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# NUMERICAL RADIUS OF KRONECKER PRODUCT OF MATRICES

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**Abstract** In this article, we present several bounds for the numerical radius of the Kronecker product of matrices to enrich our knowledge about this topic.

**Keywords** Kronecker product, numerical radius, norm, inequality.

**MSC(2010)** 47A12, 47A30, 15A60, 47A80, 15A45, 15A69.

## 1. Introduction

For positive integers $m, n$, the notation $\mathcal{M}_{m,n}$ will denote the algebra of all $m \times n$ complex matrices. If $m = n$, we simply write $\mathcal{M}_n$ or $\mathcal{M}_m$. If $A \in \mathcal{M}_n$ is such that $\langle Ax, x\rangle \geq$ for all $x \in \mathbb{C}^n$, we say that $A$ is positive semi-definite, and we write $A \geq 0$. If $A \geq 0$ and $A$ is invertible, it is said to be positive definite, and we write $A > 0$.

The Kronecker product of the matrix $A \in \mathcal{M}_{m,n}(\mathbb{C})$ with the matrix $B \in \mathcal{M}_{p,q}(\mathbb{C})$ is the $mp \times nq$ matrix, defined by the block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

The Kronecker sum of two square matrices $A \in \mathcal{M}_m$ and $B \in \mathcal{M}_n$, which is symbolized by $A \oplus B$, is the matrix in $\mathcal{M}_{mn}$ defined as

$$A \oplus B \equiv (A \otimes I_n) + (I_m \otimes B),$$

where $I_k$ represents the identity matrix of size $k \times k$ for any natural number $k$. This latter operation is related to the tensor product on Lie algebras and appears naturally in physics when considering ensembles of non-interacting systems.

It is readily seen that, in general, $A \otimes B \neq B \otimes A$. Yet, there exist permutation matrices $P$ and $Q$ such that $B \otimes A = P(A \otimes B)Q$.

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Despite its commutativity issue, the Kronecker product enjoys many interesting properties. If $A, B \in \mathcal{M}_n$, then

$$\|A \otimes B\| = \|A\| \, \|B\| \,, \tag{1.1}$$

$$(A \otimes B)^* = A^* \otimes B^*, \tag{1.2}$$

$$|A \otimes B| = |A| \otimes |B| \,, \tag{1.3}$$

where $\| \cdot \|$ denotes the operator norm, defined by $\|A\| = \sup_{\|x\|=1} \|Ax\|$, and $| \cdot |$ denotes the absolute value, defined by $|A| = (A^*A)^{\frac{1}{2}}$, in which $A^*$ is the conjugate transpose of $A$. Further, if $A, B > 0$, then

$$(A \otimes B)^r = A^r \otimes B^r; \quad (r > 0) \,. \tag{1.4}$$

If $A$, $B$, $C$, and $D$ are matrices of such size that one can form the matrix products $AC$ and $BD$, then

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{1.5}$$

which is named the mixed-product property because it mixes the ordinary matrix product and the Kronecker product.

It follows that $A \otimes B$ is invertible if and only if both $A$ and $B$ are invertible, in which case the inverse is given by

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{1.6}$$

This property follows directly from the mixed product property. By (1.4) and (1.6),

$$(A \otimes B)^r = A^r \otimes B^r; \ (r < 0) \,.$$

The Kronecker product has received considerable attention in the literature due to its significance in mathematics and mathematical physics, as seen in [11, 17, 27].

This paper presents several bounds for the numerical radius of Kronecker product quantities. Here we recall that given $A \in \mathcal{M}_n$, the numerical radius of $A$ is defined by

$$\omega(A) = \sup\{| \langle Ax, x \rangle | : x \in \mathbb{C}^n, \|x\| = 1\}.$$

This numerical quantity has considerable significance in the literature due to its importance in understanding the geometry of $\mathcal{M}_n$, in addition to its applications in operator theory and mathematical physics, to mention a few. We refer the reader to [1, 10, 16, 18–21, 23–25] as a list of recent references treating the numerical radius.

Due to the difficulty in computing the exact value of the numerical radius, it has been an important topic in the literature to find easier upper and lower bounds for $\omega(\cdot)$. A basic inequality that gives such bounds is

$$\frac{1}{2}\|A\| \le \omega(A) \le \|A\|. \tag{1.7}$$

This has been significantly improved in the literature, where numerous better bounds have been found.

Although $\omega(\cdot)$ is a norm on $\mathcal{M}_n$, it is not a matrix norm. That is, it is not sub-multiplicative. Therefore, it has also been important to study possible bounds for $\omega(AB)$, where $A, B \in \mathcal{M}_n$.

In the following lemma, we collect some related results that we will need in the sequel.

