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BILINEAR Θ-TYPE CALDERÓN-ZYGMUND OPERATOR AND ITS COMMUTATOR ON PRODUCT NON-HOMOGENEOUS GENERALIZED MORREY SPACES*

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Abstract Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space satisfying both the geometrically doubling and upper doubling conditions in the sense of Hytönen. In this setting, the authors first introduce generalized Morrey spaces $\mathcal{L}^{p,\varphi,\kappa}(\mu)$ and generalized weak Morrey spaces $W\mathcal{L}^{p,\varphi,\kappa}(\mu)$ for $p \in [1, \infty)$; second, under assumption that the dominating function λ and $(\varphi_1, \varphi_2, \varphi)$ satisfy certain conditions, the authors prove that bilinear θ-type Calderón-Zygmund operators $\widetilde{T}$ are bounded from product of spaces $\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu) \times \mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)$ into spaces $\mathcal{L}^{p,\varphi,\kappa}(\mu)$, and also bounded from product of spaces $\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu) \times \mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)$ into spaces $W\mathcal{L}^{p,\varphi,\kappa}(\mu)$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ for $1 < p_1, p_2 < \infty$; finally, the boundedness of the commutator $\widetilde{T}_{b_1,b_2}$ formed by $b_1, b_2 \in \mathrm{RBMO}(\mu)$ and $\widetilde{T}$ on spaces $\mathcal{L}^{p,\varphi,\kappa}(\mu)$ and on spaces $W\mathcal{L}^{p,\varphi,\kappa}(\mu)$ is obtained.

Keywords Non-homogeneous metric measure space, bilinear θ-type Calderón-Zygmund operator, commutator, space $\mathrm{RBMO}(\mu)$, product generalized Morrey space.

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1. Introduction

In order to unify the spaces of homogeneous type in the sense of Coifman and Weiss (see [2,3]) and non-doubling measure spaces whose measures satisfy the polynomial growth conditions (for example, see [11,15,27–30]), in 2010, Hytönen [12] first introduced a new class of metric measure spaces which satisfy the so-called geometrically doubling and upper doubling conditions, which are now called **non-homogeneous metric measure spaces** and simply denoted by $(\mathcal{X}, d, \mu)$. Since then, many papers focus on the integral operators and function spaces on $(\mathcal{X}, d, \mu)$. For example, Hytönen *et al.* [13] showed that the boundedness of a Calderón-Zygmund operator T on spaces $L^2(\mu)$ is equivalent to that of T on spaces $L^p(\mu)$ for some $p \in (1, \infty)$. In 2022, Wang and Xie [31] proved that the strongly singular integral operator T and its commutator $[b, T]$ formed by $b \in \mathrm{RBMO}(\mu)$ and T are bounded on $L^p(\mu)$ s-

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paces. In 2021, Zhao *et al.* [35] established some weak-type multiple weighted estimates for the iterated commutator $T_{\prod \vec{b}}$ generated by the multilinear Calderón-Zygmund operator T and $\vec{b} = (b_1, \cdots, b_m) \in (\widetilde{\mathrm{RBMO}}(\mu))^m$ on $(\mathcal{X}, d, \mu)$. In 2017, Lu and Tao [22] obtained the notion of generalized Morrey space on $(\mathcal{X}, d, \mu)$, and also established the boundedness of some usual operators of classical analysis (like the Hardy-Littlewood maximal operator and the Calderón-Zygmund operator) are bounded on these spaces. The more researchers about the various integral operators and function spaces on $(\mathcal{X}, d, \mu)$ can be seen [1, 4, 6, 17, 18, 21, 23, 24].

On the other hand, in 1985, Yabuta [33] first introduced the θ-type Calderón-Zygmund operator, and established the boundedness of θ-type Calderón-Zygmund operators on spaces $L^p(\mathbb{R}^n)$. Later, many researchers further studied the properties of this operator. For example, in 2023, V.S. Guliyev [7] proves that the operator T with Dini's type kernel and the commutator $T_{\vec{b}}$ generated by $\vec{b} = (b_1, \cdots, b_m) \in (\mathrm{BMO}(\mathbb{R}^n))^m$ and the T on generalized weighted Morrey. In 2021, V.S. Guliyev [8] obtained the boundedness of the operator T with Dini's type kernel and the commutator $T_{\vec{b}}$ which is generated by $\vec{b} = (b_1, \cdots, b_m) \in (\mathrm{BMO}(\mathbb{R}^n))^m$ on generalized weighted variable exponent Morrey spaces $M^{p(\cdot),\varphi}(\omega)$. At the same year, V.S. Guliyev and A.F. Ismayilova [10] showed that the operators T and $T_{\vec{b}}$ are bounded on generalized weighted Morrey spaces $M_{p,\varphi}(\omega)$. Recently, Lu in [20] has studied the weighted bounded properties for the θ-type Calderón-Zygmund operator T_θ and its commutator $T_{\theta,b}$ on weighted Morrey spaces. The more researchers on the different kinds of θ-type Calderón-Zygmund operators can be seen [16, 19, 25, 26, 32, 34, 36].

Motivated by the above results, in this paper, the authors first obtain the definitions of generalized Morrey spaces $\mathcal{L}^{p,\varphi,\kappa}(\mu)$ and generalized weak Morrey spaces $W\mathcal{L}^{p,\varphi,\kappa}(\mu)$ on $(\mathcal{X}, d, \mu)$, which are different from the definitions of generalized Morrey spaces and generalized weak Morrey spaces introduced in [22]. In other words, there are not any relations between the two classes of spaces; but, the two classes of generalized Morrey spaces can go back to the Morrey spaces introduced in [1]. Second, under assumption that the dominating function λ and the $(\varphi_1, \varphi_2, \varphi)$ satisfy certain conditions, the authors show that the bilinear θ-type Calderón-Zygmund operator $\widetilde{T}$ is bounded from the product of generalized Morrey spaces $\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu) \times \mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)$ into spaces $\mathcal{L}^{p,\varphi,\kappa}(\mu)$, and it is bounded from product spaces $\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu) \times \mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)$ into generalized weak Morrey spaces $W\mathcal{L}^{p,\varphi,\kappa}(\mu)$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ for $1 < p_1, p_2 < \infty$. Finally, the boundedness of the commutator $\widetilde{T}_{b_1,b_2}$ which is generated by $b_1, b_2 \in \mathrm{RBMO}(\mu)$ and $\widetilde{T}$ on spaces $\mathcal{L}^{p,\varphi,\kappa}(\mu)$ and on spaces $W\mathcal{L}^{p,\varphi,\kappa}(\mu)$ is also obtained.

Before stating the main results, we need to recall some necessary notions. The following definitions of geometrically doubling condition and upper doubling condition were introduced by Hytönen [12].

Definition 1.1. A metric space $(\mathcal{X}, d)$ is said to be *geometrically doubling* if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

Remark 1.1. Let $(\mathcal{X}, d)$ be a metric measure. Hytönen [12] showed that the geometrically doubling is equivalent to the following statement: for every $\epsilon \in (0, 1)$, any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$ contains at most $N_0 \epsilon^{-n_0}$ centers of disjoint balls $\{B(x_i, \epsilon r)\}_i$, here and in what follows, N_0 is as in Definition 1.1

and $n_0 := \log_2 N_0$.

Definition 1.2. A metric measure space $(\mathcal{X}, d, \mu)$ is said to be *upper doubling* if μ is a Borel measure on $\mathcal{X}$ and there exist a dominating function $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant $C_{(\lambda)}$, only depending on λ, such that, for each $x \in \mathcal{X}$, $r \to \lambda(x, r)$ is a non-decreasing and, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_{(\lambda)} \lambda(x, r/2). \tag{1.1}$$

A metric measure space $(\mathcal{X}, d, \mu)$ is called a ***non-homogeneous metric measure space*** if $(\mathcal{X}, d)$ is geometrically doubling and $(\mathcal{X}, d, \mu)$ is upper doubling.

Remark 1.2. Hytönen *et al.* [14] showed that there exists another dominating function $\widetilde{\lambda}$ such that $\widetilde{\lambda} \leq \lambda$, $C_{(\widetilde{\lambda})} \leq C_{(\lambda)}$ and, for all $x, y \in \mathcal{X}$ with $d(x, y) \leq r$,

$$\widetilde{\lambda}(x, r) \leq C_{(\lambda)} \widetilde{\lambda}(y, r). \tag{1.2}$$

Thus, in this paper, we also assume that the λ defined as in (1.1) satisfies (1.2).

The following definition of the coefficient $K_{B,S}$, which is more close to the quantity $K_{Q,R}$ introduced by Tolsa [29, 30], is from [12].

Definition 1.3. For any two balls B, S with $B \subset S \subset \mathcal{X}$, define

$$K_{B,S} := 1 + \int_{(2S)\setminus B} \frac{1}{\lambda(c_B, d(x, c_B))} \mathrm{d}\mu(x), \tag{1.3}$$

where c_B represents the center of ball B.

We now recall the following definition of (α, β)-doubling ball introduced in [12].

