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MULTIPLE PERIODIC SOLUTIONS FOR SUPERQUADRATIC AND SUBQUADRATIC SECOND-ORDER HAMILTONIAN SYSTEMS

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Abstract In this paper, a class of second-order Hamiltonian systems is studied. Under the assumptions of superquadratic and subquadratic for the nonlinear term, the existence of six periodic solutions and nine periodic solutions is obtained by using the variational method and space decomposition. Finally, two examples are given to verify the feasibility of the new criteria.

Keywords Hamiltonian systems, variational method, critical point theorem, space decomposition.

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1. Introduction and main results

Since Rabinowitz published his pioneer paper [15] in 1978, more and more mathematicians have paid more attention to the periodic solutions for the first-order or second-order Hamiltonian systems. There has been a lot of literature on the study of periodic solutions for Hamiltonian systems, such as [5, 7, 9, 10, 16, 18–23] and the references therein. In [15], Rabinowitz considered the following second-order Hamiltonian systems

$$\ddot{u} + V'(u) = 0, \quad u \in R^N. \tag{1.1}$$

He studied the existence of periodic solutions of system (1.1) under the superquadratic condition, i.e., (AR): there exist $\mu > 2$ and $L > 0$ such that $0 < \mu F(t, u) \leq (\nabla F(t, u), u)$ for $t \in [0, T]$ and $|u| \geq L$. (AR) plays an important role in showing that Palais-Smale sequence is bounded. Such condition was first introduced by Ambrosetti and Rabinowitz [1]. So it is useful in solving superlinear problems such as elliptic equations, dirac equations and wave equations. Tang and Wu [17] studied the existence of periodic solutions of system (1.1) with subquadratic and convex potentials, which unified and generalized the results in [14, 16, 18, 26]. In [11], Long proved the existence of period solution for system (1.1) without any convexity assumptions, which is one of the some papers [4, 6, 11–13, 24] on the assumptions of nonconvexity. Inspired by [11], Li [8] obtained the existence of two minimal periodic solutions of system (1.1) by using a generalized version of the Weierstrass theorem and a new space decomposition in 2021. To our best knowledge, this is the first result of the existence of multiple minimal periodic solutions for Hamiltonian systems with subquadratic potentials. However, under the assumptions of superquadratic

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potentials and subquadratic potentials for system (1.1), the existence of periodic solutions with more properties has not been obtained.

Motivated by the above mentioned work, we study the following second-order Hamiltonian systems with a parameter

$$\ddot{u}+\lambda V'(u)=0, \tag{1.2}$$

where λ is a parameter, $V\in C^1(R^N,R)$, $V(0)=0$ and $V(u)=\int_0^u V'(s)ds$.

Next, we assume the following conditions, in which condition (V_2) is the superquadratic assumption for the nonlinear term and condition (V_3) is the subquadratic assumption for the nonlinear term.

(V_1) $V(-u)=V(u)$ for any $u\in R^N$;

(V_2) $0<\mu V(u)\le V'(u)u$ for $u\ge M$, where μ and M are two positive constants and $\mu>2$;

(V_3) there is a constant $1<\beta<2$ such that $V(u)\le A|u|^\beta+p(t)$, where $p(t)\in L^1[0,T]$.

The new insights presented in the paper are as follows. Firstly, system (1.2) is a generalization of system (1.1). If $\lambda=1$, system (1.2) reduces to system (1.1). Secondly, superquadratic and subquadratic assumptions are imposed on nonlinear term, respectively. In the two cases, the existence of six periodic solutions and nine periodic solutions is obtained. Finally, comparing with [8], we also consider the existence of two odd $T/2$-antiperiodic nonconstant solutions with period T.

Our main results are as follows.

Theorem 1.1. *Assume that conditions (V_1), (V_2) hold and there exists a positive constant r_1 such that (V_4) $\int_0^T V(\sin\frac{2\pi}{T}t)dt>\frac{\pi^2}{r_1}\max_{|u|<c_1}V(u)$, where $c_1=\sqrt{\frac{(24+\pi^2)Tr_1}{24\pi^2}}$. Then for each $\lambda\in\left(\frac{\pi^2}{T\int_0^T V(\sin\frac{2\pi}{T}t)dt},\frac{r_1}{T\max_{|u|<c_1}V(u)}\right)$, system (1.2) has at least two odd $T/2$-antiperiodic nonconstant solutions with period T.*

Corollary 1.1. *Assume that conditions (V_1), (V_2) hold and there exists a positive constant r_2 such that (V_5) $\int_0^T V(\sin\frac{4\pi}{T}t)dt\ge\frac{4\pi^2}{r_2}\max_{|u|<c_2}V(u)$, where $c_2=\sqrt{\frac{Tr_2}{24}}$. Then for each $\lambda\in\left(\frac{4\pi^2}{T\int_0^T V(\sin\frac{4\pi}{T}t)dt},\frac{r_2}{T\max_{|u|<c_2}V(u)}\right)$, system (1.2) has at least two odd nonconstant periodic solutions with period $T/2$.*