**Proposition 1.1.** *Let $A, B \in \mathcal{M}_n$.*

1. *[4, Theorem 1] If $r \geq 1$, then*

$$\omega^r(AB) \leq \frac{1}{2}\left\||A^*|^{2r} + |B|^{2r}\right\|. \tag{1.8}$$

2. *[22, Theorem 2.10] If $r \geq 1$, then*

$$\omega^r(AB) \leq \frac{1}{4}\left\||A|^{2r} + |B^*|^{2r}\right\| + \frac{1}{2}\omega^r(BA). \tag{1.9}$$

3. *[5, Theorem 1] If $0 \leq \nu \leq 1$, then*

$$\omega(A) \leq \frac{1}{2}\left\||A|^{2(1-\nu)} + |A^*|^{2\nu}\right\|. \tag{1.10}$$

4. *[18, Remark 2.6]*

$$\omega^2(A) \leq \frac{1}{4}\left(\left\||A|^2 + |A^*|^2\right\| + \|\,|A|\,|A^*| + |A^*|\,|A|\,\|\right). \tag{1.11}$$

In this proposition, (1.9) refines (1.8), while (1.10) refines the second inequality in (1.7) when $\nu = \frac{1}{2}$.

Further, we will need the following two properties of the operator norm

$$\|A\| = \|A^*\| = \|\,|A|\,\| = \|\,|A^*|\,\| \tag{1.12}$$

and

$$\|\,|A|\,|B|\,\| = \|AB^*\|, \tag{1.13}$$

where $A, B \in \mathcal{M}_n$. A more delicate needed inequality asserts that if $A, B \geq 0$, then [12]

$$\|A+B\| \leq \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|^2}\right). \tag{1.14}$$

When dealing with matrix inequalities, it is keen to recall that when $A \in \mathcal{M}_n$ is Hermitian, then $A = U\text{diag}(\lambda_1, \ldots, \lambda_n)U^*$, where $U$ is unitary and $\text{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix whose diagonal entries are the eigenvalues of $A$. Now, if $f : J \to \mathbb{R}$ is a function defined on an interval $J$ that contains the eigenvalues of $A$, then we can define the new matrix $f(A)$, as follows

$$f(A) = U\text{diag}(f(\lambda_1), \ldots, f(\lambda_n))U^*.$$

The following identity holds for an arbitrary $A \in \mathcal{M}_n$ and non-negative increasing function $f$ on $[0, \infty)$

$$f(\|A\|) = \|f(|A|)\|. \tag{1.15}$$

The sole goal of this paper is to discuss inequalities that govern the numerical radius of the Kronecker product in a way that complements the existing literature. For example, we will show that

$$\omega(AB \otimes BA) \leq \frac{1}{4}\left(\|A\|^2\|B\|^2 + \|AB\|\,\|BA\|\right) + \frac{1}{2}\omega(BA \otimes AB),$$

$$\|A\otimes B+B\otimes A\|^2\le\sqrt{\left\||A|^2\otimes|B|^2+|B|^2\otimes|A|^2\right\|\left\||A^*|^2\otimes|B^*|^2+|B^*|^2\otimes|A^*|^2\right\|}$$
$$+\|A\|^2\|B\|^2\omega\left(B^*A\otimes A^*B\right),$$

and

$$\omega\left(T\otimes\widetilde{T}\right)\le\frac{1}{4}\left(\|T\|^2+\|T\|\left\||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right\|\right)+\frac{1}{2}\omega\left(\widetilde{T}\otimes T\right),$$

where $\widetilde{T}$ is the Aluthge transform of $T\in\mathcal{M}_n$. Here we recall that if $T\in\mathcal{M}_n$, then $T=U|T|$, where $U$ is a unitary matrix. When $0\le\nu\le 1$ , the $\nu-$Aluthge transform is defined by $\widetilde{T_\nu}=|T|^\nu U|T|^{1-\nu}$. When $\nu=\frac{1}{2}$, we simply write $\widetilde{T}$ instead of $\widetilde{T}_{\frac{1}{2}}$. The Aluthge transform is an important tool when studying numerical radius inequalities, as found in [26].

# 2. Main Results

In this section, we present our main results, which focus on the numerical radius of the Kronecker product and its variants. In particular, we study $\omega(A\otimes B)$, $\omega(AB\otimes BA)$, $\omega(T\otimes\widetilde{T})$, and $\|A\otimes B+B\otimes A\|$. For organizational purposes, we present our results in three subsections.

## 2.1. On $\omega(A\otimes B)$

We notice that if $A,B\in\mathcal{M}_n$, then using (1.7) and (1.1), we have $\omega(A\otimes B)\le\|A\|\,\|B\|$. In fact, we have better estimates as follows [7]

$$\omega(A\otimes B)\le\min\left(\|A\|\omega(B),\|B\|\omega(A)\right).$$

This subsection presents different upper bounds for $\omega(A\otimes B)$.

