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ALMOST-PERIODIC BIFURCATIONS FOR 2-DIMENSIONAL DEGENERATE HAMILTONIAN VECTOR FIELDS*

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Abstract In this paper, we develop almost-periodic tori bifurcation theory for 2-dimensional degenerate Hamiltonian vector fields. With KAM theory and singularity theory, we show that the universal unfolding of completely degenerate Hamiltonian $N(x, y) = x^2y+y^l$ and partially degenerate Hamiltonian $M(x, y) = x^2 + y^l$, respectively, can persist under any small almost-periodic time-dependent perturbation and some appropriate non-resonant conditions on almost-periodic frequency $\omega = (\cdots, \omega_i, \cdots)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$. We extend the analysis about almost-periodic bifurcations of one-dimensional degenerate vector fields considered in [21] to 2-dimensional degenerate vector fields. Our main results (Theorem 2.1 and Theorem 2.2) imply infinite-dimensional degenerate umbilical tori or normally parabolic tori bifurcate according to a generalised umbilical catastrophe or generalised cuspoid catastrophe under any small almost-periodic perturbation. For the proof in this paper we use the overall strategy of [21], which however has to be substantially developed to deal with the equations considered here.

Keywords Almost-periodic bifurcations, universal unfolding, singularity theory, KAM theory, infinite-dimensional degenerate tori.

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1. Introduction

It is well known that the qualitative structure of a dynamical system can be characterized by its invariant subsets, for example, the equilibria, periodic orbits, invariant tori and the stable and unstable manifolds of all these. These invariant subsets form the theoretical framework of the dynamics, and one is interested in the properties that are persistent under small perturbations. If we assume that a dynamical system depends on external parameters, then bifurcation theory describes changes in the qualitative structure of the dynamical system with small changes in these parame-

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ters. In the early 1970s, Meyer [16,17] or Broer et al [3,4] established the theory of bifurcations for equilibria and periodic solutions to some Hamiltonian systems and dissipative systems. Subsequently, many authors have been devoted to studies of quasi-periodic bifurcation for conservative and dissipative dynamical systems using KAM skill, see ([2,7,13]).

As we know, in quasi-periodic bifurcation theory, normally degenerate bifurcation is difficult to handle because standard KAM theory is not directly applicable. Generally speaking, it is necessary to establish a new KAM theory in order to study quasi-periodic bifurcation of normally degenerate dynamical systems. In normally one dimensional cases, Broer et al. ([8]) considered the quasi-periodic saddle-node bifurcation by investigating the perturbation of the vector field

$$X_{tr}^{\omega,\lambda}(x,y) = (\omega + a(\omega)y)\frac{\partial}{\partial x} + (\lambda + b(\omega)y^2)\frac{\partial}{\partial y},$$

where $(x,y) \in \mathbb{T}^n \times \mathbb{R}$, $\omega \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ is parameter. They proved an adapted KAM-theorem, to obtain a conjugacy sending the above vector field to

$$\bar{X}_{tr}^{\omega,\lambda}(x,y) = (\omega + a(\omega)y + \mathcal{O}(|y|^2 + |\lambda|^2))\frac{\partial}{\partial x} + (\lambda + b(\omega)y^2 + \mathcal{O}(|y|^3 + |\lambda|^3))\frac{\partial}{\partial y},$$

when ω satisfies Diophantine condition. Later, Wagener ([22]) considered the case of diffeomorphisms. Furthermore, higher order degenerate quasi-periodic bifurcation in one-dimensional systems can been seen in [20]. In the last years, the quasi-periodic bifurcation theory has been extended to higher dimension and higher order normally degenerate cases, and quasi-periodic analogues of cuspoid bifurcations of equilibria are also developed in normally higher dimensional cases. In 1998, Hanßmann ([12]) considered the quasi-periodic center-saddle bifurcation by investigating the perturbation of following Hamiltonian

$$X(x,y,p,q,\omega,\lambda) = \langle \omega, y\rangle + a(\omega)p^2 + \frac{b(\omega)}{3}q^3 - d\lambda q, \tag{1.1}$$

on $(x,y,p,q) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^2$, and proved the persistence of universal unfolding (1.1) under small perturbations provided that ω satisfies Diophantine condition. Broer et al. ([5,6]) considered the following Hamiltonian universal unfolding

$$N(x,y,p,q,\lambda,\omega) = \langle \omega, y\rangle + \frac{A(\omega)}{2}p^2 + \frac{B(\omega)}{l!}q^l + \sum_{j=1}^{l-2}\frac{\lambda_j}{j!}q^j \tag{1.2}$$

and

$$N(x,y,u,v,\lambda,\omega) = \langle \omega, y\rangle + \frac{A(\omega)}{2}u^2v + \frac{B(\omega)}{l!}v^l + \sum_{j=1}^{l-1}\frac{\lambda_j}{j!}v^j + \lambda_l u, \tag{1.3}$$

where $(x,y,p,q) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^2$, respectively. They proved universal unfolding (1.2) and (1.3) can persist under any small quasi-periodic perturbation provided that the frequencies satisfy Diophantine condition. Most direct results are the tori in the unperturbed system bifurcate according to a generalized cuspoid catastrophe and a generalised umbilical catastrophe, respectively.

We notice that many perturbations tend to be more irregular in nature, such as almost-periodic perturbations. Recently, many works focus on the existence of almost-periodic invariant tori for the finite-dimensional ([14, 15, 24]) and infinite-dimensional ([1, 10, 18]) almost-periodic forced systems. Thus, the our aim is *to investigate the almost-periodic bifurcations phenomena.* In 2020, W. Si, X. Xu and J. Si ([21]) studied the almost-periodic time-dependent perturbations of universal unfolding of one-dimensional vector field $\dot{x} = x^l$. More precisely, the authors considered

$$\dot{x} = M(x, \lambda) + f(t, x, \lambda, \epsilon),\ x \in \mathbb{R},\ \lambda = (\lambda_0, \cdots, \lambda_{l-2}) \in \Lambda \subset \mathbb{R}^{l-1}, \tag{1.4}$$

where $M(x, \lambda) = \sum_{j=0}^{l-2} \frac{\lambda_j}{j!} x^j + \frac{p}{l!} x^l$, Λ is a compact neighbourhood of $\lambda = 0$, $f(t, x, \lambda, 0) = 0$, f is real analytic in all variables and parameters and almost-periodic in t with frequency vector $\omega = (\ldots, \omega_i, \ldots)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$, and proved that all the bifurcation scenario in $\dot{x} = M(x, \lambda)$ can persist under a small almost-periodic perturbation when the almost-periodic frequency ω satisfies some non-resonant conditions. This shows the infinite-dimensional tori bifurcate from the almost-periodic perturbation.

In the present paper, we will develop almost-periodic bifurcation theory for two-dimensional degenerate Hamiltonian systems. We consider the almost-periodic time-dependent perturbations of universal unfolding of completely degenerate Hamiltonian $N(x, y) = x^2 y + y^l$ and partially degenerate Hamiltonian $M(x, y) = x^2 + y^l$, respectively*. More concretely, we consider two classes of Hamiltonians

$$H = N(x, y, \lambda) + P(t, x, y, \lambda, \epsilon),\ (x, y) \in \mathbb{R}^2$$

and

$$H = M(x, y, \bar{\lambda}) + P(t, x, y, \bar{\lambda}, \epsilon),\ (x, y) \in \mathbb{R}^2,$$

where $N(x, y, \lambda) = \frac{A}{2} x^2 y + \frac{B}{l!} y^l + \sum_{j=1}^{l-1} \frac{\lambda_j}{j!} y^j + \lambda_l x$, $M(x, y, \bar{\lambda}) = \frac{a}{2} x^2 + \frac{b}{l!} y^l + \sum_{j=1}^{l-2} \frac{\bar{\lambda}_j}{j!} y^j$, $\lambda = (\lambda_1, \ldots, \lambda_l) \in \breve{\Lambda} \subset \mathbb{R}^l$, $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{d-2}) \in \bar{\Lambda} \subset \mathbb{R}^{l-2}$, $\breve{\Lambda}$ and $\bar{\Lambda}$ are closed and bounded neighbourhood of origin, $P(t, x, y, \lambda, 0) = 0$, P is real analytic in all variables and parameters, and almost-periodic in t with frequency vector $\omega = (\ldots, \omega_i, \ldots)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$.

The perturbed tori are the most degenerate ones corresponding to the central singularity at $\lambda = 0$ or $\bar{\lambda} = 0$, and the remaining part of our perturbation problem asks what happens to the invariant tori of N and M that occur in the unfolding for $\lambda \neq 0$. When it is quasi-periodic case and according to [5, 6], there are (quasi-periodic) centre-saddle bifurcations, (quasi-perodic) Hamiltonian pitchfork bifurcation in universal unfoldings N and M. Degenerate umbilical tori or normally parabolic tori bifurcate according to a generalised umbilical catastrophe or generalised cuspoid catastrophe under any small quasi-periodic perturbation. Different from quasi-periodic case, the persistence of above universal unfolding becomes more difficult to handle in almost-periodic case because it is not only necessary to control the normal form with vanished linear part but also necessary to deal with the small divisor produced by the integer combination of infinite many frequencies.

