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BIFURCATIONS OF TWISTED FINE HETEROCLINIC LOOP FOR HIGH-DIMENSIONAL SYSTEMS*

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Abstract In the paper, under twisted conditions, we consider the bifurcation problem of the fine heteroclinic loop with two hyperbolic critical points for high-dimensional systems. By using the foundational solutions of the linear variational equation of the unperturbed system along the heteroclinic orbits as the local coordinate system in the small tubular neighborhood of the heteroclinic loop, we construct the Poincaré maps and obtain the bifurcation equations. Then, by considering the small nonnegative solutions of the bifurcation equations, we get the main results of the reservation of the heteroclinic orbits, the existence and existence regions, the coexistence and coexistence regions of the 1-homoclinic loop, 1-periodic orbit, 2-homoclinic loop and 2-periodic orbit. Moreover, the bifurcation surfaces and graphs are given.

Keywords Twist, fine, heteroclinic loop, bifurcation, high-dimensional system.

MSC(2010) 34C23, 34C37, 37C29.

1. Introduction and hypotheses

In the study of bifurcation theory of nonlinear dynamical systems, the bifurcation problems of homoclinic and heteroclinic loops have been becoming more and more important, some results can be found in [1, 4, 5, 11, 24, 28, 29, 32, 33, 36, 39] and their references. At the same time, research on related problems has also been widely carried out, such as traveling wave solutions, for some results, see [8, 10, 25, 30, 35] and other relevant works of their authors.

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In [40,41], Zhu and Xia studied the bifurcation problems of the non-degenerated homoclinic and heteroclinic loops by generalizing the Floquet method to built the local coordinate system and Poincaré map. By simplifing the methods of [40,41], the researchers used the foundational solutions of the linear variational systems of the unperturbed systems along the homoclinic and heteroclinic orbits to establish a local coordinate systems in the small tubular neighborhoods of the homoclinic and heteroclinic orbits, then, studied the bifurcations of non-twisted degenerated homoclinic loop in [13] and non-twisted fine heteroclinic loop in [31]. From then on, this method was used to study the bifurcation problems of homoclinic and heteroclinic loops for higher dimensional systems. Some of the relevant results can be found in [2,6,7,9,12,14–23,27,37,38] and their references.

In this paper, we study the bifurcations of twisted fine heteroclinic loop.

Consider the following C^r system

$$\dot{z} = f(z), \tag{1.1}$$

where $r \geq 5$, $z \in \mathbf{R}^{m+n}$, $m \geq 2, n \geq 2$. At first, we assume that the following hypotheses are valid.

(H1). (Hyperbolicity) $z = p_i$, $i = 1, 2$ are the hyperbolic critical points of (1.1), $f(p_i) = 0$. The stable manifold $W^s_{p_i}$ and the unstable manifold $W^u_{p_i}$ of p_i are m-dimensional and n-dimensional, respectively. Moreover, $-\rho_i$ and λ_i are the simple real eigenvalues of $D_z f(p_i)$ such that any other eigenvalue σ_i of $D_z f(p_i)$ satisfies either $\mathrm{Re}\sigma_i < -\rho_0 < -\rho_i < 0$ or $0 < \lambda_i < \lambda_0 < \mathrm{Re}\sigma_i$, where ρ_0 and λ_0 are some positive constants.

(H2). (Non-degeneration) (1.1) has a heteroclinic loop $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_i = \{z = r_i(t) : t \in R\}$, $r_1(+\infty) = r_2(-\infty) = p_2$, $r_2(+\infty) = r_1(-\infty) = p_1$. $\dim(T_{r_i(t)} W^u_{p_i} \cap T_{r_i(t)} W^s_{p_{i+1}}) = 1$, $T_{r_i(t)} W^u_{p_i}$ is the tangent space of $W^u_{p_i}$ at $r_i(t)$, $T_{r_i(t)} W^s_{p_{i+1}}$ is the tangent space of $W^s_{p_{i+1}}$ at $r_i(t)$.

(H3). (Strong inclination) $\lim\limits_{x \to +\infty} (T_{r_i(t)} W^u_{p_i} + T_{r_i(t)} W^s_{p_{i+1}}) = T_{p_{i+1}} W^{uu}_{p_{i+1}} \oplus T_{p_{i+1}} W^s_{p_{i+1}}$, $\lim\limits_{x \to -\infty} (T_{r_i(t)} W^u_{p_i} + T_{r_i(t)} W^s_{p_{i+1}}) = T_{p_i} W^u_{p_i} \oplus T_{p_i} W^{ss}_{p_i}$, where, $W^{uu}_{p_i}$ and $W^{ss}_{p_i}$ are the strong unstable manifolds and the strong stable manifolds of p_i, respectively. $T_{p_i} W^{uu}_{p_i}$ is the tangent space of $W^{uu}_{p_i}$ at p_i, $T_{p_i} W^{ss}_{p_i}$ is the tangent space of $W^{ss}_{p_i}$ at p_i, $T_{p_i} W^u_{p_i}$ is the tangent space of $W^u_{p_i}$ at p_i, $T_{p_i} W^s_{p_i}$ is the tangent space of $W^s_{p_i}$ at p_i.

Denote $e_i^{\pm} = \lim\limits_{t \to \mp\infty} \dot{r_i}(t)/|\dot{r_i}(t)|$, where $e_i^+ \in T_{p_i} W^u_{p_i}$ and $e_i^- \in T_{p_{i+1}} W^s_{p_{i+1}}$ are unit eigenvectors corresponding to λ_i and $-\rho_{i+1}$, respectively. Furthermore, $\mathrm{span}(T_{p_i} W^{uu}_{p_i}, e_i^+) = T_{p_i} W^u_{p_i}$, $\mathrm{span}(T_{p_i} W^{ss}_{p_i}, e_{i-1}^-) = T_{p_i} W^s_{p_i}$.

(H4). (Fineness) $\beta_1 = \rho_1/\lambda_1 > 1$, $\beta_2 = \rho_2/\lambda_2 < 1$, $\beta_1\beta_2 = 1$.

Now, we study the following C^r perturbed system of (1.1)

$$\dot{z} = f(z) + g(z, \mu), \tag{1.2}$$

where $\mu \in \mathbf{R}^l$, $l \geq 3$, $0 \leq |\mu| \ll 1$, $g(z, 0) = 0$, $g(p_i, \mu) = 0$, $i = 1, 2$.

