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BIFURCATION ANALYSIS IN A MODIFIED LESLIE-GOWER PREDATOR-PREY MODEL WITH BEDDINGTON-DEANGELIS FUNCTIONAL RESPONSE*

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Abstract This paper investigates the dynamics of a modified Leslie-Gower predator-prey model with Bedington-DeAngelis functional response. Some properties are explored, including positivity, dissipativity, permanence, and stability. In addition, the transcritical bifurcation and Hopf bifurcation taking d as the bifurcation parameter and Bogdanov-Takens bifurcation taking d and n as bifurcation parameters are studied. The theoretical results of this paper are verified by numerical simulation. The results show that the system has rich dynamical behaviors.

Keywords Modified Leslie-Gower predator-prey model, Beddington-DeAngelis functional response, bifurcation.

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1. Introduction

Biometrics is an interdisciplinary field of biology and mathematics, and it is an interesting field of applied science. In recent years, the dynamic relationship in the predator-prey model has attracted a great deal of attention in biological mathematics. In nature, the mode of survival of predation and prey among populations is universal. Predation is a kind of biological interaction, in which the predator organism feeds on another kind of creature called prey or other organisms. The interaction between prey and predator is common and well known in ecosystems, which is one of the important fields of mathematical biology.

Lotka [16] and Volterra [24] proposed the most basic and important Lotka-Volterra predator-prey model in 1926, which is used to describe the dynamic relationship between two populations. On the basis of this model, many mathe-

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maticians and ecologists are attracted to improve and conduct research in different fields [4,5,11,17,18,21,22].

The basic unit of the ecosystem is the predator-prey model. The typical one is as follows

$$\begin{cases} \dfrac{\mathrm{d}x}{dt} = xg(x) - y\psi(x,y), \\ \dfrac{\mathrm{d}y}{dt} = \beta y - y\phi(x,y), \end{cases} \tag{1.1}$$

where x, y denote the population densities of predator and prey respectively when $t > 0$. $g(x)$ represents the intrinsic growth rate of the prey without predators. The constant β represents the inherent growth rate of the predator, and $\psi(x,y)$ and $\phi(x,y)$ represent the functional response function of the predator to the prey.

In recent years, in order to study the dynamic behavior of the interacting predator-prey species in the ecosystem, researchers used different types of functional responses, such as Holling type, Beddington-DeAngelis type, Crowley-Martin type, Leslie-Gower type, and Hassell-Varley type, etc. In particular, Holling type II, III and IV functional response functions are prey dependent or predator dependent. The forms are as follows

$$\text{Holling type II: } \psi(x,y) = \frac{mx}{a+x},$$
$$\text{Holling type III: } \psi(x,y) = \frac{mx^2}{a+x^2},$$
$$\text{Holling type IV: } \psi(x,y) = \frac{mx}{a+x^2},$$

where x represents the population density of the prey, a and m are positive numbers, and a is a semi-saturated parameter. It can be found that the above functional response functions only depend on the prey. Later, scholars realized that the predator's predation rate depends not only on the density of the prey, but also on the predator itself. To describe the mutual interference between predators and prey, Beddington [3] and DeAngelis [8] proposed the following Beddington-DeAngelis functional response function in 1975

$$\text{Beddington-DeAngelis: } \psi(x,y) = \frac{mx}{a+bx+cy},$$

where x and y represent the number of prey and predator, respectively, and a, b, c, and m are positive parameters. There is an additional term cy in its denominator, which simulates mutual interference between predators. The Beddington-DeAngelis functional response function avoids some problems caused by the proportional-dependent functional response under low population density, and thus better explains the predator feeding on various predator-prey abundances. In 1960, Leslie and Gower established the classical Leslie-Gower model [12], which takes the following form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\left(s - \frac{\gamma y}{x}\right),$$

where s is the intrinsic growth rate of the predator, and $\frac{x}{\gamma}$ is the maximum environmental capacity of the predator population. $\phi(x,y) = \frac{\gamma y}{x}$ is called the Leslie-Gower term. Leslie studied that the environmental carrying capacity of predators is proportional to the prey population, which means that the decline of the predator population is only due to its preference for prey. They also found that if prey does not exist or is in short supply, predators may turn to other foods. Therefore, a positive parameter α is added to the denominator of the Leslie-Gower term, and forms the so-called modified Leslie-Gower term $\phi(x, y) = \frac{\gamma y}{x+\alpha}$. Then the modified Leslie-Gower model takes the following form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\left(s - \frac{\gamma y}{x+\alpha}\right),$$

and has been studied by many authors, see [1, 2, 6, 9].

In [23], bifurcation and a systematic approach for the estimation of identifiable parameters of a modified Leslie-Gower predator-prey system with Crowley-Martin functional response and prey refuge are discussed. Global asymptotic stability is discussed by applying the fluctuation lemma, and the stability of Hopf bifurcation is discussed. In [14], the dynamics of a diffusive predator-prey model with modified Leslie-Gower term and strong Allee effect on prey under homogeneous Neumann boundary condition is considered. Therefore, based on the research and analysis of the above literature, in this article, we mainly introduce the Beddington-DeAngelis functional response function and the modified Leslie-Gower term in the basic predator-prey model, study the stability of its equilibria and the bifurcation behavior under parameter conditions, and further discuss its dynamic properties.

In this paper, we consider a modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response function. In Section 2, we give a mathematical model and discuss the positivity, dissipativity, and permanence of the model. In Section 3, we discuss the existence and stability of the equilibria. In Section 4, we study various bifurcations at different equilibria, and show that the model undergoes transcritical bifurcations, Hopf bifurcations, and codimension two Bogdanov-Takens bifurcations under certain conditions. In Section 5, we give some numerical simulations to substantiate the theory findings in Section 4. Finally, we present a brief discussion in Section 6.

2. Preliminaries

In this paper, we consider a modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response function, assume that the prey follows the logistic growth, then system (1.1) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \dfrac{x}{K}\right) - \dfrac{qxy}{a+bx+cy}, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y\left(\beta - \dfrac{\gamma y}{x+\alpha}\right), \end{cases} \tag{2.1}$$

where x, y are the population densities of the prey and predator with respect to time t. All parameters are positive. r represents the internal growth rate of the bait, K represents the environmental carrying capacity of the prey, the growth rate of the prey is logistic with the carrying capacity K and the intrinsic growth rate r. q is to measure the number of prey that the predator can eat in each time unit. a is the prey density with a half-saturated attack rate, b represents the measurement of food abundance relative to the predator population, c is a measure of the intensity of competition between individuals of the predator population, β is the internal growth rate of predator population. γ is the maximum per capita reduction rate of predators and α measures the degree to which the environment protects predators. Before discussing in detail, we simplified model (2.1) by dimensionless. With the following non-dimensionalized change of variables

$$\bar{x} = \frac{x}{K}, \qquad \bar{y} = \frac{qy}{bkr}, \qquad \bar{t} = rt, \qquad d = \frac{a}{Kb},$$
$$e = \frac{rc}{q}, \qquad m = \frac{\beta}{r}, \qquad n = \frac{b\gamma}{q}, \qquad p = \frac{\alpha}{K},$$

and dropping the bars, model (2.1) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(1-x) - \dfrac{xy}{d+x+ey}, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y\left(m - \dfrac{ny}{x+p}\right), \end{cases} \tag{2.2}$$

with the initial conditions $x(0) = x_0 > 0$, $y(0) = y_0 > 0$. Where d, e, m, n, and p are positive numbers and $1 - e > 0$. Proceeding from the biological significance of the above models, we only consider system (2.2) in $\Omega = \{(x, y)|x \geq 0, y \geq 0\}$. It can be obtained that the positive invariant and bounded region of system (2.2) is $\triangle = \{(x, y)|0 \leq x \leq 1, 0 \leq y \leq \frac{m(1+p)}{n}\}$. For better qualitative analysis of model (2.2), we rewrite it as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x\left(1 - x - \dfrac{y}{d+x+ey}\right) = xf(x,y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y\left(m - \dfrac{ny}{x+p}\right) = yg(x,y), \end{cases} \tag{2.3}$$

where $f(x, y) = 1 - x - \frac{y}{d+x+ey}$, $g(x, y) = m - \frac{ny}{x+p}$. In the following, we will prove the positivity of the solution, the dissipativity and the permanence of the system (2.3). We can use the following lemma to prove the dissipativity and permanence.

Lemma 2.1 ([7]). *If* $a, b > 0$, $\frac{dX}{dt} \leq (\geq) X(t)(a - bX(t))$ *with* $X(0) > 0$, *then*

$$\limsup_{t\to+\infty} X(t) \leq \frac{a}{b} \left(\liminf_{t\to+\infty} X(t) \geq \frac{a}{b}\right).$$

The following lemma can be used to prove the bifurcation that occurs at positive equilibrium.

Lemma 2.2 (Lemma 1, [10]). *The system*

$$\begin{cases} \dfrac{dx}{dt} = y + Ax^2 + Bxy + Cy^2 + o(|x,y|^2), \\ \dfrac{dy}{dt} = Dx^2 + Exy + Fy^2 + o(|x,y|^2), \end{cases}$$

is equivalent to the system

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = Dx^2 + (E+2A)xy + o(|x,y|^2), \end{cases}$$

after some nonsingular transformations in the neighborhood of $(0, 0)$.

2.1. Positivity and dissipativity

Proposition 2.1. *All solutions of the system* (2.3) *are positive with the initial condition* $x(0) = x_0 > 0$, $y(0) = y_0 > 0$.