**Theorem 2.1.** *Let $A,B\in\mathcal{M}_n$. Then for any $0\le\nu\le 1$,*

$$\begin{aligned}&\omega(A\otimes B)\\ \le&\frac{1}{4}\left(\|A\|^{2(1-\nu)}\|B\|^{2(1-\nu)}+\|A\|^{2\nu}\|B\|^{2\nu}\right)\\ &+\frac{1}{4}\sqrt{\left(\|A\|^{2(1-\nu)}\|B\|^{2(1-\nu)}-\|A\|^{2\nu}\|B\|^{2\nu}\right)^2+4\left\||A|^{1-\nu}|A^*|^\nu\right\|^2\left\||B|^{1-\nu}|B^*|^\nu\right\|^2}.\end{aligned}$$

*In particular,*

$$\omega\left(A\otimes B\right)\le\frac{1}{2}\left(\|A\|\,\|B\|+\left\||A|^{\frac{1}{2}}|A^*|^{\frac{1}{2}}\right\|\left\||B|^{\frac{1}{2}}|B^*|^{\frac{1}{2}}\right\|\right).$$

**Proof.** By (1.10), we can write

$$\begin{aligned}\omega\left(A\otimes B\right)&\le\frac{1}{2}\left\||A\otimes B|^{2(1-\nu)}+\left|(A\otimes B)^*\right|^{2\nu}\right\|\\ &=\frac{1}{2}\left\||A\otimes B|^{2(1-\nu)}+|A^*\otimes B^*|^{2\nu}\right\|\quad\text{(by (1.2))}\\ &=\frac{1}{2}\left\|(|A|\otimes|B|)^{2(1-\nu)}+(|A^*|\otimes|B^*|)^{2\nu}\right\|\quad\text{(by (1.3))}\\ &=\frac{1}{2}\left\||A|^{2(1-\nu)}\otimes|B|^{2(1-\nu)}+|A^*|^{2\nu}\otimes|B^*|^{2\nu}\right\|\quad\text{(by (1.4))}.\end{aligned}$$

That is,

$$\omega\left(A\otimes B\right)\leq\frac{1}{2}\left\||A|^{2(1-\nu)}\otimes|B|^{2(1-\nu)}+|A^*|^{2\nu}\otimes|B^*|^{2\nu}\right\|.$$

On the other hand,

$$\begin{aligned}
&\left\||A|^{2(1-\nu)}\otimes|B|^{2(1-\nu)}+|A^*|^{2\nu}\otimes|B^*|^{2\nu}\right\|\\
&\leq\frac{1}{2}\left(\left\||A|^{2(1-\nu)}\otimes|B|^{2(1-\nu)}\right\|+\left\||A^*|^{2\nu}\otimes|B^*|^{2\nu}\right\|\right)\\
&\quad+\frac{1}{2}\sqrt{\left(\left\||A|^{2(1-\nu)}\otimes|B|^{2(1-\nu)}\right\|-\left\||A^*|^{2\nu}\otimes|B^*|^{2\nu}\right\|\right)^2+4\left\|\left(|A|^{2(1-\nu)}\otimes|B|^{2(1-\nu)}\right)^{\frac{1}{2}}\left(|A^*|^{2\nu}\otimes|B^*|^{2\nu}\right)^{\frac{1}{2}}\right\|^2}\\
&\quad\text{(by (1.14))}\\
&=\frac{1}{2}\left(\left\||A|^{2(1-\nu)}\right\|\left\||B|^{2(1-\nu)}\right\|+\left\||A^*|^{2\nu}\right\|\left\||B^*|^{2\nu}\right\|\right)\\
&\quad+\frac{1}{2}\sqrt{\left(\left\||A|^{2(1-\nu)}\right\|\left\||B|^{2(1-\nu)}\right\|-\left\||A^*|^{2\nu}\right\|\left\||B^*|^{2\nu}\right\|\right)^2+4\left\|\left(|A|^{2(1-\nu)}\otimes|B|^{2(1-\nu)}\right)^{\frac{1}{2}}\left(|A^*|^{2\nu}\otimes|B^*|^{2\nu}\right)^{\frac{1}{2}}\right\|^2}\\
&\quad\text{(by (1.1))}\\
&=\frac{1}{2}\left(\|\,|A|\,\|^{2(1-\nu)}\|\,|B|\,\|^{2(1-\nu)}+\|\,|A^*|\,\|^{2\nu}\|\,|B^*|\,\|^{2\nu}\right)\\
&\quad+\frac{1}{2}\sqrt{\left(\|\,|A|\,\|^{2(1-\nu)}\|\,|B|\,\|^{2(1-\nu)}-\|\,|A^*|\,\|^{2\nu}\|\,|B^*|\,\|^{2\nu}\right)^2+4\left\|\left(|A|^{2(1-\nu)}\otimes|B|^{2(1-\nu)}\right)^{\frac{1}{2}}\left(|A^*|^{2\nu}\otimes|B^*|^{2\nu}\right)^{\frac{1}{2}}\right\|^2}\\
&\quad\text{(by (1.15))}\\
&=\frac{1}{2}\left(\|A\|^{2(1-\nu)}\|B\|^{2(1-\nu)}+\|A\|^{2\nu}\|B\|^{2\nu}\right)\\
&\quad+\frac{1}{2}\sqrt{\left(\|A\|^{2(1-\nu)}\|B\|^{2(1-\nu)}-\|A\|^{2\nu}\|B\|^{2\nu}\right)^2+4\left\|\left(|A|^{2(1-\nu)}\otimes|B|^{2(1-\nu)}\right)^{\frac{1}{2}}\left(|A^*|^{2\nu}\otimes|B^*|^{2\nu}\right)^{\frac{1}{2}}\right\|^2}\\
&\quad\text{(by (1.12))}\\
&=\frac{1}{2}\left(\|A\|^{2(1-\nu)}\|B\|^{2(1-\nu)}+\|A\|^{2\nu}\|B\|^{2\nu}\right)\\
&\quad+\frac{1}{2}\sqrt{\left(\|A\|^{2(1-\nu)}\|B\|^{2(1-\nu)}-\|A\|^{2\nu}\|B\|^{2\nu}\right)^2+4\left\|\left(|A|^{1-\nu}\otimes|B|^{1-\nu}\right)\left(|A^*|^{\nu}\otimes|B^*|^{\nu}\right)\right\|^2}\\
&\quad\text{(by (1.4))}\\
&=\frac{1}{2}\left(\|A\|^{2(1-\nu)}\|B\|^{2(1-\nu)}+\|A\|^{2\nu}\|B\|^{2\nu}\right)\\
&\quad+\frac{1}{2}\sqrt{\left(\|A\|^{2(1-\nu)}\|B\|^{2(1-\nu)}-\|A\|^{2\nu}\|B\|^{2\nu}\right)^2+4\left\||A|^{1-\nu}|A^*|^{\nu}\otimes|B|^{1-\nu}|B^*|^{\nu}\right\|^2}\\
&\quad\text{(by (1.5))}\\
&=\frac{1}{2}\left(\|A\|^{2(1-\nu)}\|B\|^{2(1-\nu)}+\|A\|^{2\nu}\|B\|^{2\nu}\right)\\
&\quad+\frac{1}{2}\sqrt{\left(\|A\|^{2(1-\nu)}\|B\|^{2(1-\nu)}-\|A\|^{2\nu}\|B\|^{2\nu}\right)^2+4\left\||A|^{1-\nu}|A^*|^{\nu}\right\|^2\left\||B|^{1-\nu}|B^*|^{\nu}\right\|^2}\\
&\quad\text{(by (1.1))}.
\end{aligned}$$