2. Local coordinate system

Denote $\mathcal{F}(z,\mu) = f(z) + g(z,\mu)$, then, there is a sufficiently small neighborhood U_i of $z = p_i$, such that system (1.2) has the following form

$$\dot{z} = \left(\frac{D\mathcal{F}(z,\mu)}{Dz} \mid_{z=p_i} \right) z + o(z), \tag{2.1}$$

where, $\frac{D\mathcal{F}(z,\mu)}{Dz}|_{z=p_i}$ is the coefficient matrix of the linear approximation system, $o(z)$ is the high order term of z. According to the standard theory of Jordan, there is a nonsingular linear transformation such that (2.1) has the following form

$$\dot{z} = \begin{pmatrix} \lambda_i(\mu) & 0 & 0 & 0 \\ 0 & -\rho_i(\mu) & 0 & 0 \\ 0 & 0 & A_i(\mu) & 0 \\ 0 & 0 & 0 & -B_i(\mu) \end{pmatrix} z + o(z), \tag{2.2}$$

where $\lambda_i(0) = \lambda_i$, $\rho_i(0) = \rho_i$, $\mathrm{Re}\sigma(A_i(0)) > \lambda_0$, $\mathrm{Re}\sigma(-B_i(0)) < -\rho_0$, $z = (x, y, u^*, v^*)^*$, $x, y \in \mathbf{R}^1$, $u \in \mathbf{R}^{n-1}$, $v \in \mathbf{R}^{m-1}$, $*$ means transposition.

Moreover, for $|\mu|$ small enough, we can introduce a C^r transformation such that (2.2) has the following form in U_i:

$$\begin{cases} \dot{x} = [\lambda_i(\mu) + \cdots]\, x + O(u)[O(y) + O(v)], \\ \dot{y} = [-\rho_i(\mu) + \cdots]\, y + O(v)[O(x) + O(u)], \\ \dot{u} = [A_i(\mu) + \cdots]\, u + O(x)[O(y) + O(v)], \\ \dot{v} = [-B_i(\mu) + \cdots]\, v + O(y)[O(x) + O(u)]. \end{cases} \tag{2.3}$$

In other words, the unstable manifold, stable manifold, strong unstable manifold, strong stable manifold, principal stable manifold and principal unstable manifold were all straightened in U_i:

$$\begin{aligned} &W^u_{p_i} = \{z : y = 0, v = 0\}, && W^s_{p_i} = \{z : x = 0, u = 0\}, \\ &W^{uu}_{p_i} = \{z : x = 0, y = 0, v = 0\}, && W^{ss}_{p_i} = \{z : x = 0, u = 0, y = 0\}, \\ &W^u_{p_i}/W^{uu}_{p_i} = \{z : u = 0, y = 0, v = 0\}, && W^s_{p_i}/W^{ss}_{p_i} = \{z : x = 0, u = 0, v = 0\}, \end{aligned}$$

where $W^u_{p_i}/W^{uu}_{p_i} \oplus W^{uu}_{p_i} = W^u_{p_i}, W^s_{p_i}/W^{ss}_{p_i} \oplus W^{ss}_{p_i} = W^s_{p_i}$. Obviously, for the unperturbed system (1.1), the local heteroclinic orbits were all straightened in U_i:

$$\Gamma \cap W^u_{p_i} = \{z : u = 0, y = 0, v = 0\}, \quad \Gamma \cap W^s_{p_i} = \{z : x = 0, u = 0, v = 0\}.$$

Denote $r_i(t) = (r_i^x(t), r_i^y(t), (r_i^u(t))^*, (r_i^v(t))^*)^*$. Assume $r_i(-T_i^1) = (\delta, 0, 0, 0^*)^*$, $r_i(T_i^2) = (0, \delta, 0^*, 0^*)^*$, where $T_i^1 > 0$, $T_i^2 > 0$ are large enough such that $\{(x, y, u^*, v^*)^* : |x|, |y|, |u|, |v| < 2\delta\} \subset U_i$.

Consider the linear system

$$\dot{z} = Df(r_i(t))z, \tag{2.4}$$

and its adjoint system

$$\dot{z} = -(Df(r_i(t)))^* z. \tag{2.5}$$

By [42,43], we see (2.4) and (2.5) have exponential dichotomies in both $\mathbf{R}^-$ and $\mathbf{R}^+$. Similar to [22,23,31], we have the following Lemma.

Lemma 2.1. *System* (2.4) *has a fundamental solution matrix* $Z_i(t) = (z_i^1(t), z_i^2(t), z_i^3(t), z_i^4(t))$ *satisfying*

$$\begin{aligned}
&z_i^1(t) \in (T_{r_i(t)} W_{p_i}^u)^c \cap (T_{r_i(t)} W_{p_{i+1}}^s)^c,\\
&z_i^2(t) = -\dot{r}_i(t)/|\dot{r}_i^y(T_i^2)| \in T_{r_i(t)} W_{p_i}^u \cap T_{r_i(t)} W_{p_{i+1}}^s,\\
&z_i^3(t) = (z_i^{3,1}(t), \cdots, z_i^{3,n-1}(t)) \in T_{r_i(t)} W_{p_i}^u \cap (T_{r_i(t)} W_{p_{i+1}}^s)^c,\\
&z_i^4(t) = (z_i^{4,1}(t), \cdots, z_i^{4,m-1}(t)) \in (T_{r_i(t)} W_{p_i}^u)^c \cap T_{r_i(t)} W_{p_{i+1}}^s,
\end{aligned}$$

$$Z_i(-T_i^1) = \begin{pmatrix} w_i^{11} & w_i^{21} & 0 & w_i^{41} \\ w_i^{12} & 0 & 0 & w_i^{42} \\ w_i^{13} & 0 & I & w_i^{43} \\ 0 & 0 & 0 & w_i^{44} \end{pmatrix}, \quad Z_i(T_i^2) = \begin{pmatrix} 1 & 0 & w_i^{31} & 0 \\ 0 & 1 & w_i^{32} & 0 \\ 0 & 0 & w_i^{33} & 0 \\ w_i^{14} & 0 & w_i^{34} & I \end{pmatrix},$$

where $W_3^{ss} = W_1^{ss}$, $w_i^{21} < 0$, $w_i^{12} \neq 0$, $\det w_i^{44} \neq 0$, $\det w_i^{33} \neq 0$. *Moreover, for* δ *small enough,* $||w_i^{1j}(w_i^{12})^{-1}|| \ll 1$ *for* $j \neq 2$, $||w_i^{3j}(w_i^{33})^{-1}|| \ll 1$ *for* $j \neq 3$, $||w_i^{4j}(w_i^{44})^{-1}|| \ll 1$ *for* $j \neq 4$.

Thus, we select $z_i^1(t)$, $z_i^2(t)$, $z_i^3(t)$, $z_i^4(t)$ as a local coordinate system along Γ_i. Let $\Phi_i(t) = (\phi_i^1(t), \phi_i^2(t), \phi_i^3(t), \phi_i^4(t)) = (Z_i^{-1}(t))^*$ be the fundamental solution matrix of (2.5), where $\phi_i^1(t)$ is bounded and tends to zero exponentially as $t \to \pm\infty$.

Denote $w_i^{12} = \Delta_i |w_i^{12}|$, we say that Γ_i is non-twisted as $\Delta_i = 1$, and twisted as $\Delta_i = -1$. Furthermore, we say that Γ is non-twisted as $\Delta_1\Delta_2 = 1$, and twisted as $\Delta_1\Delta_2 = -1$. In this paper, we consider the bifurcations for twisted case $\Delta_1\Delta_2 = -1$.