Definition 1.4. Let $\alpha, \beta \in (1, \infty)$. A ball $B \subset \mathcal{X}$ is said to be (α, β)-doubling if $\mu(\alpha B) \leq \beta \mu(B)$.

The more detailed contexts on the (α, β)-doubling balls, we refer readers to see Lemmas 3.2 and 3.3 in [12]. In what follows, let $\nu := \log_2 C_{(\lambda)}$ and $n_0 := \log_2 N_0$. Throughout this article, for any $\alpha \in (1, \infty)$ and ball B, the smallest (α, β_α)-doubling ball of the form $\alpha^j B$ with $j \in \mathbb{N}$ is simply denoted by $\widetilde{B}^\alpha$, where

$$\beta_\alpha := \alpha^{\max\{n_0, \nu\}} + 30^{n_0} + 30^\nu := \max\{\alpha^{n_0}, \alpha^\nu\} + 30^{n_0} + 30^\nu. \tag{1.4}$$

Furthermore, in this paper, we always assume that $\alpha = 6$ in (1.4), then the $(6, \beta_6)$-doubling ball $\widetilde{B}^6$ is simply denoted by $\widetilde{B}$.

The following notion of regularized BMO spaces ($=$ RBMO(μ)) is from [12].

Definition 1.5. Let $\rho \in (1, \infty)$. A real-valued function $f \in L^1_{\mathrm{loc}}$ is said to be in the *space* RBMO(μ) if there exists a positive constant C and, for any ball $B \subset \mathcal{X}$, a number f_B such that

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - f_B| \mathrm{d}\mu(x) \leq C \tag{1.5}$$

and, for any ball B and S such that $B \subset S \subset \mathcal{X}$,

$$|f_B - f_S| \leq C K_{B,S}, \tag{1.6}$$

where f_B represents the mean value of functions f over ball B, i.e.,

$$f_B := \frac{1}{\mu(B)} \int_B f(y) \mathrm{d}\mu(y).$$

The infimum of the positive constants C satisfying (1.5) and (1.6) is defined to be the RBMO(μ) norm of f and denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$.

The following notion of bilinear θ-type Calderón-Zygmund operators is from [37].

Definition 1.6. Let θ be a nonnegative and nondecreasing function defined on (0, ∞) and satisfy

$$\int_0^1 \frac{\theta(t)}{t} \mathrm{d}t < \infty.$$

A kernel $K(\cdot,\cdot,\cdot) \in L^1_{\mathrm{loc}}\big(\mathcal{X} \times \mathcal{X} \times \mathcal{X} \setminus \{(x,x,x) : x \in \mathcal{X}\}\big)$ is called a *bilinear θ-type Calderón-Zygmund kernel* if it satisfies the following conditions:

(i) for all $x, y_1, y_2 \in \mathcal{X}$ with $x \neq y_i$ for $i \in \{1,2\}$,

$$|K(x,y_1,y_2)| \leq C \left[\sum_{i=1}^{2} \lambda(x, d(x,y_i)) \right]^{-2}; \tag{1.7}$$

(ii) there exists a positive constant $c \in (1,\infty)$ such that, for all $x, x', y_1, y_2 \in \mathcal{X}$ with $cd(x,x') \leq \max_{1\leq i\leq 2} d(x,y_i)$,

$$|K(x,y_1,y_2) - K(x',y_1,y_2)| \leq C\theta\left(\frac{d(x,x')}{\sum_{i=1}^2 d(x,y_i)}\right) \left[\sum_{i=1}^{2} \lambda(x, d(x,y_i)) \right]^{-2}; \tag{1.8}$$

(iii) there exists a positive constant $c \in (1,\infty)$ such that, for all $x, y_1, y_1', y_2 \in \mathcal{X}$ with $cd(y_1,y_1') \leq \max_{1\leq i\leq 2} d(x,y_i)$,

$$|K(x,y_1,y_2) - K(x,y_1',y_2)| \leq C\theta\left(\frac{d(y_1,y_1')}{\sum_{i=1}^2 d(x,y_i)}\right) \left[\sum_{i=1}^{2} \lambda(x, d(x,y_i)) \right]^{-2}. \tag{1.9}$$

Remark 1.3. (a) Without loss of generality, for the simplicity, we may assume in (1.8) and (1.9) that $c \equiv 2$.

(b) If we take $\theta(t) = t^\delta$ with $\delta \in (0,1]$ in (1.8) and (1.9), then the bilinear θ-type Calderón-Zygmund kernel is just the standard bilinear C-Z kernel.

Let $L^\infty_b(\mu)$ be the space of all $L^\infty(\mu)$ functions with bounded support. A bilinear operator $\widetilde{T}_\theta$ is called a *bilinear θ-type Calderón-Zygmund operator* with kernel K satisfying (1.7), (1.8) and (1.9) if for all $f_1, f_2 \in L^\infty_b(\mu)$ with $x \in \mathcal{X} \setminus (\mathrm{supp}(f_1) \bigcap \mathrm{supp}(f_2))$,

$$\widetilde{T}(f_1,f_2)(x) := \int_{\mathcal{X}^2} K(x,y_1,y_2) f_1(y_1) f_2(y_2) \mathrm{d}\mu(y_1) \mathrm{d}\mu(y_2). \tag{1.10}$$

Given $b_1, b_2 \in \mathrm{RBMO}(\mu)$, the commutator $\widetilde{T}_{b_1,b_2}$ formed by b_1, b_2 and $\widetilde{T}$ is defined by

$$\widetilde{T}_{b_1,b_2}(f_1,f_2)(x) := b_1(x) b_2(x) \widetilde{T}(f_1,f_2)(x) - b_1(x) \widetilde{T}(f_1, b_2(\cdot) f_2)(x)$$

$$- b_2(x)\widetilde{T}(b_1(\cdot)f_1, f_2)(x) + \widetilde{T}(b_1(\cdot)f_1, b_2(\cdot)f_2)(x). \quad (1.11)$$

Equivalently, $\widetilde{T}_{b_1,b_2}(f_1, f_2)(x)$ can be formally written as

$$\int_{\mathcal{X}^2} K(x, y_1, y_2)\big(b_1(x) - b_1(y_1)\big)\big(b_2(x) - b_2(y_2)\big) f_1(y_1) f_2(y_2) \mathrm{d}\mu(y_1) \mathrm{d}\mu(y_2). \quad (1.11')$$

Also, the commutators $\widetilde{T}_{b_1}$ and $\widetilde{T}_{b_2}$ are respectively defined by

$$\widetilde{T}_{b_1}(f_1, f_2)(x) := b_1(x)\widetilde{T}(f_1, f_2)(x) - \widetilde{T}(b_1(\cdot)f_1, f_2)(x) \quad (1.12)$$

and

$$\widetilde{T}_{b_2}(f_1, f_2)(x) := b_2(x)\widetilde{T}(f_1, f_2)(x) - \widetilde{T}(f_1, b_2(\cdot)f_2)(x). \quad (1.13)$$

Now we state the definition of a generalized Morrey space, which is sightly modified from [11], as follows.

Definition 1.7. Let $\kappa \geq 1$, $1 \leq q < \infty$ and $\varphi(x, r) : \mathcal{X} \times (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. Then the *generalized Morrey space* $\mathcal{L}^{p,\varphi,\kappa}(\mu)$ is defined by

$$\mathcal{L}^{p,\varphi,\kappa}(\mu) := \big\{ f \in L^p_{\mathrm{loc}}(\mu) : \|f\|_{\mathcal{L}^{p,\varphi,\kappa}(\mu)} < \infty \big\},$$

where

$$\|f\|_{\mathcal{L}^{p,\varphi,\kappa}(\mu)} := \sup_{x \in \mathcal{X}, r>0} \frac{1}{\varphi(x, \kappa r)} [\mu(B(x, \kappa r))]^{-\frac{1}{p}} \left(\int_{B(x,r)} |f(y)|^p \mathrm{d}\mu(y) \right)^{\frac{1}{p}}, \quad (1.14)$$

and $B(x, r) = \{y \in \mathcal{X} : d(x, y) < r\}$ is an open ball centered at $x \in \mathcal{X}$ with radius $r > 0$.

Also, we denote by $W\mathcal{L}^{p,\varphi,\kappa}(\mu)$ the *generalized weak Morrey space* of all locally integrable functions satisfying

$$\|f\|_{W\mathcal{L}^{p,\varphi,\kappa}(\mu)} := \sup_{x \in \mathcal{X}, r>0} [\varphi(x, \kappa r)]^{-1} [\mu(B(x, \kappa r))]^{-\frac{1}{p}} \|f\|_{WL^p(B(x,r))}, \quad (1.15)$$

where $WL^p(B(x, r))$ represents the weak L^p-spaces of measurable functions f for which

$$\|f\|_{WL^p(B(x,r))} = \|f\chi_{B(x,r)}\|_{WL^p(\mathcal{X})} := \sup_{t>0} t[\mu(\{y \in B(x, r) : |f(y)| > t\})]^{\frac{1}{p}}.$$

Remark 1.4. (i) By means of a similar method that used in the proof of Theorem 1.2 in [11] and Theorem 7 in [1], it is easy to see that the norms $\|f\|_{\mathcal{L}^{p,\varphi,\kappa}(\mu)}$ and $\|f\|_{W\mathcal{L}^{p,\varphi,\kappa}(\mu)}$ are independent of the choice of κ for $\kappa > 1$.

