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SOLVABILITY OF A FRACTIONAL BOUNDARY VALUE PROBLEM WITH P-LAPLACIAN OPERATOR ON AN INFINITE INTERVAL*

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Abstract In this paper, we extend the third order p-Laplacian boundary value problem researched by S. Iyase and O. Imaga in [11] to the fractional differential equation. Firstly, we construct a mild Banach space and establish an appropriate compactness criterion. Then applying the Schauder's fixed point theorem, we obtain a sufficient condition for existence of at least one solution to the fractional differential equation with p-Laplacian operator on an infinite interval. As an application, an example is given to illustrate our main result.

Keywords Fractional differential equation, boundary value problem, infinite interval, p-Laplacian operator.

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1. Introduction

In this paper, we consider the following fractional boundary value problem (BVP for short) with p-Laplacian operator on an infinite interval

$$\begin{cases} \mathcal{D}^{\beta}_{0+}\left(p(t)\varphi_p(\mathcal{D}^{\alpha}_{0+}u(t))\right)+f(t,u(t),\mathcal{D}^{\alpha-1}_{0+}u(t),\mathcal{D}^{\alpha}_{0+}u(t))=0, \quad t\in[0,+\infty)\\ u(0)=0, \quad \mathcal{D}^{\alpha-1}_{0+}u(0)=\sum\limits_{i=1}^{m}a_i\int_0^{\xi_i}u(t)dt,\\ \mathcal{D}^{\alpha}_{0+}u(0)=0, \quad \lim\limits_{t\to+\infty}\mathcal{D}^{\beta-1}_{0+}(p(t)\varphi_p(\mathcal{D}^{\alpha}_{0+}u(t)))=0, \end{cases} \tag{1.1}$$

where $\varphi_p(s)=|s|^{p-2}s$, $p>1$, $\varphi_p^{-1}=\varphi_q$ with $\dfrac{1}{p}+\dfrac{1}{q}=1$, $1<\alpha,\beta\le 2$, $a_i\ge 0$ and $0<\xi_1<\xi_2<\cdots<\xi_m<+\infty$, $\mathcal{D}^{\alpha}_{0+}$ is the standard Riemann-Liouville derivative, $f(t,u,v,w):[0,+\infty)\times R^3\to R$ is a continuous function, $p(t)\in C[0,+\infty)$, $p(t)>0$ for $t\in[0,+\infty)$ and $\varphi_q\left(\dfrac{t^{\beta-1}}{p(t)}\right)\in L^1[0,+\infty)$.

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The function f in BVP (1.1) is dependent on the fractional derivative $\mathcal{D}_{0+}^{\alpha-1}u(t)$, $\mathcal{D}_{0+}^{\alpha}u(t)$ and a noncompact infinite intervals, so the main difficulty in this paper is how to construct a mild Banach space and establish an appropriate compactness criterion.

Fractional differential equations have become a powerful tool in describing various phenomena in the fields such as fluid mechanics, physics, biology (see [2,10,16]). A great number of models in natural science can be represented by differential equations on infinite interval, for instance, mass transfer of non classical Newtonian fluid in rotating disk, temperature diffusion of phase change solid in heat conduction, boundary layer transport in hydrodynamics and so on (see [1]). These have prompted many authors to study the fractional boundary value problems widely (see [8,9,13,20,22]). Inspired by the Leibenson's work on p-Laplacian differential equation (see [17]), there are a lot of results concerning the existence of solutions for boundary value problems with p-Laplacian operator on finite or infinite interval in the literature(see [4,6,7,12,14,18,19,23]).

In [11], S. A. Iyase and O. F. Imaga applied the Leray-Schauder continuation principle to obtain the existence of at least one solution to the third order p-Laplacian boundary value problem on an unbounded domain

$$\begin{cases} (w(t)\varphi_p(u''(t)))' = K(t,u(t),u'(t),u''(t)),\ t\in(0,+\infty), \\ u(0)=0,\ u'(0)=\sum\limits_{i=1}^{m} a_i\int_0^{\xi_i}u(t)dt,\ \lim\limits_{t\to+\infty} w(t)\varphi_p(u''(t))=0, \end{cases}$$

where $K:[0,+\infty)\times R^3\to R$ is a Caratheodory function with respect to $L^1[0,+\infty)$, $a_i\in R(1\le i\le m)$, $0<\xi_1<\xi_2<\ldots<\xi_m<1$, $w(t)>0$, $t\in[0,+\infty)$, $w\in C[0,+\infty)\cap C^1[0,+\infty)$, $\dfrac{1}{w(t)}\in L^1[0,+\infty)$ and $\sum\limits_{i=1}^{m}a_i\xi_i\neq 2$.