3. Successor function and bifurcation equation

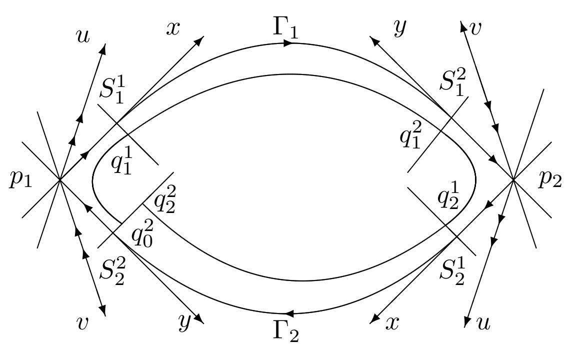

Figure 1. Heteroclinic loop

Let $h_i(t) = r_i(t) + Z_i(t)N_i$, $N_i = (n_i^1, 0, (n_i^3)^*, (n_i^4)^*)^*$, $n_i^3 = (n_i^{3,1}, \cdots, n_i^{3,n-1})^*$, $n_i^4 = (n_i^{4,1}, \cdots, n_i^{4,m-1})^*$. Define $S_i^1 = \{z = h_i(-T_i^1) : |x|, |y|, |u|, |v| < 2\delta\}$, $S_i^2 = \{z = h_i(T_i^2) : |x|, |y|, |u|, |v| < 2\delta\}$ be the cross sections of Γ_i at $t = -T_i^1$ and $t = T_i^2$, respectively. (Figure 1)

Now, we consider the map $F_i^1 : q_{i-1}^2 \in S_{i-1}^2 \mapsto q_i^1 \in S_i^1$, and the map $F_i^2 : q_i^1 \in S_i^1 \mapsto q_i^2 \in S_i^2$, where $S_0^2 = S_2^2$. Denote

$$q_i^1 = (x_i^1, y_i^1, (u_i^1)^*, (v_i^1)^*)^* = r_i(-T_i^1) + Z_i(-T_i^1)N_i^1, \quad N_i^1 = (n_i^{1,1}, 0, (n_i^{1,3})^*, (n_i^{1,4})^*)^*,$$
$$q_i^2 = (x_i^2, y_i^2, (u_i^2)^*, (v_i^2)^*)^* = r_i(T_i^2) + Z_i(T_i^2)N_i^2, \qquad N_i^2 = (n_i^{2,1}, 0, (n_i^{2,3})^*, (n_i^{2,4})^*)^*.$$

By the expressions of $Z_i(-T_i^1)$ and $Z_i(T_i^2)$, $i = 1, 2$, we get $x_i^1 \approx \delta$, $y_i^2 \approx \delta$, and

$$\begin{cases} n_i^{1,1} = (w_i^{12})^{-1}(y_i^1 - w_i^{42}(w_i^{44})^{-1}v_i^1), \\ n_i^{1,3} = u_i^1 - w_i^{13}(w_i^{12})^{-1}y_i^1 + a_i(w_i^{44})^{-1}v_i^1, \\ n_i^{1,4} = (w_i^{44})^{-1}v_i^1, \end{cases} \tag{3.1}$$

$$\begin{cases} n_i^{2,1} = x_i^2 - w_i^{31}(w_i^{33})^{-1}u_i^2, \\ n_i^{2,3} = (w_i^{33})^{-1}u_i^2, \\ n_i^{2,4} = -w_i^{14}x_i^2 + (w_i^{14}w_i^{31} - w_i^{34})(w_i^{33})^{-1}u_i^2 + v_i^2, \end{cases} \tag{3.2}$$

where, $a_i = w_i^{13}(w_i^{12})^{-1}w_i^{42} - w_i^{43}$, $i = 1, 2$.

Now, we construct the Poincare maps. By the continuity of $\beta_i(\mu) = \rho_i(\mu)/\lambda_i(\mu)$, we have $\beta_1(\mu) > 1$, $\beta_2(\mu) < 1$ for $|\mu| \ll 1$. In this paper, we just assume $\beta_1(\mu)\beta_2(\mu) = 1$. May as well denote $\beta := \beta_1(\mu) = 1/\beta_2(\mu)$.

Let τ_i be the flying time from q_{i-1}^2 to q_i^1, $s_1 = e^{-\lambda_1(\mu)\tau_1}$, $s_2 = e^{-\rho_2(\mu)\tau_2}$, then, using the linearization of (1.2) at p_i, and neglecting the higher order terms, we can get F_1^1 defined by

$$x_0^2 \approx s_1\delta, \ \ y_1^1 \approx s_1^\beta\delta, \ \ u_0^2 \approx s_1^{A_1(\mu)/\lambda_1(\mu)}u_1^1, \ \ v_1^1 \approx s_1^{B_1(\mu)/\lambda_1(\mu)}v_0^2, \tag{3.3}$$

and F_2^1 defined by

$$x_1^2 \approx s_2^\beta\delta, \ \ y_2^1 \approx s_2\delta, \ \ u_1^2 \approx s_2^{A_2(\mu)/\rho_2(\mu)}u_2^1, \ \ v_2^1 \approx s_2^{B_2(\mu)/\rho_2(\mu)}v_1^2, \tag{3.4}$$

where, (s_i, u_i^1, v_{i-1}^2), $i = 1, 2$ are called Silnikov coordinates.

Suppose that $z = h_i(t)$ is the solution of (1.2) in some small tube neighborhood of Γ_i, substituting it into (1.2) and using $\dot{r}_i(t) = f(r_i(t))$, $\dot{Z}_i(t) = Df(r_i(t))Z_i(t)$, we obtain the maps F_i^2 defined by

$$n_i^{2,j} = n_i^{1,j} + M_i^j\mu + h.o.t., \ \ j = 1, 3, 4, \tag{3.5}$$

where, $M_i^j = \int_{-\infty}^{+\infty} \phi_i^{j*}(t)g_\mu(r_i(t), 0)dt$, $i = 1, 2$, $j = 1, 3, 4$ are called Melnikov vectors.

Thus, we have defined the Poincaré maps $F_1 = F_1^2 \circ F_1^1$: $S_2^2 \mapsto S_1^2$ as

$$\begin{cases} n_1^{2,1} = (w_1^{12})^{-1}\delta s_1^\beta + M_1^1\mu + h.o.t., \\ n_1^{2,3} = u_1^1 - w_1^{13}(w_1^{12})^{-1}\delta s_1^\beta + M_1^3\mu + h.o.t., \\ n_1^{2,4} = (w_1^{44})^{-1}s_1^{B_1(\mu)/\lambda_1(\mu)}v_0^2 + M_1^4\mu + h.o.t., \end{cases} \tag{3.6}$$

and $F_2 = F_2^2 \circ F_2^1\colon S_1^2 \mapsto S_2^2$ as

$$\begin{cases} n_2^{2,1} = (w_2^{12})^{-1}\delta s_2 + M_2^1\mu + h.o.t., \\ n_2^{2,3} = u_2^1 - w_2^{13}(w_2^{12})^{-1}\delta s_2 + M_2^3\mu + h.o.t., \\ n_2^{2,4} = (w_2^{44})^{-1} s_2^{B_2(\mu)/\rho_2(\mu)} v_1^2 + M_2^4\mu + h.o.t.. \end{cases} \tag{3.7}$$