Proof. From the first equation of the system (2.3), we can see that $x = 0$ is an invariant set. That is, $x(t) > 0$ for all t with $x(0) > 0$. In the same way, from the second equation of the system (2.3), we can see that $y = 0$ is an invariant set. That is, $y(t) > 0$ for all t, if $y(0) > 0$. According to the existence and uniqueness of the solution, all the solutions of the system (2.3) to the initial conditions are positive. □

Proposition 2.2. *System* (2.3) *is dissipative.*

Proof. From Proposition 2.1, it is easy to see that variables x and y are positive. According to the first equation in (2.3), we can write

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x\left(1 - x - \frac{y}{d + x + ey}\right)\\ &\le x(1 - x).\end{aligned}$$

From Lemma 2.1, we have

$$\limsup_{t\to+\infty} x(t) \le 1 \triangleq M_1,$$

therefore, for arbitrary $\epsilon_1 > 0$, there exists $T_1 > 0$, such that for arbitrary $t \ge T_1$, the following inequality holds

$$x(t) \le M_1 + \epsilon_1.$$

According to the second equation in (2.3), we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= y\left(m - \frac{ny}{x + p}\right)\\ &\le y\left(m - \frac{ny}{M_1 + \epsilon_1 + p}\right).\end{aligned}$$

From Lemma 2.1, we have

$$\limsup_{t\to+\infty} y(t) \le \frac{m(M_1 + \epsilon_1 + p)}{n} \triangleq M_2,$$

therefore, for arbitrary $\epsilon_2 > 0$, there exists $T_2 > T_1$, such that for arbitrary $t \ge T_2$, the following inequality holds

$$y(t) \le M_2 + \epsilon_2.$$

This proves the dissipativity of system (2.3). □

2.2. Permanence

Definition 2.1. System (2.3) is permanent if there exists positive constants K_1 and $K_2 (0 < K_1 < K_2)$ such that each positive solution $(x(t, x_0, y_0), y(t, x_0, y_0))$ of system (2.3) with initial condition $(x_0, y_0) \in R_+^2$ satisfies

$$\min \left\{ \liminf_{t \to +\infty} x(t, x_0, y_0), \liminf_{t \to +\infty} y(t, x_0, y_0) \right\} \geq K_1,$$
$$\max \left\{ \limsup_{t \to +\infty} x(t, x_0, y_0), \limsup_{t \to +\infty} y(t, x_0, y_0) \right\} \leq K_2.$$

Proposition 2.3. *System* (2.3) *is permanent if*

$$1 - \frac{M_2 + \epsilon_2}{d} > 0, \ \frac{n}{m_1 - \epsilon_3 + p} > 0.$$

Proof. It is obvious that under the initial condition $x(0) > 0$ and $y(0) > 0$, the solution of system (2.3) is nonnegative. As can be seen from Proposition 2.2, for arbitrary $\epsilon_2 > 0$, there exists a T_2, such that for arbitrary $t \geq T_2$, we have $y(t) \leq M_2 + \epsilon_2$ holds.

For the first equation of (2.3), for arbitrary $t > T_2$, we can write

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x\left(1 - x - \frac{y}{d + x + ey}\right) \\ &\geq x\left(1 - x - \frac{y}{d}\right) \\ &= x\left(1 - x - \frac{M_2 + \epsilon_2}{d}\right) \\ &= x\left(1 - \frac{M_2 + \epsilon_2}{d} - x\right), \end{aligned}$$

if $1 - \frac{M_2 + \epsilon_2}{d} > 0$, then according to Lemma 2.1, we have

$$\liminf_{t \to +\infty} x(t) \geq 1 - \frac{M_2 + \epsilon_2}{d} \triangleq m_1.$$

Therefore, for arbitrary $\epsilon_3 > 0$, there exists $T_3 > T_2$, such that for arbitrary $t \geq T_3$, the following inequality holds

$$x(t) \geq m_1 - \epsilon_3.$$

For the second equation of (2.3), for arbitrary $t > T_3$, we can write

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= y\left(m - \frac{ny}{x + p}\right) \\ &\geq y\left(m - \frac{ny}{m_1 - \epsilon_3 + p}\right), \end{aligned}$$

when $\frac{n}{m_1 - \epsilon_3 + p} > 0$, through Lemma 2.1, we have

$$\liminf_{t \to +\infty} y(t) \geq \frac{m(m_1 - \epsilon_3 + p)}{n} \triangleq m_2.$$

Combining the Proposition 2.2, we can draw the following conclusions

$$\limsup_{t\to+\infty} x(t) \le 1 \triangleq M_1, \ \limsup_{t\to+\infty} y(t) \le \frac{m(M_1+\epsilon_1+p)}{n} \triangleq M_2.$$

Then there exists two constants $K_1 = \min\{m_1, m_2\}$, $K_2 = \max\{M_1, M_2\}$, such that the following inequality holds

$$\min\left\{\liminf_{t\to+\infty} x(t,x_0,y_0), \liminf_{t\to+\infty} y(t,x_0,y_0)\right\} \ge K_1,$$
$$\max\left\{\limsup_{t\to+\infty} x(t,x_0,y_0), \limsup_{t\to+\infty} y(t,x_0,y_0)\right\} \le K_2.$$

Thus, the permanence of the system (2.3) can be proved. □

3. Existence and stability of equilibrium

3.1. Existence and stability of boundary equilibrium

To find the equilibrium of the system (2.2), we give the following equation

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x\left(1 - x - \dfrac{y}{d+x+ey}\right) = 0, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y\left(m - \dfrac{ny}{x+p}\right) = 0, \end{cases} \tag{3.1}$$

from equation (3.1), we can get that the system (2.2) always has three boundary equilibria $A_1(0,0)$, $A_2(1,0)$, $A_3(0, \frac{mp}{n})$. The Jacobian matrix of the system (2.2) at any equilibrium $E(x,y)$ takes the following form

$$J(E) = \begin{bmatrix} 1 - 2x - \dfrac{dy+ey^2}{(d+x+ey)^2} & -\dfrac{dx+x^2}{(d+x+ey)^2} \\ \dfrac{ny^2}{(x+p)^2} & m - \dfrac{2ny}{x+p} \end{bmatrix}.$$

In the following, the stability of each boundary equilibrium is studied by using the above Jacobian matrix.

Theorem 3.1. *The origin $A_1(0,0)$ is always an unstable node.*

Proof. The Jacobian matrix of model (2.2) at $A_1(0,0)$ is

$$J(A_1(0,0)) = \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix},$$

$J(A_1(0,0))$ has two eigenvalues $\lambda_1 = 1 > 0$ and $\lambda_2 = m > 0$. Therefore, $A_1(0,0)$ is always an unstable node. □

Theorem 3.2. *The Boundary equilibrium $A_2(1,0)$ is always a saddle point.*

Proof. The Jacobian matrix of model (2.2) at $A_2(1,0)$ is

$$J(A_2(1,0)) = \begin{bmatrix} -1 & -\dfrac{1}{d+1} \\ 0 & m \end{bmatrix},$$

$J(A_2(1,0))$ has two eigenvalues $\lambda_1 = -1 < 0$ and $\lambda_2 = m > 0$. Therefore, $A_2(1,0)$ is always a saddle point. □

Theorem 3.3. *For the stability of $A_3(0, \frac{mp}{n})$, we have*

(1) *The Boundary equilibrium $A_3(0, \frac{mp}{n})$ is always a saddle point if $d > \dfrac{pm(1-e)}{n}$.*

(2) *The Boundary equilibrium $A_3(0, \frac{mp}{n})$ is always a stable node if $0 < d < \dfrac{pm(1-e)}{n}$.*

(3) *The Boundary equilibrium $A_3(0, \frac{mp}{n})$ is a degenerate equilibrium if $d = \dfrac{pm(1-e)}{n}$.*

Proof. The Jacobian matrix of model (2.2) at $A_3(0, \frac{mp}{n})$ is

$$J\left(A_3\left(0, \frac{mp}{n}\right)\right) = \begin{bmatrix} 1 - \dfrac{pm}{nd+epm} & 0 \\ \dfrac{m^2}{n} & -m \end{bmatrix},$$

from the Jacobian matrix, the characteristic equation at the boundary equilibrium $A_3(0, \frac{mp}{n})$ is as follows

$$\lambda^2 - \left(1 - \frac{pm}{nd+epm} - m\right)\lambda + m\left(\frac{pm}{nd+epm} - 1\right) = 0.$$

The determinant and trace of the above Jacobian matrix at the boundary equilibrium $A_3(0, \frac{mp}{n})$ are

$$\text{tr}J(A_3) = 1 - \frac{pm}{nd+epm} - m, \ \ \det J(A_3) = m\left(\frac{pm}{nd+epm} - 1\right),$$

the point $A_3(0, \frac{mp}{n})$ is a saddle point for $\det J(A_3) < 0$, and the following condition is obtained

$$d > \frac{pm(1-e)}{n} \ ,$$

the point $A_3(0, \frac{mp}{n})$ is a stable node for $\text{tr}J(A_3) < 0$ and $\det J(A_3) > 0$. This gives the following condition

$$0 < d < \frac{pm(1-e)}{n},$$

the point $A_3(0, \frac{mp}{n})$ is a degenerate equilibrium for $\det J(A_3)=0$. This gives the following condition

$$d = \frac{pm(1-e)}{n}.$$

□

Theorem 3.4. *The boundary equilibrium* $A_3(0, \frac{mp}{n})$ *of system* (2.2) *is a saddle-node, if* $d = \frac{pm(1-e)}{n}$ *and* $2pmn + (1-e)mn - 2n^2 \neq 0$.

Proof. The transformation $(X, Y) = \left(x, y - \frac{mp}{n}\right)$ changes the equilibrium from $A_3(0, \frac{mp}{n})$ to the origin, and the system (2.2) becomes

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = \left(\dfrac{n}{pm} - 1\right)X^2 - \dfrac{(1-e)n}{2pm}XY + P_1(X, Y), \\ \dfrac{\mathrm{d}Y}{\mathrm{d}t} = \dfrac{m^2}{n}X - mY - \dfrac{m^2}{np}X^2 + \dfrac{m}{p}XY - \dfrac{n}{p}Y^2 + Q_1(X, Y), \end{cases} \tag{3.2}$$

where $P_1(X, Y)$ and $Q_1(X, X)$ are terms of at least third order in X and Y. The Jacobian matrix of system (3.2) is diagonalizable with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -m$, and the eigenvectors corresponding to the above eigenvalues are $w_1 = (\frac{n}{m}, 1)^T$ and $w_2 = (0, 1)^T$. By the transformation

$$\begin{cases} x = \dfrac{m}{n}X, \\ y = -\dfrac{m}{n}X + Y, \end{cases}$$

system (3.2) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{2n^2 - 2pmn - (1-e)mn}{2pm^2}x^2 - \dfrac{(1-e)n}{2pm}xy + P_2(x, y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -my - \dfrac{2n^2 + 2m^2n - 2pmn - (1-e)mn}{2pm^2}x^2 + \dfrac{(1-e)n - 2mn}{2pm}xy \\ \qquad - \dfrac{n}{p}y^2 + Q_2(x, y), \end{cases}$$

here $P_2(x, y)$ and $Q_2(x, y)$ are terms of at least third order in x and y. Introduce new variable $\tau = -mt$, we obtain

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = \dfrac{2pmn + (1-e)mn - 2n^2}{2pm^3}x^2 + \dfrac{(1-e)n}{2pm^2}xy + P_3(x, y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = y + \dfrac{2n^2 + 2m^2n - 2pmn - (1-e)mn}{2pm^3}x^2 - \dfrac{(1-e)n - 2mn}{2pm^2}xy \\ \qquad + \dfrac{n}{pm}y^2 + Q_3(x, y), \end{cases}$$

where $P_3(x, y)$ and $Q_3(x, y)$ are terms of at least third order in x and y. When $2pmn + (1-e)mn - 2n^2 \neq 0$, by Theorem 7.1 in [25], we know that the equilibrium $A_3(0, \frac{mp}{n})$ is a saddle-node. □