Motivated by the work of S. A. Iyase and O. F. Imaga, we extend integer order in [11] to fractional order on infinite interval (see BVP (1.1)). Firstly, we construct a mild Banach space and establish an appropriate compactness criterion. Then applying the Schauder's fixed point theorem, we obtain a sufficient condition for existence of at least one solution to the fractional differential equation with p-Laplacian operator on an infinite interval. As an application, an example is given to illustrate our main result.

Throughout this paper, we give the following signs and assumptions.

(H_1) $\Gamma(\alpha+1)>\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}$.

(H_2) $p(t)\in C[0,+\infty)$, $p(t)>0$ for $t\in[0,+\infty)$ and $\varphi_q\left(\dfrac{t^{\beta-1}}{p(t)}\right)\in L^1[0,+\infty)$, $\lim\limits_{t\to+\infty}\dfrac{t^{\beta-1}}{p(t)}=0$.

We can conclude from (H_2) that $\sup\limits_{t\in[0,\infty)}\varphi_q\left(\dfrac{t^{\beta-1}}{p(t)}\right)<+\infty$.

(H_3) There exist nonnegative functions $a(t),b(t),c(t),d(t)\in L^1[0,+\infty)$ such that

$$|f(t,x,y,z)|\le a(t)+b(t)\left|\frac{x(t)}{1+t^{\alpha-1}}\right|^{p-1}+c(t)|y(t)|^{p-1}+d(t)|z(t)|^{p-1}.$$

Define

$$B=\int_0^{+\infty}(b(t)+c(t)+d(t))\,dt,\qquad \int_0^{+\infty}a(t)dt=\|a\|_1.$$

If $q<2$, we set

$$B\le\min\left\{\frac{\Gamma(\beta)\varphi_p\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}{\varphi_p(C\alpha)},\frac{\Gamma(\beta)\varphi_p\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}{\varphi_p(C\Gamma(\alpha+1))},\frac{\Gamma(\beta)}{\varphi_p(C)}\right\}.$$

If $q\ge 2$, we set

$$B\le\min\left\{\frac{2^{p-2}\Gamma(\beta)\varphi_p\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}{\varphi_p(C\alpha)},\frac{2^{p-2}\Gamma(\beta)\varphi_p\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}{\varphi_p(C\Gamma(\alpha+1))},\frac{2^{p-2}\Gamma(\beta)}{\varphi_p(C)}\right\},$$

where $C=\max\left\{\int_0^{+\infty}\varphi_q\left(\frac{t^{\beta-1}}{p(t)}\right)dt,\ \sup\limits_{t\in[0,\infty)}\varphi_q\left(\frac{t^{\beta-1}}{p(t)}\right)\right\}$.

2. Preliminaries

In this section, we will list some definitions and fundamental facts of fractional calculus theory which can be found in [15,21].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $f:(0,+\infty)\to R$ is defined as

$$I_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of a continuous function $f:(0,+\infty)\to R$ with order $\alpha>0$ is given by

$$\mathcal{D}_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t(t-s)^{n-\alpha-1}f(s)ds,$$

where $n=[\alpha]+1$, $[\alpha]$ denotes the integral part of number α, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Lemma 2.1. *Assume that $u\in C(0,\infty)\cap L^1(0,\infty)$ with a fractional derivative of order $\alpha>0$ that belongs to $C(0,\infty)\cap L^1(0,\infty)$. Then*

$$I_{0+}^{\alpha}\mathcal{D}_{0+}^{\alpha}u(t)=u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_Nt^{\alpha-N},$$

where $c_i\in R, i=1,2,\cdots,N$, N is the smallest integer greater than or equal to α.