Let $q_0^2 = q_2^2$ and $G_i(q_{i-1}^2) = (G_i^1, G_i^3, G_i^4) = F_i(q_{i-1}^2) - q_i^2$. Owing to (3.1)$\sim$(3.7), we get the successor functions G_i as following:

$$\begin{cases} G_1^1 = \delta[(w_1^{12})^{-1} s_1^\beta - s_2^\beta] + M_1^1\mu + h.o.t., \\ G_1^3 = u_1^1 - w_1^{13}(w_1^{12})^{-1}\delta s_1^\beta - (w_1^{33})^{-1} s_2^{A_2(\mu)/\rho_2(\mu)} u_2^1 + M_1^3\mu + h.o.t., \\ G_1^4 = -v_1^2 + w_1^{14}\delta s_2^\beta + (w_1^{44})^{-1} s_1^{B_1(\mu)/\lambda_1(\mu)} v_0^2 + M_1^4\mu + h.o.t., \\ G_2^1 = \delta[(w_2^{12})^{-1} s_2 - s_1] + M_2^1\mu + h.o.t., \\ G_2^3 = u_2^1 - w_2^{13}(w_2^{12})^{-1}\delta s_2 - (w_2^{33})^{-1} s_1^{A_1(\mu)/\lambda_1(\mu)} u_1^1 + M_2^3\mu + h.o.t., \\ G_2^4 = -v_0^2 + w_2^{14}\delta s_1 + (w_2^{44})^{-1} s_2^{B_2(\mu)/\rho_2(\mu)} v_1^2 + M_2^4\mu + h.o.t.. \end{cases} \tag{3.8}$$

We call the following equation the bifurcation equation.

$$(G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4) = 0. \tag{3.9}$$

Thus, there is an one to one correspondence between the 1-heteroclinic orbit, 1-homoclinic orbit, 1-periodic orbit of (1.2) and the solution $Q = (s_1, s_2, u_1^1, u_2^1, v_1^2, v_2^2)$ of (3.9) with $s_1 \geq 0$, $s_2 \geq 0$.

Obviously, the equation $(G_1^3, G_1^4, G_2^3, G_2^4) = 0$ always has a solution $u_i^1 = u_i^1(s_1, s_2, \mu)$, $v_i^2 = v_i^2(s_1, s_2, \mu)$ $i = 1, 2$ for δ, $|\mu|$, s_1, s_2 sufficiently small. Substituting it into $(G_1^1, G_2^1) = 0$, we get

$$\begin{cases} (w_1^{12})^{-1} s_1^\beta - s_2^\beta + \delta^{-1} M_1^1\mu + h.o.t. = 0, \\ (w_2^{12})^{-1} s_2 - s_1 + \delta^{-1} M_2^1\mu + h.o.t. = 0. \end{cases} \tag{3.10}$$

Thus, we only need to consider the solutions $s_1 \geq 0$, $s_2 \geq 0$ of (3.10).

4. The preservations of heteroclinic orbits and bifurcations of 1-homoclinic loop and 1-periodic orbit

Denote $R_1^2 = \{\mu : M_1^1\mu > 0,\ \Delta_2 M_2^1\mu < 0,\ |\mu| \ll 1\}$, $R_2^1 = \{\mu : \Delta_1 M_1^1\mu < 0,\ M_2^1\mu > 0,\ |\mu| \ll 1\}$.

Theorem 4.1. *Suppose that the hypotheses* **(H1)**$\sim$**(H4)** *are valid,* $\mathrm{rank}(M_1^1, M_2^1) = 2$, $|\mu| \ll 1$, *then*

(i) There exists a $(l-1)$-dimensional surface L_i defined by

$$M_i^1\mu + h.o.t. = 0, i = 1, 2, \tag{4.1}$$

such that (1.2) *has a heteroclinic orbit joining* p_1 *and* p_2 *near* Γ_i *if and only if* $\mu \in L_i$, $i = 1, 2$, *that is, the heteroclinic orbit* Γ_i *is reserved for* $\mu \in L_i$. *Moreover,* (1.2) *has a heteroclinic loop near* Γ *if and only if* $\mu \in L_{12} = L_1 \cap L_2$ *which is a* $(l-2)$*-dimensional surface with normal plane span*$\{M_1^1, M_2^1\}$ *at* $\mu = 0$, *that is, the heteroclinic loop* Γ *is reserved for* $\mu \in L_{12}$.

(ii) *There exists a* $(l-1)$*-dimensional surface* $L_1^2 \subset R_1^2$ *which is defined by*

$$(-\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.)^\beta = \delta^{-1}M_1^1\mu + h.o.t., \tag{4.2}$$

and tangent to L_1 *at* $\mu = 0$ *such that* (1.2) *has a unique homoclinic loop* Γ_1^2 *connecting* p_1 *for* $\mu \in L_1^2$. *Meanwhile, there also exists a* $(l-1)$*-dimensional surface* $L_2^1 \subset R_2^1$ *which is defined by*

$$(\delta^{-1}M_2^1\mu + h.o.t.)^\beta = -\delta^{-1}w_1^{12}M_1^1\mu + h.o.t., \tag{4.3}$$

and tangent to L_1 *at* $\mu = 0$ *such that* (1.2) *has a unique homoclinic loop* Γ_2^1 *connecting* p_2 *for* $\mu \in L_2^1$.

Proof. Substitute $s_1 > 0$, $s_2 > 0$ (or $s_1 = 0$, $s_2 > 0$, or $s_1 > 0$, $s_2 = 0$) into (3.10), and then, by some simple calculations, the conclusions can be obtained. □

Now, we consider the 1-periodic orbits bifurcating from Γ. That is, consider the solutions of (3.10) which satisfy $s_1 > 0$, $s_2 > 0$.

If (3.10) has solutions $s_1 > 0$, $s_2 > 0$, then, we have

$$-\left(w_2^{12}(s_1 - \delta^{-1}M_2^1\mu + h.o.t.)\right)^\beta + (w_1^{12})^{-1}s_1^\beta + \delta^{-1}M_1^1\mu + h.o.t. = 0, \tag{4.4}$$

$$(w_1^{12})^{-1}\left((w_2^{12})^{-1}s_2 + \delta^{-1}M_2^1\mu + h.o.t.\right)^\beta - s_2^\beta + \delta^{-1}M_1^1\mu + h.o.t. = 0. \tag{4.5}$$

Denote $\mathcal{L}_1(s_1)$ and $\mathcal{L}_2(s_2)$ as the left hands of (4.4) and (4.5), respectively.

(A1). $\Delta_1 = 1$, $\Delta_2 = -1$

In this case, $R_1^2 = \{\mu : M_1^1\mu > 0, M_2^1\mu > 0, |\mu| \ll 1\}$, $R_2^1 = \{\mu : M_1^1\mu < 0, M_2^1\mu > 0, |\mu| \ll 1\}$. We divide the parameter space into the following subregions(Figure 5):

$(R_1^2)_0 \subset R_1^2$ bounded by L_2 and L_1^2, $(R_1^2)_1 \subset R_1^2$ bounded by L_1^2 and L_1.
$(R_2^1)_0 \subset R_2^1$ bounded by L_2 and L_2^1, $(R_2^1)_1 \subset R_2^1$ bounded by L_2^1 and L_1.
$R_0 = \{\mu : M_2^1\mu < 0, |\mu| \ll 1\}$.