So,

$$\omega\left(A\otimes B\right)$$

$$\leq \frac{1}{4}\left(\|A\|^{2(1-\nu)}\|B\|^{2(1-\nu)}+\|A\|^{2\nu}\|B\|^{2\nu}\right)$$
$$+\frac{1}{4}\sqrt{\left(\|A\|^{2(1-\nu)}\|B\|^{2(1-\nu)}-\|A\|^{2\nu}\|B\|^{2\nu}\right)^{2}+4\left\||A|^{1-\nu}|A^{*}|^{\nu}\right\|^{2}\left\||B|^{1-\nu}|B^{*}|^{\nu}\right\|^{2}},$$

which proves the first desired inequality. Letting $\nu=\frac{1}{2}$ in the first inequality implies the second. This completes the proof. □

**Remark 2.1.** If $A, B \geq 0$, then [2, Theorem IX.2.1] implies

$$\|A^{r}B^{r}\| \leq \|AB\|^{r};\ \ (0 \leq r \leq 1)\,. \tag{2.1}$$

Thus, for any $A, B \in \mathcal{M}_n$, Theorem 2.1 implies

$$\begin{aligned}\omega\left(A\otimes B\right) &\leq \frac{1}{2}\left(\|A\|\,\|B\|+\left\||A|^{\frac{1}{2}}|A^{*}|^{\frac{1}{2}}\right\|\,\left\||B|^{\frac{1}{2}}|B^{*}|^{\frac{1}{2}}\right\|\right)\\ &\leq \frac{1}{2}\left(\|A\|\,\|B\|+\sqrt{\|\,|A|\,|A^{*}|\,\|\,\|\,|B|\,|B^{*}|\,\|}\right)\quad \text{(by (2.1))}\\ &= \frac{1}{2}\left(\|A\|\,\|B\|+\sqrt{\|A^{2}\|\,\|B^{2}\|}\right)\quad \text{(by (1.13))}\\ &\leq \|A\|\,\|B\|.\end{aligned}$$

From Remark 2.1, we deduce that

$$\omega(A\otimes B) \leq \frac{1}{2}\left(\|A\|\,\|B\|+\sqrt{\|\,|A|\,|A^{*}|\,\|\,\|\,|B|\,|B^{*}|\,\|}\right).$$

Squaring this inequality and applying the arithmetic-geometric mean inequality imply

$$\omega^{2}(A\otimes B) \leq \frac{1}{2}\left(\|A\|^{2}\|B\|^{2}+\|\,|A|\,|A^{*}|\,\|\,\|\,|B|\,|B^{*}|\,\|\right).$$

In the following theorem, we present a refinement of this latter inequality.