Lemma 2.2. *Suppose* $(H_1),(H_2)$ *hold and* $y\in L^1[0,+\infty)$, *then the following fractional BVP*

$$\begin{cases} \mathcal{D}_{0+}^{\beta}\left(p(t)\varphi_p(\mathcal{D}_{0+}^{\alpha}u(t))\right)+y(t)=0,\\ u(0)=0,\quad \mathcal{D}_{0+}^{\alpha-1}u(0)=\sum\limits_{i=1}^{m}a_i\int_0^{\xi_i}u(t)dt,\\ \mathcal{D}_{0+}^{\alpha}u(0)=0,\quad \lim\limits_{t\to+\infty}\mathcal{D}_{0+}^{\beta-1}(p(t)\varphi_p(\mathcal{D}_{0+}^{\alpha}u(t)))=0,\end{cases}\tag{2.1}$$

has a unique solution

$$\begin{aligned} u(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi_q\left(\frac{1}{p(s)}\int_0^{\infty}G(s,\tau)y(\tau)d\tau\right)ds\\ &+\frac{t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}\\ &\times\sum_{i=1}^{m}a_i\int_0^{\xi_i}(\xi_i-s)^{\alpha}\varphi_q\left(\frac{1}{p(s)}\int_0^{\infty}G(s,\tau)y(\tau)d\tau\right)ds,\end{aligned}\tag{2.2}$$

where

$$G(t,s)=\frac{1}{\Gamma(\beta)}\begin{cases} t^{\beta-1}-(t-s)^{\beta-1}, & 0\le s\le t,\\ t^{\beta-1}, & 0\le t\le s<+\infty.\end{cases}\tag{2.3}$$

Proof. From Lemma 2.1 and the equation (2.1), we obtain

$$\begin{aligned} p(t)\varphi_p(\mathcal{D}_{0+}^{\alpha}u(t))=&I_{0+}^{\beta}y(t)+c_1t^{\beta-1}+c_2t^{\beta-2}\\ =&-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}y(s)ds+c_1t^{\beta-1}+c_2t^{\beta-2},\end{aligned}$$

where $c_1,c_2\in R$. From $\mathcal{D}_{0+}^{\alpha}u(0)=0$, we get $c_2=0$. Thus,

$$\mathcal{D}_{0+}^{\beta-1}\left(p(t)\varphi_p(\mathcal{D}_{0+}^{\alpha}u(t))\right)=-\int_0^t y(s)ds+c_1\Gamma(\beta).$$

According to the condition $\lim\limits_{t\to+\infty}\mathcal{D}_{0+}^{\beta-1}(p(t)\varphi_p(\mathcal{D}_{0+}^{\alpha}u(t)))=0$, we have

$$c_1=\frac{1}{\Gamma(\beta)}\int_0^{+\infty}y(s)ds.$$

Hence,

$$\begin{aligned} p(t)\varphi_p(\mathcal{D}_{0+}^{\alpha}u(t))&=-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}y(s)ds+\frac{t^{\beta-1}}{\Gamma(\beta)}\int_0^{+\infty}y(s)ds\\ &=\int_0^{+\infty}G(t,s)y(s)ds.\end{aligned}$$

So

$$\mathcal{D}_{0+}^{\alpha}u(t)=\varphi_q\left(\frac{1}{p(t)}\int_0^{+\infty}G(t,s)y(s)ds\right).$$

From Lemma 2.1, we derive

$$\begin{aligned} u(t) &= I^{\alpha}_{0+}\varphi_q\left(\frac{1}{p(t)}\int_0^{+\infty} G(t,s)y(s)ds\right) + c_3t^{\alpha-1} + c_4t^{\alpha-2} \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\left(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)y(\tau)d\tau\right)ds + c_3t^{\alpha-1} + c_4t^{\alpha-2}, \end{aligned}$$

where $c_3, c_4 \in R$. By $u(0) = 0$, we obtain $c_4 = 0$. So we have

$$\mathcal{D}^{\alpha-1}_{0+}u(t) = \int_0^t \varphi_q\left(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)y(\tau)d\tau\right)ds + c_3\Gamma(\alpha).$$