(A2). $\Delta_1 = -1$, $\Delta_2 = 1$

In this case, $R_1^2 = \{\mu : M_1^1\mu > 0, M_2^1\mu < 0, |\mu| \ll 1\}$, $R_2^1 = \{\mu : M_1^1\mu > 0, M_2^1\mu > 0, |\mu| \ll 1\}$. We divide the parameter space into the following subregions(Figure 6):

$(R_1^2)_0 \subset R_1^2$ bounded by L_1 and L_1^2, $(R_1^2)_1 \subset R_1^2$ bounded by L_1^2 and L_2.
$(R_2^1)_0 \subset R_2^1$ bounded by L_1 and L_2^1, $(R_2^1)_1 \subset R_2^1$ bounded by L_2^1 and L_2.
$R_0 = \{\mu : M_1^1\mu < 0, |\mu| \ll 1\}$.

Theorem 4.2. *Suppose that the hypotheses* **(H1)**$\sim$**(H4)** *and* **(A1)** *or* **(A2)** *hold, then* *(i)* *system* (1.2) *has not any 1-periodic and 1-homoclinic loop connecting* p_1 *near* Γ *as* $\mu \in R_0$.

(ii) *If* $\mu \in (R_1^2)_0$, *then* $\mathcal{L}_1(s_1) = 0$ *has no any small non-negative solution, that is, system* (1.2) *has not any 1-periodic and 1-homoclinic loop connecting* p_1 *near* Γ *as* $\mu \in (R_1^2)_0$.

(iii) *If* $\mu \in L_1^2$, *then* $\mathcal{L}_1(s_1) = 0$ *has no any small positive solution except* $s_1 = 0$, *that is, system* (1.2) *has exactly one 1-homoclinic loop connecting* p_1 *near* Γ *as* $\mu \in L_1^2$, *but no any simple 1-periodic orbit.*

(iv) *If* $\mu \in (R_1^2)_1$, *then* $\mathcal{L}_1(s_1) = 0$ *has exactly one small positive solution, that is, system* (1.2) *has exactly one simple 1-periodic orbit near* Γ *as* $\mu \in (R_1^2)_1$.

Proof. We only consider the case **(A1)** $\Delta_1 = 1$**,** $\Delta_2 = -1$, another case **(A2)** $\Delta_1 = -1$**,** $\Delta_2 = 1$ is similar.

Obviously, if $s_1 \geq 0$, $s_2 \geq 0$, $w_2^{12} < 0$ and $M_2^1\mu < 0$, then, the left hand of the second equation of (3.10) will be negative, so, (3.10) has not any nonnegative solution $s_1 \geq 0$, $s_2 \geq 0$ if $w_2^{12} < 0$ and $M_2^1\mu < 0$. (i) proved.

In R_1^2, we consider the nonnegative solutions of equation (4.4). By (4.2) we have

$$\begin{cases} \mathcal{L}_1(0) > 0, \ \ \mu \in (R_1^2)_0, \\ \mathcal{L}_1(0) = 0, \ \ \mu \in L_1^2, \\ \mathcal{L}_1(0) < 0, \ \ \mu \in (R_1^2)_1. \end{cases}$$

Due to

$$\dot{\mathcal{L}}_1(s_1) = -\beta w_2^{12} \left(w_2^{12}(s_1 - \delta^{-1} M_2^1 \mu + h.o.t.) \right)^{\beta - 1} + \beta (w_1^{12})^{-1} s_1^{\beta - 1} > 0,$$

so, the function $\mathcal{L}_1(s_1)$ is monotonically increasing with respect to s_1. Thus, we get the three conclusions (ii) (iii) and (iv) (See Figure 2, 3, 4). $\square$

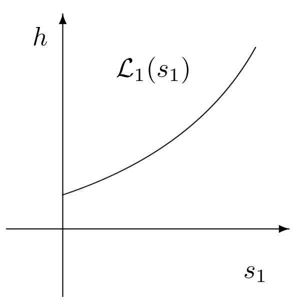

Figure 2. $\mu \in (R_1^2)_0$

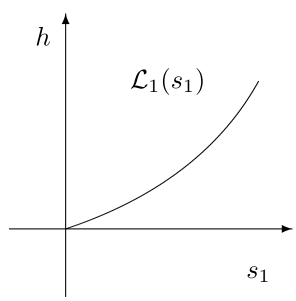

Figure 3. $\mu \in L_1^2$

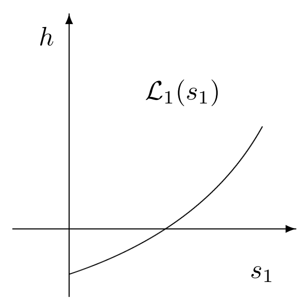

Figure 4. $\mu \in (R_1^2)_1$

Theorem 4.3. *Suppose that the hypotheses* **(H1)**$\sim$**(H4)** *and* **(A1)** *or* **(A2)** *hold, then*

(i) *System* (1.2) *has not any 1-periodic and 1-homoclinic loop connecting* p_2 *near* Γ *as* $\mu \in R_0$.

(ii) *If* $\mu \in (R_2^1)_0$, *then* $\mathcal{L}_2(s_2) = 0$ *has no any small non-negative solution, that is, system* (1.2) *has not any 1-periodic and 1-homoclinic loop connecting* p_2 *near* Γ *as* $\mu \in (R_2^1)_0$.

(iii) *If $\mu \in L_2^1$, then $\mathcal{L}_2(s_2) = 0$ has no any small positive solution except $s_2 = 0$, that is, system* (1.2) *has exactly one 1-homoclinic loop connecting p_2 near Γ as $\mu \in L_2^1$, but no any simple 1-periodic orbit.*

(iv) *If $\mu \in (R_2^1)_1$, then $\mathcal{L}_2(s_2) = 0$ has exactly one small positive solution, that is, system* (1.2) *has exactly one simple 1-periodic orbit near Γ as $\mu \in (R_2^1)_1$.*

Proof. The proof is similar to that of Theorem 4.2, we omit the details. $\square$

By Theorem 4.1, 4.2 and 4.3, we get the bifurcation graphs. (See Figure 5, 6)