**Theorem 2.2.** *Let $A, B \in \mathcal{M}_n$. Then*

$$\omega^{2}\left(A\otimes B\right) \leq \frac{1}{2}\left(\|A\|^{2}\|B\|^{2}+\omega\left(|A|\,|A^{*}|\otimes|B|\,|B^{*}|\right)\right).$$

**Proof.** Replacing $A$ by $A\otimes B$ in (1.11) implies

$$\begin{aligned}&\omega^{2}(A\otimes B)\\ &\leq \frac{1}{4}\left(\|\,|A\otimes B|^{2}+|(A\otimes B)^{*}|^{2}\|+\|\,|A\otimes B|\,|(A\otimes B)^{*}|+|(A\otimes B)^{*}|\,|A\otimes B|\,\|\right).\end{aligned} \tag{2.2}$$

The identities (1.2), (1.1) and (1.12) imply

$$\|\,|A\otimes B|^{2}+|(A\otimes B)^{*}|^{2}\| \leq 2\|A\|^{2}\|B\|^{2}. \tag{2.3}$$

On the other hand, (1.2), (1.3) and (1.5) imply

$$|A\otimes B|\,|(A\otimes B)^{*}|+|(A\otimes B)^{*}|\,|A\otimes B| = |A|\,|A^{*}|\otimes|B|\,|B^{*}|+|A^{*}|\,|A|\otimes|B^{*}|\,|B|\,, \tag{2.4}$$

which is Hermitian, by (1.2). Therefore

$$\|\,|A|\,|A^{*}|\otimes|B|\,|B^{*}|+|A^{*}|\,|A|\otimes|B^{*}|\,|B|\,\|$$

$$
\begin{aligned}
&=\omega\left(|A|\left|A^*\right| \otimes|B|\left|B^*\right|+\left|A^*\right||A| \otimes\left|B^*\right||B|\right) \\
&\leq\omega\left(|A|\left|A^*\right| \otimes|B|\left|B^*\right|\right)+\omega\left(\left|A^*\right||A| \otimes\left|B^*\right||B|\right) \\
&\quad \text{(by the triangle inequality)} \\
&=\omega\left(|A|\left|A^*\right| \otimes|B|\left|B^*\right|\right)+\omega\left(\left(\left|A^*\right||A| \otimes\left|B^*\right||B|\right)^*\right) \\
&\quad (\text{since } \omega(T)=\omega(T^*)) \\
&=\omega\left(|A|\left|A^*\right| \otimes|B|\left|B^*\right|\right)+\omega\left(\left(\left|A^*\right||A|\right)^* \otimes\left(\left|B^*\right||B|\right)^*\right) \\
&\quad \text{(by (1.2))} \\
&=2\omega\left(|A|\left|A^*\right| \otimes|B|\left|B^*\right|\right).
\end{aligned}
$$

This latter inequality, (2.4) and (2.3) imply the desired result upon substitution in (2.2). □

## 2.2. On $\omega(AB \otimes BA)$

In this subsection, we focus on finding upper bounds for $\omega(AB \otimes BA)$, with some applications on the Aluthge transform.

**Theorem 2.3.** *Let $A, B \in \mathcal{M}_n$. Then for any $r \geq 1$,*

$$
\omega^r(AB \otimes BA) \leq \frac{1}{4}\left(\|A\|^{2r}\|B\|^{2r}+\left\||A|^r|B^*|^r\right\|\left\||B|^r|A^*|^r\right\|\right)+\frac{1}{2}\omega^r(BA \otimes AB).
$$

*In particular,*

$$
\omega(AB \otimes BA) \leq \frac{1}{4}\left(\|A\|^2\|B\|^2+\|AB\|\,\|BA\|\right)+\frac{1}{2}\omega(BA \otimes AB).
$$

**Proof.** It follows from (1.9) that

$$
\begin{aligned}
&\omega^r((A \otimes B)(B \otimes A)) \\
&=\omega^r(AB \otimes BA) \quad \text{(by (1.5))} \\
&\leq \frac{1}{4}\left\||A \otimes B|^{2r}+\left|(B \otimes A)^*\right|^{2r}\right\|+\frac{1}{2}\omega^r((B \otimes A)(A \otimes B)) \\
&=\frac{1}{4}\left\||A \otimes B|^{2r}+\left|(B \otimes A)^*\right|^{2r}\right\|+\frac{1}{2}\omega^r(BA \otimes AB) \quad \text{(by (1.5))} \\
&=\frac{1}{4}\left\||A \otimes B|^{2r}+\left|B^* \otimes A^*\right|^{2r}\right\|+\frac{1}{2}\omega^r(BA \otimes AB) \quad \text{(by (1.2))} \\
&=\frac{1}{4}\left\|(|A| \otimes|B|)^{2r}+\left(\left|B^*\right| \otimes\left|A^*\right|\right)^{2r}\right\|+\frac{1}{2}\omega^r(BA \otimes AB) \quad \text{(by (1.3))} \\
&=\frac{1}{4}\left\||A|^{2r} \otimes|B|^{2r}+\left|B^*\right|^{2r} \otimes\left|A^*\right|^{2r}\right\|+\frac{1}{2}\omega^r(BA \otimes AB) \quad \text{(by (1.4))}.
\end{aligned}
$$

That is,

$$
\omega^r(AB \otimes BA) \leq \frac{1}{4}\left\||A|^{2r} \otimes|B|^{2r}+\left|B^*\right|^{2r} \otimes\left|A^*\right|^{2r}\right\|+\frac{1}{2}\omega^r(BA \otimes AB). \tag{2.5}
$$