3.2. Existence and stability of positive equilibrium

Next, we will study the positive equilibrium of the system (2.2), whose coordinates x and y satisfy the following equation

$$\begin{cases} 1 - x - \dfrac{y}{d + x + ey} = 0, \\ m - \dfrac{ny}{x + p} = 0. \end{cases} \tag{3.3}$$

The system (2.2) has positive equilibria is equivalent to

$$F(x) = k_2x^2 + k_1x + k_0$$

has positive zeros, where

$$k_2 = -(n + em),\ k_1 = n + em - nd - epm - m,\ k_0 = nd + epm - pm.$$

It can be seen from the above formula that $F(x) = 0$ has at most two positive real roots. Let's study the existence of the positive real roots of $F(x) = 0$, denote

$$\Delta = k_1^2 - 4k_2k_0,\quad x_{1,2} = \frac{-k_1 \pm \sqrt{\Delta}}{2k_2}\ (x_1 < x_2),$$
$$\bar{x} = \frac{-k_1}{2k_2},\quad y_i = \frac{m(x_i + p)}{n}\ (i = 1, 2).$$

Lemma 3.1. *For the existence of positive roots of $F(x) = 0$, we have the following conclusion:*

(i) If $\Delta > 0$, $k_0 < 0$, $\frac{k_1}{2k_2} < 0$, the equation has two distinct roots x_1, x_2;

(ii) If $\Delta = 0$, $k_0 < 0$, $\frac{k_1}{2k_2} < 0$, the equation has two identical roots $x_1 = x_2$;

(iii) If $k_0 = 0$, $\frac{k_1}{2k_2} < 0$, the equation always has a root x_2;

(iv) If $k_0 > 0$, the equation always has a root x_2;

(v) If $\Delta < 0$, the equation has no roots.

According to Lemma 3.1, when the parameters satisfy condition (i), system (2.2) has two positive equilibria $E_1^*(x_1, y_1)$ and $E_2^*(x_2, y_2)$; if the parameter satisfies condition (ii), then two positive equilibria $E_1^*(x_1, y_1)$ and $E_2^*(x_2, y_2)$ of system (2.2) collide with each other, and the only positive equilibrium is $\bar{E}(\bar{x}, \bar{y})$; when the parameters satisfy conditions (iii) and (iv), system (2.2) has a positive equilibrium $E_2^*(x_2, y_2)$; when the parameters satisfy conditions (v), system (2.2) has no positive equilibria.

The graph of the equation system (3.3) under different parameters is shown in Figure 1. The green and blue curves correspond to the first and second equations of (3.3), respectively. The two lines have positive intersections, meaning that $F(x) = 0$ has positive roots, which means that the system (2.2) has positive equilibria. According to Lemma 3.1 we choose the parameters $m = 0.2$, $n = 0.6$, $p = 0.4$, for other parameters, we have

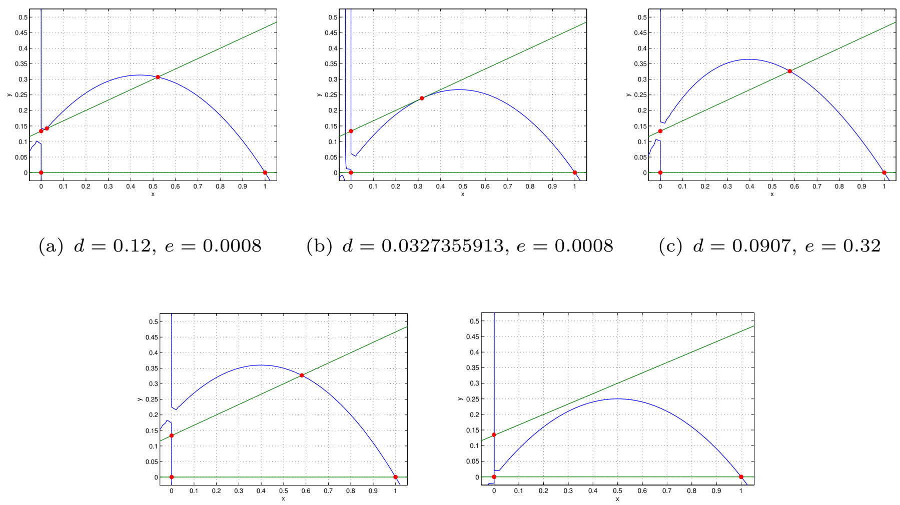
(a) $d = 0.12$, $e = 0.0008$, the parameter satisfies condition (i), the system (2.2) has two positive equilibria;

(b) $d = 0.0327355913$, $e = 0.0008$, the parameter satisfies condition (ii), the system (2.2) has two identical positive equilibria;

(c) $d = 0.0907$, $e = 0.32$, the parameters satisfy conditions (iii), the system (2.2) has a positive equilibrium;

(d) $d = 0.2$, $e = 0.008$, the parameters satisfy conditions (iv), the system (2.2) has a positive equilibrium;

(e) $d = 0.0000052$, $e = 0.0008$, the parameter satisfies condition (v), the system (2.2) has no positive equilibria,



(a) $d = 0.12$, $e = 0.0008$ (b) $d = 0.0327355913$, $e = 0.0008$ (c) $d = 0.0907$, $e = 0.32$

(d) $d = 0.2$, $e = 0.008$ (e) $d = 0.0000052$, $e = 0.0008$

Figure 1. Number of positive equilibrium under parameter conditions.

which is illustrated in Figure 1.

Remark 3.1. We express any positive equilibrium of system (2.2) as $E^*(x^*, y^*)$, where the coordinate $x^* = \xi$ and $y^* = \frac{m(\xi+p)}{n}$.

Theorem 3.5. *$E^*(x^*, y^*)$ is locally asymptotically stable if*

$$d > \frac{m(1-2\xi)(\xi+p) - em(\xi+p)(1-\xi)^2 - m^2(\xi+p)}{n(1-\xi)^2},$$

and

$$d > \frac{m(1-2\xi)(\xi+p)^2 - em(\xi+p)^2(1-\xi)^2 - n\xi^2(1-\xi)^2}{n(\xi+p)(1-\xi)^2 + n\xi(1-\xi)^2}.$$

Proof. The Jacobian matrix of system (2.2) at $E^*(x^*, y^*)$ is

$$J(E^*(x^*, y^*)) = \begin{bmatrix} \frac{m(1-2\xi)(\xi+p)-[nd+em(\xi+p)](1-\xi)^2}{m(\xi+p)} & -\frac{n^2\xi(d+\xi)(1-\xi)^2}{m^2(\xi+p)^2} \\ \frac{m^2}{n} & -m \end{bmatrix}.$$

Then, the characteristic equation of the positive equilibrium $E^*(x^*, y^*)$ is $\lambda^2 - T\lambda + D = 0$, where

$$T = \frac{m(1-2\xi)(\xi+p) - [nd + em(\xi+p)](1-\xi)^2 - m^2(\xi+p)}{m(\xi+p)},$$
$$D = \frac{m(2\xi-1)(\xi+p)^2 + [nd + em(\xi+p)](\xi+p)(1-\xi)^2 + n\xi(d+\xi)(1-\xi)^2}{(\xi+p)^2},$$

from the above expressions of T and D, we can know that if

$$d > \frac{m(1-2\xi)(\xi+p) - em(\xi+p)(1-\xi)^2 - m^2(\xi+p)}{n(1-\xi)^2},$$

and

$$d > \frac{m(1-2\xi)(\xi+p)^2 - em(\xi+p)^2(1-\xi)^2 - n\xi^2(1-\xi)^2}{n(\xi+p)(1-\xi)^2 + n\xi(1-\xi)^2},$$

then $-T > 0, D > 0$. By Routh-Hurwitz criteria, positive equilibrium $E^*(x^*, y^*)$ is locally asymptotically stable. □

Theorem 3.6. *The positive equilibrium $E^*(x^*, y^*)$ is a degenerate cusp, if*

$$d = \frac{m\xi^2(1-2\xi) - em\xi^2(1-\xi)^2 - m^2\xi^2}{me\xi(1-\xi)^2 + m^2(2\xi+p) - m\xi(1-2\xi)},$$

and

$$n = \frac{me\xi(\xi+p)(1-\xi)^2 + m^2(2\xi+p)(\xi+p) - m\xi(1-2\xi)(\xi+p)}{\xi^2(1-\xi)^2}.$$

It is codimension 2 when $K_1K_2 \neq 0$, and it is at least codimension 3 when $K_1K_2 = 0$, where $K_1 = \frac{-b_4 i^3 + (a_4 - b_3)i^2 + (a_3 - b_5)i}{s}$ and $K_2 = (2a_4 + b_3)i + a_3 + 2b_5$.

Proof. The Jacobian matrix of system (2.2) at $E^*(x^*, y^*)$ is

$$J(E^*(x^*, y^*)) = \begin{bmatrix} \frac{m(\xi+p)(1-2\xi)-(nd+em\xi+emp)(1-\xi)^2}{m(\xi+p)} & -\frac{n^2\xi(d+\xi)(1-\xi)^2}{m^2(\xi+p)^2} \\ \frac{m^2}{n} & -m \end{bmatrix}.$$

If

$$d = \frac{m\xi^2(1-2\xi) - em\xi^2(1-\xi)^2 - m^2\xi^2}{me\xi(1-\xi)^2 + m^2(2\xi+p) - m\xi(1-2\xi)},$$
$$n = \frac{me\xi(\xi+p)(1-\xi)^2 + m^2(2\xi+p)(\xi+p) - m\xi(1-2\xi)(\xi+p)}{\xi^2(1-\xi)^2},$$

then $T = 0$ and $D = 0$, therefore, the characteristic values are $\lambda_1 = \lambda_2 = 0$. Let's do some transformations first, considering the expression $(d + x + ey)(x + p) > 0$, by transformation $\mathrm{d}t = (d + x + ey)(x + p)\mathrm{d}\tau$, system (2.2) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(1-x)(d+x+ey)(x+p) - xy(x+p), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = my(d+x+ey)(x+p) - ny^2(d+x+ey). \end{cases} \tag{3.4}$$

Under linear transformation

$$\begin{cases} x = X + \xi, \\ y = Y + \dfrac{m(\xi+p)}{n}, \end{cases}$$

the positive equilibrium $E^*(x^*, y^*)$ of system (2.2) was changed to the origin, and we have