Corollary 1.2. *Assume that conditions (V_1), (V_2) hold and there exists a positive constant r_3 such that (V_6) $\int_0^T V(\cos\frac{2\pi}{T}t)dt\ge\frac{\pi^2}{r_3}\max_{|u|<c_3}V(u)$, where $c_3=\sqrt{\frac{(24+\pi^2)Tr_3}{24\pi^2}}$. Then for each $\lambda\in\left(\frac{\pi^2}{T\int_0^T V(\cos\frac{2\pi}{T}t)dt},\frac{r_3}{T\max_{|u|<c_3}V(u)}\right)$, system (1.2) has at least two even $T/2$-antiperiodic nonzero solutions with period T.*

Remark 1.1. Assume that conditions (V_1), (V_2), (V_4), (V_5) and (V_6) hold. Then for each $\lambda\in\Big(\max\{\frac{\pi^2}{T\int_0^T V(\sin\frac{2\pi}{T}t)dt},\frac{4\pi^2}{T\int_0^T V(\sin\frac{4\pi}{T}t)dt},\frac{\pi^2}{T\int_0^T V(\cos\frac{2\pi}{T}t)dt}\}$, $\min\{\frac{r_1}{T\max_{|u|<c_1}V(u)},\frac{r_2}{T\max_{|u|<c_2}V(u)},\frac{r_3}{T\max_{|u|<c_3}V(u)}\}\Big)$, system (1.2) has at least six periodic solutions.

Theorem 1.2. *Assume that there is a positive constant r_4 and a function $v\in X$ with $\Phi(v)>2k_1$, where $k_1=\sqrt{\frac{(24+\pi^2)Tr_4}{24\pi^2}}$. Suppose that conditions (V_1), (V_3) and (V_7) $\int_0^T V(\sin\frac{2\pi}{T}t)dt>\frac{3\pi^2}{2r_4}\max_{|u|<k_1}V(u)$ hold. Then, for each $\lambda\in$*

$\left(\frac{3\pi^2}{2T\int_0^T V(\sin\frac{2\pi}{T}t)dt}, \frac{r_4}{T\max_{|u|<k_1} V(u)}\right)$, *the system (1.2) has at least three odd $T/2$-antiperiodic solutions with period T.*

Corollary 1.3. *Assume that there is a positive constant r_5 and a function $v \in X$ with $\Phi(v) > 2k_2$, where $k_2 = \sqrt{\frac{Tr_5}{24}}$. In addition, suppose that the conditions (V_1), (V_3) and (V_8) $\int_0^T V(\sin\frac{4\pi}{T}t)dt > \frac{6\pi^2}{r_5}\max_{|u|<k_2} V(u)$ hold. Then, for each $\lambda \in \left(\frac{6\pi^2}{T\int_0^T V(\sin\frac{4\pi}{T}t)dt}, \frac{r_5}{T\max_{|u|<k_2} V(u)}\right)$, the system (1.2) has at least three odd periodic solutions with period $T/2$.*

Corollary 1.4. *Assume that there is a positive constant r_6 and a function $v \in X$ with $\Phi(v) > 2k_3$, where $k_3 = \sqrt{\frac{(24+\pi^2)Tr_6}{24\pi^2}}$. In addition, suppose that the conditions (V_1), (V_3) and (V_9) $\int_0^T V(\cos\frac{2\pi}{T}t)dt > \frac{3\pi^2}{2r_6}\max_{|u|<k_3} V(u)$ hold. Then, for each $\lambda \in \left(\frac{3\pi^2}{2T\int_0^T V(\cos\frac{2\pi}{T}t)dt}, \frac{r_6}{T\max_{|u|<k_3} V(u)}\right)$, the system (1.2) has at least three even $T/2$-antiperiodic solutions with period T.*

Remark 1.2. Assume that conditions (V_1), (V_3), (V_7), (V_8) and (V_9) hold. Then for each $\lambda \in \Big(\max\{\frac{3\pi^2}{2T\int_0^T V(\sin\frac{2\pi}{T}t)dt}, \frac{6\pi^2}{T\int_0^T V(\sin\frac{4\pi}{T}t)dt}, \frac{3\pi^2}{2T\int_0^T V(\cos\frac{2\pi}{T}t)dt}\}$, $\min\{\frac{r_4}{T\max_{|u|<k_1} V(u)}, \frac{r_5}{T\max_{|u|<k_2} V(u)}, \frac{r_6}{T\max_{|u|<k_3} V(u)}\}\Big)$, system (1.2) has at least nine periodic solutions.

2. Preliminaries

In this section, we recall some essential definitions and several lemmas.

Let us consider the space $X = H_T^1 = W^{1,2}(S_T, R^N)$ with the norm $\|u\| = \left(\int_0^T |u|^2 + |\dot{u}|^2 dt\right)^{\frac{1}{2}}$, where $S_T = R/(TZ)$, $T > 0$, Z is the integer. It is well known that X is a reflexive Banach space. We can split X into $X = X_T \bigoplus Y_T$, where $X_T = \{u \in H_T^1 \big| u(-t) = -u(t)\}$ and $Y_T = \{u \in H_T^1 \big| u(-t) = u(t)\}$. X_T and Y_T are closed subspaces of X and Y, then they are reflexive Banach spaces. Moreover, we define

$$\begin{aligned} X_T^1 &= \{u \in X_T \big| u(t) = -u(t - T/2)\} \quad \text{and} \quad X_T^2 = \{u \in X_T \big| u(t) = u(t - T/2)\}, \\ Y_T^1 &= \{u \in Y_T \big| u(t) = -u(t - T/2)\} \quad \text{and} \quad Y_T^2 = \{u \in Y_T \big| u(t) = u(t - T/2)\}, \end{aligned}$$