(ii) If we take $(\mathcal{X}, d, \mu) := (\mathbb{R}^n, |\cdot|, \mathrm{d}x)$ and $\kappa \equiv 1$, then it is easy to see that generalized Morrey spaces $\mathcal{L}^{p,\varphi,\kappa}(\mu)$ and generalized weak Morrey spaces $W\mathcal{L}^{p,\varphi,\kappa}(\mu)$ defined as in Definition 1.7 are just the generalized Morrey space $M^{p,\varphi}(\mathbb{R}^n)$ and the weak generalized Morrey space $WM^{p,\varphi}(\mathbb{R}^n)$ introduced by Guliyev (see [9]).

(iii) If we take $(\mathcal{X}, d, \mu) := (\mathbb{R}^n, |\cdot|, \mu)$, then generalized Morrey spaces $\mathcal{L}^{p,\varphi,\kappa}(\mu)$ and generalized weak Morrey spaces $W\mathcal{L}^{p,\varphi,\kappa}(\mu)$ are just the generalized Morrey space $M^{p,\varphi}(k, \mu)$ and weak generalized Morrey space $WM^{p,\varphi}(k, \mu)$ with nondoubling measure (see [11]).

(iv) When $\varphi(x, \kappa r) := [\mu(B(x, \kappa r))]^{-\frac{1}{p}}$, then $\mathcal{L}^{p,\varphi,\kappa}(\mu) = L^p(\mu)$ and $W\mathcal{L}^{p,\varphi,\kappa}(\mu) = WL^p(\mu)$.

(v) If we take $\varphi(x,\kappa r)=[\mu(B(x,\kappa r))]^{-\frac{1}{q}}$ with $1<p\le q<\infty$ in (1.13), then the generalized Morrey space $\mathcal{L}^{p,\varphi,\kappa}(\mu)$ is just the Morrey space $M_p^q(\mu)$ introduced in [1].

The following definition of an ϵ-weak reverse doubling condition is from [4].

Definition 1.8. Let $\epsilon\in(0,\infty)$. A dominating function λ is said to satisfy the *ϵ-weak reverse doubling condition* if, for all $r\in(0,2\mathrm{diam}(\mathcal{X}))$ and $a\in(1,2\mathrm{diam}(\mathcal{X})/r)$, there exists some number $C(a)\in[1,\infty)$, depending only on a and $\mathcal{X}$, such that, for all $x\in\mathcal{X}$,

$$\lambda(x,ar)\ge C(a)\lambda(x,r) \tag{1.16}$$

and, moreover,

$$\sum_{k=1}^{\infty}\frac{1}{[C(a^k)]^{\epsilon}}<\infty. \tag{1.17}$$

The main results of this paper are stated as follows.

Theorem 1.1. *Let K satisfy (1.7), (1.8) and (1.9), $1<p_1,p_2<\infty$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, and the functions $\varphi,\varphi_i(i=1,2):\mathcal{X}\times(0,\infty)\to(0,\infty)$ be Lebesgue measurable functions with satisfying*

$$\sum_{k=0}^{\infty}\prod_{i=1}^{2}\varphi_i(x,6^{k+1}r)[\mu(B(x,6^{k+1}r))]^{\frac{1}{p_i}}\le C\varphi(x,6r)[\mu(B(x,6r))]^{\frac{1}{p}}. \tag{1.18}$$

Suppose that $\widetilde{T}$ is defined as in (1.10) and the λ satisfies $\frac{1}{p}$-weak reverse doubling. Then there exists a constant $C>0$ such that, for all $f_i\in\mathcal{L}^{p_i,\varphi_i,\kappa}(\mu)$ with $i=1,2$,

$$\|\widetilde{T}(f_1,f_2)\|_{\mathcal{L}^{p,\varphi,\kappa}(\mu)}\le C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}.$$

Theorem 1.2. *Let K satisfy (1.7), (1.8) and (1.9), $1<p_1,p_2<\infty$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\varphi,\varphi_i(i=1,2):\mathcal{X}\times(0,\infty)\to(0,\infty)$ be Lebesgue measurable functions satisfying (1.18). Suppose that $\widetilde{T}$ is defined as in (1.10) and the λ satisfies $\frac{1}{p}$-weak reverse doubling. Then there exists a constant $C>0$ such that, for all $f_i\in\mathcal{L}^{p_i,\varphi_i,\kappa}(\mu)$ with $i=1,2$,*

$$\|\widetilde{T}(f_1,f_2)\|_{W\mathcal{L}^{p,\varphi,\kappa}(\mu)}\le C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}.$$

Theorem 1.3. *Let $b_1,b_2\in\mathrm{RBMO}(\mu)$, K satisfy (1.7), (1.8) and (1.9), $1<p_1,p_2<\infty$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and φ, $\varphi_i(i=1,2):\mathcal{X}\times(0,\infty)\to(0,\infty)$ be Lebesgue measurable functions with satisfying*

$$\sum_{k=0}^{\infty}(k+1)\prod_{i=1}^{2}\varphi_i(x,6^{k+1}r)[\mu(B(x,6^{k+1}r))]^{\frac{1}{p_i}}\le C\varphi(x,6r)[\mu(B(x,6r))]^{\frac{1}{p}}. \tag{1.19}$$

Suppose that $\widetilde{T}$ is defined as in (1.10) and the λ satisfies $\frac{1}{p}$-weak reverse doubling. Then there exists a constant $C>0$ such that, for all $f_i\in\mathcal{L}^{p_i,\varphi_i,\kappa}(\mu)$ with $i=1,2$,

$$\|\widetilde{T}_{b_1,b_2}(f_1,f_2)\|_{\mathcal{L}^{p,\varphi,\kappa}(\mu)}\le C\prod_{i=1}^{2}\|b_i\|_{\mathrm{RBMO}(\mu)}\|f_i\|_{\mathcal{L}^{p_i,\varphi_i,\kappa}(\mu)}.$$

Theorem 1.4. *Let* $b_1, b_2 \in \mathrm{RBMO}(\mu)$*, K satisfy* (1.7), (1.8) *and* (1.9), $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\varphi, \varphi_i (i = 1, 2) : \mathcal{X} \times (0, \infty) \to (0, \infty)$ *be Lebesgue measurable functions with satisfying* (1.19)*. Suppose that* $\widetilde{T}$ *is defined as in* (1.10) *and the* λ *satisfies* $\frac{1}{p}$*-weak reverse doubling. Then there exists a constant* $C > 0$ *such that, for all* $f_i \in \mathcal{L}^{p_i,\varphi_i,\kappa}(\mu)$ *with* $i = 1, 2$,

$$\|\widetilde{T}_{b_1,b_2}(f_1, f_2)\|_{W\mathcal{L}^{p,\varphi,\kappa}(\mu)} \leq C \prod_{i=1}^{2} \|b_i\|_{\mathrm{RBMO}(\mu)} \|f_i\|_{\mathcal{L}^{p_i,\varphi_i,\kappa}(\mu)}.$$

Remark 1.5. Once Theorem 1.1 and Theorem 1.3 are proved, it is easy to see that Theorems 1.2 and 1.4 also hold. Hence, in this paper, we only focus the proof of Theorem 1.1 and Theorem 1.3.

Finally, we make some conventions on nations. Throughout this paper, we always denote by C, c or $\widetilde{c}$ is a positive constant being independent of the main parameter, but may vary from line to line. Furthermore, we use $C_{(\alpha)}$ to denote a positive constant depending on the main parameter α. Given any $p \in [1, \infty)$, we denote p' as its conjugate index, i.e., $p' := p/(p-1)$. Also, for any measurable subset $E \subset \mathcal{X}$, χ_E denotes its characteristic function. For any ball B and $f \in L^1_{\mathrm{loc}}(\mu)$, $m_B(f)$ also represents the mean value of f over B, namely,

$$m_B(f) := \frac{1}{\mu(B)} \int_B f(x) \mathrm{d}\mu(x).$$

2. Preliminaries

To prove the main results of this paper, we should recall some necessary results.

Lemma 2.1 (Lemmas 5.1 and 5.2, [12]). *Let* $(\mathcal{X}, d, \mu)$ *be a non-homogeneous metric measure space.*

(i) *For all balls* $B \subset R \subset S$, $K_{B,R} \leq K_{B,S}$.

(ii) *For any* $\rho \in [1, \infty)$*, there exists a positive constant* $C_{(\rho)}$*, depending on* ρ*, such that, for all balls* $B \subset S$ *with* $r_S \leq \rho r_B$, $K_{B,S} \leq C_{(\rho)}$.