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = a_1X + a_2Y + a_3XY + a_4X^2 + a_5X^3 + a_6X^2Y + o(|X,Y|^4), \\ \dfrac{\mathrm{d}Y}{\mathrm{d}t} = b_1X + b_2Y + b_3XY + b_4X^2 + b_5Y^2 + b_6X^2Y + b_7XY^2 + o(|X,Y|^4), \end{cases} \tag{3.5}$$

where

$$\begin{aligned} a_1 =& \frac{(-4n-3em)\xi^3 + [3n((1-d-p)+2(e-1)m-5emp]\xi^2}{n} \\ &+ \frac{[2n(d+p-dp)+mp(3e-2ep-3)]\xi + mp(ep-p)+ndp}{n}, \\ a_2 =& -e\xi^3 + (e-ep-1)\xi^2 + p(e-1)\xi, \ a_3 = -3e\xi^2 + 2(e-ep-1)\xi + ep - p, \\ a_4 =& \frac{-6em\xi^2 + [-12n(1+d+p)+2m(e-4ep-1)]\xi}{2n} \\ &+ \frac{2n(d+p-dp)+2mp(e-ep-1)}{2n}, \\ a_5 =& \frac{-6(nd+em\xi+emp+4n\xi+np-n)}{n}, \ a_6 = 2(e-3e\xi-ep-1), \\ b_1 =& \frac{(em^3+m^2n)\xi^2 + m^2(dn+pn+2pem)\xi + m^2p(emp+nd)}{n^2}, \\ b_2 =& \frac{(-m^2ne-mn^2)\xi^2 + mn(-2emp-dn-pn)\xi - mnp(emp+nd)}{n^2}, \\ b_3 =& \frac{2em^2\xi + m(2emp-np+nd)}{n}, \ b_4 = \frac{m^2\xi+m^2p}{n}, \\ b_5 =& \frac{-n(2em+n)\xi - n(2emp+dn)}{n}, \ b_6 = 2m, \ b_7 = 2(me-n). \end{aligned}$$

Let $B = (v_1, v_2) = \begin{bmatrix} i & s \\ 1 & 0 \end{bmatrix}$, v_1, v_2 are the generalized eigenvectors of the Jacobian matrix $J(E^*(x^*, y^*))$ for zero eigenvalues, where $i = \frac{i_1}{i_2}$, $s = \frac{i+1}{a_1+b_1}$. The values of i_1 and i_2 are shown in the appendix. Under the transformation $(X, Y)^T = B(x, y)^T$, that is

$$\begin{cases} x = Y, \\ y = \dfrac{1}{s}X - \dfrac{i}{s}Y, \end{cases}$$

then system (3.5) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{dt} = y + h_1xy + h_2x^2 + h_3y^2 + o(|x,y|^3), \\ \dfrac{\mathrm{d}y}{dt} = l_1xy + l_2x^2 + l_3y^2 + o(|x,y|^3), \end{cases} \tag{3.6}$$

where $h_1 = b_3s + 2b_4is, h_2 = b_3i + b_4i^2 + b_5, h_3 = b_4s^2, l_1 = a_3 + 2a_4i - b_3i - 2b_4i^2$, $l_2 = \frac{-b_4i^3+(a_4-b_3)i^2+(a_3-b_5)i}{s}, l_3 = a_4s - b_4si$, and we choose not to present here for the complexity.

By Lemma 2.2, system (3.6) is equivalent to the following system in the small domain near $(0,0)$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = K_1x^2 + K_2xy + o(|x,y|^3), \end{cases} \tag{3.7}$$

where $K_1 = \frac{-b_4i^3+(a_4-b_3)i^2+(a_3-b_5)i}{s}$, $K_2 = (2a_4+b_3)i + a_3 + 2b_5$. According to literature [20], we can conclude that if $K_1K_2 \neq 0$, the positive equilibrium $E^*(x^*, y^*)$ is a cusp of codimension 2, and if $K_1K_2 = 0$, it is a cusp of at least codimension 3. □

4. Bifurcation analysis of model (2.2)

In this section, we study various bifurcations of the system (2.2) at positive equilibrium.

Theorem 4.1. *System* (2.2) *undergoes transcritical bifurcation at* $A_3(0, \frac{mp}{n})$*, if*

$$d = d^{tc} = \frac{mp(1-e)}{n},$$

and

$$n + me - 2mp - m \neq 0.$$

Proof. According to Theorem 3.3, when $d = \frac{mp(1-e)}{n}$, the eigenvalues of the characteristic equation of the Jacobian matrix $J(A_3)$ of the system (2.2) at A_3 are

$$\lambda_1 = 0,\ \lambda_2 = -m.$$

The eigenvectors corresponding to the zero eigenvalue of $J(A_3)$ and $(J(A_3))^T$ are $v = (\frac{n}{m}, 1)^T$ and $w = (1,0)^T$ respectively. Let

$$F(X) = \begin{bmatrix} x(1-x) - \dfrac{xy}{d+x+ey} \\ y\left(m - \dfrac{ny}{x+p}\right) \end{bmatrix},\ X = (x,y) \in \Omega,\ F \in R^2,$$

by calculation, we have

$$F_d(A_3, d^{tc}) = (0,0)^T,$$

$$DF_d(A_3, d^{tc}) = \begin{bmatrix} \dfrac{n}{mp} & 0 \\ 0 & 0 \end{bmatrix},$$

$$D^2F(A_3, d^{tc})(v,v) = \begin{bmatrix} \dfrac{n(n+me-2mp-m)}{m^2p} \\ 0 \end{bmatrix},$$

from the above formula, we can get

$$w^TF_d(A_3, d^{tc}) = 0,$$

$$w^T[DF_d(A_3, d^{tc})v] = \frac{n^2}{m^2p},$$
$$w^T[D^2F(A_3, d^{tc})(v,v)] = \frac{n(n+me-2mp-m)}{m^2p}.$$

When $d = d^{tc} = \frac{mp(1-e)}{n}$ and $n + me - 2mp - m \neq 0$, we calculate that

$$w^T F_d(A_3, d^{tc}) = 0,$$
$$w^T[DF_d(A_3, d^{tc})v] \neq 0,$$
$$w^T[D^2F(A_3, d^{tc})(v,v)] \neq 0.$$

By Sotomayor's theorem [20], we know that when $d = d^{tc} = \frac{mp(1-e)}{n}$ and $n + me - 2mp - m \neq 0$, system (2.2) undergoes transcritical bifurcation. □

Theorem 4.2. *System* (2.2) *undergoes Hopf bifurcation at* $E^*(x^*, y^*)$, *if*

$$d = d^* = \frac{m(1-2\xi)(\xi+p) - em(\xi+p)(1-\xi)^2 - m^2(\xi+p)}{n(1-\xi)^2},$$

and

$$-e\xi^3 + 2(e-1)\xi^2 + (1-2m-e)\xi - mp > 0.$$

Proof. The eigenvalues of the Jacobian matrix of system (2.2) at $E^*(x,y)$ are

$$\lambda_{1,2} = \frac{T(d) \pm \sqrt{T^2(d) - 4\omega^2(d)}}{2}, \ \omega^2(d) = D(d),$$

where

$$T(d) = \frac{m(1-2\xi)(\xi+p) - [nd + em(\xi+p)](1-\xi)^2 - m^2(\xi+p)}{m(\xi+p)},$$
$$\omega^2(d) = \frac{m(2\xi-1)(\xi+p)^2 + [nd + em(\xi+p)](\xi+p)(1-\xi)^2 + n\xi(d+\xi)(1-\xi)^2}{(\xi+p)^2}.$$

When

$$d = d^* = \frac{m(1-2\xi)(\xi+p) - em(\xi+p)(1-\xi)^2 - m^2(\xi+p)}{n(1-\xi)^2},$$

then $T(d) = 0$, and

$$\omega^2(d^*) = \frac{m(\xi+p)[-e\xi^3 + 2(e-1)\xi^2 + (1-2m-e)\xi - mp] + n\xi^2(1-\xi)^2}{(\xi+p)^2}.$$

When

$$-e\xi^3 + 2(e-1)\xi^2 + (1-2m-e)\xi - mp > 0,$$

then $\omega^2(d^*) > 0$ and $\lambda_{1,2} = \pm i\omega(d^*)$. Let $M = \frac{T(d)}{2}$, we have

$$\left.\frac{\mathrm{d}M}{\mathrm{d}d}\right|_{d=d^*} = \frac{T'(d^*)}{2} = \frac{-n(1-\xi)^2}{2m(\xi+p)} \neq 0.$$

Through verification of the above transversality condition, we can know that system (2.2) undergoes the Hopf bifurcation at $E^*(x^*, y^*)$. To determine the stability and direction of the bifurcation periodic solution, we need to calculate the first Lyapunov coefficient σ. The system (3.5) is obtained through a series of transformations and the specific process is shown in Theorem 3.6.

If the above two conditions are satisfied, then the eigenvalues of the system (3.5) are $\lambda = \pm i\eta$, where

$$\eta = \frac{\sqrt{m(\xi+p)[-e\xi^3 + 2(e-1)\xi^2 + (1-2m-e)\xi - mp] + n\xi^2(1-\xi^2)}}{\xi + p}.$$

Let $u = u_1 + iu_2$ be the eigenvector corresponding to the eigenvalue $\lambda = i\eta$, where u_1 and u_2 are real vectors. By calculation, we can get $u_1 = (q, 1)^T$, and $u_2 = (\frac{\eta}{r_1}, 0)^T$, where

$$q = \frac{(m^2ne + mn^2)\xi^2 + mn(2emp + dn + pn)\xi + mnp(emp + nd)}{(em^3 + m^2n)\xi^2 + m^2(dn + pn + 2pem)\xi + m^2p(emp + nd)},$$
$$r_1 = \frac{(em^3 + m^2n)\xi^2 + m^2(dn + pn + 2pem)\xi + m^2p(emp + nd)}{n^2}.$$