By using the condition $\mathcal{D}^{\alpha-1}_{0+}u(0) = \sum\limits_{i=1}^{m} a_i \int_0^{\xi_i} u(t)dt$, we conclude

$$\begin{aligned} c_3\Gamma(\alpha) =& \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\left(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)y(\tau)d\tau\right)dsdt \\ &+ \sum_{i=1}^{m} a_i \int_0^{\xi_i} c_3t^{\alpha-1}dt \\ =& \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} a_i \int_0^{\xi_i}\int_s^{\xi_i} (t-s)^{\alpha-1}\varphi_q\left(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)y(\tau)d\tau\right)dtds \\ &+ c_3\sum_{i=1}^{m} a_i\frac{\xi_i^{\alpha}}{\alpha} \\ =& \frac{1}{\alpha\Gamma(\alpha)}\sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i-s)^{\alpha}\varphi_q\left(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)y(\tau)d\tau\right)ds \\ &+ \frac{c_3}{\alpha}\sum_{i=1}^{m} a_i\xi_i^{\alpha}, \end{aligned}$$

thus,

$$c_3 = \frac{1}{\Gamma(\alpha)\left(\Gamma(\alpha+1) - \sum\limits_{i=1}^{m} a_i\xi_i^{\alpha}\right)}\sum_{i=1}^{m} a_i\int_0^{\xi_i}(\xi_i-s)^{\alpha}\varphi_q\left(\frac{1}{p(s)}\int_0^{\infty} G(s,\tau)y(\tau)d\tau\right)ds.$$

So (2.2) holds. This completes the proof. □

For $s, t \in [0, +\infty)$, it is easy to see that $G(t, s)$ defined by (2.3) is continuous and the following inequality holds

$$0 \le G(t,s) \le \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad \text{for } t, s \in [0, +\infty). \tag{2.4}$$

Define the spaces

$$X = \left\{u(t) \in C[0, +\infty) : \sup_{t\in[0,+\infty)} \frac{|u(t)|}{1 + t^{\alpha-1}} < +\infty\right\}$$

with the norm $\|u\|_\infty = \sup\limits_{t\in[0,+\infty)} \dfrac{|u(t)|}{1+t^{\alpha-1}}$ and

$$Y = \Bigg\{ u(t) \in X : \mathcal{D}_{0+}^{\alpha} u(t) \in C[0,+\infty), \sup_{t\in[0,+\infty)} |\mathcal{D}_{0+}^{\alpha-1} u(t)| < +\infty, \sup_{t\in[0,+\infty)} |\mathcal{D}_{0+}^{\alpha} u(t)| < +\infty \Bigg\}$$

with the norm $\|u\| = \max\{\|u\|_\infty, \|\mathcal{D}_{0+}^{\alpha-1}u\|_\infty, \|\mathcal{D}_{0+}^{\alpha}u\|_\infty\}$, where $\|\mathcal{D}_{0+}^{\alpha-1}u\|_\infty = \sup\limits_{t\in[0,+\infty)} |\mathcal{D}_{0+}^{\alpha-1}u(t)|$, $\|\mathcal{D}_{0+}^{\alpha}u\|_\infty = \sup\limits_{t\in[0,+\infty)} |\mathcal{D}_{0+}^{\alpha}u(t)|$.

Lemma 2.3. *$(X, \|\cdot\|_\infty)$ and $(Y, \|\cdot\|)$ are Banach spaces.*

Proof. Let $\{u_n\}_{n=1}^{n=\infty}$ be a Cauchy sequence in the space $(X, \|\cdot\|_\infty)$, then there exists $u(t)$, such that for any $t \in [0,+\infty)$, $\lim\limits_{n\to+\infty} u_n = u$ and $u(t) \in C[0,+\infty)$. Thus $\lim\limits_{n\to+\infty} \dfrac{u_n(t)}{1+t^{\alpha-1}} = \dfrac{u(t)}{1+t^{\alpha-1}}$, $\sup\limits_{t\in[0,+\infty)} \dfrac{|u(t)|}{1+t^{\alpha-1}} < +\infty$ and $u(t) \in X$. Therefore, $(X, \|\cdot\|_\infty)$ is a Banach space.