where $X_T = X_T^1 \bigoplus X_T^2$ and $Y_T = Y_T^1 \bigoplus Y_T^2$. Obviously, for $x_1 \in X_T^1, x_2 \in X_T^2, y_1 \in Y_T^1$ and $y_2 \in Y_T^2$, we have the following Fourier expansions

$$\begin{aligned} x_1 &= \sum_{k=0}^{+\infty} b_{2k+1}\sin((2k+1)\omega t) \quad \text{and} \quad x_2 = \sum_{k=1}^{+\infty} b_{2k}\sin(2k\omega t), \\ y_1 &= \sum_{k=0}^{+\infty} a_{2k+1}\cos((2k+1)\omega t) \quad \text{and} \quad y_2 = \sum_{k=0}^{+\infty} a_{2k}\cos(2k\omega t), \end{aligned}$$

where $\omega = \frac{2\pi}{T}$. In these spaces X_T^1, X_T^2 and Y_T^1, we define the norms as follows $\|u\|_{X_T^1} = \|u\|_{X_T^2} = \|u\|_{Y_T^1} = \left(\int_0^T |\dot{u}|^2 dt\right)^{\frac{1}{2}}$, and these norms are equivalent to the normal norm $\|u\|$. In addition, take $\|u\|_{Y_T^2} = \left(\int_0^T |u|^2 + |\dot{u}|^2 dt\right)^{\frac{1}{2}} = \|u\|$. We define energy functional $I_\lambda : X \to R^N$ by

$$I_\lambda(u) = \frac{1}{2}\int_0^T |\dot{u}|^2 dt - \lambda \int_0^T V(u(t))dt. \tag{2.1}$$

And $I_\lambda(u)$ can also be represented as $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$, where the functionals $\Phi(u), \Psi(u) : X \to R^N$ are defined as follows

$$\Phi(u) = \frac{1}{2}\int_0^T |\dot{u}|^2 dt \tag{2.2}$$

and

$$\Psi(u) = \int_0^T V(u(t))dt. \tag{2.3}$$

Obviously, $I_\lambda(u)$ is a *Gâteaux* differentiable functional and its *Gâteaux* derivation is continuous in u. So its *Fréchet* derivative at the point u is

$$\langle I_\lambda'(u), v\rangle = \int_0^T (\dot{u}(t), \dot{v}(t))dt - \lambda \int_0^T V'(u(t))v(t)dt. \tag{2.4}$$

Definition 2.1. A function $u \in X$ is said to be a weak solution of system (1.2) , if u satisfies $\langle I_\lambda'(u), v\rangle = 0$ for all $v \in X$.

Definition 2.2. A funtion u is said to be a classical solution of system (1.2), if $u \in C^2(R, R)$ satisfies equations in system (1.2).

The following two lemmas are the latest Two-Critical-Point-Theorem [2] and Three-Critical-Point-Theorem [3], which are used to prove Theorem 1.1 and Theorem 1.2.

Lemma 2.1 ([2]). *Let X be a reflexive real Banach space and let $\Phi, \Psi : X \to R$ be two functionals of class C^1 such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r \in R$ and $\tilde{u} \in X$ with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{2.5}$$

and for each $\lambda \in \wedge = \left(\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}\right)$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the Palais-Smale Condition ((PS)-condition) and it is unbounded from below. Then, for each $\lambda \in \wedge$, the functional $I_\lambda(x)$ admits at least two nonzero critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.

Lemma 2.2 ([3]). *Let X be a reflexive real Banach space and let $\Phi : X \to R$ be a coercive and continuous Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*. In addition, let $\Psi : X \to R$ be a continuous Gâteaux differentiable functional whose derivative is compact with $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exists a positive constant r and an element $v \in X$ with $2r < \Phi(v)$, such that*

(a_1) $\frac{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}{r} < \frac{2\Psi(v)}{3\Phi(v)}$;

(a_2) *for all* $\lambda \in \left(\frac{3\Phi(v)}{2\Psi(v)}, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}\right)$, *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \left(\frac{3\Phi(v)}{2\Psi(v)}, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}\right)$, *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points.*

For the next two lemmas, we can refer to [8, Lemma 2.2] and [4, Lemma 1.2].

Lemma 2.3 ([4, 8]). *Suppose that condition* (V_1) *holds. Then we have* $I_\lambda \in C^1(X_T, R)$, *and* $u \in X_T$ *is a critical point of* I_λ *restricted to* X_T *if and only if it is a* C^2*-solution of system (1.2) (The result still holds if we replace* X_T *with* Y_T*.)*

Lemma 2.4 ([4, 8]). *Suppose that* (V_1) *holds. Then one has*
(i) $x^* \in X_T^1(X_T^2)$ *is a critical point of* I_λ *restricted to* $X_T^1(X_T^2)$ *if and only if it is a critical point of* I_λ *in* X_T, *that is,* x^* *is an odd* C^2*-solution of system (1.2).*
(ii) $y^* \in Y_T^1(Y_T^2)$ *is a critical point of* I_λ *restricted to* $Y_T^1(Y_T^2)$ *if and only if it is a critical point of* I_λ *in* Y_T, *that is,* y^* *is an odd* C^2*-solution of system (1.2).*

Next, in order to obtain the main conclusions, it is necessary to prove the following lemmas.