*Let $X_H : \mathbb{R}^n \to \mathbb{R}^n$ be the vector field corresponding to Hamiltonian H and $X_H(0) = 0$. We call Hamiltonian H is completely degenerate if Jacobi matrix $DX_H(0) = 0$ and Hamiltonian H is partially degenerate if Jacobi matrix $DX_H(0) \neq 0$ and $\det(DX_H(0)) = 0$.

In this paper, we will prove both infinite-dimensional invariant tori and the bifurcation scenario in $H = N(x, y, \lambda)$ and $H = M(x, y, \bar{\lambda})$ can persist under a small almost-periodic perturbation when the almost-periodic frequency ω satisfies some non-resonant conditions.

The method we adopt is first introduced by Pöschel([19]), which deals with small divisors with spatial structure in infinite dimensional Hamiltonian systems. We generalize it to deal with normally degenerate problems. Although in our proof we use the overall strategy of [17], however it has to be substantially developed to deal with the equations considered here.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and definitions applied in the sequent and give the main results. In Section 3, we give the proof of Theorem 2.1. In section 4, we give an outline of proof of Theorem 2.2.

2. Preliminary and main results

2.1. Some definitions and notations

Some of the notations and definitions given in this section are also given in [21]. For convenience of readers and the integrity of this paper, we restate them here.

Definition 2.1. Let X be a complex Banach space. A function $f : U \subset X \to \mathbb{C}$, where U is an open subset of X, is called analytic if f is continuous on U, and $f|_{U\cap X_1}$ is analytic in the classical sense as a function of several complex variables for each finite dimensional subspace X_1 of X.

Definition 2.2 ([9, 11])**.** A function $f : \mathbb{R} \to \mathbb{R}$ is called an almost periodic function, if it is continuous and, for any $\epsilon > 0$, the ϵ-translation set of f,

$$T(f, \epsilon) = \{\tau \in \mathbb{R}; |f(t + \tau) - f(t)| < \epsilon, t \in \mathbb{R}\}$$

is a relative dense set on $\mathbb{R}$ (i.e., there exists $l > 0$, such that for any $a \in \mathbb{R}$, $[a, a + l] \cap T(f, \epsilon) \neq 0$).

We denote by AP the set of analytic almost periodic functions. If $f \in AP$, then $f\exp(-\mathrm{i}l\cdot) \in AP$ for any real number l. Define

$$a(l, f) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)e^{-\mathrm{i}lt}dt.$$

The set

$$\Lambda(f) = \{l \in \mathbb{R} : a(l, f) \neq 0\}$$

is called the set of Fourier exponents of f, the numbers $a(l, f)$, $l \in \Lambda(f)$, the Fourier coefficients and Fourier series of f is designated by

$$f(t) \sim \sum_{l\in\Lambda(f)} a(l, f)e^{\mathrm{i}lt}.$$

Throughout the paper, we study a class of special Bohr almost periodic function with the frequency $\omega = (\cdots, \omega_i, \cdots)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$, that is we always assume that the analytic almost periodic function f has the Fourier exponents $\cup_{i\in\mathbb{Z}}\Omega_i$, where each

Ω_i is a real number set which is expanded integrally by $\omega^i := (\omega_{i_1}, \cdots, \omega_{i_{j(i)}})$ with $\mathrm{Elm}(\omega) \subset \cup_{i\in\mathbb{Z}} \mathrm{Elm}(\omega^i)$. Here $\mathrm{Elm}(\cdot)$ represents the elements of set $\cdot$. Thus, almost periodic function f can be expressed as

$$f(t) = \sum_{i\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^{j(i)}} a(\langle k, \omega^i\rangle, f) e^{\mathrm{i}\langle k,\omega^i\rangle t}. \tag{2.1}$$

Next, we will give more general definition of this kind of special almost-periodic function. Let $\mathcal{S}$ is a family of finite subsets A of $\mathbb{Z}$ which has the following spatial structure: $\mathbb{Z} \subset \cup_{A\in\mathcal{S}} A$ and the union of any two sets in $\mathcal{S}$ is again in $\mathcal{S}$, i.e.,

$$\forall A, B \in \mathcal{S},\ A \cap B \neq \emptyset \ \Rightarrow\ A \cup B \in \mathcal{S}. \tag{2.2}$$

For infinite dimensional integer vector $k = (\cdots, k_i, \cdots)_{i\in\mathbb{Z}}$, we denote its support

$$\mathrm{supp}k = \{i : k_i \neq 0\}.$$

Definition 2.3. A function $f : \mathbb{R} \to \mathbb{R}$ is called real analytic almost periodic with the frequency $\omega = (\cdots, \omega_i, \cdots)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$, if there exists a real analytic function

$$F : \theta = (\cdots, \theta_i, \cdots) \in \mathbb{R}^{\mathbb{Z}}/(2\pi\mathbb{Z})^{\mathbb{Z}} \to \mathbb{R}$$

which admits a uniformly converging Fourier series expansion

$$F(\theta) = \sum_{A\in\mathcal{S}} F_A(\theta),$$

where $\mathcal{S}$ has spatial structure (2.2) and

$$F_A(\theta) = \sum_{\mathrm{supp}k\subset A} f_{A,k} e^{\mathrm{i}\langle k,\theta\rangle},$$

with $\langle k, \theta\rangle = \sum_{i\in\mathbb{Z}} k_i\theta_i$, such that $f(t) = F(\omega t)$ for all $t \in \mathbb{R}$, where F is bounded in a complex neighborhood

$$\mathbb{T}^{\mathbb{Z}}_s := \left\{\theta = (\cdots, \theta_\lambda, \cdots) \in \mathbb{C}^{\mathbb{Z}}/(2\pi\mathbb{Z})^{\mathbb{Z}},\ \sup_{\lambda\in\mathbb{Z}} |\mathrm{Im}\theta_\lambda| \leq s\right\}$$

for some s. Here $F(\theta)$ is called the shell function of $f(t)$.

Thus, $f(t)$ can be represented as $f(t) = \sum_{A\in\mathcal{S}} f_A(t)$ with

$$f_A(t) = \sum_{\mathrm{supp}k\subset A} f_{A,k} e^{\mathrm{i}\langle k,\omega\rangle t}.$$

According to Definition 2.3, we can observe that $\mathcal{S} = \{(\lambda_1, \cdots, \lambda_{j(i)}) : i \in \mathbb{Z}\}$ in Fourier series expansion (2.1).

From the definitions of the support $\mathrm{supp}k$ and the set A, we know that $f_A(t)$ is a real analytic quasi-periodic function with the frequency $\omega_A = \{\omega_i : i \in A\}$. Therefore, the almost periodic function $f(t)$ can be represented as the sum of countably many quasi-periodic functions $f_A(t)$ formally.

Actually, if an almost-periodic function satisfies Definition 2.3, then it must satisfies Definition 2.2 introduced by Bohr([9, 11]).

Definition 2.4. A polynomial $F(x, y, \lambda_1, \ldots, \lambda_k)$ is said to be quasi-homogeneous of order d with weight $(\alpha_x, \alpha_y, \alpha_1, \ldots, \alpha_k)$ if

$$F(e^{\alpha_x \zeta}x, e^{\alpha_y \zeta}y, e^{\alpha_1 \zeta}\lambda_1, \ldots, e^{\alpha_k \zeta}\lambda_k) \equiv e^{d\zeta}F(x, y, \lambda_1, \ldots, \lambda_k),$$

where d is a positive integer and ζ is a real number.

Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ and $h = (h_1, \ldots, h_k) \in \mathbb{Z}_+^k$. If $F(x, y, \lambda)$ is a quasi-homogeneous polynomial of order d with weight $(\alpha_x, \alpha_y, \alpha_1, \ldots, \alpha_k)$, then F can write as

$$F(x, y, \lambda) = \sum_{\|(i,j,h)\|=d} f_{ijh}x^i y^j \lambda^h$$

and the definition of $\|(i, j, h)\|$ is as follows:

$$\|(i, j, h)\| := \alpha_x i + \alpha_y j + \alpha_1 h_1 + \ldots + \alpha_k h_k.$$

Let M be a compact set in $\mathbb{C}^{k+2}$ and define $D = \mathbb{T}_s^{\mathbb{Z}} \times M$. If $F(\theta, x, y, \lambda)$ is an analytic function defined on D, then we can expand $F(\theta, x, y, \lambda)$ into a Fourier-Taylor series

$$F(\theta, x, y, \lambda) = \sum_{A\in\mathcal{S}} F_A(\theta, x, y, \lambda) = \sum_{A\in\mathcal{S}} \sum_{\mathrm{supp}k\subset A} f_{A,k}(x, y, \lambda)e^{\mathrm{i}\langle k,\theta\rangle}$$

with

$$f_{A,k}(x, y, \lambda) = \sum_{d\geq 0} f_{A,kd} = \sum_{d\geq 0} \left(\sum_{\|(i,j,h)\|=d} f_{A,kijh}x^i y^j \lambda^h \right).$$

Furthermore, we write

$$_{<l}\{f\} := \sum_{A\in\mathcal{S}} \sum_{\mathrm{supp}k\subset A} \sum_{0\leq d\leq l} f_{A,kd}e^{\mathrm{i}\langle k,\theta\rangle}, \qquad _{>l}\{f\} := \sum_{A\in\mathcal{S}} \sum_{\mathrm{supp}k\subset A} \sum_{d\geq l+1} f_{A,kd}e^{\mathrm{i}\langle k,\theta\rangle}$$

and denote the norm of F by

$$\|F\|_D^m = \sum_{A\in\mathcal{S}} \sum_{\mathrm{supp}k\subset A} \|f_{A,k}(x, \lambda)\|_M e^{|k|s}e^{[A]m}.$$