Let $A = (u_1, u_2) = \begin{bmatrix} q & \frac{\eta}{r_1} \\ 1 & 0 \end{bmatrix}$, under the transformation $(X, Y)^T = A(x, y)^T$, namely

$$\begin{cases} x = Y, \\ y = \dfrac{r_1}{\eta}X - \dfrac{r_1 q}{\eta}Y, \end{cases}$$

the system (3.5) has the following normal form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \eta y + c_1xy + c_2x^2 + c_3y^2 + c_4x^3 + c_5x^2y + c_6xy^2 + o(|x,y|^4), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -\eta x + d_1xy + d_2x^2 + d_3y^2 + d_4x^3 + d_5y^3 + d_6x^2y + d_7xy^2 + o(|x,y|^4), \end{cases} \tag{4.1}$$

where

$$\begin{aligned}
c_1 =& \frac{\eta b_3 + 2q\eta b_4}{r_1}, \quad c_2 = b_3q + b_4q^2 + b_5, \quad c_3 = \frac{b_{4\eta^2}}{r_1^2}, \quad c_4 = b_6q^2 + b_7q, \\
c_5 =& \frac{2q\eta b_6 + \eta b_7}{r_1}, \quad c_6 = \frac{\eta^2 b_6}{r_1^2}, \quad d_1 = \frac{2a_4q\eta + \eta a_3 - b_3q\eta - 2q^2b_4\eta}{\eta}, \\
d_2 =& \frac{r_1(a_3q + a_4q^2 - b_3q^2 - b_4q^3 - b_5q)}{\eta}, \\
d_3 =& \frac{\eta(r_1 - b_4q)}{r_1}, \quad d_4 = \frac{r_1(a_5q^3 + a_6q^2 - b_6q^3 - b_7q^2)}{\eta}, \quad d_5 = \frac{\eta^2}{r_1^2}, \\
d_6 =& 3q^2a_5 + 2a_6q - 2q^2b_6 - qb_7, \quad d_7 = \frac{\eta(3a_5q + a_6 - b_6q)}{r_1}.
\end{aligned}$$

By using the formula in [19,20], the first Lyapunov coefficient σ is calculated as follows

$$\sigma = \frac{3\pi}{2}[3(c_4 + d_5) + (c_6 + d_6) - 2(c_2d_2 - c_3d_3) + c_1(c_3 + d_2) - d_1(d_3 + d_2)].$$

Due to the complexity of c_i and $d_i (i = 1, 2, 3, 4, 5, 6, 7)$, we cannot determine the sign of σ and whether the value of σ is zero. However, the following three conditions can be used to determine the type of Hopf bifurcation of the positive equilibrium $E^*(x^*, y^*)$ of the system (2.2).
(1) If $\sigma < 0$, system (2.2) undergoes supercritical Hopf bifurcation at $E^*(x^*, y^*)$;
(2) If $\sigma > 0$, system (2.2) undergoes subcritical Hopf bifurcation at $E^*(x^*, y^*)$;
(3) If $\sigma = 0$, system (2.2) undergoes degenerate Hopf bifurcation at $E^*(x^*, y^*)$. □

Theorem 4.3. *System* (2.2) *undergoes cusp bifurcation at* $E^*(x^*, y^*)$, *if*

$$d = \frac{m(1-2\xi)(\xi+p)^2 - em(\xi+p)^2(1-\xi)^2 - n\xi^2(1-\xi)^2}{n(\xi+p)(1-\xi)^2 + n\xi(1-\xi)^2},$$

and

$$b_7k_1 + \frac{1}{k_3}(a_5k_1^3 + a_6k_1^2 - b_7k_1^2) \neq 0.$$

Proof. According to Theorem 4.2, translate the positive equilibrium $E^*(x^*, y^*)$ to the origin, system (2.2) becomes equation (3.4). Assuming

$$d = \frac{m(1-2\xi)(\xi+p)^2 - em(\xi+p)^2(1-\xi)^2 - n\xi^2(1-\xi)^2}{n(\xi+p)(1-\xi)^2 + n\xi(1-\xi)^2},$$

then $T \neq 0$ and $D = 0$. For the convenience of subsequent calculations, let

$$k_{11} = m(1-2\xi)(\xi+p)^2 - em(\xi+p)^2(1-\xi)^2 - n\xi^2(1-\xi)^2,$$
$$k_{12} = n(\xi+p)(1-\xi)^2 + n\xi(1-\xi)^2.$$

Therefore, we can obtain that the eigenvalues of system (2.2) at the positive equilibrium $E^*(x^*, y^*)$ are $\lambda_1 = 0$ and

$$\lambda_2 = \frac{[n\xi^2 - em\xi(\xi+p)](1-\xi)^2 + (\xi+p)[m\xi(1-2\xi) - 2m^2\xi - m^2p]}{m(\xi+p)(2\xi+p)},$$

the eigenvectors corresponding to the above eigenvalues are $w_1 = (k_1, 1)^T$ and $w_2 = (k_2, 1)^T$, where

$$k_1 = \frac{k_{12}n(me\xi^2 + n\xi^2 + 2emp\xi + np\xi + mp^2e) + k_{11}n^2(\xi+p)}{k_{12}m[em\xi^2 + n\xi^2 + (pn + 2pem)\xi + mp^2e] + k_{11}mn(\xi+p)},$$
$$k_2 = \frac{n^2k_{12}\{[n\xi^2 - em\xi(\xi+p)](1-\xi)^2 + (\xi+p)[m\xi(1-2\xi) - 2m^2\xi - m^2p]\}}{m(\xi+p)(2\xi+p)\{k_{12}m^2[em\xi^2 + n\xi^2 + (pn+2pem)\xi + mp^2e] + k_{11}m^2n(\xi+p)\}} - k_1.$$

Let $C = (w_1, w_2) = \begin{bmatrix} k_1 & k_2 \\ 1 & 1 \end{bmatrix}$, by the transformation $(X, Y)^T = C(x, y)^T$, denote $k_3 = k_1 - k_2$, that is

$$\begin{cases} x = \dfrac{1}{k_3}X - \dfrac{k_2}{k_3}Y, \\ y = -\dfrac{1}{k_3}X + \dfrac{k_1}{k_3}Y, \end{cases}$$

so equation (3.5) is transformed into the following form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = o_1x^2 + o_2xy + o_3y^2 + o_4x^3 + o_5x^2y + o_6xy^2 + o_7y^3 + o(|x,y|^4), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \lambda_2 y + q_1x^2 + q_2xy + q_3y^2 + q_4x^3 + q_5x^2y + q_6xy^2 + q_7y^3 + o(|x,y|^4), \end{cases} \tag{4.2}$$

where

$$\begin{aligned}
o_1 =& b_3k_1 + b_4k_1^2 + b_5 + \frac{1}{k_3}(a_3k_1 + a_4k_1^2 - b_3k_1^2 - b_4k_1^3 - b_5k_1),\\
o_2 =& b_3k_1 + b_3k_2 + 2b_4k_1k_2 + 2b_5 + \frac{1}{k_3}(a_3k_1 + a_3k_2 + 2a_4k_1k_2 - b_3k_1^2 - b_3k_1k_2\\
&- 2b_4k_1^2k_2 - 2b_5k_1),\\
o_3 =& b_3k_2 + b_4k_2^2 + b_5 + \frac{1}{k_3}(a_3k_2 + a_4k_2^2 - b_3k_1k_2 - b_4k_1k_2^2 - b_5k_1),\\
o_4 =& b_7k_1 + \frac{1}{k_3}(a_5k_1^3 + a_6k_1^2 - b_7k_1^2),\\
o_5 =& 2b_7k_1 + b_7k_2 + \frac{1}{k_3}(3a_5k_1^2k_2 + 2a_6k_1k_2 + a_6k_1^2 - 2b_7k_1^2 - b_7k_1k_2),\\
o_6 =& b_7k_1 + 2b_7k_2 + \frac{1}{k_3}(3a_5k_1k_2^2 + a_6k_2^2 + 2a_6k_1k_2 - b_7k_1^2 - 2b_7k_1k_2),\\
o_7 =& b_7k_2 + \frac{1}{k_3}(a_5k_2^3 + a_6k_2^2 - b_7k_1k_2),\\
q_1 =& -\frac{1}{k_3}(a_3k_1 + a_4k_1^2 - b_3k_1^2 - b_4k_1^3 - b_5k_1),\\
q_2 =& -\frac{1}{k_3}(a_3k_1 + a_3k_2 + 2a_4k_1k_2 - b_3k_1^2 - b_3k_1k_2 - 2b_4k_1^2k_2 - 2b_5k_1),\\
q_3 =& -\frac{1}{k_3}(a_3k_2 + a_4k_2^2 - b_3k_1k_2 - b_4k_1k_2^2 - b_5k_1),\\
q_4 =& -\frac{1}{k_3}(a_5k_1^3 + a_6k_1^2 - b_7k_1^2),\\
q_5 =& -\frac{1}{k_3}(3a_5k_1^2k_2 + 2a_6k_1k_2 + a_6k_1^2 - 2b_7k_1^2 - b_7k_1k_2),\\
q_6 =& -\frac{1}{k_3}(3a_5k_1k_2^2 + a_6k_2^2 + 2a_6k_1k_2 - b_7k_1^2 - 2b_7k_1k_2),\\
q_7 =& -\frac{1}{k_3}(a_5k_2^3 + a_6k_2^2 - b_7k_1k_2).
\end{aligned}$$

The center manifold of system (4.2) near the origin is

$$y = -\frac{q_1}{\lambda_2}x^2 + P(x),$$

where $P(x)$ are terms of at least third order in x. Substitute the above equation into the first equation of equation (4.2), and obtain the equation induced by this equation on the x-axis as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = o_1x^2 + Q(x),$$

where $Q(x)$ are terms of at least second order in x.

If $o_1 = 0$ and $q_1 = 0$, then the center manifold of system (4.2) near the origin becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = o_4 x^3 + Q(x),$$

where $Q(x)$ are terms of at least third order in x.