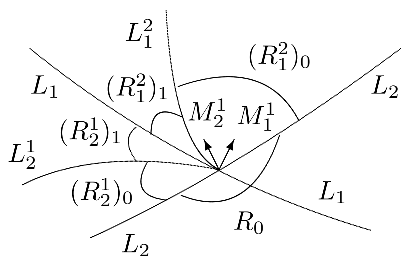

Figure 5. $\Delta_1 = 1$, $\Delta_2 = -1$

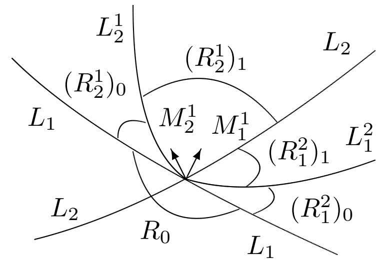

Figure 6. $\Delta_1 = -1$, $\Delta_2 = 1$

5. Bifurcations of 2-homoclinic loop and 2-periodic orbit

Let τ_1, τ_2, τ_3, τ_4 be the flying times from $q_0^2(x_0^2, y_0^2, (u_0^2)^*, (v_0^2)^*) \in S_2^2$ to $q_1^1(x_1^1, y_1^1, (u_1^1)^*, (v_1^1)^*) \in S_1^1$, $q_1^2(x_1^2, y_1^2, (u_1^2)^*, (v_1^2)^*) \in S_1^2$ to $q_2^1(x_2^1, y_2^1, (u_2^1)^*, (v_2^1)^*) \in S_2^1$, $q_2^2(x_2^2, y_2^2, (u_2^2)^*, (v_2^2)^*) \in S_2^2$ to $q_3^1(x_3^1, y_3^1, (u_3^1)^*, (v_3^1)^*) \in S_1^1$, $q_3^2(x_3^2, y_3^2, (u_3^2)^*, (v_3^2)^*) \in S_1^2$ to $q_4^1(x_4^1, y_4^1, (u_4^1)^*, (v_4^1)^*) \in S_2^1$, respectively. $s_1 = e^{-\lambda_1(\mu)\tau_1}$, $s_2 = e^{-\rho_2(\mu)\tau_2}$, $s_3 = e^{-\lambda_1(\mu)\tau_3}$, $s_4 = e^{-\rho_2(\mu)\tau_4}$. (See Figure 7)

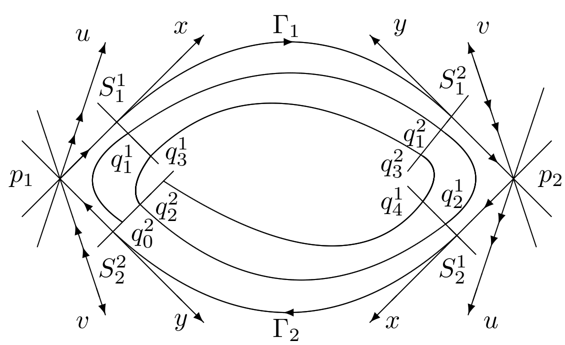

Figure 7. Two cycle mapping

Let $q_4^2 = q_0^2$. By some calculation, we get the following bifurcation equation.

$$(G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4, G_3^1, G_3^3, G_3^4, G_4^1, G_4^3, G_4^4) = 0, \tag{5.1}$$

where, the successor functions $G_i = (G_i^1, G_i^3, G_i^4)$, $i = 1, 2, 3, 4$ as followings.

$$\begin{cases} G_1^1 = \delta[(w_1^{12})^{-1}s_1^\beta - s_2^\beta] + M_1^1\mu + h.o.t., \\ G_1^3 = u_1^1 - w_1^{13}(w_1^{12})^{-1}\delta s_1^\beta - (w_1^{33})^{-1}s_2^{A_2(\mu)/\rho_2(\mu)}u_2^1 + M_1^3\mu + h.o.t., \\ G_1^4 = -v_1^2 + w_1^{14}\delta s_2^\beta + (w_1^{44})^{-1}s_1^{B_1(\mu)/\lambda_1(\mu)}v_0^2 + M_1^4\mu + h.o.t., \end{cases}$$

$$\begin{cases} G_2^1 = \delta[(w_2^{12})^{-1}s_2 - s_3] + M_2^1\mu + h.o.t., \\ G_2^3 = u_2^1 - w_2^{13}(w_2^{12})^{-1}\delta s_2 - (w_2^{33})^{-1}s_3^{A_1(\mu)/\lambda_1(\mu)}u_3^1 + M_2^3\mu + h.o.t., \\ G_2^4 = -v_2^2 + w_2^{14}\delta s_3 + (w_2^{44})^{-1}s_2^{B_2(\mu)/\rho_2(\mu)}v_1^2 + M_2^4\mu + h.o.t., \end{cases}$$

$$\begin{cases} G_3^1 = \delta[(w_1^{12})^{-1}s_3^\beta - s_4^\beta] + M_1^1\mu + h.o.t., \\ G_3^3 = u_3^1 - w_1^{13}(w_1^{12})^{-1}\delta s_3^\beta - (w_1^{33})^{-1}s_4^{A_2(\mu)/\rho_2(\mu)}u_4^1 + M_1^3\mu + h.o.t., \\ G_3^4 = -v_3^2 + w_1^{14}\delta s_4^\beta + (w_1^{44})^{-1}s_3^{B_1(\mu)/\lambda_1(\mu)}v_2^2 + M_1^4\mu + h.o.t., \end{cases}$$

$$\begin{cases} G_4^1 = \delta[(w_2^{12})^{-1}s_4 - s_1] + M_2^1\mu + h.o.t., \\ G_4^3 = u_4^1 - w_2^{13}(w_2^{12})^{-1}\delta s_4 - (w_2^{33})^{-1}s_1^{A_1(\mu)/\lambda_1(\mu)}u_1^1 + M_2^3\mu + h.o.t., \\ G_4^4 = -v_0^2 + w_2^{14}\delta s_1 + (w_2^{44})^{-1}s_4^{B_2(\mu)/\rho_2(\mu)}v_3^2 + M_2^4\mu + h.o.t.. \end{cases}$$

Thus, there is an one to one correspondence between the 2-heteroclinic loop, 2-homoclinic and 2-periodic orbit of (1.2) and the solution $Q = (s_1, s_2, s_3, s_4, u_1^1, u_2^1, u_3^1, u_4^1, v_1^2, v_0^2, v_2^2, v_3^2)$ of (5.1) with $s_1 \geq 0$, $s_2 \geq 0$, $s_3 \geq 0$, $s_4 \geq 0$.

It is easy to see that the equation $(G_1^3, G_1^4, G_2^3, G_2^4, G_3^3, G_4^3, G_4^3, G_4^4) = 0$ always has a solution $(u_1^1, u_2^1, u_3^1, u_4^1, v_1^2, v_0^2, v_2^2, v_3^2) = (u_1^1, u_2^1, u_3^1, u_4^1, v_1^2, v_0^2, v_2^2, v_3^2)(s_1, s_2, s_3, s_4, \mu)$ for δ, $|\mu|$, s_1, s_2, s_3, s_4 sufficiently small. Substituting it into $(G_1^1, G_2^1, G_3^1, G_4^1) = 0$, we have the bifurcation equation as following.

$$\begin{cases} (w_1^{12})^{-1}s_1^\beta - s_2^\beta + \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ (w_2^{12})^{-1}s_2 - s_3 + \delta^{-1}M_2^1\mu + h.o.t. = 0, \\ (w_1^{12})^{-1}s_3^\beta - s_4^\beta + \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ (w_2^{12})^{-1}s_4 - s_1 + \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases} \tag{5.2}$$

Thus, we only need to consider the solutions $s_1 \geq 0$, $s_2 \geq 0$, $s_3 \geq 0$, $s_4 \geq 0$ of (5.2).