Here and below, $\|\cdot\|_M$ stands for the supremum norm on the set M. For all above bounded real analytic functions, we can define a Banach space

$$C_m^\omega(D) = \{f \ : \ \|f\|_D^m < \infty\}.$$

For $F(\theta, x, y, \lambda) \in C_m^\omega(D)$, we denote the average of $F(\theta, x, y, \lambda)$ by $[F(\theta, x, y, \lambda)] = \sum_{A\in\mathcal{S}}[F_A(\theta, x, y, \lambda)]$ where $[F_A(\theta, x, y, \lambda)]$ represents the average of quasi-periodic function $F_A(\theta, x, y, \lambda)$. Define a nonnegative weight function

$$[\cdot] : A \mapsto [A] = 1 + \sum_{i\in A} log^\rho(1 + |i|),$$

where $\rho > 2$ is a constant. Next, we define the strongly non-resonant conditions of the almost-periodic frequency $\omega = (\cdots, \omega_i, \cdots)_{i\in\mathbb{Z}}$. Denote

$$\mathbb{Z}_{\mathcal{S}}^{\mathbb{Z}} = \{k = (\cdots, k_i, \cdots) \in \mathbb{Z}^{\mathbb{Z}} : \mathrm{supp}k \subset A, A \in \mathcal{S}\}.$$

For $k \in \mathbb{Z}_{\mathcal{S}}^{\mathbb{Z}}$, we define the weight of its support

$$[[k]] = \min_{\mathrm{supp} k \subset A \in \mathcal{S}} [A].$$

Then the non-resonant conditions read

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{\Delta([[k]])\Delta(|k|)}, \ 0 \neq k \in \mathbb{Z}_{\mathcal{S}}^{\mathbb{Z}}, \tag{2.3}$$

where $\gamma > 0$, $|k| = \sum_i |k_i|$ and Δ is some fixed approximation function. A function $\Delta : [1, \infty) \to [1, \infty)$ is called an approximation function, if Δ is non-decreasing and $\Delta(1) = 1$;

$$\int_1^\infty \frac{\ln \Delta(t)}{t^2} dt < \infty. \tag{2.4}$$

In the following we will give a criterion for the existence of strongly non-resonant frequencies. It is based on growth conditions on the distribution function

$$N_i(t) = \mathrm{card}\{A \in \mathcal{S} : \mathrm{card}(A) = i, [A] \leq t\},$$

for $i \geq 1$ and $t \geq 0$.

Lemma 2.1. *There exist a constant N_0 and an approximation function Φ such that*

$$N_i(t) \leq \begin{cases} 0, & t < t_i, \\ N_0 \Phi(t), & t \geq t_i \end{cases}$$

with a sequence of real numbers t_i satisfying

$$i \log^{\rho - 1} i \leq t_i \sim i \log^\rho i$$

for i large with some exponent $\rho - 1 > 1$. Here we say $a_i \sim b_i$, if there are two constants c, C such that $ca_i \leq b_i \leq Ca_i$ and c, C are independent of i.

For a rigorous proof of Lemma 2.1, the reader is referred to Pöschel ([19]) and Huang et al. ([15]), we omit it here. According to Lemma 2.1, there exist an approximation function Δ and a probability measure $\mathfrak{U}$ on the parameter space $\mathbb{R}^{\mathbb{Z}}$ with support at any prescribed point such that the measure of the set of ω satisfying the following inequalities

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{\Delta([[k]])\Delta(|k|)}, \ \gamma > 0, \ \text{for all } k \in \mathbb{Z}_{\mathcal{S}}^{\mathbb{Z}}$$

is positive for a suitably small γ, the proof can be found in Pöschel ([19]), we omit it here.

Throughout this paper, we denote by c, C the universal positive constants if we do not care their values, denote the absolute value (or norm of vector, or norm of matrix) by $|\cdot|$. In the sequel, we still denote the shell of a quasi-periodic function $h(t)$ by $h(\omega t)$, for the sake of simplicity.

2.2. Main results

We work in extended phase space $\mathbb{T}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^2$. The general question we focus on is what remains of the integrable bifurcations when perturbing to nearly integrable Hamiltonian families

$$H(\theta, J, x, y, \lambda) = N(J, x, y, \lambda) + P(\theta, x, y, \lambda, \epsilon) \tag{2.5}$$

and Hamiltonian families

$$H(\theta, J, x, y, \bar{\lambda}) = M(J, x, y, \bar{\lambda}) + P(\theta, x, y, \bar{\lambda}, \epsilon), \tag{2.6}$$

where $(\theta, J, x, y) \in \mathbb{T}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^2$, symplectic form is $\sum_{i\in\mathbb{Z}} d\theta_i \wedge dJ_i + dx \wedge dy$, $\lambda = (\lambda_1, \dots, \lambda_l) \in \breve{\Lambda} \subset \mathbb{R}^l$, $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_{d-2}) \in \bar{\Lambda} \subset \mathbb{R}^{l-2}$, $\breve{\Lambda}$ and Λ are closed and bounded neighbourhood of zero,

$$N(J, x, y, \lambda) = \langle \omega, J \rangle + \frac{A}{2} x^2 y + \frac{B}{l!} y^l + \sum_{j=1}^{l-1} \frac{\lambda_j}{j!} y^j + \lambda_l x$$

and

$$M(J, x, y, \bar{\lambda}) = \langle \omega, J \rangle + \frac{a}{2} x^2 + \frac{b}{l!} y^l + \sum_{j=1}^{l-2} \frac{\bar{\lambda}_j}{j!} y^j \tag{2.7}$$

with A, B, a and b being fixed constants.

We have the following theorems.

Theorem 2.1. *Consider the Hamiltonian system (2.5). Suppose that*
(i) *ω satisfies the non-resonant conditions (2.3);*
(ii) *$P(\theta, x, y, \lambda, \epsilon) = \mathcal{O}(\epsilon)$;*
Then for sufficiently small positive constant ϵ, there exists a close to identity transformation Φ which transforms the Hamiltonian system (2.5) into the following form

$$H_\infty(\theta, J, \tilde{x}, \tilde{y}, \tilde{\lambda}) = N_\infty(J, \tilde{x}, \tilde{y}, \tilde{\lambda}) + P_\infty(\theta, \tilde{x}, \tilde{y}, \tilde{\lambda})$$

such that
(i) *Φ is symplectic, real analytic for θ and C^∞-smooth for $(\tilde{x}, \tilde{y}, \tilde{\lambda})$;*
(ii) *The new normal form $N_\infty(J, \tilde{x}, \tilde{y}, \tilde{\lambda})$ has the same form as $N(J, x, y, \lambda)$;*
(iii) *$\frac{\partial^{i+j+|h|} P_\infty}{\partial \tilde{x}^i \partial \tilde{y}^j \partial \tilde{\lambda}^h}(\theta, 0, 0, 0) = 0$ for all $\theta \in \mathbb{T}^{\mathbb{Z}}$ and all i, j, $h = (h_1, \cdots, h_l)$ satisfy*

$$(l-1)i + 2j + (2l-2)h_1 + (2l-4)h_2 + \cdots + 2h_{l-1} + (l+1)h_l + \leq 2l.$$

Theorem 2.2. *Consider the Hamiltonian system (2.6). Suppose that*
(i) *ω satisfies the non-resonant condition (2.3);*
(ii) *$P(\theta, x, y, \bar{\lambda}, \epsilon) = \mathcal{O}(\epsilon)$;*
Then for sufficiently small positive constant ϵ, there exists a close to identity transformation Φ which transforms the Hamiltonian system (2.6) into the following form

$$H_\infty(\theta, J, \tilde{x}, \tilde{y}, \tilde{\lambda}) = M_\infty(J, \tilde{x}, \tilde{y}, \tilde{\lambda}) + P_\infty(\theta, \tilde{x}, \tilde{y}, \tilde{\lambda})$$

such that
(i) *Φ is symplectic, real analytic for θ and C^∞-smooth for $(\tilde{x}, \tilde{y}, \tilde{\lambda})$;*

(ii) *The new normal form* $M_\infty(J,\tilde{x},\tilde{y},\tilde{\lambda})$ *has the same form as* $M(J,x,y,\bar{\lambda})$;

(iii) $\frac{\partial^{i+j+|h|}P_\infty}{\partial\tilde{x}^i\partial\tilde{y}^j\partial\tilde{\lambda}^h}(\theta,0,0,0)=0$ *for all* $\theta\in\mathbb{T}^{\mathbb{Z}}$ *and all* i, j, $h=(h_1,\cdots,h_{l-2})$ *satisfy*

$$li+2j+(2l-2)h_1+(2l-4)h_2+\cdots+4h_{l-2}\le 2l.$$

Remark 2.1. The formulations of Theorems 2.1 and 2.2 are based on KAM theory on infinite dimensional invariant tori for almost-periodic forced Hamiltonian systems. For the proofs we use the overall strategy of [21], which however has to be substanially developed in techniques to deal with (2.5) and (2.6). In addition, Theorem 2.1 can be regarded as a generalization of the work in [6] from the case of quasi-periodic perturbations to the case of almost-periodic perturbations and Theorem 2.2 can be regarded as a generalization of the work in [21] from 1-dimensional almost-periodic systems to 2-dimensional almost-periodic systems.