Furthermore, Let $\{u_n\}_{n=1}^{n=\infty}$ be a Cauchy sequence in the space $(Y, \|\cdot\|)$. Obviously, $\{u_n\}_{n=1}^{n=\infty}$ is also a Cauchy sequence in the space $(X, \|\cdot\|_\infty)$, and hence, there exists $u(t) \in X$ such that $\lim\limits_{n\to+\infty} \dfrac{u_n(t)}{1+t^{\alpha-1}} = \dfrac{u(t)}{1+t^{\alpha-1}}$. Similarly, there exist $v(t), w(t) \in C[0,+\infty)$ such that

$$\lim_{n\to+\infty} \mathcal{D}_{0+}^{\alpha-1} u_n(t) = v(t), \lim_{n\to+\infty} \mathcal{D}_{0+}^{\alpha} u_n(t) = w(t)$$

and

$$\sup_{t\in[0,+\infty)} |v(t)| < +\infty, \sup_{t\in[0,+\infty)} |w(t)| < +\infty.$$

Next we will prove that $v(t) = \mathcal{D}_{0+}^{\alpha-1}u(t)$, $w(t) = \mathcal{D}_{0+}^{\alpha}u(t)$.

Let $\dfrac{M_0}{2} = \sup\limits_{t\in[0,+\infty)} \dfrac{|u(t)|}{1+t^{\alpha-1}}$, then for $\dfrac{M_0}{2} > 0$, there exists $N > 0$ such that

$$\left| \frac{u_n(t)}{1+t^{\alpha-1}} - \frac{u(t)}{1+t^{\alpha-1}} \right| < \frac{M_0}{2}, \ \forall t \in [0,+\infty) \ \text{ and } n > N.$$

Define $M_i = \sup\limits_{t\in[0,+\infty)} \dfrac{|u_i(t)|}{1+t^{\alpha-1}}$, $i = 1, 2, \cdots, N$. Let $M = \max\{M_i, i = 0, 1, 2, \cdots, N\}$, so $\dfrac{|u_n(t)|}{1+t^{\alpha-1}} < M, n = 1, 2, \cdots$. Thus, for any $t \in [0,+\infty)$ and $1 < \alpha \le 2$, we have

$$\begin{aligned} \left| \int_0^t (t-s)^{1-\alpha} u_n(s) ds \right| &= \left| \int_0^t (t-s)^{1-\alpha} (1+s^{\alpha-1}) \frac{u_n(s)}{1+s^{\alpha-1}} ds \right| \\ &\le M \left| \int_0^t (t-s)^{1-\alpha} (1+s^{\alpha-1}) ds \right| \\ &= M \left[t^{2-\alpha} \int_0^1 (1-\tau)^{1-\alpha} d\tau + t \int_0^1 \tau^{\alpha-1} (1-\tau)^{1-\alpha} d\tau \right] \end{aligned}$$

$$= \frac{M}{2-\alpha}t^{2-\alpha} + B(\alpha, 2-\alpha)Mt,$$

where $B(\alpha, 2-\alpha)$ is the Beta function. By Lebesgue's dominated convergence theorem and the uniform convergence of $\{\mathcal{D}_{0+}^{\alpha-1}u_n(t)\}_{n=1}^{n=\infty}$ and $\{\mathcal{D}_{0+}^{\alpha}u_n(t)\}_{n=1}^{n=\infty}$, we can obtain that

$$\begin{aligned} v(t) = \lim_{n\to+\infty} \mathcal{D}_{0+}^{\alpha-1}u_n(t) = \lim_{n\to+\infty} \frac{d}{dt}\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}u_n(s)ds \\ = \frac{d}{dt}\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}u(s)ds \\ = \mathcal{D}_{0+}^{\alpha-1}u(t), \end{aligned}$$

and

$$\begin{aligned} w(t) = \lim_{n\to+\infty} \mathcal{D}_{0+}^{\alpha}u_n(t) = \lim_{n\to+\infty} \frac{d^2}{dt^2}\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}u_n(s)ds \\ = \frac{d^2}{dt^2}\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}u(s)ds \\ = \mathcal{D}_{0+}^{\alpha}u(t). \end{aligned}$$

Hence, we conclude that $(Y, \|\cdot\|)$ is a Banach space. This completes the proof of Lemma 2.3. □

Since $[0, +\infty)$ is not compact, the Arzela-Ascoli theorem fails to work in Y, we need a modified compactness criterion to the operator. We shall rely on the following a specific compactness criterion. More general cases can be found in [1], [24] and the references therein.