Case 1. If (5.2) has a solution $s_1 > 0, s_2 > 0, s_3 = s_4 = 0$, then (5.2) becomes

$$\begin{cases} s_2^\beta = (w_1^{12})^{-1}s_1^\beta + \delta^{-1}M_1^1\mu + h.o.t., \\ (w_2^{12})^{-1}s_2 + \delta^{-1}M_2^1\mu + h.o.t. = 0, \\ \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ s_1 = \delta^{-1}M_2^1\mu + h.o.t.. \end{cases} \tag{5.3}$$

Substitute the second, third and fourth formulas of (5.3) into the first formula, we get

$$\left((w_1^{12})^{\frac{1}{\beta}}w_2^{12}+1\right)\left(\delta^{-1}M_2^1\mu+h.o.t.\right)+h.o.t.=0.$$

If $(w_1^{12})^{\frac{1}{\beta}}w_2^{12}+1\neq 0$, we get $M_2^1\mu+h.o.t.=0$, combined with (5.3), we get $s_1=0$, $s_2=0$, this means that the 2-heteroclinic loop is just the 1-heteroclinic loop for $\mu\in L_1\cap{}_2$, so, system (1.2) has not any 2-heteroclinic loop near Γ.

Case 2. It is obvious that system (1.2) has the 2-homoclinic loop joining p_1 near Γ if and only if (5.2) has the solution $s_1>0, s_2>0, s_3=0, s_4>0$; system (1.2) has the 2-homoclinic loop joining p_2 near Γ if and only if (5.2) has the solution $s_1>0, s_2>0, s_3>0, s_4=0$.

Theorem 5.1. *Suppose that hypotheses* **(H1)**$\sim$**(H4)**, **(A1)***(or* **(A2)***) are valid, then, in* $(R_1^2)_1$*, there exist a* $(l-1)$*-dimensional surface* L_3^{124} *which is tangent to* L_1 *at point* $\mu=0$*, such that system* (1.2) *has one* 2*-homoclinic loop connecting* p_1 *near* Γ *for* $\mu\in L_3^{124}$*,* $|\mu|\ll 1$*. (See Figure 8, 9)*

Proof. If (5.2) has a solution $s_1>0, s_2>0, s_3=0, s_4>0$, then (5.2) becomes

$$\begin{cases} s_2^{\beta}=(w_1^{12})^{-1}s_1^{\beta}+\delta^{-1}M_1^1\mu+h.o.t.,\\ s_2=-\delta^{-1}w_2^{12}M_2^1\mu+h.o.t.,\\ s_4^{\beta}=\delta^{-1}M_1^1\mu+h.o.t.,\\ s_1=(w_2^{12})^{-1}s_4+\delta^{-1}M_2^1\mu+h.o.t.. \end{cases} \tag{5.4}$$

So, we have $M_1^1\mu>0, \Delta_2M_2^1\mu<0$, and

$$\begin{aligned}&(-\delta^{-1}w_2^{12}M_2^1\mu+h.o.t.)^{\beta}\\ &=(w_1^{12})^{-1}\left[(w_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\frac{1}{\beta}}+\delta^{-1}M_2^1\mu+h.o.t.\right]^{\beta}+\delta^{-1}M_1^1\mu+h.o.t..\end{aligned} \tag{5.5}$$

Denote L_3^{124} is the $(l-1)$-dimensional surface defined by (5.5) in (R_1^2), then, L_3^{124} is tangent to L_1 at point $\mu=0$, and (1.2) has one 2-homoclinic loop connecting p_1 near Γ for $\mu\in L_3^{124}$, $|\mu|\ll 1$. Moreover

If $\Delta_1=1$, $\Delta_2=-1$, then, by (5.5), we have

$$\delta^{-1}M_1^1\mu\|_{L_3^{124}}<(-\delta^{-1}w_2^{12}M_2^1\mu+h.o.t.)^{\beta}=\delta^{-1}M_1^1\mu\|_{L_1^2}.$$

This means that L_3^{124} is located in the open region $(R_1^2)_1$.

Similarly, if $\Delta_1=-1$, $\Delta_2=1$, then, by (5.5), we have

$$\delta^{-1}M_1^1\mu\|_{L_3^{124}}>(-\delta^{-1}w_2^{12}M_2^1\mu+h.o.t.)^{\beta}=\delta^{-1}M_1^1\mu\|_{L_1^2}.$$

This means that L_3^{124} is located in the open region $(R_1^2)_1$.

Obviously, L_3^{124} is tangent to L_1 at point $\mu=0$ for the reason that $\beta>1$. $\square$

Theorem 5.2. *Suppose that hypotheses* **(H1)**$\sim$**(H4)**, **(A1)***(or* **(A2)***) are valid, then, in* $(R_2^1)_1$*, there exist a* $(l-1)$*-dimensional surface* L_4^{123} *which is tangent to* L_1 *at point* $\mu=0$*, such that system* (1.2) *has one* 2*-homoclinic loop connecting* p_2 *near* Γ *for* $\mu\in L_4^{123}$*,* $|\mu|\ll 1$*.*

Proof. The proof is similar to that of Theorem 5.1, we omit the details. □

Case 3. Denote D_2 is the open region which is bounded by L_3^{124} and L_4^{123}. Where, for the case $\Delta_1 = 1$, $\Delta_2 = -1$, the vector M_1^1 points into D_2 from L_4^{123}, and points to the outside of D_2 from L_3^{124}, for the case $\Delta_1 = -1$, $\Delta_2 = 1$, the vector M_1^1 points into D_2 from both of L_4^{123} and L_3^{124}.

Theorem 5.3. *System* (1.2) *has a 2-periodic orbit in the neighborhood of* Γ *for* $\mu \in D_2$. *(See Figure 8, 9)*

Proof. **(i)** If (5.2) has a solution $s_1 > 0, s_2 > 0, s_3 > 0, s_4 > 0$, then, in the neighborhood of L_3^{124}, we have

$$\begin{aligned} s_3 =& (w_2^{12})^{-1}\left\{(w_1^{12})^{-1}\left[(w_2^{12})^{-1}\Big((w_1^{12})^{-1}s_3^{\beta}+\delta^{-1}M_1^1\mu\Big)^{\frac{1}{\beta}}+\delta^{-1}M_2^1\mu\right]^{\beta}+\delta^{-1}M_1^1\mu\right\}^{\frac{1}{\beta}} \\ &+ \delta^{-1}M_2^1\mu + h.o.t.. \end{aligned}$$