Remark 2.2. According to the final normal form $N_\infty(J,\tilde{x},\tilde{y},\tilde{\lambda})=\langle\omega,J\rangle+\frac{\tilde{A}}{2}x^2y+\frac{\tilde{B}}{l!}y^l+\sum_{j=1}^{l-1}\frac{\tilde{\lambda}_j}{j!}y^j+\tilde{\lambda}_l x$ in Theorem 2.1, we know that H_∞ has infinite-dimensional tori at $(J,x,y,\tilde{\lambda})=(0,0,0,0)$. Furthermore, $\tilde{\lambda}=\lambda+e_P(\epsilon)$ where $e_P(\epsilon)=\mathcal{O}(\epsilon)$ depends on the perturbation $P(\theta,x,y,\lambda)$, and $e_P(\epsilon)$ is given by (3.4). That is, original Hamiltonian H also has infinite-dimensional tori if $\lambda+e_P(\epsilon)=0$. The same judgment also applies to $M(J,x,y,\bar{\lambda})$ in Theorem 2.2.

Remark 2.3. A motivating example in practical application background besides its mathematical interest is the almost-periodic forced Duffing-van der Pol oscillators. For example, the nonlinear forced oscillator

$$\ddot{x}+x^l=\epsilon f(t,x,\epsilon), \tag{2.8}$$

where $x,t\in\mathbb{R}$ and $\epsilon>0$ is a small parameter, $f(t,x,\epsilon)$ is real analytic with regard to all variables and almost-periodic in t with frequency vector $\omega=(\cdots,\omega_i,\cdots)_{i\in\mathbb{Z}}\in\mathbb{R}^{\mathbb{Z}}$. Its Hamiltonian reads

$$H(\theta,J,x,y,\epsilon)=\langle\omega,J\rangle+\frac{y^2}{2}+\frac{x^{l+1}}{l+1}+\epsilon P(\theta,x,\epsilon),\ (\theta,J,x,y)\in\mathbb{T}^{\mathbb{Z}}\times\mathbb{R}^{\mathbb{Z}}\times\mathbb{R}^2, \tag{2.9}$$

where $P(\theta,x,\epsilon)=-\int_0^x f(\theta,u,\epsilon)du$. According to Theorem 2.2, almost-periodic response solutions exist when $e_P(\epsilon)=0$.

3. Proof of Theorem 2.1

In this section, we give the proof of Theorem 2.1 detailedly. Taylor expansion of each function in (2.5) is regard as the infinite sum of quasi-homogeneous polynomials in (x,y,λ) with weight $(l-1,2;2l-2,\cdots,2,l+1)$. For given $s,r>0$, we define the set $D(s,r)=\mathbb{T}_s^d\times W(r)$, where $W(r)=B(r)\times U(r)$, $B(r)=\{J:|J|_w\le r^{2l}\}$ and

$$U(r)=\{(x,y,\lambda):|x|\le r^{l-1},|y|<r^2,|\lambda_j|\le r^{2l-2j},|\lambda_l|\le r^{l+1},j=1,\cdots,l-1\}.$$

We define the norm

$$|J|_w=\sum_{i\in\mathbb{Z}}|J_i|e^{w[i]},$$

where $0<w\le m_*$ is a fixed constant, and the weights at the individual lattice are defined by $[i]=\min_{i\in A\in\mathcal{S}}[A]$.

3.1. Iterative lemma

Let Δ be an approximation function as defined in (2.3) and let $\bar{\Lambda}(t) = t\Delta^2(t)$. For $m, s, r, \epsilon > 0$, we set $m_0 = m, s_0 = s, r_0 = r$ and $\epsilon_0 = \epsilon$. Let $\Lambda_0 \geq \bar{\Lambda}(1) = \Delta(1)$. We note that the universal constant C appeared in (3.5) is independent of q and iterative step ν. We choose $0 < a, b < 1$, $0 < q < 1$, $0 < \delta \leq 1/2$ such that

$$1 - a + b + Cq^{2l+1} \leq q^{2l+\alpha} \leq \delta^{\frac{2(2l+\alpha)}{\alpha}},$$

where $0 < \alpha < 1$. Then, let $\Lambda_0 \geq (\delta^{-1}q^{\frac{\alpha}{2}})^{-1}$ and $\tau_0 := \bar{\Lambda}^{-1}(\Lambda_0)$ be large enough such that

$$\frac{\log(1-a)}{\log(\delta^{-1}q^{\frac{\alpha}{2}})} \int_{\tau_0}^{\infty} \frac{\ln \bar{\Lambda}(t)}{t^2} dt < \max\{\frac{s_0}{2}, \frac{m_0}{2}\}.$$

Next, we define the sequences $(\epsilon_\nu)_{\nu\geq 0}, (\Lambda_\nu)_{\nu\geq 0}, (\tau_\nu)_{\nu\geq 0}, (\sigma_\nu)_{\nu\geq 0}$, $(m_\nu)_{\nu\geq 0}$,$(s_\nu)_{\nu\geq 0}$, $(r_\nu)_{\nu\geq 0}$, $(r_\nu^1)_{\nu\geq 0}$, $(r_\nu^2)_{\nu\geq 0}$, $(r_\nu^3)_{\nu\geq 0}$ and $(r_\nu^4)_{\nu\geq 0}$ in the following manner:

$$\begin{cases} \epsilon_\nu = \epsilon_0 q^{(2l+\alpha)\nu}, \ \Lambda_\nu = \left(\delta/q^{\frac{\alpha}{2}}\right)^\nu \Lambda_0, \ \tau_\nu = \bar{\Lambda}^{-1}(\Lambda_\nu), \\ 1 - a = e^{-\tau_\nu \sigma_\nu}, s_{\nu+1} = s_\nu - \sigma_\nu, \ r_\nu^1 = q r_\nu, \\ r_\nu^2 = \frac{1}{2} r_\nu^1, \ r_\nu^3 = \frac{1}{2} r_\nu^2, \ r_\nu^4 = \frac{1}{2} r_\nu^3, \ r_{\nu+1} = \frac{1}{2} r_\nu^4, \ m_{\nu+1} = m_\nu - \sigma_\nu. \end{cases}$$

We can see that $r_\nu = \frac{q}{8} r_{\nu-1} = r_0 (\frac{q}{8})^\nu$. Similar to [21], we can prove $s_\nu \to s_* \geq s_0/2$ and $m_\nu \to m_* \geq m_0/2$.

We suppose that after ν steps, the transformed system defined in the domain $D(s_\nu, r_\nu)$ is of the form

$$H_\nu(\theta, J, x, y, , \lambda^\nu) = N_\nu(J, x, y, \lambda^\nu) + P_\nu(\theta, x, y, \lambda^\nu), \qquad (Eq)_\nu$$

where

$$N_\nu(J, x, y, \lambda^\nu) = \langle \omega, J \rangle + \frac{A_\nu}{2} x^2 y + \frac{B_\nu}{l!} y^l + \sum_{j=1}^{l-1} \frac{\lambda_j^\nu}{j!} y^j + \lambda_l^\nu x.$$

We have the following lemma.

Lemma 3.1. *Let us consider a family of the Hamiltonian systems $(Eq)_{\breve{l}}$ ($\breve{l} = 0, 1, \ldots, \nu$) in the domain $D(s_{\breve{l}}, r_{\breve{l}})$. Assume that*

($\breve{l}$.1) *The function $P_{\breve{l}} \in C^\omega_{m_{\breve{l}}}(D(s_{\breve{l}}, r_{\breve{l}}))$ and the following estimate holds true:*

$$\|P_{\breve{l}}\|^{m_{\breve{l}}}_{D(s_{\breve{l}}, r_{\breve{l}})} \leq \epsilon_{\breve{l}},$$

($\breve{l}$.2)

$$|A_{\breve{l}} - A_{\breve{l}-1}| \leq C\epsilon_0 q^{(\breve{l}-1)\frac{\alpha}{2}},$$
$$|B_{\breve{l}} - B_{\breve{l}-1}| \leq C\epsilon_0 q^{(\breve{l}-1)\frac{\alpha}{2}}.$$

Then, for sufficient small ϵ_0, there is a symplectic change of variables

$$\Phi_\nu : \ (\theta, x_+, y_+, \lambda^{\nu+1}) \in D(s_{\nu+1}, r_{\nu+1}) \to (\theta, x, y, \lambda^\nu) \in D(s_\nu, r_\nu)$$

such that

$$H_{\nu+1} = H_\nu \circ \Phi_\nu = N_{\nu+1} + P_{\nu+1} \qquad (Eq)_{\nu+1}$$

defined on $D(s_{\nu+1}, r_{\nu+1})$ *and conditions* $(\breve{l}.1)$ *and* $(\breve{l}.2)$ *are fulfilled by replacing* $\breve{l}$ *by* $\nu+1$. *Moreover*

$$\left\|\frac{\partial^{i+j+|h|} P_{\nu+1}}{\partial x_+^i \partial y_+^j \partial (\lambda^{\nu+1})^h}\right\|_{D(s_{\nu+1}, r_{\nu+1})}^{m_{\nu+1}} \le C\epsilon_0 q^{(l+\frac{\alpha}{2}-d)(\nu+1)},$$

where $d := (l-1)i + 2j + (2l-2)h_1 + \cdots + 2h_{l-1} + (l+1)h_l \le 2l$.

Proof. We divide the proof into several parts.