Assuming $b_7 k_1 + \frac{1}{k_3}(a_5 k_1^3 + a_6 k_1^2 - b_7 k_1^2) \neq 0$, then there is $o_4 \neq 0$, according to reference [19], when the previous assumption holds, system (2.2) undergoes cusp bifurcation at $E^*(x^*, y^*)$. □

Theorem 4.4. When $K_1 K_2 \neq 0$ in Theorem 3.6 holds, system (2.2) undergoes Bogdanov-Takens bifurcation in a small neighborhood of $E^*(x^*, y^*)$, if

$$d_{BT} = \frac{m\xi^2(1-2\xi) - em\xi^2(1-\xi)^2 - m^2\xi^2}{me\xi(1-\xi)^2 + m^2(2\xi+p) - m\xi(1-2\xi)},$$

and

$$n_{BT} = \frac{me\xi(\xi+p)(1-\xi)^2 + m^2(2\xi+p)(\xi+p) - m\xi(1-2\xi)(\xi+p)}{\xi^2(1-\xi)^2}.$$

Proof. We first provide the perturbations of parameters d and n near d_{BT} and n_{BT}, have $d = d_{BT} + \epsilon_1$ and $n = n_{BT} + \epsilon_2$, ϵ_1 and ϵ_2 are parameters in the small neighborhood of $(0, 0)$. Then system (2.2) is transformed into the following model

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(1-x) - \dfrac{xy}{d_{BT} + \epsilon_1 + x + ey}, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y\left(m - \dfrac{(n_{BT} + \epsilon_2)y}{x+p}\right). \end{cases} \tag{4.3}$$

Under linear translation

$$\begin{cases} x = X + x^*, \\ y = Y + y^*, \end{cases}$$

the positive equilibrium of system (2.2) was changed from $E^*(x^*, y^*)$ to the origin, and we have

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = h_{10} + h_{11}X + h_{12}Y + h_{13}X^2 + h_{14}XY + h_{15}X^3 + h_{16}X^2Y + P(X,Y), \\ \dfrac{\mathrm{d}Y}{\mathrm{d}t} = l_{10} + l_{11}X + l_{12}Y + l_{13}X^2 + l_{14}XY + l_{15}Y^2 + l_{16}X^2Y + l_{17}XY^2 + Q(X,Y), \end{cases} \tag{4.4}$$

where $P(X, Y)$ and $Q(X, Y)$ are terms of at least fourth order in X and Y, and

$$\begin{aligned} h_{10} &= (1-x^*)(x^*+p)x^*\epsilon_1, \\ h_{11} &= (d_{BT} + \epsilon_1 + ey^* + x^*)[(x^*+p)(1-2x^*) + (1-x^*)x^*] + (x^*+p)[x^* - x^{*2} \\ &\quad - y^* - x^*y^*], \\ h_{12} &= (x^*+p)[ex^*(1-x^*) - x^*], \\ h_{13} &= 2[(d_{BT} + \epsilon_1 + ey^* + x^*)(1 - 3x^* - p) + (x^*+p)(1-2x^*) + (1-x^*)x^* - y^*], \\ h_{14} &= ep(1-x^*) - (x^*+p)(ex^*+1) - x^*, \end{aligned}$$

$h_{15} = -6(d_{BT} + \epsilon_1 + ey^* + 4x^* + p - 1),$
$h_{16} = 2[e(1 - 3x^* - p) - 1],$
$l_{10} = my^*\epsilon_1(x^* + p) - \epsilon_2 y^{*2}(d_{BT} + \epsilon_1 + ey^* + x^*) - ny^{*2}\epsilon_1,$
$l_{11} = my^*(d_{BT} + \epsilon_1 + ey^* + 2x^* + p) - (n_{BT} + \epsilon_2)y^{*2},$
$l_{12} = m(x^*+p)(d_{BT}+\epsilon_1+2ey^*+x^*)-(n_{BT}+\epsilon_2)y^*[2(d_{BT}+\epsilon_1+ey^*+x^*)+ey^*],$
$l_{13} = 2my^*, \quad l_{14} = m(d_{BT} + \epsilon_1 + 2ey^* + 2x^* + p) - 2y^*(n_{BT} + \epsilon_2),$
$l_{15} = 2me(x^* + p) - 2(n_{BT} + \epsilon_2)(d_{BT} + \epsilon_1 + ey^* + x^*) - 4y^*e(n_{BT} + \epsilon_2),$
$l_{16} = 2m, \quad l_{17} = 2(me - n_{BT} - \epsilon_2).$

Let

$$\begin{cases} x = X, \\ y = h_{10} + h_{11}X + h_{12}Y + h_{13}X^2 + h_{14}XY + h_{15}X^3 + h_{16}X^2Y + P(X,Y), \end{cases} \tag{4.5}$$

then the system (4.4) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = L_{10} + L_{11}x + L_{12}y + L_{13}x^2 + L_{14}xy + L_{15}y^2 + L_{16}x^3 + L_{17}x^2y \\ \qquad + Q_1(x,y,\epsilon_1,\epsilon_2), \end{cases} \tag{4.6}$$

where $Q_1(x, y, \epsilon_1, \epsilon_2)$ are terms of at least fourth order in x and y, its coefficients depend smoothly on ϵ_1 and ϵ_2, and

$$L_{10} = \frac{h_{10}^2 l_{15}}{h_{12}} - h_{10}l_{12},$$
$$L_{11} = \frac{2h_{10}h_{11}h_{12}l_{15} + h_{10}^2 h_{12}l_{17} - h_{10}h_{12}h_{14}l_{12} + h_{10}^2 h_{14}l_{15}}{h_{12}^2} - h_{10}l_{14} + h_{12}l_{11} - h_{11}l_{12},$$
$$L_{12} = \frac{-2h_{10}h_{12}l_{15} - h_{10}h_{12}h_{14} + h_{11}h_{12}^2}{h_{12}^2} + l_{12},$$
$$L_{13} = \frac{h_{11}^2 h_{12}l_{15} - h_{11}h_{12}h_{14}l_{12}}{h_{12}^2} - h_{10}l_{16} + h_{12}l_{13} + h_{14}l_{11} + h_{16} - h_{11}l_{14} + \frac{2h_{10}h_{11}h_{12}l_{17} - h_{10}h_{12}h_{14}l_{14} + 2h_{10}h_{11}h_{14}l_{15} + h_{10}^2 h_{14}l_{17}}{h_{12}^2},$$
$$L_{14} = \frac{-2h_{10}h_{12}l_{17} - 2h_{10}h_{14}l_{15} - 2h_{10}h_{12}h_{16} - 2h_{11}h_{12}l_{15} - h_{11}h_{12}h_{14} + h_{12}h_{14}l_{12}}{h_{12}^2} + 2h_{13} + l_{14},$$
$$L_{15} = \frac{h_{14} + l_{15}}{h_{12}},$$
$$L_{16} = \frac{2h_{10}h_{11}h_{14}l_{17} - h_{10}h_{12}h_{14}l_{16} + h_{11}^2 h_{12}l_{17} + h_{11}^2 h_{14}l_{15}}{h_{12}^2} + h_{14}l_{13} - h_{11}l_{16},$$
$$L_{17} = \frac{-2h_{11}h_{12}l_{17} + h_{12}h_{14}l_{14} - 2h_{11}h_{14}l_{15} - 2h_{11}h_{12}h_{16} - 2h_{10}h_{14}l_{17}}{h_{12}^2} + 3h_{15} + l_{16}.$$

Next, introduce a new time variable τ such that $\mathrm{d}t = (1 - L_{15}x)\mathrm{d}\tau$, and use t to denote τ after the transformation, so equation (4.6) can be written as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (1 - L_{15}x)y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = (1 - L_{15}x)(L_{10} + L_{11}x + L_{12}y + L_{13}x^2 + L_{14}xy + L_{15}y^2 + L_{16}x^3 \\ \qquad + L_{17}x^2y + Q_1(x, y, \epsilon_1, \epsilon_2)). \end{cases} \tag{4.7}$$

Let $X = x$ and $Y = (1 - L_{15}x)y$, so equation (4.7) can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = Y, \\ \dfrac{\mathrm{d}Y}{\mathrm{d}t} = L_{20} + L_{21}X + L_{22}Y + L_{23}X^2 + L_{24}XY + Q_2(X, Y, \epsilon_1, \epsilon_2), \end{cases} \tag{4.8}$$

where $Q_2(X, Y, \epsilon_1, \epsilon_2)$ are terms of at least third order in X and Y, its coefficients depend smoothly on ϵ_1 and ϵ_2, and

$$\begin{aligned} L_{20} =& L_{10},\ L_{21} = L_{11} - 2L_{10}L_{15},\ L_{22} = L_{12}, \\ L_{23} =& L_{13} - 2L_{11}L_{15} + L_{10}L_{15}^2,\ L_{24} = L_{14} - L_{12}L_{15}. \end{aligned}$$

Due to the complexity of the expression for L_{23}, it is not possible to directly determine its symbol. Therefore, in this article, we assume $L_{23} < 0$ and discuss it under this assumption.

If $L_{23} < 0$, then we introduce new variable $x = X$, $y = \frac{Y}{\sqrt{-L_{23}}}$ and $\tau = \sqrt{-L_{23}}t$, and use t to denote τ after the transformation, so equation (4.8) can be transformed into

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = I_{10} + I_{11}x + I_{12}y - x^2 + I_{14}xy + Q_3(x, y, \epsilon_1, \epsilon_2), \end{cases} \tag{4.9}$$

where $Q_3(x, y, \epsilon_1, \epsilon_2)$ are terms of at least third order in x and y, its coefficients depend smoothly on ϵ_1 and ϵ_2, and

$$I_{10} = -\frac{L_{20}}{L_{23}},\ I_{11} = -\frac{L_{21}}{L_{23}},\ I_{12} = \frac{L_{22}}{\sqrt{-L_{23}}},\ I_{14} = \frac{L_{24}}{\sqrt{-L_{23}}}.$$

Next, let $X = x - \frac{I_{11}}{2}$ and $Y = y$, the above equation (4.9) is transformed into

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = Y, \\ \dfrac{\mathrm{d}Y}{\mathrm{d}t} = I_{20} + I_{22}Y - X^2 + I_{24}XY + Q_4(X, Y, \epsilon_1, \epsilon_2), \end{cases} \tag{4.10}$$

where $Q_4(X, Y, \epsilon_1, \epsilon_2)$ are terms of at least third order in X and Y, its coefficients depend smoothly on ϵ_1 and ϵ_2, and

$$I_{20} = I_{10} + \frac{I_{11}^2}{4},\ I_{22} = I_{12} + \frac{I_{11}I_{14}}{2},\ I_{24} = I_{14}.$$

When $L_{24} \neq 0$ holds, we have $I_{24} = I_{14} = \frac{L_{24}}{\sqrt{-L_{23}}} \neq 0$. Make the following transformation, let $x = -I_{24}^2 X$, $y = I_{24}^3 Y$ and $\tau = -\frac{1}{I_{24}} t$, so system (4.10) can be transformed into

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \varrho_1 + \varrho_2 y + x^2 + xy + Q_5(x, y, \epsilon_1, \epsilon_2), \end{cases} \tag{4.11}$$

where $Q_5(x, y, \epsilon_1, \epsilon_2)$ are terms of at least third order in x and y, its coefficients depend smoothly on ϵ_1 and ϵ_2, and

$$\varrho_1 = -I_{20} I_{24}^4, \ \varrho_2 = -I_{22} I_{24}. \tag{4.12}$$

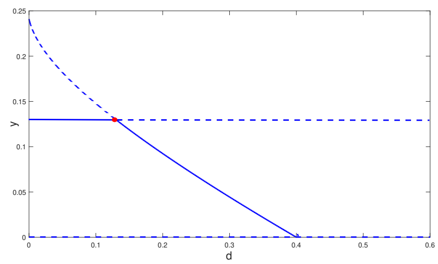
According to reference [10, 20], we know that if $\left| \frac{\partial(\varrho_1, \varrho_2)}{\partial(\epsilon_1, \epsilon_2)} \right|_{\epsilon_1=\epsilon_2=0} \neq 0$, then it can be inferred that parameter transformation (4.12) is a homeomorphism in a small domain $(0, 0)$. Therefore, when (ϵ_1, ϵ_2) is in a small neighborhood of the $(0, 0)$, system (4.3) will undergo Bogdanov-Takens bifurcation. □

5. Numerical simulations

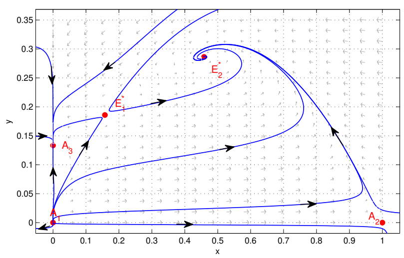
In this section, we will give some numerical simulations to verify the findings of this paper. We give the bifurcation diagram, phase diagram, and time series diagram of model (2.2) under different parameter conditions. For the bifurcation diagram, the equilibrium is stable (unstable) on the solid (dotted) line.