Applying the Taylor expansion formula at $s_3^{\beta} = 0$ for the right hand, we get

$$\begin{aligned} s_3 =& K_3 \bullet s_3^{\beta} + (w_2^{12})^{-1}\left\{(w_1^{12})^{-1}\left[(w_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\frac{1}{\beta}}+\delta^{-1}M_2^1\mu\right]^{\beta}+\delta^{-1}M_1^1\mu\right\}^{\frac{1}{\beta}} \\ &+\delta^{-1}M_2^1\mu + h.o.t., \end{aligned} \tag{5.6}$$

where,

$$K_3 = \frac{\frac{1}{\beta}\left[(w_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\frac{1}{\beta}}+\delta^{-1}M_2^1\mu\right]^{\beta-1}}{(\delta^{-1}M_1^1\mu)^{1-\frac{1}{\beta}}(w_2^{12})^2(w_1^{12})^2\left\{(w_1^{12})^{-1}\left[(w_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\frac{1}{\beta}}+\delta^{-1}M_2^1\mu\right]^{\beta}+\delta^{-1}M_1^1\mu\right\}^{1-\frac{1}{\beta}}}.$$

Because of $\beta > 1$, we have $\left\{(w_1^{12})^{-1}\left[(w_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\frac{1}{\beta}}+\delta^{-1}M_2^1\mu\right]^{\beta}+\delta^{-1}M_1^1\mu\right\}^{\frac{1}{\beta}}+$ $\delta^{-1}M_2^1\mu$ is $O\left((\mu)^{\frac{1}{\beta}}\right)$ order, thus, we can take (5.6) as the following.

$$s_3 = K_3 \bullet s_3^{\beta} + (w_2^{12})^{-1}\left\{(w_1^{12})^{-1}(w_2^{12})^{-\beta}+1\right\}^{\frac{1}{\beta}}(\delta^{-1}M_1^1\mu)^{\frac{1}{\beta}} + h.o.t.. \tag{5.7}$$

Differentiating (5.7), and denoting the gradient of $s_3(\mu)$ with respect to μ by $(s_3)_\mu$, we get

$$\begin{aligned} (s_3)_\mu =& \beta K_3 \bullet s_3^{\beta-1}(s_3)_\mu \\ &+ \frac{1}{\beta}(w_2^{12})^{-1}\left\{(w_1^{12})^{-1}(w_2^{12})^{-\beta}+1\right\}^{\frac{1}{\beta}}(\delta^{-1}M_1^1\mu)^{\frac{1}{\beta}-1}\delta^{-1}M_1^1 + h.o.t.. \end{aligned}$$

Thus, if $\mu \in L_3^{124}$, then,

$$(s_3)_\mu \|_{s_3=0} = \tfrac{1}{\beta}(w_2^{12})^{-1}\left\{(w_1^{12})^{-1}(w_2^{12})^{-\beta}+1\right\}^{\frac{1}{\beta}}(\delta^{-1}M_1^1\mu)^{\frac{1}{\beta}-1}\delta^{-1}M_1^1 + h.o.t.. \tag{5.8}$$

(5.8) means that $s_3 = s_3(\mu)$ increases along the direction $w_2^{12}M_1^1$ in the small neighborhood of $\mu \in L_3^{124}$.

(ii) If μ is situated in the neighborhood of L_4^{123}, the proof is similar to that of **(i)**. □

By the Theorem 5.1, 5.2 and 5.3, we get the bifurcation graphs as following. (See Figure 8, 9)

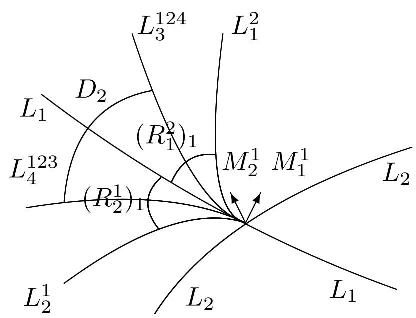

Figure 8. $\Delta_1 = 1$, $\Delta_2 = -1$

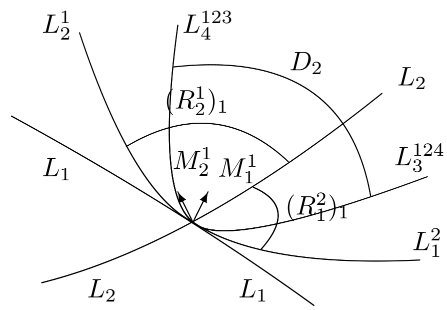

Figure 9. $\Delta_1 = -1$, $\Delta_2 = 1$

6. Conclusion

Note that $\mu \in (R_1^2)_1^1$ is the open sub-region of $(R_1^2)_1$ which is bounded by L_1^2 and L_3^{124}, $\mu \in (R_2^1)_1^1$ is the open sub-region of $(R_2^1)_1$ which is bounded by L_2^1 and L_4^{123}. Based on Theorems 4.1, 4.2, 4.3, and 5.1, 5.2 and 5.3, we obtain the following conclusions:

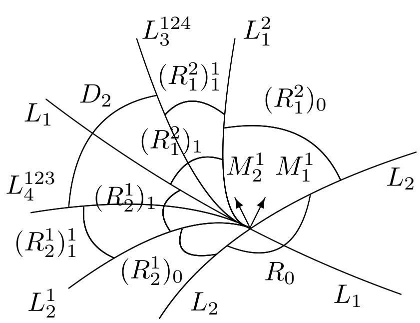

Figure 10. Bifurcation graph for $\Delta_1 = 1$, $\Delta_2 = -1$

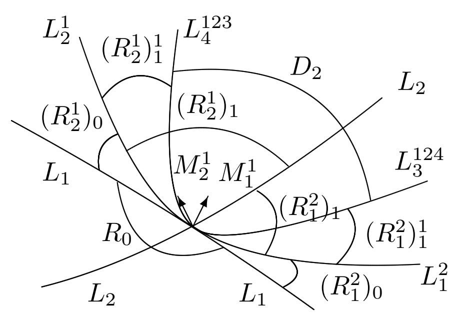

Figure 11. Bifurcation graph for $\Delta_1 = -1$, $\Delta_2 = 1$

Theorem 6.1. *Suppose that the hypotheses* **(H1)**$\sim$**(H4)** *and* **(A1)** *or* **(A2)** *hold, then*

(i) system (1.2) *has not any 1-periodic orbit and 1-homoclinic loop near* Γ *as* $\mu \in R_0 \bigcup (R_1^2)_0 \bigcup (R_2^1)_0$.

(ii) system (1.2) *has one 1-homoclinic loop connecting* p_2 *near* Γ *as* $\mu \in L_2^1$, *but no any simple 1-periodic orbit.*

(iii) system (1.2) *has one 1-homoclinic loop connecting* p_1 *near* Γ *as* $\mu \in L_1^2$, *but no any simple 1-periodic orbit.*

(iv) *system* (1.2) *has one simple 1-periodic orbit near* Γ *as* $\mu \in (R_1^2)_1^1 \bigcup (R_2^1)_1^1$.

(v) *system* (1.2) *has one 1-periodic orbit and one* 2*-homoclinic loop connecting* p_1 *near* Γ *for* $\mu \in L_3^{124}$,

(vi) *system* (1.2) *has one 1-periodic orbit and one* 2*-homoclinic loop connecting* p_2 *near* Γ *for* $\mu \in L_4^{123}$,

(vii) *System* (1.2) *has one 1-periodic orbit and one 2-periodic orbit in the neighborhood of* Γ *for* $\mu \in D_2$.

About the bifurcation graphs, see Figure 10 and 11.

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