A. **Solving linear homological equations.** $P_\nu(\theta, x, y, \lambda^\nu)$ admits spatial expansion

$$\begin{aligned} P_\nu(\theta, x, y, \lambda^\nu) &= \sum_{A\in\mathcal{S}} \sum_{\mathrm{supp}k\subset A} P^\nu_{A,k}(x, y, \lambda^\nu) e^{\mathrm{i}\langle k,\theta\rangle} \\ &= \sum_{A\in\mathcal{S}} \sum_{\mathrm{supp}k\subset A} \sum_{d\ge 0} \left(\sum_{\|(i,j,h)\|=d} P^\nu_{A,kijh} x^i y^j (\lambda^\nu)^h \right) e^{\mathrm{i}\langle k,\theta\rangle}. \end{aligned}$$

Then, we truncate $P_\nu(\theta, x, y, \lambda^\nu) = \widetilde{P}_\nu(\theta, x, y, \lambda^\nu) + \widehat{P}_\nu(\theta, x, y, \lambda^\nu)$ with

$$\begin{aligned} &\widetilde{P}_\nu(\theta, x, y, \lambda^\nu) \\ &= \sum_{\substack{A\in\mathcal{S} \\ 0<[A]\le\tau_\nu}} \sum_{\substack{\mathrm{supp}k\subset A \\ 0\le|k|\le\tau_\nu}} (1-(1-a)e^{\sigma_\nu[A]})(1-(1-a)e^{\sigma_\nu|k|}) P^\nu_{A,k}(x, y, \lambda^\nu) e^{\mathrm{i}\langle k,\theta\rangle} \end{aligned}$$

and

$$\begin{aligned} &\widehat{P}_\nu(\theta, x, y, \lambda^\nu) \\ &= \sum_{\substack{A\in\mathcal{S} \\ [A]>\tau_\nu}} \sum_{\mathrm{supp}k\subset A} P^\nu_{A,k}(x, \lambda^\nu) e^{\mathrm{i}\langle k,\theta\rangle} \\ &\quad + \sum_{\substack{A\in\mathcal{S} \\ 0<[A]\le\tau_\nu}} \sum_{\mathrm{supp}k\subset A} (1-a)e^{\sigma_\nu[A]} P^\nu_{A,k}(x, y, \lambda^\nu) e^{\mathrm{i}\langle k,\theta\rangle} \\ &\quad + \sum_{\substack{A\in\mathcal{S} \\ 0<[A]\le\tau_\nu}} \sum_{\substack{\mathrm{supp}k\subset A \\ |k|>\tau_\nu}} (1-(1-a)e^{\sigma_\nu[A]}) P^\nu_{A,k}(x, y, \lambda^\nu) e^{\mathrm{i}\langle k,\theta\rangle} \\ &\quad + \sum_{\substack{A\in\mathcal{S} \\ 0<[A]\le\tau_\nu}} \sum_{\substack{\mathrm{supp}k\subset A \\ 0\le|k|\le\tau_\nu}} (1-(1-a)e^{\sigma_\nu[A]})(1-a)e^{\sigma_\nu|k|} P^\nu_{A,k}(x, y, \lambda^\nu) e^{\mathrm{i}\langle k,\theta\rangle}. \end{aligned}$$

Then, we have the following estimate

$$\begin{aligned} &\|\widehat{P}_\nu\|^{m_{\nu+1}}_{D(s_{\nu+1}, r_\nu)} \\ &\le \sum_{\substack{A\in\mathcal{S} \\ [A]>\tau_\nu}} \sum_{\mathrm{supp}k\subseteq A} |P^\nu_{A,k}| e^{|k|s_\nu} e^{[A]m_{\nu+1}} \\ &\quad + \sum_{\substack{A\in\mathcal{S} \\ 0<[A]\le\tau_\nu}} \sum_{\mathrm{supp}k\subseteq A} (1-a)e^{\sigma_\nu[A]} |P^\nu_{A,k}| e^{|k|s_\nu} e^{[A]m_{\nu+1}} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{A\in\mathcal{S}\\0<[A]\le\tau_\nu}}\sum_{\substack{\mathrm{supp}k\subseteq A\\|k|>\tau_\nu}}(1-(1-a)e^{\sigma_\nu[A]})|P^\nu_{A,k}|e^{|k|(s_\nu-\sigma_\nu)}e^{[A]m_\nu}\\
&+\sum_{\substack{A\in\mathcal{S}\\0<[A]\le\tau_\nu}}\sum_{\substack{\mathrm{supp}k\subseteq A\\0\le|k|\le\tau_\nu}}(1-(1-a)e^{\sigma_\nu[A]})(1-a)e^{\sigma_\nu|k|}|P^\nu_{A,k}|e^{|k|(s_\nu-\sigma_\nu)}e^{[A]m_\nu}\\
\le&\sum_{\substack{A\in\mathcal{S}\\ [A]>\tau_\nu}}\sum_{\mathrm{supp}k\subseteq A}(1-a)|P^\nu_{A,k}|e^{|k|s_\nu}e^{[A]m_\nu}\\
&+\sum_{\substack{A\in\mathcal{S}\\0<[A]\le\tau_\nu}}\sum_{\mathrm{supp}k\subseteq A}(1-a)|P^\nu_{A,k}|e^{|k|s_\nu}e^{[A]m_\nu}\\
&+\sum_{\substack{A\in\mathcal{S}\\0<[A]\le\tau_\nu}}\sum_{\substack{\mathrm{supp}k\subseteq A\\|k|>\tau_\nu}}a(1-a)|P^\nu_{A,k}|e^{|k|s_\nu}e^{[A]m_\nu}\\
&+\sum_{\substack{A\in\mathcal{S}\\0<[A]\le\tau_\nu}}\sum_{\substack{\mathrm{supp}k\subseteq A\\0\le|k|\le\tau_\nu}}a(1-a)|P^\nu_{A,k}|e^{|k|s_\nu}e^{[A]m_\nu}\\
\le&\, 2(1-a)\|P_\nu\|^{m_\nu}_{D(s_\nu,r_\nu)}
\end{aligned}
$$

because $e^{-\tau_\nu\sigma_\nu}=1-a$, function $(1-(1-a)e^{t\sigma_\nu})$ under the sup is monotonically decreasing for $0\le t\le\tau_\nu$ and equals a at $t=0$. On the other hand, the polynomial rest $\widetilde{P}_\nu$ is bounded on a larger domain. Indeed,

$$
\begin{aligned}
&\|\widetilde{P}_\nu\|^{m_\nu}_{D(s_\nu,r_\nu)}\\
\le&\sum_{\substack{A\in\mathcal{S}\\0<[A]\le\tau_\nu}}\sum_{\substack{\mathrm{supp}k\subseteq A\\0\le|k|\le\tau_\nu}}(1-(1-a)e^{\sigma_\nu[A]})(1-(1-a)e^{\sigma_\nu|k|})|P^\nu_{A,k}|e^{|k|s_\nu}e^{[A]m_\nu}\\
\le&\, a^2\|P_\nu\|^{m_\nu}_{D(s_\nu,r_\nu)}.
\end{aligned}
$$

Then we rewrite H_ν as

$$H_\nu=N_\nu+\widetilde{P}_\nu+\widehat{P}_\nu.$$

In the following, we will look for a coordinate transformation ϕ_ν defined in a domain $D\subset D(s_\nu,r_\nu)$, which is written as the time-1-map of the flow $X^t_{F_\nu}$ of a Hamiltonian vector field X_{F_ν} :

$$\phi_\nu=X^t_{F_\nu}\mid_{t=1}.$$

Then ϕ_ν is symplectic. Moreover, we may expand $H_\nu\circ\phi_\nu=H_\nu\circ X^t_{F_\nu}\mid_{t=1}$ with respect to t at 0 using Taylor's formula. Recall that

$$\frac{d}{dt}G\circ X^t_{F_\nu}=\{G,F_\nu\}\circ X^t_{F_\nu},$$

the Poisson bracket of G and F_ν evaluated at $X^t_{F_\nu}$. Thus, we may write

$$
\begin{aligned}
H_\nu\circ\phi_\nu&=(N_\nu+\widetilde{P}_\nu)\circ\phi_\nu+\widehat{P}_\nu\circ\phi_\nu\\
&=N_\nu+\widetilde{P}_\nu+\{N_\nu,F_\nu\}+\{\widetilde{P}_\nu,F_\nu\}\\
&\quad+\int_0^1(1-t)\{\{N_\nu+\widetilde{P}_\nu,F_\nu\},F_\nu\}\circ X^t_{F_\nu}dt+\widehat{P}_\nu\circ\phi_\nu.
\end{aligned}
$$

The point is to find F_ν such that $N_\nu+_{<2l}\{[\widetilde{P}_\nu]\}=N^*_\nu$ is again a normal form. This amounts to solving the linear equation

$$_{<2l}\{\widetilde{P}_\nu-[\widetilde{P}_\nu]+\{N_\nu,F_\nu\}\}=0 \tag{3.1}$$

for F_ν. Then the Hamiltonian H_ν becomes

$$H^*_\nu = N^*_\nu + P^*_\nu,$$

where

$$\begin{aligned} P^*_\nu &= {}_{>2l}\{\widetilde{P}_\nu - [\widetilde{P}_\nu] + \{N_\nu, F_\nu\}\} + \{\widetilde{P}_\nu, F_\nu\} \\ &\quad + \int_0^1 (1-t)\{\{N_\nu + \widetilde{P}_\nu, F_\nu\}, F_\nu\} \circ X^t_{F_\nu} dt + \widehat{P}_\nu \circ \phi_\nu. \end{aligned}$$