(iii) *For any* $\alpha \in (1, \infty)$*, there exists a positive constant* $C_{(\alpha)}$*, depending on* α*, such that, for all balls* B, $K_{B,\widetilde{B}^{\alpha}} \leq C_{(\alpha)}$.

(iv) *There exists a positive constant c such that, for all balls* $B \subset R \subset S$,

$$K_{B,S} \leq K_{B,R} + cK_{R,S}.$$

In particular, if B and R are concentric, then $c = 1$.

(v) *There exists a positive constant* $\widetilde{c}$ *such that, for all balls* $B \subset R \subset S$, $K_{R,S} \leq \widetilde{c}K_{B,S}$*; moreover, if B and R are concentric, then* $K_{R,S} \leq K_{B,S}$.

Lemma 2.2 (Proposition 2.10, [12]). *Let* $\rho \in (1, \infty)$ *and* $f \in L^1_{\mathrm{loc}}(\mu)$*. The following statements are mutually equivalent:*

(i) $f \in \mathrm{RBMO}(\mu)$*;*

(ii) *there exists a positive constant C such that, for all balls B,*

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - m_{\widetilde{B}}(f)| \mathrm{d}\mu(x) \leq C$$

and, for all doubling balls $B \subset S$,

$$|m_{\widetilde{B}}(f) - m_{\widetilde{S}}(f)| \leq CK_{B,S}. \tag{2.1}$$

Corollary 2.1 (Corollary 2.3, [5]). *If $f \in \mathrm{RBMO}(\mu)$, then there exists a positive constant C such that, for any ball B, $\rho \in (1,\infty)$ and $r \in [1,\infty)$,*

$$\left(\frac{1}{\mu(\rho B)}\int_B |f(x) - m_{\widetilde{B}}(f)|^r \mathrm{d}\mu(x)\right)^{\frac{1}{r}} \le C\|f\|_{\mathrm{RBMO}(\mu)}. \tag{2.2}$$

Moreover, the infimum of positive constants C satisfying both (2.2) and (2.1) is an equivalent $\mathrm{RBMO}(\mu)$-norm of f.

Lemma 2.3 (Theorem 1.5, [37]). *Let $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and K satisfy (1.7), (1.8) and (1.9). Suppose that $\widetilde{T}$ defined as in (1.10) is bounded from product of spaces $L^1(\mu) \times L^1(\mu)$ into spaces $L^{\frac{1}{2},\infty}(\mu)$. Then there exists some constant C such that, for all $f_1 \in L^{p_1}(\mu)$ and $f_2 \in L^{p_2}(\mu)$,*

$$\|\widetilde{T}(f_1,f_2)\|_{L^p(\mu)} \le C\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.$$

Lemma 2.4 (Theorem 10, [32]). *Let $b_1, b_2 \in \mathrm{RBMO}(\mu)$ and $\widetilde{T}$ be as in (1.10) with kernel K satisfying (1.7), (1.8) and (1.9), which is bounded from product of spaces $L^{p_1}(\mu) \times L^{p_2}(\mu)$ into spaces $L^p(\mu)$ for all $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then the commutator $\widetilde{T}_{b_1,b_2}$ satisfies that there exists a positive constant C such that, for all $f_1 \in L^{p_1}(\mu)$ and $f_2 \in L^{p_2}(\mu)$,*

$$\|\widetilde{T}_{b_1,b_2}(f_1,f_2)\|_{L^p(\mu)} \le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.$$

3. Proof of Theorem 1.1

Proof. With loss of generality, we may assume that $\kappa = 6$ in (1.14). Represent functions f_i as

$$f_i := f_i^1 + f_i^\infty := f_i\chi_{6B} + f_i\chi_{\mathcal{X}\setminus(6B)}, \quad i = 1, 2, \tag{3.1}$$

where $B = B(x,r)$ represents the open ball centered at x with radius r. Then write

$$\begin{aligned}
&\|\widetilde{T}(f_1,f_2)\|_{\mathcal{L}^{p,\varphi,\kappa}(\mu)}\\
=&\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}(f_1,f_2)\|_{L^p(B(x,r))}\\
\le&\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}(f_1^1,f_2^1)\|_{L^p(B(x,r))}\\
&+\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}(f_1^1,f_2^\infty)\|_{L^p(B(x,r))}\\
&+\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}(f_1^\infty,f_2^1)\|_{L^p(B(x,r))}\\
&+\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}(f_1^\infty,f_2^\infty)\|_{L^p(B(x,r))}\\
=&\mathrm{D}_1+\mathrm{D}_2+\mathrm{D}_3+\mathrm{D}_4.
\end{aligned}$$

From $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, (1.14), (1.18) and Lemma 2.3, it then follows that

$$\mathrm{D}_1 \le \sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}(f_1^1,f_2^1)\|_{L^p(\mathcal{X})}$$

$$
\begin{aligned}
&\le C \sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|f_1\|_{L^{p_1}(B(x,6r))}\|f_2\|_{L^{p_2}(B(x,6r))}\\
&\le C \sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\varphi_1(x,36r)\varphi_2(x,36r)[\mu(B(x,36r))]^{\frac{1}{p_1}+\frac{1}{p_2}}\\
&\quad\times[\varphi_1(x,36r)]^{-1}[\mu(B(x,36r))]^{-\frac{1}{p_1}}\|f_1\|_{L^{p_1}(B(x,6r))}\\
&\quad\times[\varphi_2(x,36r)]^{-1}[\mu(B(x,36r))]^{-\frac{1}{p_2}}\|f_2\|_{L^{p_2}(B(x,6r))}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sup_{x\in\mathcal{X},r>0}\frac{\prod_{i=1}^{2}\varphi_i(x,36r)[\mu(B(x,36r))]^{\frac{1}{p_i}}}{\varphi(x,6r)[\mu(B(x,6r))]^{\frac{1}{p}}}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}.
\end{aligned}
$$

To estimate D_2, we first need to consider $|\widetilde{T}(f_1^1,f_2^\infty)(y)|$ for $y\in B(x,r)$. By applying (1.7), (1.14), the Hölder inequality and (1.17), we have

$$
\begin{aligned}
&|\widetilde{T}(f_1^1,f_2^\infty)(y)|\\
&\le C\int_{\mathcal{X}^2}\frac{|f_1^1(z_1)||f_2^\infty(z_2)|}{[\lambda(y,d(y,z_1))+\lambda(y,d(y,z_2))]^2}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
&\le C\int_{6B}|f_1(z_1)|\mathrm{d}\mu(z_1)\int_{\mathcal{X}\setminus(6B)}\frac{|f_2(z_2)|}{[\lambda(y,d(y,z_2))]^2}\mathrm{d}\mu(z_2)\\
&\le C\int_{6B}|f_1(z_1)|\mathrm{d}\mu(z_1)\sum_{k=1}^{\infty}\frac{1}{[\lambda(x,6^kr)]^2}\int_{6^{k+1}B}|f_2(z_2)|\mathrm{d}\mu(z_2)\\
&\le C\sum_{k=1}^{\infty}\frac{1}{[\lambda(x,6^kr)]^2}\prod_{i=1}^{2}\int_{6^{k+1}B}|f_i(z_i)|\mathrm{d}\mu(z_i)\\
&\le C\sum_{k=1}^{\infty}\frac{1}{[\lambda(x,6^kr)]^2}\prod_{i=1}^{2}\left(\int_{6^{k+1}B}|f_i(z_i)|^{p_i}\mathrm{d}\mu(z_i)\right)^{\frac{1}{p_i}}[\mu(6^{k+1}B)]^{1-\frac{1}{p_i}}\\
&\le C\sum_{k=1}^{\infty}\frac{[\mu(6^{k+1}B)]^{2-\frac{1}{p}}}{[\lambda(x,6^kr)]^2}\prod_{i=1}^{2}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}\\
&\quad\times[\varphi_i(x,6^{k+2}r)]^{-1}[\mu(B(x,6^{k+2}r))]^{-\frac{1}{p_i}}\left(\int_{6^{k+1}B}|f_i(z_i)|^{p_i}\mathrm{d}\mu(z_i)\right)^{\frac{1}{p_i}}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sum_{k=1}^{\infty}\frac{[\lambda(x,6^{k+1}r)]^{2-\frac{1}{p}}}{[\lambda(x,6^kr)]^2}\\
&\quad\times\prod_{i=1}^{2}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sum_{k=1}^{\infty}\frac{\prod_{i=1}^{2}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[\lambda(x,6^kr)]^{\frac{1}{p}}}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sum_{k=1}^{\infty}\frac{\prod_{i=1}^{2}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[C(6^k)]^{\frac{1}{p}}[\lambda(x,r)]^{\frac{1}{p}}},
\end{aligned}
$$

further, via (1.1), (1.14) and (1.18), we obtain

$$\begin{aligned}
\mathrm{D}_2 =& \sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}(f_1^1,f_2^\infty)\|_{L^p(B(x,r))}\\
\le& C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\\
&\times[\mu(B(x,r))]^{\frac{1}{p}}\sum_{k=1}^\infty\frac{\prod_{i=1}^2\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[C(6^k)]^{\frac{1}{p}}[\lambda(x,r)]^{\frac{1}{p}}}\\
\le& C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sup_{x\in\mathcal{X},r>0}\sum_{k=1}^\infty\frac{\prod_{i=1}^2\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{\varphi(x,6r)[\mu(B(x,6r))]^{\frac{1}{p}}}\\
\le& C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}.
\end{aligned}$$