Lemma 2.5. *If* $u \in X_T^1$ *(or* $u \in Y_T^1$*), then* $\int_0^T |u(t)|^2dt \le \frac{T^2}{4\pi^2}\int_0^T |\dot u(t)|^2dt$ *and* $\|u\|_\infty^2 \le \frac{T}{2\pi^2}(1+\frac{\pi^2}{24})\int_0^T |\dot u(t)|^2dt$. *In addition, if* $u \in \tilde{X_T^2} = \{u \in X_T^2 : \int_0^T u(t)dt = 0\}$, *we have* $\int_0^T |u(t)|^2dt \le \frac{T^2}{16\pi^2}\int_0^T |\dot u(t)|^2dt$ *and* $\|u\|_\infty^2 \le \frac{T}{48}\int_0^T |\dot u(t)|^2dt$.

Proof. If $u \in X_T^1$, we have

$$u(t) = \sum_{k=0}^{+\infty} b_{2k+1}\sin((2k+1)\omega t). \tag{2.6}$$

The Parseval equality implies that

$$\int_0^T |u(t)|^2dt = \frac{T}{2}\sum_{k=0}^{+\infty}|b_{2k+1}|^2. \tag{2.7}$$

Since

$$\dot u(t) = \sum_{k=0}^{+\infty}(2k+1)\omega\cdot b_{2k+1}\cos((2k+1)\omega t) = \sum_{k=0}^{+\infty}\frac{2(2k+1)\pi}{T}b_{2k+1}\cos((2k+1)\omega t), \tag{2.8}$$

by (2.6)-(2.8), we have

$$\int_0^T |\dot u(t)|^2dt = \sum_{k=0}^{+\infty}\frac{2(2k+1)^2\pi^2}{T}|b_{2k+1}|^2 \ge \frac{4\pi^2}{T^2}\sum_{k=0}^{+\infty}\frac{T}{2}|b_{2k+1}|^2 = \frac{4\pi^2}{T^2}\int_0^T |u(t)|^2dt. \tag{2.9}$$

By Cauchy-Schwarz inequality, we obtain

$$|u(t)|^2 \le \left(\sum_{k=0}^{+\infty}|b_{2k+1}|\right)^2 = \left(\sum_{k=0}^{+\infty}\frac{T}{2(2k+1)^2\pi^2}\right)\left(\sum_{k=0}^{+\infty}\frac{2(2k+1)^2\pi^2}{T}|b_{2k+1}|^2\right), \tag{2.10}$$

where

$$\sum_{k=0}^{+\infty}\frac{T}{2(2k+1)^2\pi^2}=\frac{T}{2\pi^2}\sum_{k=0}^{+\infty}\frac{1}{(2k+1)^2}=\frac{T}{2\pi^2}\left(1+\sum_{k=1}^{+\infty}\frac{1}{(2k+1)^2}\right)$$
$$\leq\frac{T}{2\pi^2}\left(1+\sum_{k=1}^{+\infty}\frac{1}{4k^2}\right)=\frac{T}{2\pi^2}\left(1+\frac{\pi^2}{4\times 6}\right)=\frac{T}{2\pi^2}\left(1+\frac{\pi^2}{24}\right). \tag{2.11}$$

Put (2.9) and (2.11) into (2.10), we get

$$|u(t)|^2\leq\frac{T}{2\pi^2}\left(1+\frac{\pi^2}{24}\right)\int_0^T|\dot{u}(t)|^2dt.$$

If $u\in X_T^2$, and $\int_0^T u(t)dt=0$

$$u(t)=\sum_{k=1}^{+\infty}b_{2k}\sin(2k\omega t). \tag{2.12}$$

By Parseval equality, we have

$$\int_0^T|u(t)|^2dt=\frac{T}{2}\sum_{k=1}^{+\infty}|b_{2k}|^2. \tag{2.13}$$

Since

$$\dot{u}(t)=\sum_{k=1}^{+\infty}2k\omega b_{2k}\cos(2k\omega t)=\sum_{k=1}^{+\infty}\frac{4k\pi}{T}b_{2k}\cos(2k\omega t), \tag{2.14}$$

we have

$$\int_0^T|\dot{u}(t)|^2dt=\sum_{k=1}^{+\infty}\frac{8k^2\pi^2}{T}|b_{2k}|^2\geq\frac{16\pi^2}{T^2}\sum_{k=1}^{+\infty}\frac{T}{2}|b_{2k}|^2=\frac{16\pi^2}{T^2}\int_0^T|u(t)|^2dt. \tag{2.15}$$

According to Cauchy-Schwarz inequality and combining (2.12) with (2.15), we obtain

$$|u(t)|^2\leq\left(\sum_{k=1}^{+\infty}|b_{2k}|\right)^2\leq\left(\sum_{k=1}^{+\infty}\frac{T}{8k^2\pi^2}\right)\left(\sum_{k=1}^{+\infty}\frac{8k^2\pi^2}{T}|b_{2k}|^2\right)$$
$$=\left(\frac{T}{8\pi^2}\sum_{k=1}^{+\infty}\frac{1}{k^2}\right)\left(\sum_{k=1}^{+\infty}\frac{8k^2\pi^2}{T}|b_{2k}|^2\right)\leq\frac{T}{48}\int_0^T|\dot{u}(t)|^2dt.$$