On the other hand, (1.14) implies

$$
\left\||A|^{2r} \otimes|B|^{2r}+\left|B^*\right|^{2r} \otimes\left|A^*\right|^{2r}\right\|
$$

$$\begin{aligned}
&\le \frac{1}{2}\left(\left\||A|^{2r}\otimes|B|^{2r}\right\|+\left\||B^*|^{2r}\otimes|A^*|^{2r}\right\|\right)\\
&\quad+\frac{1}{2}\sqrt{\left(\left\||A|^{2r}\otimes|B|^{2r}\right\|-\left\||B^*|^{2r}\otimes|A^*|^{2r}\right\|\right)^2+4\left\|\left(|A|^{2r}\otimes|B|^{2r}\right)^{\frac{1}{2}}\left(|B^*|^{2r}\otimes|A^*|^{2r}\right)^{\frac{1}{2}}\right\|^2}\\
&= \frac{1}{2}\left(\left\||A|^{2r}\right\|\left\||B|^{2r}\right\|+\left\||B^*|^{2r}\right\|\left\||A^*|^{2r}\right\|\right)\\
&\quad+\frac{1}{2}\sqrt{\left(\left\||A|^{2r}\right\|\left\||B|^{2r}\right\|-\left\||B^*|^{2r}\right\|\left\||A^*|^{2r}\right\|\right)^2+4\left\|\left(|A|^{2r}\otimes|B|^{2r}\right)^{\frac{1}{2}}\left(|B^*|^{2r}\otimes|A^*|^{2r}\right)^{\frac{1}{2}}\right\|^2}\\
&\quad\text{(by (1.1))}\\
&= \frac{1}{2}\left(\|\,|A|\,\|^{2r}\|\,|B|\,\|^{2r}+\|\,|B^*|\,\|^{2r}\|\,|A^*|\,\|^{2r}\right)\\
&\quad+\frac{1}{2}\sqrt{\left(\|\,|A|\,\|^{2r}\|\,|B|\,\|^{2r}-\|\,|B^*|\,\|^{2r}\|\,|A^*|\,\|^{2r}\right)^2+4\left\|\left(|A|^{2r}\otimes|B|^{2r}\right)^{\frac{1}{2}}\left(|B^*|^{2r}\otimes|A^*|^{2r}\right)^{\frac{1}{2}}\right\|^2}\\
&\quad\text{(by (1.15))}\\
&= \|A\|^{2r}\|B\|^{2r}+\|(|A|^r\otimes|B|^r)(|B^*|^r\otimes|A^*|^r)\| \quad \text{(by (1.12))}\\
&= \|A\|^{2r}\|B\|^{2r}+\||A|^r|B^*|^r\otimes|B|^r|A^*|^r\| \quad \text{(by (1.4))}\\
&= \|A\|^{2r}\|B\|^{2r}+\||A|^r|B^*|^r\|\,\||B|^r|A^*|^r\| \quad \text{(by (1.5))}
\end{aligned}$$

i.e.,

$$\left\||A^*|^{2r}\otimes|B^*|^{2r}+|B|^{2r}\otimes|A|^{2r}\right\|\le\|A\|^{2r}\|B\|^{2r}+\||A|^r|B^*|^r\|\,\||B|^r|A^*|^r\|. \quad (2.6)$$

Now, (2.5), together with (2.6), implies the first inequality.

The second inequality follows from the first inequality by letting $r=1$. Indeed,

$$\begin{aligned}
\omega(AB\otimes BA)&\le\frac{1}{4}\left(\|A\|^2\|B\|^2+\||A|\,|B^*|\|\,\||B|\,|A^*|\|\right)+\frac{1}{2}\omega(BA\otimes AB)\\
&=\frac{1}{4}\left(\|A\|^2\|B\|^2+\|AB\|\,\|BA\|\right)+\frac{1}{2}\omega(BA\otimes AB) \quad \text{(by (1.13))}.
\end{aligned}$$

This completes the proof. $\square$

We employ Theorem 2.3 to obtain the following version for the Aluthge transform.

**Corollary 2.1.** *Let $T\in\mathcal{M}_n$. Then for any $r\ge1$ and $0\le\nu\le1$,*

$$\omega^r\left(T\otimes\widetilde{T_\nu}\right)\le\frac{1}{4}\left(\|T\|^{2r}+\|T\|^r\left\||T|^{r\nu}|T^*|^{r(1-\nu)}\right\|\right)+\frac{1}{2}\omega^r\left(\widetilde{T_\nu}\otimes T\right).$$

*In particular,*

$$\omega\left(T\otimes\widetilde{T}\right)\le\frac{1}{4}\left(\|T\|^2+\|T\|\left\||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right\|\right)+\frac{1}{2}\omega\left(\widetilde{T}\otimes T\right).$$