With an argument similar to that used in the estimate of D_2, it is easy to obtain that

$$\mathrm{D}_3\le C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}.$$

Now let us estimate D_4. For any $y\in B(x,r)$, by applying the (1.7), (1.14), the Hölder inequality and (1.17), we deduce

$$\begin{aligned}
&|\widetilde{T}(f_1^\infty,f_2^\infty)(y)|\\
\le& C\int_{\mathcal{X}^2}\frac{|f_1^\infty(z_1)||f_2^\infty(z_2)|}{[\lambda(y,d(y,z_1))+\lambda(y,d(y,z_1))]^2}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
\le& C\int_{\mathcal{X}^2\setminus(6B)^2}\frac{|f_1(z_1)||f_2(z_2)|}{[\lambda(x,d(x,z_1))+\lambda(x,d(x,z_1))]^2}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
\le& C\int_{\mathcal{X}^2\setminus(6B)^2}\prod_{i=1}^2\frac{|f_i(z_i)|}{\lambda(x,d(x,z_i))}\mathrm{d}\mu(z_i)\\
\le& C\sum_{k=1}^\infty\prod_{i=1}^2\frac{1}{\lambda(x,6^kr)}\left(\int_{6^{k+1}B}|f_i(z_i)|^{p_i}\mathrm{d}\mu(z_i)\right)^{\frac{1}{p_i}}[\mu(6^{k+1}B)]^{1-\frac{1}{p_i}}\\
\le& C\sum_{k=1}^\infty\prod_{i=1}^2\frac{1}{\lambda(x,6^kr)}[\mu(6^{k+1}B)]^{1-\frac{1}{p_i}}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}\\
&\times[\varphi_i(x,6^{k+2}r)]^{-1}[\mu(B(x,6^{k+2}r))]^{-\frac{1}{p_i}}\left(\int_{6^{k+1}B}|f_i(z_i)|^{p_i}\mathrm{d}\mu(z_i)\right)^{\frac{1}{p_i}}\\
\le& C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sum_{k=1}^\infty\prod_{i=1}^2\frac{\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[\lambda(x,6^kr)]^{\frac{1}{p_i}}}\\
\le& C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sum_{k=1}^\infty\frac{\prod_{i=1}^2\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[C(6^k)]^{\frac{1}{p}}[\lambda(x,r)]^{\frac{1}{p}}}\\
\le& C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sum_{k=1}^\infty\frac{\prod_{i=1}^2\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[\lambda(x,r)]^{\frac{1}{p}}},
\end{aligned}$$

further, from (1.1), (1.14) and (1.18), it then follows that

$$\begin{aligned}
\mathrm{D}_4 =& \sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}(f_1^\infty,f_2^\infty)\|_{L^p(B(x,r))}\\
\leq& C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\\
&\times[\mu(B(x,r))]^{\frac{1}{p}}\sum_{k=1}^{\infty}\frac{\prod_{i=1}^{2}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[\lambda(x,r)]^{\frac{1}{p}}}\\
\leq& C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sup_{x\in\mathcal{X},r>0}\sum_{k=1}^{\infty}\frac{\prod_{i=1}^{2}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{\varphi(x,6r)[\mu(B(x,6r))]^{\frac{1}{p}}}\\
\leq& C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}.
\end{aligned}$$

Which, combining the estimates of $\mathrm{D}_1,\mathrm{D}_2$ and D_3, the proof of Theorem 1.1 is finished. □

4. Proof of Theorem 1.3

Proof. Let $\kappa=6$ in (1.14), and decompose f_i using the same form in (3.1) as

$$f_i := f_i^1 + f_i^\infty,\quad i=1,2,$$

where $f_i^1=f\chi_{6B}$ and $B=B(x,r)$ is the open ball. Then, via the Minkowski inequality, we have

$$\begin{aligned}
&\|\widetilde{T}_{b_1,b_2}(f_1,f_2)\|_{\mathcal{L}^{p,\varphi,\kappa}(\mu)}\\
=&\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}_{b_1,b_2}(f_1,f_2)\|_{L^p(B(x,r))}\\
\leq&\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}_{b_1,b_2}(f_1^1,f_2^1)\|_{L^p(B(x,r))}\\
&+\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}_{b_1,b_2}(f_1^1,f_2^\infty)\|_{L^p(B(x,r))}\\
&+\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}_{b_1,b_2}(f_1^\infty,f_2^1)\|_{L^p(B(x,r))}\\
&+\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}_{b_1,b_2}(f_1^\infty,f_2^\infty)\|_{L^p(B(x,r))}\\
:=&\mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3+\mathrm{E}_4.
\end{aligned}$$

From $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, (1.14), (1.19) and Lemma 2.4, it then follows that

$$\begin{aligned}
\mathrm{E}_1\leq&\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}_{b_1,b_2}(f_1^1,f_2^1)\|_{L^p(\mathcal{X})}\\
\leq& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\sup_{x\in\mathcal{X},r>0}\frac{\varphi_1(x,36r)[\mu(B(x,36r))]^{\frac{1}{p_1}+\frac{1}{p_2}}\varphi_2(x,36r)}{\varphi(x,6r)[\mu(B(x,6r))]^{\frac{1}{p}}}\\
&\times[\varphi_1(x,36r)]^{-1}[\mu(B(x,36r))]^{-\frac{1}{p_1}}\left(\int_{B(x,6r)}|f_1(z_1)|^{p_1}\mathrm{d}\mu(z_1)\right)^{\frac{1}{p_1}}
\end{aligned}$$

$$
\begin{aligned}
&\times[\varphi_2(x,36r)]^{-1}[\mu(B(x,36r))]^{-\frac{1}{p_2}}\left(\int_{B(x,6r)}|f_2(z_2)|^{p_2}\mathrm{d}\mu(z_2)\right)^{\frac{1}{p_2}}\\
\le& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\\
&\times\sup_{x\in\mathcal{X},r>0}\frac{\prod_{i=1}^{2}\varphi_i(x,36r)[\mu(B(x,36r))]^{\frac{1}{p_i}}}{\varphi(x,6r)[\mu(B(x,6r))]^{\frac{1}{p}}}\\
\le& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\\
&\times\sup_{x\in\mathcal{X},r>0}\frac{\varphi(x,6r)[\mu(B(x,36r))]^{\frac{1}{p}}}{\varphi(x,6r)[\mu(B(x,6r))]^{\frac{1}{p}}}\\
\le& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}.
\end{aligned}
$$

To estimate E_2, we first consider $|\widetilde{T}_{b_1,b_2}(f_1^1,f_2^\infty)(y)|$ for $y\in B(x,r)$. By applying (1.7), (1.14), the Hölder inequality, (1.17) and Corollary 2.1, we have