If $u\in Y_T^1$, we get

$$u(t)=\sum_{k=0}^{+\infty}a_{2k+1}\cos((2k+1)\omega t). \tag{2.16}$$

According to Parseval equality, it is obvious that

$$\int_0^T|u(t)|^2dt=\frac{T}{2}\sum_{k=0}^{+\infty}|a_{2k+1}|^2. \tag{2.17}$$

According to

$$\begin{aligned}\dot{u}(t) &= -\sum_{k=0}^{+\infty}(2k+1)\omega \cdot a_{2k+1}\sin((2k+1)\omega t)\\ &= -\sum_{k=0}^{+\infty}\frac{2(2k+1)\pi}{T}a_{2k+1}\sin((2k+1)\omega t),\end{aligned} \tag{2.18}$$

we have

$$\int_0^T |\dot{u}(t)|^2 dt = \sum_{k=0}^{+\infty}\frac{2(2k+1)^2\pi^2}{T}|a_{2k+1}|^2 \geq \frac{4\pi^2}{T^2}\sum_{k=0}^{+\infty}\frac{T}{2}|a_{2k+1}|^2 = \frac{4\pi^2}{T^2}\int_0^T |u(t)|^2 dt. \tag{2.19}$$

By Cauchy-Schwarz inequality, (2.11) and (2.19), we get

$$\begin{aligned}|u(t)|^2 \leq \left(\sum_{k=0}^{+\infty}|a_{2k+1}|\right)^2 &= \left(\sum_{k=0}^{+\infty}\frac{T}{2(2k+1)^2\pi^2}\right)\left(\sum_{k=0}^{+\infty}\frac{2(2k+1)^2\pi^2}{T}|a_{2k+1}|^2\right)\\ &\leq \frac{T}{2\pi^2}\left(1+\frac{\pi^2}{24}\right)\int_0^T |\dot{u}(t)|^2 dt.\end{aligned} \tag{2.20}$$

□

Lemma 2.6. *The spaces $X_T^1, \tilde{X_T^2}$ and Y_T^1 are compactly embedded to $C[0,T]$, i.e., $X_T^1 \hookrightarrow\hookrightarrow C[0,T]$, $\tilde{X_T^2} \hookrightarrow\hookrightarrow C[0,T]$and $Y_T^1 \hookrightarrow\hookrightarrow C[0,T]$.*

Proof. In order to prove that space X_T^1 is compactly embedded to $C[0,T]$, it is sufficient to prove that space X_T^1 is continuously embedded to space X since $X \hookrightarrow\hookrightarrow C[0,T]$. From Lemma 2.5, one has

$$\|u\|^2 = \|\dot{u}\|_{L^2}^2 + \|u\|_{L^2}^2 \leq \|\dot{u}\|_{L^2}^2 + \frac{T^2}{4\pi^2}\|\dot{u}\|_{L^2}^2 = \frac{T^2+4\pi^2}{4\pi^2}\|\dot{u}\|_{L^2}^2 = \frac{T^2+4\pi^2}{4\pi^2}\|u\|_{X_T^1}^2$$

for $u \in X_T^1$, which means space X_T^1 is continuously embedded to space X. Therefore space X_T^1 is compactly embedded to $C[0,T]$. In the same way , we have $\tilde{X_T^2} \hookrightarrow\hookrightarrow C[0,T]$ and $Y_T^1 \hookrightarrow\hookrightarrow C[0,T]$. □

Lemma 2.7. *The spaces $X_T^1, \tilde{X_T^2}$ and Y_T^1 are reflexive real Banach spaces.*

Proof. It is enough to show that X_T^1 is a closed subspace of X_T. Let $\{u_n\} \subset X_T^1$ and $u_n \to u_0$ as $n \to \infty$. Next we show $u_0 \in X_T^1$. Since $\{u_n\} \subset X_T^1$ and $u_n \to u_0$ as $n \to \infty$, then $u_n(t) = -u_n(t - \frac{T}{2})$ and $\|u_n - u_0\|_{X_T^1} \to 0$ as $n \to \infty$. By Lemma 2.5 , $\|u_n - u_0\|_\infty \to 0$ as $n \to \infty$, which means $u_n(t) \to u_0(t)$ as $n \to \infty$, $t \in [0,T]$. So $u_0(t) = -u_0(t - \frac{T}{2})$. That is , $u_0 \in X_T^1$. Similarly, the spaces $\tilde{X_T^2}$ and Y_T^1 are reflexive real Banach spaces. □

Lemma 2.8. *If (V_2) holds and $\lambda > 0$, then the functional I_λ is unbound from below on X_T^1(or $\tilde{X_T^2}$,Y_T^1), and it satisfies the (PS)-condition on X_T^1(or $\tilde{X_T^2}$,Y_T^1).*

Proof. Firstly, we discuss whether I_λ is unbound from below. By (V_2), one knows there exist two constants $\alpha, \beta > 0$ such that $V(u) \geq \alpha|u|^\mu - \beta$, where $\mu > 2$. For some $u_0 \in X_T^1/\{0\}, l \in R$, we obtain

$$I_\lambda(lu_0) = \frac{1}{2}\int_0^T |l\dot{u}_0|^2 dt - \lambda \int_0^T V(lu_0)dt.$$
$$\leq \frac{l^2}{2}\|\dot{u}_0\|_{L^2}^2 - \lambda l^\mu \int_0^T \alpha |u_0|^\mu dt + \lambda\beta T \to -\infty.$$

Thus, the energy functional I_λ is unbound from below.