Here

$$_{<2l}\{[\widetilde{P}_\nu]\} = \sum_{j=1}^{l} G^\nu_j(\lambda^\nu) y^j + \sum_{j=0}^{[\frac{l+1}{2}]} Q^\nu_j(\lambda^\nu) x y^j + R^\nu(\lambda^\nu) x^2 + \sum_{A\in\mathcal{S}} P^\nu_{A,0210} x^2 y$$

with

$$\begin{aligned} G^\nu_j(\lambda^\nu) &= \sum_{A\in\mathcal{S}} \sum_{2(l-1)h_1+\cdots+2h_{l-1}+(l+1)h_l \le 2l-2j} P^\nu_{A,00jh}(\lambda^\nu)^h, \\ Q^\nu_j(\lambda^\nu) &= \sum_{A\in\mathcal{S}} \sum_{2(l-1)h_1+\cdots+2h_{l-1}+(l+1)h_l \le l+1-2j} P^\nu_{A,01jh}(\lambda^\nu)^h, \\ R^\nu(\lambda^\nu) &= \sum_{A\in\mathcal{S}} \sum_{(2l-1)h_1+\cdots+2h_{l-1}+(l+1)h_l \le 2} P^\nu_{A,020h}(\lambda^\nu)^h. \end{aligned}$$

Now we solve homological equation (3.1). For each $A \in \mathcal{S}$, we have

$$_{<2l}\{\widetilde{P}^\nu_A - [\widetilde{P}^\nu_A] + \{N_\nu, F^\nu_A\}\} = 0.$$

For each supp$k \subset A$, the same order of quasi-homogeneous polynomial of function F^ν_A can be defined inductively by

$$\mathrm{i}\langle k, \omega\rangle F^\nu_{A,kd} = \widetilde{P}^\nu_{A,kd} + \{\breve{N}_\nu, F^\nu_{A,k,d+1-l}\},$$

which implies

$$F^\nu_{A,kd} = \frac{1}{\mathrm{i}\langle k,\omega\rangle}\widetilde{P}^\nu_{A,kd} + \sum_{i=1}^{3} \frac{1}{(\mathrm{i}\langle k,\omega\rangle)^{i+1}} \underbrace{\{\breve{N}_\nu, \cdots, \{\breve{N}_\nu,}_{i} \widetilde{P}^\nu_{A,k,d-i(l-1)} \underbrace{\}\cdots\}}_{i},$$

where $\breve{N}_\nu = N_\nu - \langle\omega, J\rangle$. Thus, we obtain

$$\begin{aligned} \|F_\nu\|^{m_\nu}_{D(s_\nu,r^1_\nu)} &= \| \sum_{\substack{A\in\mathcal{S}\\ 0<[A]\le\tau_\nu}} \sum_{\substack{\mathrm{supp}k\subset A\\ 0<|k|\le\tau_\nu}} \sum_{d\le 2l} \sum_{i=0}^{3} \frac{1}{(\mathrm{i}\langle k,\omega\rangle)^{i+1}} \\ &\qquad \underbrace{\{\breve{N}_\nu, \cdots, \{\breve{N}_\nu,}_{i} \widetilde{P}^\nu_{A,k,d-i(l-1)} \underbrace{\}\cdots\}}_{i} e^{\mathrm{i}\langle k,\theta\rangle} \|^{m_\nu}_{D(s_\nu,r^1_\nu)} \\ &\le C \sum_{A\in\mathcal{S}} \sum_{\substack{\mathrm{supp}k\subset A\\ 0<|k|\le\tau_\nu}} \sum_{i=0}^{3} \Delta^{i+1}(\tau_\nu) \|\widetilde{P}^\nu_{A,k}\|_{W(r_\nu)} e^{|k|s_\nu} e^{[A]m_\nu} \end{aligned}$$

$$\leq Ca^2\Delta^4(\tau_\nu)\epsilon_\nu$$
$$\leq C\epsilon_0\sigma_\nu\delta^\nu q^{(2l+\frac{\alpha}{2})\nu}.$$

According to the above computation, one can check that F_ν is independent of J. By Cauchy's estimate

$$\|\frac{\partial^{i+j+|h|}F_\nu}{\partial x^i\partial y^j\partial(\lambda^\nu)^h}\|^{m_\nu}_{D(s_\nu,r_\nu^2)} \leq Cr_\nu^{-d}\epsilon_0\delta^\nu q^{(2l+\frac{\alpha}{2})\nu} \tag{3.2}$$

if $\|(i,j,h)\| \leq d$. On the domain $D(s_{\nu-1},r_\nu^2)$, we have

$$|\frac{\partial F_\nu}{\partial\theta}|_{m_\nu} = \sum_i |\frac{\partial F_\nu}{\partial\theta_i}|e^{m_\nu[i]} \leq \sum_{i\in A}\sum_{A\in\mathcal{S}}\|\frac{\partial F_A^\nu}{\partial\theta_i}\|_{D(s_{\nu-1},r_\nu^2)}e^{m_\nu[A]} \leq C\epsilon_0\sigma_\nu\delta^\nu q^{(2l+\frac{\alpha}{2})\nu}.$$

Define

$$|X_{F_\nu}|^{m_\nu}_{D(s_{\nu+1},r_\nu^2)}$$
$$:= \max\left\{r_\nu^{-2l}|\frac{\partial F_\nu}{\partial\theta}|_{m_\mu}, r_\nu^{-2l+2}\|\frac{\partial F_\nu}{\partial y}\|^{m_\nu}_{D(s_{\nu+1},r_\nu^2)}, r_\nu^{-l+1}\|\frac{\partial F_\nu}{\partial x}\|^{m_\nu}_{D(s_{\nu+1},r_\nu^2)}\right\},$$
$$\Uparrow D_\mu X_{F_\nu}\Uparrow^{m_\nu}_{D(s_{\nu+1},r_\nu^2)} := \max_{i+j+|h|\leq\mu}\left\{\|\frac{\partial^{i+j+|h|}Y}{\partial x^i\partial y^j\partial(\lambda^\nu)^h}\|^{m_\nu}_{D(s_{\nu+1},r_\nu^2)}\right\},$$

for all $\mu \geq 1$, where Y stands for one of $\frac{\partial F_\nu}{\partial\theta}$, $\frac{\partial F_\nu}{\partial y}$ and $\frac{\partial F_\nu}{\partial x}$. By (3.2) and Cauchy estimation, we have

$$|X_{F_\nu}|^{m_\nu}_{D(s_{\nu+1},r_\nu^2)} \leq Cr_\nu^{-2l}\epsilon_0\delta^\nu q^{(2l+\frac{\alpha}{2})\nu},\ \ \Uparrow D_\mu X_{F_\nu}\Uparrow^{m_\nu}_{D(s_{\nu+1},r_\nu^2)} \leq Cr_\nu^{-2l}\epsilon_0\delta^\nu q^{(2l+\frac{\alpha}{2})\nu}.$$

Hence, the flow $X^t_{F_\nu}$ of X_{F_ν} satisfies

$$|X^t_{F_\nu} - id|^{m_\nu}_{D(s_{\nu+1},r_\nu^2)} \leq Cr_\nu^{-2l}\epsilon_0\delta^\nu q^{(2l+\frac{\alpha}{2})\nu},$$

which implies $X^t_{F_\nu}(D(s_{\nu+1},r_\nu^2)) \subset D(s_\nu,r_\nu^1)$, for all $-1 \leq t \leq 1$. The norm for ϕ_ν is defined by

$$|\phi_\nu|^{m_\nu}_{C^{ij}(D(s_{\nu+1},r_\nu^2))} := \max_{0\leq t\leq 1}\|\frac{\partial^{i+j}\phi^t_{F_\nu}}{\partial x^i\partial y^j}\|^{m_\nu}_{D(s_\nu,r_\nu^2)}.$$

Thus we have

$$|\phi_\nu - id|^{m_\nu}_{C^{ij}(D(s_{\nu+1},r_\nu^2))} \leq Cr_\nu^{-2l}\epsilon_0\delta^\nu q^{(2l+\frac{\alpha}{2})\nu}, \tag{3.3}$$

for any given i and j.