$$
\begin{aligned}
&|\widetilde{T}_{b_1,b_2}(f_1^1,f_2^\infty)(y)|\\
\le& C\int_{\mathcal{X}^2}\frac{|b_1(y)-b_1(z_1)||b_2(y)-b_2(z_2)||f_1^1(z_1)||f_2^\infty(z_2)|}{[\lambda(y,d(y,z_1))+\lambda(y,d(y,z_2))]^2}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
\le& C\int_{6B}|b_1(y)-b_1(z_1)||f_1(z_1)|\mathrm{d}\mu(z_1)\int_{\mathcal{X}\setminus(6B)}\frac{|b_2(y)-b_2(z_2)||f_2(z_2)|}{[\lambda(y,d(y,z_2))]^2}\mathrm{d}\mu(z_2)\\
\le& C\int_{6B}|b_1(y)-b_1(z_1)||f_1(z_1)|\mathrm{d}\mu(z_1)\\
&\times\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus(6^kB)}\frac{|b_2(y)-b_2(z_2)||f_2(z_2)|}{[\lambda(x,d(x,z_2))]^2}\mathrm{d}\mu(z_2)\\
\le& C\int_{6B}|b_1(y)-b_1(z_1)||f_1(z_1)|\mathrm{d}\mu(z_1)\\
&\times\sum_{k=1}^{\infty}\frac{1}{[\lambda(x,6^kr)]^2}\int_{6^{k+1}B}|b_2(y)-b_2(z_2)||f_2(z_2)|\mathrm{d}\mu(z_2)\\
\le& C\sum_{k=1}^{\infty}\frac{1}{[\lambda(x,6^kr)]^2}\prod_{i=1}^{2}\int_{6^{k+1}B}|b_i(y)-b_i(z_i)||f_i(z_i)|\mathrm{d}\mu(z_i)\\
\le& C\sum_{k=1}^{\infty}\frac{1}{[\lambda(x,6^kr)]^2}\prod_{i=1}^{2}\Bigg(|b_i(y)-m_{6B}(b_i)|\int_{6^{k+1}B}|f_i(z_i)|\mathrm{d}\mu(z_i)\\
&+\int_{6^{k+1}B}|b_i(z_i)-m_{6B}(b_i)||f_i(z_i)|\mathrm{d}\mu(z_i)\Bigg)\\
\le& C\sum_{k=1}^{\infty}\frac{1}{[\lambda(x,6^kr)]^2}\prod_{i=1}^{2}\Bigg(|b_i(y)-m_{6B}(b_i)|\int_{6^{k+1}B}|f_i(z_i)|\mathrm{d}\mu(z_i)\\
&+|(b_i)_{6^{k+1}B}-m_{6B}(b_i)|\int_{6^{k+1}B}|f_i(z_i)|\mathrm{d}\mu(z_i)\\
&+\int_{6^{k+1}B}|b_i(z_i)-m_{6^{k+1}B}(b_i)||f_i(z_i)|\mathrm{d}\mu(z_i)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\leq & C \sum_{k=1}^{\infty} \frac{1}{[\lambda(x, 6^k r)]^2} \prod_{i=1}^{2} \Bigg\{ |b_i(y) - m_{6B}(b_i)| [\mu(B(x, 6^{k+1} r))]^{1-\frac{1}{p_i}} \\
& \times \left(\int_{6^{k+1}B} |f_i(z_i)|^{p_i} \mathrm{d}\mu(z_i) \right)^{\frac{1}{p_i}} \\
& + k \|b_i\|_{\mathrm{RBMO}(\mu)} \left(\int_{6^{k+1}B} |f_i(z_i)|^{p_i} \mathrm{d}\mu(z_i) \right)^{\frac{1}{p_i}} [\mu(B(x, 6^{k+1} r))]^{1-\frac{1}{p_i}} \\
& + \left(\int_{6^{k+1}B} |f_i(z_i)|^{p_i} \mathrm{d}\mu(z_i) \right)^{\frac{1}{p_i}} \left(\int_{6^{k+1}B} |b_i(z_i) - m_{6^{k+1}B}(b_i)|^{p_i'} \mathrm{d}\mu(z_i) \right)^{\frac{1}{p_i'}} \Bigg\} \\
\leq & C \sum_{k=1}^{\infty} \frac{1}{[\lambda(x, 6^k r)]^2} \prod_{i=1}^{2} \Bigg\{ |b_i(y) - m_{6B}(b_i)| [\mu(B(x, 6^{k+1} r))]^{1-\frac{1}{p_i}} \varphi_i(x, 6^{k+2} r) \\
& \times \frac{[\mu(B(x, 6^{k+2} r))]^{\frac{1}{p_i}}}{[\mu(B(x, 6^{k+2} r))]^{\frac{1}{p_i}}} [\varphi_i(x, 6^{k+2} r)]^{-1} \left(\int_{6^{k+1}B} |f_i(z_i)|^{p_i} \mathrm{d}\mu(z_i) \right)^{\frac{1}{p_i}} \\
& + k \|b_i\|_{\mathrm{RBMO}(\mu)} [\mu(B(x, 6^{k+1} r))]^{1-\frac{1}{p_i}} \varphi_i(x, 6^{k+2} r) [\mu(B(x, 6^{k+2} r))]^{\frac{1}{p_i}} \\
& \times [\varphi_i(x, 6^{k+2} r)]^{-1} [\mu(B(x, 6^{k+2} r))]^{-\frac{1}{p_i}} \left(\int_{6^{k+1}B} |f_i(z_i)|^{p_i} \mathrm{d}\mu(z_i) \right)^{\frac{1}{p_i}} \\
& + \varphi_i(x, 6^{k+2} r) [\mu(B(x, 6^{k+2} r))]^{\frac{1}{p_i}} [\mu(2 \times 6^{k+1} r)]^{1-\frac{1}{p_i}} \\
& \times [\varphi_i(x, 6^{k+2} r)]^{-1} [\mu(B(x, 6^{k+2} r))]^{-\frac{1}{p_i}} \left(\int_{6^{k+1}B} |f_i(z_i)|^{p_i} \mathrm{d}\mu(z_i) \right)^{\frac{1}{p_i}} \\
& \times \left(\frac{1}{\mu(2 \times 6^{k+1} r)} \int_{6^{k+1}B} |b_i(z_i) - m_{6^{k+1}B}(b_i)|^{p_i'} \mathrm{d}\mu(z_i) \right)^{\frac{1}{p_i'}} \Bigg\} \\
\leq & C \|f_1\|_{\mathcal{L}^{p_1, \varphi_1, \kappa}(\mu)} \|f_2\|_{\mathcal{L}^{p_2, \varphi_2, \kappa}(\mu)} \sum_{k=1}^{\infty} \frac{[\mu(2 \times 6^{k+1} r)]^{2-\frac{1}{p}}}{[\lambda(x, 6^k r)]^2} \prod_{i=1}^{2} \varphi_i(x, 6^{k+2} r) \\
& \times [\mu(B(x, 6^{k+2} r))]^{\frac{1}{p_i}} \left\{ |b_i(y) - m_{6B}(b_i)| + k \|b_i\|_{\mathrm{RBMO}(\mu)} + \|b_i\|_{\mathrm{RBMO}(\mu)} \right\} \\
\leq & C \|f_1\|_{\mathcal{L}^{p_1, \varphi_1, \kappa}(\mu)} \|f_2\|_{\mathcal{L}^{p_2, \varphi_2, \kappa}(\mu)} \sum_{k=1}^{\infty} \frac{[\lambda(x, 6^{k+1} r)]^{2-\frac{1}{p}}}{[\lambda(x, 6^k r)]^2} \prod_{i=1}^{2} \varphi_i(x, 6^{k+2} r) \\
& \times [\mu(B(x, 6^{k+2} r))]^{\frac{1}{p_i}} \left\{ |b_i(y) - m_{6B}(b_i)| + k \|b_i\|_{\mathrm{RBMO}(\mu)} \right\} \\
\leq & C \|f_1\|_{\mathcal{L}^{p_1, \varphi_1, \kappa}(\mu)} \|f_2\|_{\mathcal{L}^{p_2, \varphi_2, \kappa}(\mu)} \sum_{k=1}^{\infty} \frac{\prod_{i=1}^{2} \varphi_i(x, 6^{k+2} r) [\mu(B(x, 6^{k+2} r))]^{\frac{1}{p_i}}}{[C(6^k)]^{\frac{1}{p}} [\lambda(x, r)]^{\frac{1}{p}}} \\
& \times \left\{ |b_i(y) - m_{6B}(b_i)| + k \|b_i\|_{\mathrm{RBMO}(\mu)} \right\},
\end{aligned}
$$

where we have used the following fact (see [5])

$$
|m_{6B}(b_i) - m_{6^{k+1}B}(b_i)| \leq Ck \|b_i\|_{\mathrm{RBMO}(\mu)}. \tag{4.1}
$$

Furthermore, via (1.1), (1.14) and (1.19), we obtain that

$$
\begin{aligned}
\mathrm{E}_2 &= \sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}_{b_1,b_2}(f_1^1,f_2^\infty)\|_{L^p(B(x,r))} \\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)} \\
&\quad\times \sup_{x\in\mathcal{X},r>0}\sum_{k=1}^{\infty} k\frac{\prod_{i=1}^{2}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}[\mu(B(x,r))]^{\frac{1}{p}}}{\varphi(x,6r)[\mu(B(x,6r))]^{\frac{1}{p}}[\lambda(x,r)]^{\frac{1}{p}}} \\
&\quad + C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)} \\
&\quad\times \sup_{x\in\mathcal{X},r>0}\sum_{k=1}^{\infty}\frac{\prod_{i=1}^{2}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{\varphi(x,6r)[\lambda(x,r)]^{\frac{1}{p}}[\mu(B(x,6r))]^{\frac{1}{p}}} \\
&\quad\times\left(\int_{B(x,r)}|b_i(y)-(b_i)_{6B}|^p\mathrm{d}\mu(y)\right)^{\frac{1}{p}} \\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)} \\
&\quad\times \sup_{x\in\mathcal{X},r>0}\sum_{k=1}^{\infty}(k+1)\frac{\prod_{i=1}^{2}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{\varphi(x,6r)[\mu(B(x,6r))]^{\frac{1}{p}}} \\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}.
\end{aligned}
$$

With an argument similar to that used in the estimate of E_2, it is easy to obtain that

$$\mathrm{E}_3 \le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}.$$

To estimate E_4, write

$$
\begin{aligned}
\mathrm{E}_4 &= \sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\|\widetilde{T}_{b_1,b_2}(f_1^\infty,f_2^\infty)\|_{L^p(B(x,r))} \\
&\le \sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}} \\
&\quad\times\|(b_1-m_{6B}(b_1))(b_2-m_{6B}(b_2))\widetilde{T}(f_1^\infty,f_2^\infty)\|_{L^p(B(x,r))} \\
&\quad+\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}} \\
&\quad\times\|(b_1-m_{6B}(b_1))\widetilde{T}(f_1^\infty,(b_2-m_{6B}(b_2))f_2^\infty)\|_{L^p(B(x,r))} \\
&\quad+\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}} \\
&\quad\times\|(b_2-m_{6B}(b_2))\widetilde{T}((b_1-m_{6B}(b_1))f_1^\infty,f_2^\infty)\|_{L^p(B(x,r))} \\
&\quad+\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}} \\
&\quad\times\|\widetilde{T}((b_1-m_{6B}(b_1))f_1^\infty,(b_2-m_{6B}(b_2))f_2^\infty)\|_{L^p(B(x,r))} \\
&= \mathrm{E}_{41}+\mathrm{E}_{42}+\mathrm{E}_{43}+\mathrm{E}_{44}.
\end{aligned}
$$

