f'\biggl(tb+(1-t)\frac{a+b}{2}\biggr)\biggr|^q\,\mathrm{d}\,t\\
&\le\int_0^1\biggl|\frac{t}{2}-\frac{1}{3}\biggr|\biggl[t^{\alpha s}|f'(b)|^q+m(1-t^{\alpha})^s\biggl|f'\biggl(\frac{a+b}{2m}\biggr)\biggr|^q\biggr]\,\mathrm{d}\,t\\
&=S_1(\alpha,s)|f'(b)|^q+mS_2(\alpha,s)\biggl|f'\biggl(\frac{a+b}{2m}\biggr)\biggr|^q
\end{aligned}\tag{3.4}
$$

and

$$
\int_0^1\biggl|\frac{t}{2}-\frac{1}{3}\biggr|\biggl|f'\biggl(ta+(1-t)\frac{a+b}{2}\biggr)\biggr|^q\,\mathrm{d}\,t
$$

$$\begin{aligned}&\le \int_0^1 \left|\frac{t}{2}-\frac{1}{3}\right| \left[t^{\alpha s}|f'(a)|^q + m\big(1-t^{\alpha}\big)^s \left|f'\left(\frac{a+b}{2m}\right)\right|^q\right] \mathrm{d}\, t\\ &= S_1(\alpha,s)|f'(a)|^q + mS_2(\alpha,s)\left|f'\left(\frac{a+b}{2m}\right)\right|^q. \end{aligned} \tag{3.5}$$

Substituting the results (3.3), (3.4), and (3.5) into (3.2) gives the desired inequality (3.1). The proof of Theorem 3.1 is thus complete. □

Using the method in the proof of Theorem 3.1, we can obtain the following inequality for (α, s)-convex functions.

Theorem 3.2. *For some fixed $\alpha \in (0,1]$ and $s \in (-1,1]$, let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1([a,b])$ for $a, b \in I$ with $a < b$. If $|f'|^q$ is (α, s)-convex on $\big[a,b\big]$ for $q \ge 1$, then*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\ &\le \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-1/q} \times \left\{\left[S_1(\alpha,s)|f'(a)|^q + S_2(\alpha,s)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.\\ &\quad \left. + \left[S_1(\alpha,s)|f'(b)|^q + S_2(\alpha,s)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\},\end{aligned}$$

where $S_1(\alpha, s)$ and $S_2(\alpha, s)$ are defined as in Theorem 3.1.

In Theorem 3.2, if $\alpha = 1$, then

Corollary 3.1. *For some fixed $s \in (-1,1]$, let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1([a,b])$ for $a, b \in I$ with $a < b$. If $|f'|^q$ is extended s-convex on $\big[a,b\big]$ for $q \ge 1$, then*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\ &\le \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-1/q}\left[\left(\frac{[2^{s+3}-(1-s)3^{s+1}]|f'(a)|^q+[(2s+1)3^{s+1}+2]\big|f'\big(\frac{a+b}{2}\big)\big|^q}{2\times 3^{s+2}(s+1)(s+2)}\right)^{1/q}\right.\\ &\quad \left. + \left(\frac{[2^{s+3}-(1-s)3^{s+1}]|f'(b)|^q+[(2s+1)3^{s+1}+2]\big|f'\big(\frac{a+b}{2}\big)\big|^q}{2\times 3^{s+2}(s+1)(s+2)}\right)^{1/q}\right].\end{aligned} \tag{3.6}$$

In particular, if $q = 1$, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\ &\le \frac{b-va}{2}\,\frac{\begin{pmatrix}[2^{s+3}-(1-s)3^{s+1}]\big[|f'(a)|+|f'(b)|\big]\\ +2[(2s+1)3^{s+1}+2]\big|f'\big(\frac{a+b}{2}\big)\big|\end{pmatrix}}{2\times 3^{s+2}\big(s+1\big)\big(s+2\big)}.\end{aligned} \tag{3.7}$$

Remark 3.1. If $\big|f'\big(\frac{a+b}{2}\big)\big| = 0$, the estimation accuracy of the inequality (3.6) is better than that of the inequality (1.3). If $2\big|f'\big(\frac{a+b}{2}\big)\big| \le |f'(a)| + |f'(b)|$, the estimation accuracy of the inequality (3.7) is better than that of the inequality (1.2). This can be demonstrated by the following computation:

$$\begin{aligned}
&\frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+2}(s+1)(s+2)}[|f'(a)|+|f'(b)|]\\
&-\frac{2^{s+3}-(1-s)3^{s+1}}{4\times 3^{s+2}(s+1)(s+2)}[|f'(a)|+|f'(b)|]\\
&-v\frac{(2s+1)3^{s+1}+2}{2\times 3^{s+2}(s+1)(s+2)}\left|f'\left(\frac{a+b}{2}\right)\right|\\
\geq&\begin{pmatrix} 2\times 5^{s+2}-4\times 6^{s+1}-2\times 3^{s+2}\\ +2-2\times 4^{s}-3\times 6^{s}s-2^{s+1}\end{pmatrix}\frac{|f'(a)|+|f'(b)|}{6^{s+2}(s+1)(s+2)}\\
\geq&\, 0.
\end{aligned}$$