Secondly, we prove that I_λ satisfies the (PS)-condition. Let $\{u_n\} \subset X_T^1$ be a sequence such that $|I_\lambda(u_n)| < M$ and $\langle I'_\lambda(u_n), u_n\rangle \to 0$ as $n \to \infty$. For n large enough, by (V_2), we evaluate

$$\begin{aligned} M + \frac{1}{\mu}\|u_n\|_{X_T^1} &\geq I_\lambda(u_n) - \frac{1}{\mu}\langle I'_\lambda(u_n), u_n\rangle \\ =&(\frac{1}{2} - \frac{1}{\mu})\|u_n\|_{X_T^1}^2 - \lambda(\int_0^T V(u_n)dt - \frac{1}{\mu}\int_0^T V'(u_n)u_n dt) \\ \geq&(\frac{1}{2} - \frac{1}{\mu})\|u_n\|_{X_T^1}^2, \end{aligned}$$

where $\mu > 2$. Thus $\{u_n\}$ is bounded in X_T^1. Since X_T^1 is a reflexive Banach space, the fact that $\{u_n\}$ is bounded in X_T^1 means that one has weakly convergent subsequence $\{u_{n_m}\}$ such that $u_{n_m} \rightharpoonup u$ in X_T^1. Moreover, one has

$$\begin{aligned} &\langle I'_\lambda(u_{n_m}) - I'_\lambda(u), u_{n_m} - u\rangle \to 0 \\ =&\|u_{n_m} - u\|_{X_T^1}^2 - \lambda\int_0^T (V'(u_{n_m}) - V'(u))(u_{n_m}(t) - u(t))dt. \end{aligned}$$

By Lemma 2.6, one has $(X_T^1, \|\cdot\|) \hookrightarrow\hookrightarrow C([0,T])$, which means

$$\int_0^T (V'(u_{n_m}) - V'(u))(u_{n_m}(t) - u(t))dt \to 0$$

as $m \to \infty$ and $\|u_{n_m} - u\|_{X_T^1}^2 \to 0$ as $m \to +\infty$. Therefore I_λ satisfies (PS)-condition. Using the same proof method, if $u \in \tilde{X_T^2}$ or $u \in Y_T^1$, then one knows that the functional I_λ is unbound from below and satisfies the (PS)-condition. □

Lemma 2.9. *Φ is coercive on X_T^1(or $\tilde{X_T^2}$,Y_T^1) and Φ' has a continuous inverse on $(X_T^1)^*$(or$(\tilde{X_T^2})^*$,$(Y_T^1)^*$).*

Proof. By (2.2), it is obvious that Φ is coercive. Moreover, from [[25], Theorem 26], Φ' will have a continuous inverse on $(X_T^1)^*$ if Φ' is coercive and continuous monotone. Firstly, we know $\langle \Phi'(u), u\rangle = \int_0^T |\dot{u}(t)|^2 dt = \|u\|_{X_T^1}^2$, which yields that Φ' is coercive. Secondly, in consideration of

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle = \|u(t) - v(t)\|_{X_T^1}^2,$$

we get Φ' is continuous monotone. Hence, Lemma 2.9 holds. □

Lemma 2.10. $\Psi' : X_T^1 \to (X_T^1)^*$ *is compact with* $\inf_{u\in X_T^1} \Phi(u) = \Phi(0) = \Psi(0) = 0.$

Proof. Firstly, since $V(0)=0$, it is clear that $\inf_{u\in X_T^1}\Phi=\Phi(0)=\Psi(0)=0$. Secondly, let $\{u_n\}\subset X_T^1$ be a sequence such that $u_n\rightharpoonup u$ as $n\to\infty$. Since $X_T^1\hookrightarrow\hookrightarrow C[0,T]$, we know $\{u_n\}$ is uniformly convergent to u in $C[0,T]$ as $n\to\infty$. According to the fact that $V(u)\in C^1(R^N,R)$, we get $\lim_{n\to\infty}V'(u_n)=V'(u)$, which obtains

$$\lim_{n\to\infty}\sup_{v\in X_T^1}\frac{<\Psi'(u_n)-\Psi'(u),v>}{\|v\|_{X_T^1}}=\lim_{n\to\infty}\sup_{v\in X_T^1}\frac{\int_0^T(V'(u_n)-V'(u),v)dt}{\|v\|_{X_T^1}}=0.$$

Therefore Ψ' is strongly continuous in X_T^1. By [[25], Proposition 26.2], Ψ' is compact. Similarly Ψ' is compact in spaces $\tilde{X}_T^2$ and Y_T^1. In addition, we have $\inf_{u\in\tilde{X}_T^2}\Phi(u)=\Phi(0)=\Psi(0)=0$ and $\inf_{u\in Y_T^1}\Phi(u)=\Phi(0)=\Psi(0)=0$. □