From the Hölder inequality, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and (2.2), it then follows that

$$\mathrm{E}_{41} = \sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}$$

$$
\begin{aligned}
&\times \|(b_1-m_{6B}(b_1))(b_2-m_{6B}(b_2))\widetilde{T}(f_1^\infty,f_2^\infty)\|_{L^p(B(x,r))}\\
\le& C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\\
&\times\sum_{k=1}^\infty\frac{\prod_{i=1}^2\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[\lambda(x,r)]^{\frac{1}{p}}}\\
&\times \|(b_1-m_{6B}(b_1))(b_2-m_{6B}(b_2))\|_{L^p(B(x,r))}\\
\le& C\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\\
&\times\sum_{k=1}^\infty\frac{\prod_{i=1}^2\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[\lambda(x,r)]^{\frac{1}{p}}}\\
&\times\left(\int_{B(x,r)}|b_1(y)-m_{6B}(b_1)|^{p_1}\mathrm{d}\mu(y)\right)^{\frac{1}{p_1}}\\
&\times\left(\int_{B(x,r)}|b_2(y)-m_{6B}(b_2)|^{p_2}\mathrm{d}\mu(y)\right)^{\frac{1}{p_2}}\\
\le& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}\\
&\times\sum_{k=1}^\infty\frac{\prod_{i=1}^2\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[\mu(B(x,6r))]^{\frac{1}{p}}[\lambda(x,r)]^{\frac{1}{p}}}[\mu(36B)]^{\frac{1}{p}}\\
\le& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\\
&\times\sup_{x\in\mathcal{X},r>0}\sum_{k=1}^\infty\frac{\prod_{i=1}^2\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{\varphi(x,6r)[\mu(B(x,6r))]^{\frac{1}{p}}}\\
\le& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}.
\end{aligned}
$$

For any $y\in B(x,r)$, by applying (1.7), (1.14), the Hölder inequality, (1.17) and (4.1), we obtain

$$
\begin{aligned}
&|\widetilde{T}(f_1^\infty,(b_2-m_{6B}(b_2))f_2^\infty)(y)|\\
\le& C\int_{\mathcal{X}^2\setminus(6B)^2}\frac{|f_1(z_1)||b_2(z_2)-m_{6B}(b_2)||f_2(z_2)|}{[\lambda(y,d(y,z_1))+\lambda(y,d(y,z_2))]^2}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
\le& C\sum_{k=1}^\infty\int_{(6^{k+1}B)^2\setminus(6^kB)^2}\frac{|f_1(z_1)||b_2(z_2)-m_{6B}(b_2)||f_2(z_2)|}{[\lambda(x,d(x,z_1))+\lambda(x,d(x,z_2))]^2}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
\le& C\sum_{k=1}^\infty|m_{6B}(b_2)-m_{6^{k+1}B}(b_2)|\int_{(6^{k+1}B)^2\setminus(6^kB)^2}\frac{|f_1(z_1)||f_2(z_2)|\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)}{[\lambda(x,d(x,z_1))+\lambda(x,d(x,z_2))]^2}\\
&+C\sum_{k=1}^\infty\int_{(6^{k+1}B)^2\setminus(6^kB)^2}\frac{|f_1(z_1)||b_2(z_2)-m_{6^{k+1}B}(b_2)||f_2(z_2)|}{[\lambda(x,d(x,z_1))+\lambda(x,d(x,z_2))]^2}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
\le& C\sum_{k=1}^\infty|m_{6B}(b_2)-m_{6^{k+1}B}(b_2)|\prod_{i=1}^2\int_{6^{k+1}B\setminus(6^kB)}\frac{|f_i(z_i)|}{\lambda(x,d(x,z_i))}\mathrm{d}\mu(z_i)
\end{aligned}
$$

$$
\begin{aligned}
&+ C\sum_{k=1}^{\infty}\int_{6^{k+1}B\backslash(6^kB)}\frac{|f_1(z_1)|}{\lambda(x,d(x,z_1))}\mathrm{d}\mu(z_1)\\
&\times\int_{6^{k+1}B\backslash(6^kB)}\frac{|b_2(z_2)-m_{6^{k+1}B}(b_2)||f_2(z_2)|}{\lambda(x,d(x,z_2))}\mathrm{d}\mu(z_2)\\
\le& C\sum_{k=1}^{\infty}|m_{6B}(b_2)-m_{6^{k+1}B}(b_2)|\prod_{i=1}^{2}\frac{1}{\lambda(x,6^kr)}\int_{6^{k+1}B}|f_i(z_i)|\mathrm{d}\mu(z_i)\\
&+ C\sum_{k=1}^{\infty}\frac{1}{\lambda(x,6^kr)}\int_{6^{k+1}B}|f_1(z_1)|\mathrm{d}\mu(z_1)\\
&\times\frac{1}{\lambda(x,6^kr)}\int_{6^{k+1}B}|b_2(z_2)-m_{6^{k+1}B}(b_2)||f_2(z_2)|\mathrm{d}\mu(z_2)\\
\le& C\sum_{k=1}^{\infty}|m_{6B}(b_2)-m_{6^{k+1}B}(b_2)|\prod_{i=1}^{2}\frac{[\mu(B(x,6^{k+1}r))]^{1-\frac{1}{p_i}}}{\lambda(x,6^kr)}\\
&\times\left(\int_{6^{k+1}B}|f_i(z_i)|^{p_i}\mathrm{d}\mu(z_i)\right)^{\frac{1}{p_i}}\\
&+ C\sum_{k=1}^{\infty}\frac{1}{\lambda(x,6^kr)}\left(\int_{6^{k+1}B}|f_1(z_1)|^{p_1}\mathrm{d}\mu(z_1)\right)^{\frac{1}{p_1}}[\mu(B(x,6^{k+1}r))]^{1-\frac{1}{p_1}}\\
&\times\frac{1}{\lambda(x,6^kr)}\left(\int_{6^{k+1}B}|f_2(z_2)|^{p_2}\mathrm{d}\mu(z_2)\right)^{\frac{1}{p_2}}\\
&\times\left(\int_{6^{k+1}B}|b_2(z_2)-m_{6^{k+1}B}(b_2)|^{p_2'}\mathrm{d}\mu(z_2)\right)^{\frac{1}{p_2'}}\\
\le& C\|b_2\|_{\mathrm{RBMO}(\mu)}\sum_{k=1}^{\infty}k\prod_{i=1}^{2}\frac{[\mu(B(x,6^{k+1}r))]^{1-\frac{1}{p_i}}}{\lambda(x,6^kr)}\left(\int_{6^{k+1}B}|f_i(z_i)|^{p_i}\mathrm{d}\mu(z_i)\right)^{\frac{1}{p_i}}\\
&+ C\|b_2\|_{\mathrm{RBMO}(\mu)}\sum_{k=1}^{\infty}\prod_{i=1}^{2}\frac{[\mu(B(x,6^{k+2}r))]^{1-\frac{1}{p_i}}}{\lambda(x,6^kr)}\left(\int_{6^{k+1}B}|f_i(z_i)|^{p_1}\mathrm{d}\mu(z_i)\right)^{\frac{1}{p_1}}\\
\le& C\|b_2\|_{\mathrm{RBMO}(\mu)}\sum_{k=1}^{\infty}(k+1)\prod_{i=1}^{2}\frac{[\mu(B(x,6^{k+1}r))]^{1-\frac{1}{p_i}}}{\lambda(x,6^kr)}\left(\int_{6^{k+1}B}|f_i(z_i)|^{p_i}\mathrm{d}\mu(z_i)\right)^{\frac{1}{p_i}}\\
\le& C\|b_2\|_{\mathrm{RBMO}(\mu)}\sum_{k=1}^{\infty}(k{+}1)\prod_{i=1}^{2}\frac{[\mu(B(x,6^{k+1}r))]^{1-\frac{1}{p_i}}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{\lambda(x,6^kr)}\\
&\times[\varphi_i(x,6^{k+2}r)]^{-1}[\mu(B(x,6^{k+2}r))]^{-\frac{1}{p_i}}\left(\int_{6^{k+1}B}|f_i(z_i)|^{p_i}\mathrm{d}\mu(z_i)\right)^{\frac{1}{p_i}}\\
\le& C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\\
&\times\sum_{k=1}^{\infty}(k+1)\prod_{i=1}^{2}\frac{\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}[\lambda(x,6^{k+1}r)]^{1-\frac{1}{p_i}}}{\lambda(x,6^kr)}\\
\le& C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}
\end{aligned}
$$