Theorem 3.3. *For some fixed $(\alpha,m)\in(0,1]^2$ and $s\in(-1,1]$, let $f:\left[0,\frac{b}{m}\right]\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L_1([a,\frac{b}{m}])$ for $b>a\geq 0$. If $|f'|^q$ is (α,s,m)-convex on $\left[0,\frac{b}{m}\right]$ for $q>1$, then*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\
&\leq\frac{b-a}{2}\left[\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}\right]^{1/p}\left\{\left[\frac{|f'(b)|^q}{\alpha s+1}+\frac{m}{\alpha}B\left(\frac{1}{\alpha},s+1\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right.\\
&\quad\left.+\left[\frac{|f'(a)|^q}{\alpha s+1}+\frac{m}{\alpha}B\left(\frac{1}{\alpha},s+1\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right\},
\end{aligned}\tag{3.8}$$

where $B(x,y)$ is defined as in Theorem 3.1 and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By Lemma 2.1 and the Hölder integral inequality [14], we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\
&\leq\frac{b-a}{2}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|+\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|\right]\mathrm{d}\,t\\
&\leq\frac{b-a}{2}\left[\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^p\mathrm{d}\,t\right]^{1/p}\left\{\left[\int_0^1\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right.\\
&\quad\left.+\left[\int_0^1\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right\}.
\end{aligned}\tag{3.9}$$

Since $|f'|^q$ is (α,s,m)-convex on $\left[0,\frac{b}{m}\right]$, we acquire

$$\begin{aligned}
\int_0^1\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}\,t&\leq\int_0^1\left[t^{\alpha s}|f'(b)|^q+m(1-t^\alpha)^s\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]\mathrm{d}\,t\\
&=\frac{|f'(b)|^q}{\alpha s+1}+\frac{m}{\alpha}B\left(\frac{1}{\alpha},s+1\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q
\end{aligned}\tag{3.10}$$

and

$$\int_0^1\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}\,t\leq\frac{|f'(a)|^q}{\alpha s+1}+\frac{m}{\alpha}B\left(\frac{1}{\alpha},s+1\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q.\tag{3.11}$$

It's easy to calculate that

$$\int_0^1 \left|\frac{t}{2}-\frac{1}{3}\right|^p \mathrm{d}\,t = \frac{2(1+2^{p+1})}{6^{p+1}(p+1)}. \tag{3.12}$$

Substitution of (3.10), (3.11), and (3.12) into (3.9) gives the desired (3.8). The proof of Theorem 3.3 is thus complete. □

By similar approach as in the proof of Theorem 3.3, we arrive at the following theorem.

Theorem 3.4. *For some fixed $\alpha \in (0,1]$ and $s \in (-1,1]$, let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1([a,b])$ for $a, b \in I$ with $a < b$. If $|f'|^q$ is (α, s)-convex on $\big[a,b\big]$ for $q > 1$, then*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\ &\le \frac{b-a}{2}\left[\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}\right]^{1/p}\left\{\left[\frac{|f'(b)|^q}{\alpha s+1}+\frac{1}{\alpha}B\left(\frac{1}{\alpha},s+1\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.\\ &\quad\left.+\left[\frac{|f'(a)|^q}{\alpha s+1}+\frac{1}{\alpha}B\left(\frac{1}{\alpha},s+1\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\},\end{aligned}$$

where $B(x,y)$ is defined as in Theorem 3.1 and $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 3.2. *Under conditions of Theorem 3.4 with $\alpha = 1$, we have*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\ &\le \frac{b-a}{2}\left[\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}\right]^{1/p}\left\{\left[\frac{|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right]^{1/q}\right.\\ &\quad\left.+\left[\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right]^{1/q}\right\},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

4. Conclusions

In this paper, we presented several integral inequalities of the Simpson type for functions whose derivatives are (α, s, m)-convex. The key tool is the identity in Lemma 2.1 and Hölder's integral inequality. The main results are contained in four theorems and two corollaries established in the third section.

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