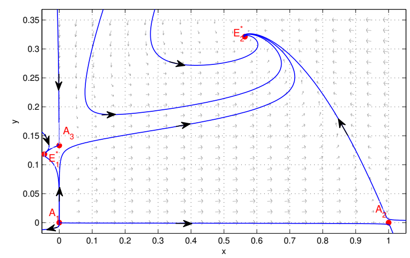
Figure 2 (a) shows the transcritical bifurcation diagram in $A_3(0, \frac{mp}{n})$. According to Figure 2 (a), we know that under conditions $e = 0.07, m = 0.2, n = 0.6, p = 0.4$, the positive equilibrium bifurcates from $A_3(0, \frac{mp}{n})$ for $d = 0.124$. (b) is the phase diagram corresponding to (a) under the above conditions when $d = 0.05$, at this time, A_3 is a stable node and E_1^* is a saddle point. (c) is the phase diagram corresponding to (a) at $d = 0.15$, in this case, A_3 is a saddle point and E_1^* is a stable node. The stability of these two equilibria changes, and with decreasing d, the positive equilibrium E_1^* becomes unstable and A_3 becomes stable, at this time, the whole system tends to the boundary equilibrium A_3, the prey gradually goes extinct, and the system also tends to the positive equilibrium E_2^*, the prey and the predator reach the equilibrium state of coexistence.

Figure 3 shows the Hopf bifurcation diagram around $E^*(0.10978, 0.6797)$ about d under conditions $e = 0.2, m = 0.02, n = 0.015, p = 0.4$. The Hopf bifurcation is supercritical for the first coefficient $\sigma < 0$, when the Hopf bifurcation curve crosses the vertical line $d \approx 0.517801$ to the right, system (2.2) changes from a stable limit cycle containing an unstable equilibrium to a stable equilibrium. Figure 3 (b), (c), (e) and (f) show the phase diagram and corresponding time diagram of Hopf bifurcation when $d = 0.53$. (c) amplified phase portrait of (b), (d) corresponding phase diagram of the system when $d = 0.51$. In Figure 3 (b) and (c), the system (2.2) has a positive equilibrium E_2^*, E_2^* is an unstable focus and a stable limit cycle surrounds E_2^*, for any initial value, the solution will tend to the limit cycle, that is, the predator and prey will oscillate periodically. In Figure 3 (d), the system (2.2) is stable and tends to the positive equilibrium E_2^*.



(a) $e = 0.07, m = 0.2, n = 0.6, p = 0.4$





(b) $d = 0.05, e = 0.07, m = 0.2, n = 0.6, p = 0.4$ (c) $d = 0.15, e = 0.07, m = 0.2, n = 0.6, p = 0.4$

Figure 2. (a) the transcritical bifurcation diagram around $A_3(0, \frac{mp}{n})$ with parameters $e = 0.07, m = 0.2, n = 0.6, p = 0.4$, (b) the phase diagram corresponding to (a) at $d = 0.05$, (c) the phase diagram corresponding to (a) at $d = 0.15$.

The phase diagram of the Bogdanov-Takens bifurcation of codimension 2 with $m = 0.2, p = 0.3, e = 0.12$ is shown in Figure 4. Figure 4 (a) shows that under the parameter conditions $d_{BT} = 0.08975, e = 0.12, m = 0.2, n_{BT} = 0.3899, p = 0.3$, the system (4.3) has no positive equilibrium when $(\epsilon_1, \epsilon_2) = (-0.00775, 0.0001)$. We can see that for almost all initial value densities, the system tends to the boundary equilibrium A_3, at which time the prey species will tend to extinction. (b) have an unstable focus and saddle point when $(\epsilon_1, \epsilon_2) = (0.00002, 0.0112)$. (c) have an unstable limit cycle when $(\epsilon_1, \epsilon_2) = (0.0008, 0.0112)$. (d) have an unstable homoclinic cycle surrounding stable hyperbolic focus when $(\epsilon_1, \epsilon_2) = (0.00325, 0.0112)$ and (e) have a stable focus when $(\epsilon_1, \epsilon_2) = (0.00525, 0.0112)$.

6. Conclusions

In this paper, we studied the dynamic behavior of a modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response. We discuss the positivity, dissipation, and permanence of the solutions of the model and analyze the existence and stability of the equilibria under different parameter conditions. We observe that for any parameter value, the system will not collapse because the origin is always unstable. The stability of the positive equilibrium is discussed by the Routh Hurwitz criteria. We take the prey density d with the semi-saturated attack rate and the maximum per capita reduction rate n of the predator as bifurcation parameters, and discuss the codimension one bifurcation and the codimension two

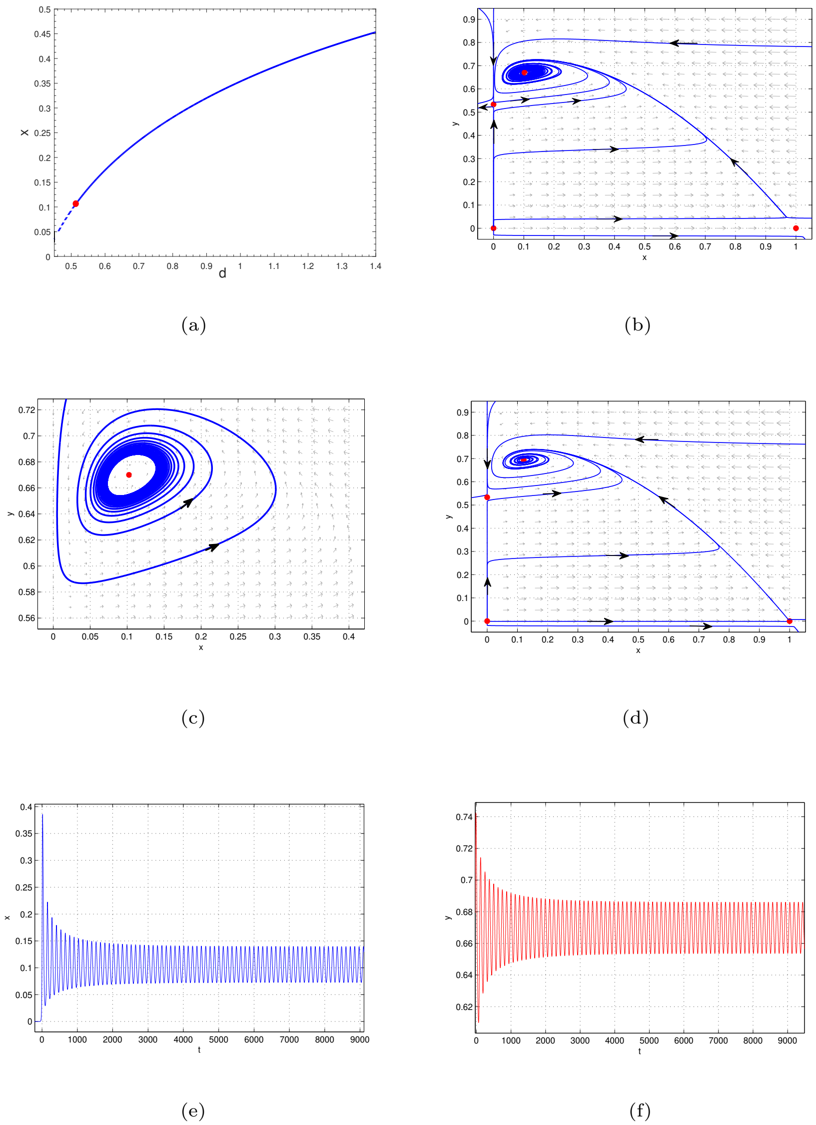
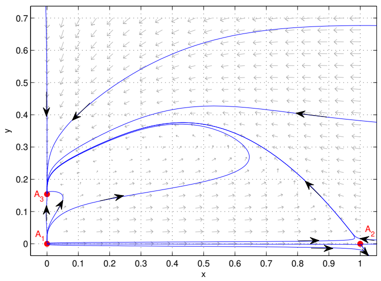
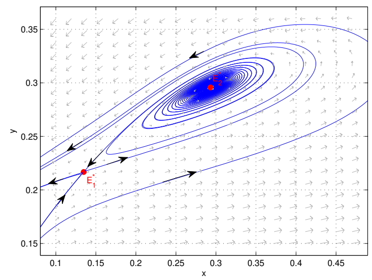


Figure 3. (a) Hopf bifurcation around $E^*(0.10978, 0.6797)$ under conditions $e = 0.2, m = 0.02, n = 0.015, p = 0.4$. (b) (c) (e) (f) the phase diagram of the limit cycle and the corresponding time diagram of Hopf bifurcation when $d = 0.53$. (d) the phase diagram of the system when $d = 0.51$.