**Proof.** Let $T=U|T|$ be the polar decomposition of $T$. Let $A=U|T|^{1-\nu}$ and $B=|T|^\nu$, in Theorem 2.3. Then $AB=T$, $BA=\widetilde{T_\nu}$, $|B|^2=|B^*|^2=|T|^{2\nu}$. Using the basic properties of the polar decomposition [6, p.58], we have

$$|A^*|^2=U|T|^{2(1-\nu)}U^*=|T^*|^{2(1-\nu)},$$

and

$$|A|^2 = |T|^{1-\nu} U^* U |T|^{1-\nu} = |T|^{2(1-\nu)}.$$

Now from the inequality (2.5), we reach

$$\omega^r \left( T \otimes \widetilde{T_\nu} \right) \leq \frac{1}{4} \left\| |T|^{2r(1-\nu)} \otimes |T|^{2r\nu} + |T|^{2r\nu} \otimes |T^*|^{2r(1-\nu)} \right\| + \frac{1}{2} \omega^r \left( \widetilde{T_\nu} \otimes T \right).$$

By (1.14), we see that

$$\left\| |T|^{2r(1-\nu)} \otimes |T|^{2r\nu} + |T|^{2r\nu} \otimes |T^*|^{2r(1-\nu)} \right\| \leq \|T\|^{2r} + \|T\|^r \left\| |T|^{r\nu} |T^*|^{r(1-\nu)} \right\|,$$

which completes the proof. □

## 2.3. On $\|A \otimes B + B \otimes A\|$

In [8, Theorem 2.8], it has been shown that if $A, B \in \mathcal{M}_n$ are positive semi-definite, then

$$\|A \otimes B + B \otimes A\| \leq \|A\| \, \|B\| + \left\| A^{\frac{1}{2}} B^{\frac{1}{2}} \right\|^2.$$

For general matrices $A, B \in \mathcal{M}_n$, it was shown in [8, Theorem 2.13] that

$$\|A \otimes B + B \otimes A\| \leq \|A\| \, \|B\| + \max(\|AB^*\|, \|A^*B\|).$$

We begin this subsection with the following lemma that we will need to prove a new upper bound for $\|A \otimes B + B \otimes A\|$.

**Lemma 2.1.** *Let $A, B \in \mathcal{M}_n$. Then for any $0 \leq \nu \leq 1$,*

$$\|A + B\|^2 \leq \sqrt{\left\| |A|^{4\nu} + |B|^{4\nu} \right\| \left\| |A^*|^{4(1-\nu)} + |B^*|^{4(1-\nu)} \right\|} + \|A\| \, \|B\| \, \omega(B^* A).$$

**Proof.** Let $x, y \in \mathbb{C}^n$ be any unit vectors. Then

$$\begin{aligned}
& |\langle (A+B) x, y \rangle|^2 \\
& \leq (|\langle Ax, y\rangle| + |\langle Bx, y\rangle|)^2 \quad \text{(by the triangle inequality)} \\
& = |\langle Ax, y\rangle|^2 + |\langle Bx, y\rangle|^2 + 2 |\langle Ax, y\rangle| \, |\langle Bx, y\rangle| \\
& \leq \left\langle |A|^{2\nu} x, x \right\rangle \left\langle |A^*|^{2(1-\nu)} y, y \right\rangle + \left\langle |B|^{2\nu} x, x \right\rangle \left\langle |B^*|^{2(1-\nu)} y, y \right\rangle \\
& \quad + \|Ax\| \, \|Bx\| \, |\langle Ax, Bx\rangle| \\
& \quad \text{(by the Buzano's inequality [3])} \\
& \leq \sqrt{\left( \left\langle |A|^{2\nu} x, x \right\rangle^2 + \left\langle |B|^{2\nu} x, x \right\rangle^2 \right) \left( \left\langle |A^*|^{2(1-\nu)} y, y \right\rangle^2 + \left\langle |B^*|^{2(1-\nu)} y, y \right\rangle^2 \right)} \\
& \quad + \|Ax\| \, \|Bx\| \, |\langle Ax, Bx\rangle| \\
& \quad \text{(by the Cauchy-Schwarz inequality)} \\
& \leq \sqrt{\left\langle \left( |A|^{4\nu} + |B|^{4\nu} \right) x, x \right\rangle \left\langle \left( |A^*|^{4(1-\nu)} + |B^*|^{4(1-\nu)} \right) y, y \right\rangle} \\
& \quad + \|Ax\| \, \|Bx\| \, |\langle Ax, Bx\rangle| \\
& \quad \text{(by the McCarty inequality)}
\end{aligned}$$

$$\leq \sqrt{\left\| |A|^{4\nu} + |B|^{4\nu} \right\| \left\| |A^*|^{4(1-\nu)} + |B^*|^{4(1-\nu)} \right\|} + \|A\| \, \|B\| \, \omega\left(B^* A\right).$$

That is,

$$|\langle (A+B)\,x, y\rangle|^2 \leq \sqrt{\left\| |A|^{4\nu} + |B|^{4\nu} \right\| \left\| |A^*|^{4(1-\nu)} + |B^*|^{4(1-\nu)} \right\|} + \|A\| \, \|B\| \, \omega\left(B^* A\right).$$

Taking the supremum over unit vectors $x, y \in \mathbb{C}^n$ in the above inequality, we get

$$\|A+B\|^2 \leq \sqrt{\left\| |A|^{4\nu} + |B|^{4\nu} \right\| \left\| |A^*|^{4(1-\nu)} + |B^*|^{4(1-\nu)} \right\|} + \|A\| \, \|B\| \, \omega\left(B^* A\right),$$

as expected. □

Now we show our main result concerning a possible upper bound of $\|A \otimes B + B \otimes A\|$.