B. **Transformation of N_ν^* into normal form.** One can apply singularity theory to normalize N_ν^*. First we use the shear transformation

$$\phi_1^\nu : \begin{cases} y_1 = y, \\ x_1 = x + (A_\nu + 2\sum_{A\in\mathcal{S}} P^\nu_{A,0210})^{-1}\sum_{j=1}^{[\frac{l+1}{2}]} Q_j^\nu(\lambda^\nu)y^{j-1} \end{cases}$$

to kill the crossing terms $\sum_{j=1}^{[\frac{l+1}{2}]} Q_j^\nu(\lambda^\nu)xy^{j-1}$. we can get

$$N_\nu^* \circ \phi_1^\nu = \langle\omega, J\rangle + \frac{A_{\nu+1}}{2}x_1^2y_1 + \frac{B_{\nu+1}}{l!}y_1^l + R^\nu(\lambda^\nu)x_1^2 + (\lambda_l^\nu + Q_0^\nu(\lambda^\nu))x_1$$

$$
\begin{aligned}
&+\sum_{j=1}^{l-1}(\frac{\lambda_j^\nu}{j!}+G_j^\nu(\lambda^\nu)-\frac{\lambda_j^\nu}{A_{\nu+1}}Q_{j+1}^\nu(\lambda^\nu))y_1^j\\
&-\frac{Q_0^\nu(\lambda^\nu)+2R^\nu(\lambda^\nu)x_1}{A_{\nu+1}}\sum_{j=1}^{[\frac{l+1}{2}]}Q_j^\nu(\lambda^\nu)y_1^{j-1}\\
&+\frac{A_{\nu+1}y_1-2R^\nu(\lambda^\nu)}{2A_{\nu+1}^2}\left(\sum_{j=1}^{[\frac{l+1}{2}]}Q_j^\nu(\lambda^\nu)y_1^{j-1}\right)^2\\
:=\ & N_\nu^{**}+\breve{P}_\nu
\end{aligned}
$$

where $A_{\nu+1}=A_\nu+2\sum_{A\in\mathcal{S}}P^\nu_{A,0210}$ and $B_{\nu+1}=B_\nu+l!\sum_{A\in\mathcal{S}}P^\nu_{A,00l0}$. Using the transformation

$$
\phi_2^\nu:\begin{cases} y_2=y_1+\frac{2}{A_{\nu+1}}R^\nu(\lambda^\nu),\\ x_2=x_1,\end{cases}
$$

we can eliminate the term $R^\nu(\lambda^\nu)x_1^2$. We have $N_\nu^*\circ\phi_1^\nu\circ\phi_2^\nu=N_{\nu+1}+\breve{P}_\nu\circ\phi_2^\nu$. We also have the following parameter transformation

$$
\begin{aligned}
\lambda_i^{\nu+1}&=\lambda_i^\nu+i!G_i^\nu(\lambda^\nu)+\frac{\lambda_l^\nu}{A_{\nu+1}}Q_{i+1}^\nu(\lambda^\nu)\\
&\quad+\sum_{j=i+1}^{l-1}\frac{(-1)^{j-i}}{(j-i)!A_{\nu+1}^{j-i}}(2R(\lambda_\nu))^{j-i}\\
&\quad\times\left(\lambda_j^\nu+j!G_j^\nu(\lambda^\nu)-\frac{\lambda_l^\nu}{A_{\nu+1}}Q_{j+1}^\nu(\lambda^\nu)\right), \qquad (3.4)\\
\lambda_l^{\nu+1}&=\lambda_l^\nu+Q_0(\lambda^\nu).
\end{aligned}
$$

Finally, we get

$$
H_{\nu+1}=N_{\nu+1}+\breve{P}_\nu\circ\phi_2^\nu+P_\nu^*\circ\phi_1^\nu\circ\phi_2^\nu.
$$

C. **Estimation for new perturbation**

By Cauchy's estimate, we have

$$
|P^\nu_{00210}|\le C\frac{\epsilon_\nu}{r_\nu^{2l}},\ |P^\nu_{000l0}|\le C\frac{\epsilon_\nu}{r_\nu^{2l}},
$$

and

$$
\begin{aligned}
&\|2A_{\nu+1}^{-1}R^\nu(\lambda^\nu)\|_{U(r_\nu)}\le C\frac{\epsilon_\nu}{r_\nu^{2l-2}},\\
&\|i!G_i^\nu(\lambda^\nu)-\lambda_l^\nu A_{\nu+1}^{-1}Q_{i+1}^\nu(\lambda^\nu)\|_{U(r_\nu)}\le C\frac{\epsilon_\nu}{r_\nu^{2i}},\\
&\|Q_0^\nu(\lambda^\nu)\|_{U(r_\nu)}\le C\frac{\epsilon_\nu}{r_\nu^{l-1}}.
\end{aligned}
$$

Thus, we have $\phi_1^\nu(D(s_{\nu+1},r_\nu^3))\subset D(s_{\nu+1},r_\nu^2)$ and $\phi_2^\nu(D(s_{\nu+1},r_\nu^4))\subset D(s_{\nu+1},r_\nu^3)$. And we have the following estimates

$$
\|\breve{P}_\nu\circ\phi_2^\nu\|^{m_\nu}_{D(s_{\nu+1},r_\nu^4)}\le\frac{\epsilon_\nu^2}{r_\nu^{2l}}\le C\epsilon_0\epsilon_\nu,
$$

$$\begin{aligned}
&\|_{>2l}\{\widetilde{P}_\nu - [\widetilde{P}_\nu] + \{N_\nu, F_\nu\}\} \circ \phi_1^\nu \circ \phi_2^\nu\|^{m_\nu}_{D(s_{\nu+1}, r_\nu^4)} \le Cq^{2l+1}\epsilon_\nu, \qquad (3.5)\\
&\|\{\widetilde{P}_\nu, F_\nu\}\|^{m_\nu}_{D(s_{\nu+1}, r_\nu^4)} \le C\epsilon_0\epsilon_\nu,\\
&\|\int_0^1 (1-t)\{\{N_\nu + \widetilde{P}_\nu, F_\nu\}, F_\nu\} \circ \phi^t_{F_\nu} \circ \phi_1^\nu \circ \phi_2^\nu dt\|^{m_\nu}_{D(s_{\nu+1}, r_\nu^4)} \le C\epsilon_0\epsilon_\nu,\\
&\|\widehat{P}_\nu \circ \phi^1_{F_\nu} \circ \phi_1^\nu \circ \phi_2^\nu\|^{m_\nu}_{D(s_{\nu+1}, r_\nu^4)} \le (1-a)\epsilon_\nu.
\end{aligned}$$

The new perturbation is

$$P_{\nu+1} = \breve{P}_\nu \circ \phi_2^\nu + P_\nu^* \circ \phi_1^\nu \circ \phi_2^\nu.$$

Thus,

$$\|P_{\nu+1}\|^{m_\nu}_{D(s_{\nu+1}, r_\nu^4)} \le (1 - a + b + Cq^{l+1})\epsilon_\nu \le \epsilon_{\nu+1}.$$

By Cauchy's estimate, we have

$$\|\frac{\partial^{i+j+|h|} P_{\nu+1}}{\partial x_+^i \partial y_+^j \partial(\lambda^{\nu+1})^h}\|^{m_\nu}_{D(s_{\nu+1}, r_{\nu+1})} \le C\epsilon_0 q^{(l+\frac{\alpha}{2}-d)(\nu+1)}$$

with $d := \|(i,j,h)\| \le 2l$. This completes the proof of Lemma 3.1. □

3.2. Convergence of KAM iteration

Let us take $N_0 = N$ and $P_0 = P$. It is easy to check that system (2.5) satisfies all hypotheses of Lemma 3.1 with $\nu = 0$. Thus, there is a sequence

$$\Phi_\nu = \phi_\nu \circ \phi_1^\nu \circ \phi_2^\nu : \ D(s_{\nu+1}, r_{\nu+1}) \to D(s_\nu, r_\nu) \qquad (3.6)$$

such that

$$H_\nu \circ \Phi_\nu = H_{\nu+1}.$$

By (3.3), we have

$$\|\Phi_\nu - id\|^{m_\nu}_{D(s_{\nu+1}, r_{\nu+1})} \le C\epsilon_0 q^{(\frac{\alpha}{2})\nu}, \qquad (3.7)$$

$$\|D\Phi_\nu - Id\|^{m_\nu}_{D(s_{\nu+1}, r_{\nu+1})} \le C\epsilon_0 q^{(\frac{\alpha}{2})\nu}. \qquad (3.8)$$

Let

$$\Phi^\nu := \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : \ D(s_{\nu+1}, r_{\nu+1}) \to D(s_0, r_0).$$

For sufficiently small $\epsilon_0 > 0$, it follows from the inequality (3.7) and (3.8) that

$$\|D\Phi_j(\Phi_{j+1} \circ \cdots \circ \Phi_\nu)\|^{m_\nu}_{D(s_{\nu+1}, r_{\nu+1})} \le 1 + q^{\frac{\alpha j}{2}}, \quad j = 0, 1, \ldots, \nu - 1.$$

We infer

$$\|D\Phi^{\nu-1}\|^{m_\nu}_{D(s_\nu, r_\nu)} \le \prod_{\nu \ge 0}(1 + q^{\frac{\alpha\nu}{2}}) \le e^{\frac{1}{1-q^{\frac{\alpha}{2}}}},$$

and hence

$$\begin{aligned}
\|\Phi^\nu - \Phi^{\nu-1}\|^{m_\nu}_{D(s_{\nu+1}, r_{\nu+1})} &= \|\Phi^{\nu-1}(\Phi_\nu) - \Phi^{\nu-1}\|^{m_\nu}_{D(s_{\nu+1}, r_{\nu+1})}\\
&\le \|D\Phi^{\nu-1}\|^{m_\nu}_{D(s_\nu, r_\nu)} \|\Phi_\nu - id\|^{m_\nu}_{D(s_{\nu+1}, r_{\nu+1})}
\end{aligned}$$

$$\leq e^{\frac{1}{1-q^{\frac{\alpha}{2}}}} q^{\frac{\alpha\nu}{2}}.$$

The same inequality holds for $\nu = 0$ if we define $\Phi^{-1} := id$. By composition, $\Phi^{\nu} := \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{\nu}$, we obtain a coordinate transformation that turns the given $H_0 = N_0 + P_0$ into $H_{\nu} = N_{\nu} + P_{\nu}$. Our aim is to find a 'limit' Φ^{∞} which transforms the original system into $H_{\infty} = N_{\infty} + P_{\infty}$.