3. Proof of main results

Proof of Theorem 1.1. By (2.2) and Lemma 2.5, we deduce that

$$\begin{aligned}\Phi^{-1}(-\infty,r_1)&=\{u\in X_T^1|\Phi(u)<r_1\}=\{u\in X_T^1|\|\dot{u}\|_{L^2}^2<2r_1\}\\&\subseteq\{u\in X_T^1|\|u\|_\infty^2<\frac{(24+\pi^2)Tr_1}{24\pi^2}\}=\{u\in X_T^1|\|u\|_\infty<c_1\}.\end{aligned}$$

Therefore

$$\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)<\sup_{\|u\|_\infty<c_1}\int_0^T V(u(t))dt<T\max_{|u|<c_1}V(u).$$

Take $\tilde{u}=\sin(\frac{2\pi}{T}t)$, we have

$$\Phi(\tilde{u})=\frac{1}{2}\int_0^T|\dot{\tilde{u}}|^2dt=\frac{2\pi^2}{T^2}\int_0^T(\cos\frac{2\pi}{T}t)^2dt=\frac{\pi^2}{T}.$$

From (V_4), we get

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}=\frac{T\int_0^T V(\sin\frac{2\pi}{T}t)dt}{\pi^2}>\frac{T\max_{|u|<c_1}V(u)}{r_1}>\frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{r_1}.$$

Hence, inequality (2.5) of Lemma 2.1 is verified. Combining Lemma 2.1, Lemma 2.8 and Lemma 2.10, for each $\lambda_1\in\left(\frac{\pi^2}{T\int_0^T V(\sin\frac{2\pi}{T}t)dt},\frac{r_1}{T\max_{|u|<c_1}V(u)}\right)$, we obtain that system (1.2) has two nonzero critical points $u_{\lambda,1}$, $u_{\lambda,2}$.

In addition, it is obvious that $X_T^1\cap R^N=\{0\}$ by the definition of X_T^1. So $u_{\lambda,1}$ and $u_{\lambda,2}$ are not constants. By Lemma 2.4 , we conclude that there are at least two odd $T/2$-antiperiodic nonconstant solutions with period T of system (1.2). □

Proof of Theorem 1.2. By (2.2) and Lemma 2.5, one has

$$\begin{aligned}\Phi^{-1}(-\infty,r_4)&=\{u\in X_T^1|\Phi(u)<r_4\}=\{u\in X_T^1|\|\dot{u}\|_{L^2}^2<2r_4\}\\&\subseteq\{u\in X_T^1|\|u\|_\infty^2<\frac{(24+\pi^2)Tr_4}{24\pi^2}\}=\{u\in X_T^1|\|u\|_\infty<k_1\},\end{aligned}$$

which deduces that

$$\sup_{u\in\Phi^{-1}(-\infty,r_4)}\Psi(u) < T\max_{|u|<k_1} V(u).$$

Let $v=\sin(\frac{2\pi}{T}t)$. One has

$$\Phi(v)=\frac{1}{2}\int_0^T |\dot{v}|^2dt=\frac{2\pi^2}{T^2}\int_0^T(\cos\frac{2\pi}{T}t)^2dt=\frac{\pi^2}{T}.$$

From (V_7), it follows

$$\frac{2\Psi(v)}{3\Phi(v)}=\frac{2T\int_0^T V(\sin\frac{2\pi}{T}t)dt}{3\pi^2}>\frac{T\max_{|u|<k_1}V(u)}{r_4}>\frac{\sup_{u\in\Phi^{-1}(-\infty,r_4)}\Psi(u)}{r_4}.$$

Then, combining (2.1) and (V_3), we have

$$\begin{aligned}I_\lambda(u)=\Phi(u)-\lambda\Psi(u)&=\frac{1}{2}\int_0^T|\dot{u}|^2dt-\lambda\int_0^T V(u)dt\\ &\geq\frac{1}{2}\int_0^T|\dot{u}|^2dt-\lambda\int_0^T A|u|^\beta+p(t)dt,\end{aligned}$$

which means $I_\lambda(u)\to+\infty$ as $\|u\|_X\to+\infty$. Thus, $\Phi-\lambda\Psi$ is coercive. In addition, by Lemma 2.2, Lemma 2.4, Lemma 2.9 and Lemma 2.10, the system (1.2) has at least three odd $T/2$-antiperiodic solutions with period T. □

Remark 3.1. Similar to the proof of Theorem 1.1 and Theorem 1.2, we get Corollary 1.1, Corollary 1.2, Corollary 1.3 and Corollary 1.4.