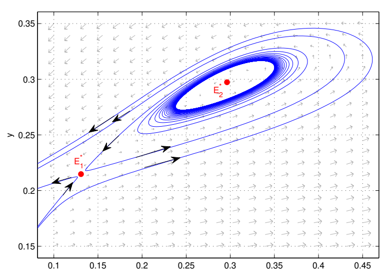
bifurcation experienced by the system (2.1) under different parameter conditions. Under the conditions of different parameters, the model shows transcritical bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation. These bifurcations are very important in ecology. transcritical bifurcation transforms the extinction equilibrium of prey into unstable equilibrium, and the unstable coexistence equilibrium into stable equilibrium. The local existence of limit cycles under different conditions is observed by Hopf bifurcation, and the stability of limit cycles is verified by numerical simulation. We also observed that the size of the limit cycle changes with



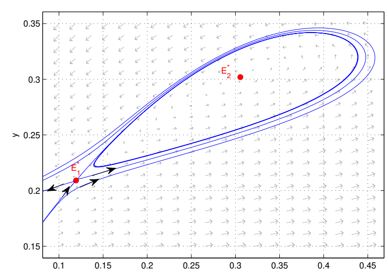
(a) $\epsilon_1 = -0.00775$, $\epsilon_2 = 0.0001$



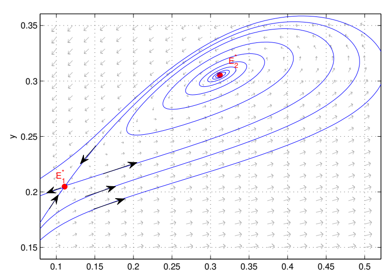
(b) $\epsilon_1 = 0.00002$, $\epsilon_2 = 0.0112$



(c) $\epsilon_1 = 0.0008$, $\epsilon_2 = 0.0112$



(d) $\epsilon_1 = 0.00325$, $\epsilon_2 = 0.0112$



(e) $\epsilon_1 = 0.00525$, $\epsilon_2 = 0.0112$

Figure 4. Phase diagram near Bogdanov-Takens bifurcation point when $d_{BT} = 0.08975, e = 0.12, m = 0.2, n_{BT} = 0.3899, p = 0.3$. (a) no positive equilibria when $(\epsilon_1, \epsilon_2) = (-0.00775, 0.0001)$, (b) there is an unstable focus when $(\epsilon_1, \epsilon_2) = (0.00002, 0.0112)$, (c) there is an unstable limit cycle when $(\epsilon_1, \epsilon_2) = (0.0008, 0.0112)$, (d) there is an unstable homoclinic cycle when $(\epsilon_1, \epsilon_2) = (0.00325, 0.0112)$, (e) have a stable focus when $(\epsilon_1, \epsilon_2) = (0.00525, 0.0112)$.

the change of the bifurcation parameters, and the stability of the positive equilibrium also changes. We verify the bifurcation of the system by numerical simulation, which effectively reflects the dynamic behavior of the system.

In recent years, on the basis of some ordinary differential equations, more and more researchers have introduced diffusion and delay to the basic model and carried out a lot of studies. The literature [15] proposes a predator-prey model with herd behavior and prey-taxis, analyzes the stability and bifurcation of the positive equilibrium of the model, and the results show that the prey-predator can generate spatial patterns. In [13], a density predator-prey model with Crowley-Martin functional response and two time delays is investigated, the authors analyze the stability and bifurcation of the equilibria and discuss the direction of Hopf bifurcation and the stability of bifurcation period solutions by means of the methods of normal form and center manifold theorem. Inspired by the above two articles, we find that further improvements can be made to the model, and it is worthwhile to investigate further the dynamic properties.

Appendix. i_1 and i_2 in the proof of Theorem 3.6

$$
\begin{aligned}
i_2 =&(-m^3e^3\xi^6 + m^3e^3\xi^4p - m^3e^3\xi^5p - 2e^3m^3\xi^5)(1-\xi)^6(\xi+p)^2\\
&- 3m^4\xi^4p(1-2\xi)^2(\xi+p)(2\xi+p) - 6m^4e^2\xi^5(\xi+p)^2(1-\xi)^4(2\xi+p)\\
&+ (5m^4e^2\xi^4 + 4m^4e^2\xi^3p - 5m^4e^2\xi^4p)(\xi+p)^2(1-\xi)^4(2\xi+p)\\
&+ (3m^3e^2\xi^6 - 3m^3e^2\xi^5)(\xi+p)^2(1-\xi)^4(1-2\xi)\\
&+ (-3m^3e^2\xi^4p + 3m^3e^2\xi^5p + 2m^4\xi^5e)(\xi+p)^2(1-\xi)^4(1-2\xi)\\
&+ (-9m^5e\xi^4 + 7m^5e\xi^3 - 7m^5e\xi^3p + 3m^5e\xi^2p)(\xi+p)^2(1-\xi)^2(2\xi+p)^2\\
&+ (12m^4e\xi^5 - 8m^4e\xi^3p)(\xi+p)^2(2\xi+p)(1-2\xi)(1-\xi)^2\\
&+ (10m^4e\xi^4p - 10m^4e\xi^4 + m^3\xi^2)(\xi+p)^2(2\xi+p)(1-2\xi)(1-\xi)^2\\
&+ (-3m^3e\xi^6 + 3m^3e\xi^5)(1-2\xi)^2(\xi+p)^2(1-\xi)^2\\
&+ (3em^3\xi^4p - 3m^3e\xi^5p - 4m^4e\xi^5p - m^4\xi^5)(1-2\xi)^2(\xi+p)^2(1-\xi)^2\\
&+ (2m^5ep\xi^2 + 4m^5\xi^4 + 3m^6\xi^4 - 2m^6\xi^3 + 2pm^6\xi^3 - pm^6\xi^2)(2\xi+p)^2(\xi+p)^2\\
&+ (-4m^6\xi^3 + 2m^6p\xi - 3m^6\xi^2p)(2\xi+p)^3(\xi+p)^2)\\
&+ (5m^5\xi^4 - 9m^5\xi^3 - 5m^5\xi^2p + 7m^5\xi^3p)(\xi+p)^2(2\xi+p)^2(1-2\xi)\\
&+ (-6m^4\xi^5 + 5m^4\xi^4 + 4m^4\xi^3p - 2m^4\xi^4p)(1-2\xi)^2(\xi+p)^2(2\xi+p)\\
&+ (m^3\xi^6 - m^3\xi^5 - m^3\xi^4p - 2pm^3\xi^5)(1-2\xi)^3(\xi+p)^2\\
&+ (-6m^5\xi^5 + 4m^5\xi^4 - 4pm^5\xi^4 + 2m^5\xi^3p)(\xi+p)^2(2\xi+p)(1-2\xi)\\
&+ (3m^4\xi^6e^2 - 2m^4e^2\xi^5 + 2pm^4e^2\xi^5 - pm^4e^2\xi^4 - m^3e\xi^3)(\xi+p)^2(1-\xi)^4\\
&+ (16m^5e\xi^5 - 4m^5e\xi^4 + 4m^5e\xi^4p - 2m^5e\xi^3p - m^4\xi^2)(\xi+p)^2(1-\xi)^2(2\xi+p)\\
&+ (-6m^4e\xi^6 + 4m^4e\xi^5 + 2m^4e\xi^4p + m^3\xi^3)(\xi+p)^2(1-\xi)^2(1-2\xi)\\
&+ 3m^6\xi^2(2\xi+p)^3(\xi+p)^2 + (2pe^3m^3\xi^5 - m^2e^2\xi^3)(1-\xi)^6(\xi+p)^2\\
&+ (2m^3\xi^6 - 2m^4\xi^5 + 2pm^4\xi^5 - m^4p\xi^4 + 3m^4\xi^6)(1-2\xi)^2(\xi+p)^2\\
&+ (-2emp^2\xi - 3mp\xi + mp^2e - mp^2 + m^2\xi^2 + m^2p\xi)\\
&* [m^2e^2\xi^4(\xi+p)(1-\xi)^6 + 2m^3e\xi^3(2\xi+p)(\xi+p)(1-\xi)^4 - 2m^2e\xi^5(1-\xi)^4\\
&+ m^4\xi^2(2\xi+p)^2(\xi+p)(1-\xi)^2 - 2m^3\xi^3(2\xi+p)(1-2\xi)(\xi+p)(1-\xi)^2\\
&+ m^2\xi^4(1-\xi)^2(\xi+p)(1-2\xi)^2 - 2m^2e\xi^4p(1-2\xi)(1-\xi)^4 + 4m^2e\xi^6(1-\xi)^4]
\end{aligned}
$$

$$
\begin{aligned}
&+(-3em\xi^3+2em\xi^2-2m\xi^2-5emp\xi^2+3emp\xi)\\
&*[m^2e^2\xi^4(\xi+p)(1-\xi)^6+2m^3e\xi^3(2\xi+p)(\xi+p)(1-\xi)^4-2m^2e\xi^5(1-\xi)^4\\
&+m^4\xi^2(2\xi+p)^2(\xi+p)(1-\xi)^2-2m^3\xi^3(2\xi+p)(1-2\xi)(\xi+p)(1-\xi)^2\\
&+m^2\xi^4(1-\xi)^2(\xi+p)(1-2\xi)^2-2m^2e\xi^4p(1-2\xi)(1-\xi)^4+4m^2e\xi^6(1-\xi)^4]\\
&+2m^5\xi^3(1-2\xi)(2\xi+p)^2(\xi+p)^2+3m^3\xi^5p(1-2\xi)^3(\xi+p)^3\\
&+2pem^3\xi^5[me\xi(1-\xi)^6+m^2(2\xi+p)(1-\xi)^4-m\xi(1-2\xi)(1-\xi)^4]\\
&+(em^3\xi^6+m^3ep^2\xi^4)[me\xi(1-\xi)^6+m^2(2\xi+p)(1-\xi)^4-m\xi(1-2\xi)(1-\xi)^4].
\end{aligned}
$$

$$
\begin{aligned}
i_1=&\{[e\xi^3-(e-ep-m-1)\xi^2+(mp-p(e-1)\xi)]\times[m^2e^2\xi^2(1-\xi)^4(\xi+p)\\
&-2m^2e\xi^2(1-\xi)^2(1-2\xi)(\xi+p)-2m^3\xi(1-2\xi)(\xi+p)(2\xi+p)\\
&+m^2\xi^2(1-2\xi)^2(\xi+p)+2m^3e\xi(1-\xi)^2(2\xi+p)(\xi+p)+m^4(2\xi+p)^2(\xi+p)]\\
&+[2m^3e\xi^3(\xi+p)^2(1-\xi)^2(1-2\xi)+m^4\xi^2(2\xi+p)(\xi+p)^2(1-2\xi)\\
&-m^3\xi^3(1-2\xi)^2(\xi+p)^2-m^3e^2\xi^3(\xi+p)^2(1-\xi)^4\\
&-m^4e\xi^2(2\xi+p)(\xi+p)^2(1-\xi)^2-m^4e\xi^3(\xi+p)^2(1-\xi)^2\\
&-m^5\xi^2(2\xi+p)(\xi+p)^2+m^4\xi^3(\xi+p)^2(1-2\xi)]+(m^2e\xi^2+2m^2ep\xi+em^2p^2)\\
&\times[me\xi^3(1-\xi)^4+m^2\xi^2(2\xi+p)(1-\xi)^2-m\xi^3(1-2\xi)(1-\xi)^2]\}\\
&\times[me\xi(\xi+p)(1-\xi)^2+m^2(2\xi+p)(\xi+p)-m\xi(1-2\xi)(\xi+p)].
\end{aligned}
$$

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