However, in the present situation the coordinate changes Φ_{ν} do not form a group. Indeed, the bifurcating tori require higher order terms, which in turn have to be dealt with by both ϕ_{ν}, ϕ_1^{ν} and ϕ_1^{ν} defined in (3.6). The problem is now that one cannot restrict oneself to the fixed weighted order $2l$ in $(x, y; \lambda)$ imposed by the type of bifurcation, as the composition of Φ_{ν} and $\Phi_{\nu+1}$ itself would increase this order to $4l$. Therefore, we have to pass to a polynomial truncation of fixed degree in order to define Φ^{∞} by means of limits of coefficient functions. This the 'truncated transformations' Υ_{ν} has to satisfy the following condition: 1. the 'truncated transformations' Υ_{ν} have to be symplecto-morphisms as well. 2. The estimates implied by lemma 3.1 should remain valid after the transformed system $H_0 \circ \Phi^{\nu}$ are replaced by the system $H_0 \circ \Upsilon_{\nu}$.

Since $\Phi^{\nu} : (\theta, J, x, y) \to (\theta, \mathcal{J}, X, Y)$ is a symplecto-morphism for fixed λ_j, we can define the generation function $S_{\nu}(\theta, \mathcal{J}, X, y)$. In view of the first condition we do not simply truncate Φ^{ν}, but truncate a generating function S_{ν} to define Υ_{ν} as follows. Obviously, S_{ν} is close and one-valued. Because of the second condition we define the truncation $\tilde{S}_{\nu}$ of S_{ν} to be of order $l + 1$ in $(\mathcal{J}, X, y)$. Furthermore, we drop all terms that involve more than one derivative with respect to parameters λ_j. On the other hand, we do not truncate in θ. To be precise, we write

$$\Phi^{\nu}(\theta, J, x, y, \lambda^{\nu}) = ((\theta, J, x, y) + W_{\nu}(\theta, x, y, \lambda^{\nu}), \lambda^{\nu} + \tilde{\Lambda}_{\nu}(\lambda^{\nu})),$$

where $W_{\nu}(\theta, x, y, \lambda^{\nu}) = (0, W_{\nu}^2, W_{\nu}^3) = (0, \Phi_2^{\nu}(\theta, J, x, y, \lambda^{\nu}) - J, \Phi_3^{\nu}(\theta, J, x, y, \lambda^{\nu}) - J)$ with $\Phi_i^{\nu}(\theta, x, y, \lambda^{\nu})$, $i = 1, 2, 3$, being the i component of vector function $\Phi^{\nu}(\theta, J, x, y, \lambda^{\nu})$. And let $\mathcal{F} : D(s_{\nu}, r_{\nu}) \to D(s_0, r_0)$ denote the transformation of (θ, J, x, y) into

$$(\theta, J + W_{\nu}^2(\theta, x, y, \lambda^{\nu}), x + W_{\nu}^3(\theta, x, y, \lambda^{\nu}), y, \lambda^{\nu}) = (\theta, \mathcal{J}, X, y, \lambda^{\nu}),$$

and $\mathcal{G}_{\nu} = (\mathcal{F}_{\nu})^{-1}$. The truncations $\tilde{S}_{\nu}$ are polynomials in $\mathcal{J}, X, Y$ and λ^{ν} and the coefficients of which are holomorphic functions in θ. To truncate we write S_{ν} as a Taylor series at $\mathcal{F}_{\nu}(\theta, 0, 0, 0, 0) =: (\theta, \mathcal{J}_{\nu}, X_{\nu}, 0, 0)$. Therefore, we define

$$\tilde{S}_{\nu}(\theta, \mathcal{J}, X, y, \lambda^{\nu}) = \sum_{|d|_{\infty}+i+j=0}^{l+1} \sum_{|h|=0}^{\min(|d|_{\infty}+i+j,1)} S_{\nu}^{dijh}(\theta)(\mathcal{J} - \mathcal{J}_{\nu})^d (x - X_{\nu})^i y^j (\lambda^{\nu})^h,$$

where

$$S_{\nu}^{dijh}(\theta) = \frac{\partial^{|h|+i+j+|d|_{\infty}}}{\partial \mathcal{J}^d \partial X^i \partial y^j \partial(\lambda^{\nu})^h} S^{\nu}(\theta, \mathcal{J}_{\nu}, X_{\nu}, 0, 0).$$

Under the conditions of lemma 3.1 and Whitney's Extension Theorem [23], the sequence $\tilde{S}_{\nu}(\theta, \mathcal{J}, X, y, \lambda^{\nu})$ of truncations is uniformly convergent on $\mathbb{T}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^2$. Thus, we can get $\tilde{S}_{\infty}$ for all $\lambda \in \mathbb{R}^l$. Furthermore, we can get Φ^{∞} by $\tilde{S}_{\infty}$. Letting $P_{\infty} = H_0 \circ \Phi^{\infty} - N_{\infty}$, we have, according to our choice of the truncations $\tilde{S}_{\nu}$ of S_{ν} at order $l + 1$,

$$P_{\infty A, ijh} = \lim_{\nu \to \infty} P_{\nu A, ijh}$$

as long as $|h| \le 1$ and $|h|+i+j \le 1$. In particular, we can conclude that these all vanish for weighted order $\|(i,j,h)\| \le 2l$. This completes the proof.

4. Outline of proof of Theorem 2.2

In this section, we only give the main points on the proof of Theorem 2.2, the details of which are similar to the proof of Theorem 2.1. In other words, this section is a road map through technical aspects of proof. In the proof of Theorem 2.2, the weight of quasi-homogeneous polynomials is different from that in proof of Theorem 2.1.

In this proof, Taylor expansion of each function in (2.6) is regard as the infinite sum of quasi-homogeneous polynomials in $(x, y, \bar{\lambda})$ with weight $(l, 2; 2l-2, \cdots, 4)$. For given $s, r > 0$, we define the set $D(s,r) = \mathbb{T}_s^d \times \bar{W}(r)$, where $\bar{W}(r) = \bar{B}(r) \times \bar{U}(r)$, $\bar{B}(r) = \{J : |J|_w \le r^{2l}\}$ and

$$\bar{U}(r) = \{(x, y, \bar{\lambda}) : |x| \le r^l, |y| < r^2, |\bar{\lambda}_j| \le r^{2l-2j}, j = 1, \cdots, l-2\}.$$

We use KAM iteration to prove Theorem 2.2. At the ν-th step in KAM, Hamiltonian (2.6) becomes

$$H_\nu = M_\nu + P_\nu, \tag{4.1}$$

where

$$M_\nu = \langle \omega, J \rangle + \frac{a_\nu}{2} x^2 + \frac{b_\nu}{l!} y^l + \sum_{j=1}^{l-2} \frac{\bar{\lambda}_j^\nu}{j!} y^j. \tag{4.2}$$

One can use a coordinate transformation defined in a domain $D \subset D(s_\nu, r_\nu)$, which is written as the time-1-map of the flow $X_{F_\nu}^t$ of a Hamiltonian vector field X_{F_ν} :

$$\phi_\nu = X_{F_\nu}^t |_{t=1},$$

to conjugate system (4.1) to

$$H_\nu^* = M_\nu^* + P_\nu^*. \tag{4.3}$$

In (4.3), what add to normal form M_ν^* is

$$_{<2l}\{[\widetilde{P}_\nu]\} = \sum_{j=1}^{l} G_j^\nu(\bar{\lambda}^\nu) y^j + \sum_{j=0}^{[\frac{l}{2}]} Q_j^\nu(\bar{\lambda}^\nu) x y^j + \sum_{A \in \mathcal{S}} P_{A,0200}^\nu x^2,$$

where

$$G_j^\nu(\bar{\lambda}^\nu) = \sum_{A \in \mathcal{S}} \sum_{2(l-1)h_1 + \cdots + 2h_{l-2} \le 2l-2j} P_{A,00jh}^\nu (\bar{\lambda}^\nu)^h,$$
$$Q_j^\nu(\bar{\lambda}^\nu) = \sum_{A \in \mathcal{S}} \sum_{2(l-1)h_1 + \cdots + 2h_{l-2} \le l-2j} P_{A,01jh}^\nu (\bar{\lambda}^\nu)^h.$$

Thus, $M_\nu^* = M_\nu +_{<2l} \{[\widetilde{P}_\nu]\}$.

In what follows, we give how to apply singular theory to renormalize the normal form M_ν^*. Used transformation φ_1^ν, φ_2^ν defined in

$$\varphi_1^\nu : \begin{cases} y_1 = y, \\ x_1 = x + \frac{\sum_{j=1}^{[\frac{l}{2}]} Q_j^\nu(\bar{\lambda}^\nu)}{a_\nu + 2\sum_{A\in\mathcal{S}} P_{A,0200}^\nu} y^{j-1}, \end{cases} \quad \text{and} \quad \varphi_2^\nu : \begin{cases} y_2 = y_1 + \frac{(l-1)!}{b_{\nu+1}} G_{l-1}^\nu(\bar{\lambda}^\nu), \\ x_2 = x_1, \end{cases}$$

M_ν^* can be renormalized into $M_{\nu+1}$. In concretely, readers can see the proof in [5].

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