4. Main examples

Example 4.1. Consider the second order Hamiltonian systems (1.2), where $V(u)=u^{100}$. Let $c_1=c_2=c_3=\frac{1}{100}$, $T=2\pi$ and $r_1=r_3=\frac{12\pi c_1^2}{24+\pi^2}, r_2=\frac{12c_2^2}{\pi}$. Therefore, We have $\frac{\pi^2}{r}\max_{|u|<c_1}V(u)=\frac{24\pi+\pi^3}{12c_1^2}\max_{|u|<c_1}V(u)<\frac{24\pi+\pi^3}{12}(\frac{1}{100})^{98}$. If $v=\sin\frac{2\pi}{T}t\in X_T^1$, then $\int_0^{2\pi}(\sin t)^{100}dt>\frac{24\pi+\pi^3}{12}(\frac{1}{100})^{98}$. Therefore for $v\in X_T^1$, the condition (V_4) of Theorem 1.1 is satisfied. According to Theorem 1.1, for each $\lambda_1\in\left(\frac{\pi}{2\int_0^{2\pi}(\sin t)^{100}dt},\frac{6(100)^{98}}{24+\pi^2}\right)$, system (1.2) has at least two odd $T/2$-antiperiodic nonconstant solutions with period T. In addition, we take $v=\sin 2t$ in $\tilde{X_T^2}$, then $\int_0^{2\pi}(\sin 2t)^{100}dt>\frac{\pi^3}{12}(\frac{1}{100})^{98}$. Therefore for $v\in\tilde{X_T^2}$, the condition (V_5) of Theorem 1.1 is satisfied. According to Theorem 1.1, for each $\lambda_2\in\left(\frac{\pi}{2\int_0^{2\pi}(\sin 2t)^{100}dt},\frac{6(100)^{98}}{\pi^2}\right)$, system (1.2) has at least two odd nonconstant periodic solutions with period $T/2$. If $v=\cos\frac{2\pi}{T}t\in X_T^1$, then $\int_0^{2\pi}(\cos t)^{100}dt>\frac{24\pi+\pi^3}{12}(\frac{1}{100})^{98}$. Therefore for $v\in Y_T^1$, the condition (V_6) of Theorem 1.1 is satisfied. According to Theorem 1.1, for each $\lambda_3\in\left(\frac{\pi}{2\int_0^{2\pi}(\cos t)^{100}dt},\frac{6(100)^{98}}{24+\pi^2}\right)$, system (1.2) has at least two even $T/2$-antiperiodic nonzero solutions with period T. Therefore, for $\lambda\in\left(\frac{\pi}{2\int_0^{2\pi}(\sin t)^{100}dt},\frac{6(100)^{98}}{24+\pi^2}\right)$, system (1.2) has at least six periodic solutions.

Example 4.2. Consider the second order Hamiltonian systems (1.2), where $V(u) = u^4$. Let $k_1 = k_2 = k_3 = \frac{1}{100}$, $T = 2\pi$, $r_4 = r_6 = \frac{12\pi k_1^2}{24+\pi^2}$, $r_5 = \frac{12k_2^2}{\pi}$, $A = 2$, $\beta = \frac{4}{3}$, $p(t) = u^4$. Take $v(t) = \sin t(t \in (\frac{\pi}{4} + k\pi, \frac{3\pi}{4} + k\pi))$ in X_T^1, which means that $\Phi(v) > 2k_1$ and $V(u) \leq A|u|^\beta + p(t)$. If $t \in (\frac{\pi}{4} + k\pi, \frac{3\pi}{4} + k\pi)$, we have $v(t) > \frac{\sqrt{2}}{2}$. Further we have$\int_0^{2\pi} V(\sin t) > \frac{\pi}{2} > \frac{3\pi^2}{2r_4} \max_{|u|<k_1} V(u)$, which means condition (V_7) of Theorem 1.2 is satisfied. Therefore for $\lambda_4 \in \left(\frac{3\pi}{4\int_0^{2\pi}(\sin t)^4 dt}, \frac{6(100)^2}{24+\pi^2}\right)$, the system (1.2) has at least three odd $T/2$-antiperiodic solutions with period T. In addition, we take $v = \sin 2t$ in $\tilde{X_T^2}$, which means that $\Phi(v) > 2k_2$ and $V(u) \leq A|u|^\beta + p(t)$. If $t \in (\frac{\pi}{8} + k\pi, \frac{3\pi}{8} + k\pi)$, we have $v(t) > \frac{\sqrt{2}}{2}$. Further we have$\int_0^{2\pi} V(\sin 2t) > \frac{\pi}{2} > \frac{6\pi^2}{r_5} \max_{|u|<k_2} V(u)$, which means condition (V_8) of Theorem 1.2 is satisfied. Therefore for $\lambda_5 \in \left(\frac{3\pi}{\int_0^{2\pi}(\sin 2t)^4 dt}, \frac{6(100)^2}{\pi^2}\right)$, the system (1.2) has at least three odd periodic solutions with period $T/2$. Take $v = \cos t(t \in (-\frac{\pi}{4}+k\pi, \frac{\pi}{4}+k\pi))$ in Y_T^1, which means that $\Phi(v) > 2k_3$ and $V(u) \leq A|u|^\beta + p(t)$. If $t \in (-\frac{\pi}{4} + k\pi, \frac{\pi}{4} + k\pi)$, we have $v(t) > \frac{\sqrt{2}}{2}$. Further we have$\int_0^{2\pi} V(\sin t) > \frac{\pi}{2} > \frac{3\pi^2}{2r_6} \max_{|u|<k_3} V(u)$, which means condition (V_9) of Theorem 1.2 is satisfied. Therefore for $\lambda_6 \in \left(\frac{3\pi}{4\int_0^{2\pi}(\cos t)^4 dt}, \frac{6(100)^2}{24+\pi^2}\right)$, the system (1.2) has at least three even $T/2$-antiperiodic solutions with period T. Therefore, for $\lambda \in \left(\frac{3\pi}{\int_0^{2\pi}(\sin 2t)^4 dt}, \frac{6(100)^{98}}{24+\pi^2}\right)$, system (1.2) has at least nine periodic solutions.

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