$$\times \sum_{k=1}^{\infty}(k+1)\frac{\prod_{i=1}^{2}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[\lambda(x,6^k r)]^{\frac{1}{p}}},$$

further, from (1.14), (1.19) and (2.2), it then follows that

$$\begin{aligned}
\mathrm{E}_{42} =& \sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}}\\
&\times \|(b_1-m_{6B}(b_1))\widetilde{T}(f_1^{\infty},(b_2-m_{6B}(b_2))f_2^{\infty})\|_{L^p(B(x,r))}\\
\leq& C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}\\
&\times [\mu(B(x,6r))]^{-\frac{1}{p}}\sum_{k=1}^{\infty}(k+1)\frac{\prod_{i=1}^{2}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[\lambda(x,6^k r)]^{\frac{1}{p}}}\\
&\times \left(\int_{B(x,r)}|b_1(y)-m_{6B}(b_1)|^p\mathrm{d}\mu(y)\right)^{\frac{1}{p}}\\
\leq& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}\\
&\times \sup_{x\in\mathcal{X},r>0}\sum_{k=1}^{\infty}(k+1)\frac{\prod_{i=1}^{2}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{\varphi(x,6r)[\mu(B(x,6r))]^{\frac{1}{p}}}\\
\leq& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}.
\end{aligned}$$

Similarly, we have

$$\mathrm{E}_{43} \leq C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}.$$

Finally, let us show E_{44}. For any $y \in B(x,r)$, by applying (1.7), the Hölder inequality, (1.14), (1.17), (2.2) and (4.1), we have

$$\begin{aligned}
&|\widetilde{T}((b_1-m_{6B}(b_1))f_1^{\infty},(b_2-m_{6B}(b_2))f_2^{\infty})(y)|\\
\leq& C\int_{\mathcal{X}^2\setminus(6B)^2}\prod_{i=1}^{2}\frac{|b_i(z_i)-m_{6B}(b_i)||f_1(z_i)|}{\lambda(y,d(y,z_i))}\mathrm{d}\mu(z_i)\\
\leq& C\sum_{k=1}^{\infty}\int_{(6^{k+1}B)^2\setminus(6^kB)^2}\prod_{i=1}^{2}\frac{|b_i(z_i)-m_{6B}(b_i)||f_1(z_i)|}{\lambda(x,d(x,z_i))}\mathrm{d}\mu(z_i)\\
\leq& C\sum_{k=1}^{\infty}\prod_{i=1}^{2}\frac{1}{\lambda(x,6^k r)}\int_{6^{k+1}B}|b_i(z_i)-m_{6B}(b_i)||f_1(z_i)|\mathrm{d}\mu(z_i)\\
\leq& C\sum_{k=1}^{\infty}\prod_{i=1}^{2}\frac{1}{\lambda(x,6^k r)}\left(\int_{6^{k+1}B}|b_i(z_i)-m_{6^{k+1}B}(b_i)||f_1(z_i)|\mathrm{d}\mu(z_i)\right.\\
&\left.+|m_{6^{k+1}B}(b_i)-m_{6B}(b_i)|\int_{6^{k+1}B}|f_1(z_i)|\mathrm{d}\mu(z_i)\right)\\
\leq& C\sum_{k=1}^{\infty}\prod_{i=1}^{2}\frac{1}{\lambda(x,6^k r)}\left\{\left(\int_{6^{k+1}B}|f_1(z_i)|^{p_i}\mathrm{d}\mu(z_i)\right)^{\frac{1}{p_i}}\right.
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_{6^{k+1}B} |b_i(z_i) - m_{6^{k+1}B}(b_i)|^{p_i'} \mathrm{d}\mu(z_i) \right)^{\frac{1}{p_i'}} \\
&+ k\|b_i\|_{\mathrm{RBMO}(\mu)}[\mu(B(x,6^{k+1}r))]^{1-\frac{1}{p_i}} \left(\int_{6^{k+1}B} |f_1(z_i)|^{p_i} \mathrm{d}\mu(z_i) \right)^{\frac{1}{p_i}} \Bigg\} \\
\le & C \sum_{k=1}^{\infty} \prod_{i=1}^{2} \frac{1}{\lambda(x,6^k r)} \Bigg(\int_{6^{k+1}B} |b_i(z_i) - m_{6^{k+1}B}(b_i)||f_1(z_i)| \mathrm{d}\mu(z_i) \\
&+ |m_{6^{k+1}B}(b_i) - m_{6B}(b_i)| \int_{6^{k+1}B} |f_1(z_i)| \mathrm{d}\mu(z_i) \Bigg) \\
\le & C \sum_{k=1}^{\infty} \prod_{i=1}^{2} \frac{1}{\lambda(x,6^k r)} \Bigg\{ \varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}} [\mu(B(x,2\times 6^{k+1}r))]^{1-\frac{1}{p_i}} \\
&\times [\varphi_i(x,6^{k+2}r)]^{-1}[\mu(B(x,6^{k+2}r))]^{-\frac{1}{p_i}} \left(\int_{6^{k+1}B} |f_1(z_i)|^{p_i} \mathrm{d}\mu(z_i) \right)^{\frac{1}{p_i}} \\
&\times \left(\frac{1}{\mu(2B(x,6^{k+1}r))} \int_{6^{k+1}B} |b_i(z_i) - m_{6^{k+1}B}(b_i)|^{p_i'} \mathrm{d}\mu(z_i) \right)^{\frac{1}{p_i'}} \\
&+ k\|b_i\|_{\mathrm{RBMO}(\mu)}[\mu(B(x,6^{k+1}r))]^{1-\frac{1}{p_i}} \varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}} \\
&\times [\varphi_i(x,6^{k+2}r)]^{-1}[\mu(B(x,6^{k+2}r))]^{-\frac{1}{p_i}} \left(\int_{6^{k+1}B} |f_1(z_i)|^{p_i} \mathrm{d}\mu(z_i) \right)^{\frac{1}{p_i}} \Bigg\} \\
\le & C \sum_{k=1}^{\infty} \prod_{i=1}^{2} \frac{1}{\lambda(x,6^k r)} \Bigg\{ \varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}} [\mu(B(x,2\times 6^{k+1}r))]^{1-\frac{1}{p_i}} \\
&\times \|b_i\|_{\mathrm{RBMO}(\mu)} \|f_i\|_{\mathcal{L}^{p_i,\varphi_i,\kappa}(\mu)} + k\|b_i\|_{\mathrm{RBMO}(\mu)} \|f_i\|_{\mathcal{L}^{p_i,\varphi_i,\kappa}(\mu)} \\
&\times [\mu(B(x,6^{k+1}r))]^{1-\frac{1}{p_i}} \varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}} \Bigg\} \\
\le & C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)} \\
&\times \sum_{k=1}^{\infty} \prod_{i=1}^{2} (k+1) \frac{\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}} [\mu(B(x,2\times 6^{k+1}r))]^{1-\frac{1}{p_i}}}{\lambda(x,6^k r)} \\
\le & C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)} \\
&\times \sum_{k=1}^{\infty} \prod_{i=1}^{2} (k+1) \frac{\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[\lambda(x,6^k r)]^{\frac{1}{p_i}}} \\
\le & C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)} \\
&\times \sum_{k=1}^{\infty} (k^2+2k+1) \frac{\prod_{i=1}^{2} \varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[\lambda(x,6^k r)]^{\frac{1}{p}}} \\
\le & C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)} \\
&\times \sum_{k=1}^{\infty} (k+1) \frac{\prod_{i=1}^{2} \varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}}{[\lambda(x,6^k r)]^{\frac{1}{p}}},
\end{aligned}
$$

further, from (1.1), (1.14) and (1.19), we obtain that

$$\begin{aligned}
\mathrm{E}_{44} = &\sup_{x\in\mathcal{X},r>0}[\varphi(x,6r)]^{-1}[\mu(B(x,6r))]^{-\frac{1}{p}} \\
&\qquad \times \|\widetilde{T}((b_1-m_{6B}(b_1))f_1^{\infty},(b_2-m_{6B}(b_2))f_2^{\infty})\|_{L^p(B(x,r))} \\
\leq &C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)} \\
&\times \sup_{x\in\mathcal{X},r>0}\sum_{k=1}^{\infty}(k+1)\frac{\prod_{i=1}^{2}\varphi_i(x,6^{k+2}r)[\mu(B(x,6^{k+2}r))]^{\frac{1}{p_i}}[\mu(B(x,r))]^{\frac{1}{p}}}{\varphi(x,6r)[\mu(B(x,6r))]^{\frac{1}{p}}[C(6^k)]^{\frac{1}{p}}[\lambda(x,r)]^{\frac{1}{p}}} \\
\leq &C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\varphi_1,\kappa}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\varphi_2,\kappa}(\mu)}.
\end{aligned}$$

Which, combining the above whole estimates, we finish the proof of Theorem 1.3. □

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