**Theorem 2.4.** *Let $A, B \in \mathcal{M}_n$. Then for any $0 \leq \nu \leq 1$,*

$$\begin{aligned} &\|A \otimes B + B \otimes A\|^2 \\ &\leq \sqrt{\left\| |A|^{4\nu} \otimes |B|^{4\nu} + |B|^{4\nu} \otimes |A|^{4\nu} \right\| \left\| |A^*|^{4(1-\nu)} \otimes |B^*|^{4(1-\nu)} + |B^*|^{4(1-\nu)} \otimes |A^*|^{4(1-\nu)} \right\|} \\ &\quad + \|A\|^2 \|B\|^2 \omega\left(B^* A \otimes A^* B\right). \end{aligned}$$

*In particular,*

$$\begin{aligned} &\|A \otimes B + B \otimes A\|^2 \\ &\leq \sqrt{\left\| |A|^2 \otimes |B|^2 + |B|^2 \otimes |A|^2 \right\| \left\| |A^*|^2 \otimes |B^*|^2 + |B^*|^2 \otimes |A^*|^2 \right\|} \\ &\quad + \|A\|^2 \|B\|^2 \omega\left(B^* A \otimes A^* B\right). \end{aligned} \tag{2.7}$$

**Proof.** We have by Lemma 2.1,

$$\begin{aligned} &\|A \otimes B + B \otimes A\|^2 \\ &\leq \sqrt{\left\| |A \otimes B|^{4\nu} + |B \otimes A|^{4\nu} \right\| \left\| \left|(A \otimes B)^*\right|^{4(1-\nu)} + \left|(B \otimes A)^*\right|^{4(1-\nu)} \right\|} \\ &\quad + \|A \otimes B\| \, \|B \otimes A\| \, \omega\left((B \otimes A)^* (A \otimes B)\right) \\ &= \sqrt{\left\| |A \otimes B|^{4\nu} + |B \otimes A|^{4\nu} \right\| \left\| |A^* \otimes B^*|^{4(1-\nu)} + |B^* \otimes A^*|^{4(1-\nu)} \right\|} \\ &\quad + \|A \otimes B\| \, \|B \otimes A\| \, \omega\left((B^* \otimes A^*)(A \otimes B)\right) \quad \text{(by (1.2))} \\ &= \sqrt{\left\| |A \otimes B|^{4\nu} + |B \otimes A|^{4\nu} \right\| \left\| |A^* \otimes B^*|^{4(1-\nu)} + |B^* \otimes A^*|^{4(1-\nu)} \right\|} \\ &\quad + \|A\|^2 \|B\|^2 \omega\left((B^* \otimes A^*)(A \otimes B)\right) \quad \text{(by (1.1))} \\ &= \sqrt{\left\| (|A| \otimes |B|)^{4\nu} + (|B| \otimes |A|)^{4\nu} \right\| \left\| (|A^*| \otimes |B^*|)^{4(1-\nu)} + (|B^*| \otimes |A^*|)^{4(1-\nu)} \right\|} \\ &\quad + \|A\|^2 \|B\|^2 \omega\left((B^* \otimes A^*)(A \otimes B)\right) \quad \text{(by (1.3))} \\ &= \sqrt{\left\| |A|^{4\nu} \otimes |B|^{4\nu} + |B|^{4\nu} \otimes |A|^{4\nu} \right\| \left\| |A^*|^{4(1-\nu)} \otimes |B^*|^{4(1-\nu)} + |B^*|^{4(1-\nu)} \otimes |A^*|^{4(1-\nu)} \right\|} \end{aligned}$$

$$+\left\|A\right\|^2\left\|B\right\|^2\omega\left(\left(B^*\otimes A^*\right)\left(A\otimes B\right)\right)\quad \text{(by (1.4))}$$
$$=\sqrt{\left\||A|^{4\nu}\otimes|B|^{4\nu}+|B|^{4\nu}\otimes|A|^{4\nu}\right\|\left\||A^*|^{4(1-\nu)}\otimes|B^*|^{4(1-\nu)}+|B^*|^{4(1-\nu)}\otimes|A^*|^{4(1-\nu)}\right\|}$$
$$+\left\|A\right\|^2\left\|B\right\|^2\omega\left(B^*A\otimes A^*B\right)\quad \text{(by (1.5))}$$

which completes the proof. □

**Remark 2.2.** If we replace $A$ by $A^*$ and $B$ by $B^*$, in (2.7), we get

$$\begin{aligned}&\left\|A\otimes B+B\otimes A\right\|^2\\ \leq&\left\||A|^2\otimes|B|^2+|B|^2\otimes|A|^2\right\|\left\||A^*|^2\otimes|B^*|^2+|B^*|^2\otimes|A^*|^2\right\|\\ &+\left\|A\right\|^2\left\|B\right\|^2\min\left(\omega\left(B^*A\otimes A^*B\right),\omega\left(BA^*\otimes AB^*\right)\right).\end{aligned}$$

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