Lemma 2.4. *Assume V is bounded in Y, then V is relatively compact in Y if the following conditions hold:*

(i) for any $u(t) \in V$, $\frac{u(t)}{1+t^{\alpha-1}}, \mathcal{D}_{0+}^{\alpha-1}u(t)$, and $\mathcal{D}_{0+}^{\alpha}u(t)$ are equicontinuous on any compact interval of $[0, +\infty)$.

(ii) for any $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 0$ such that

$$\left|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon, \quad \left|\mathcal{D}_{0+}^{\alpha-1}u(t_1) - \mathcal{D}_{0+}^{\alpha-1}u(t_2)\right| < \varepsilon$$
$$\text{and } \left|\mathcal{D}_{0+}^{\alpha}u(t_1) - \mathcal{D}_{0+}^{\alpha}u(t_2)\right| < \varepsilon, \tag{2.5}$$

for any $u(t) \in V$ and $t_1, t_2 \geq T$.

Proof. Obviously, it is sufficient to prove that V is totally bounded. By condition (ii), we divide $[0, +\infty)$ into $[0, T]$ and $[T, +\infty)$. Set $V_{[0,T]} = \{u(t) : u(t) \in V, t \in [0, T]\}$ with the norm $\|u\|_T = \sup_{t\in[0,T]} \frac{|u(t)|}{1+t^{\alpha-1}}$, then $V_{[0,T]}$ is a Banach space. From condition (i) and the Arzela-Ascoli theorem, we can get that $V_{[0,T]}$ is relatively compact. Thus $V_{[0,T]}$ is totally bounded, i.e., for any $\varepsilon > 0$, there exist finitely many balls $B_\varepsilon(u_i)$ such that $V_{[0,T]} \subset \bigcup_{i=1}^{n} B_\varepsilon(u_i)$, where

$$B_\varepsilon(u_i) = \left\{u(t) \in V_{[0,T]} : \|u - u_i\|_T = \sup_{t\in[0,T]}\left|\frac{u(t)}{1+t^{\alpha-1}} - \frac{u_i(t)}{1+t^{\alpha-1}}\right| < \varepsilon\right\}.$$

Similarly, the space $V_{[0,T]}^{\alpha-1} = \{\mathcal{D}_{0+}^{\alpha-1}u(t) : u(t) \in V_{[0,T]},\ t \in [0,T]\}$ with the norm $\|\mathcal{D}_{0+}^{\alpha-1}u\|_T = \sup_{t\in[0,T]} |\mathcal{D}_{0+}^{\alpha-1}u(t)|$ is a Banach space and can be covered by finitely many balls $B_\varepsilon(\mathcal{D}_{0+}^{\alpha-1}v_j)$, i.e.,

$$V_{[0,T]}^{\alpha-1} \subset \bigcup_{j=1}^{m} B_\varepsilon(\mathcal{D}_{0+}^{\alpha-1}v_j),$$

where

$$B_\varepsilon(\mathcal{D}_{0+}^{\alpha-1}v_j) = \{\mathcal{D}_{0+}^{\alpha-1}u \in V_{[0,T]}^{\alpha-1} : \|u - v_j\|_T = \sup_{t\in[0,T]} \left|\mathcal{D}_{0+}^{\alpha-1}u(t) - \mathcal{D}_{0+}^{\alpha-1}v_j(t)\right| < \varepsilon\}.$$

Again, the space

$$V_{[0,T]}^{\alpha} = \left\{\mathcal{D}_{0+}^{\alpha}u(t) : u(t) \in V_{[0,T]}, \mathcal{D}_{0+}^{\alpha-1}u(t) \in V_{[0,T]}^{\alpha-1},\ t \in [0,T]\right\}$$

with the norm $\|\mathcal{D}_{0+}^{\alpha}u\|_T = \sup_{t\in[0,T]} |\mathcal{D}_{0+}^{\alpha}u(t)|$ is a Banach space and can be covered by finitely many balls $B_\varepsilon(\mathcal{D}_{0+}^{\alpha}w_k)$, i.e.,

$$V_{[0,T]}^{\alpha} \subset \bigcup_{k=1}^{l} B_\varepsilon(\mathcal{D}_{0+}^{\alpha}w_k),$$

where

$$B_\varepsilon(\mathcal{D}_{0+}^{\alpha}w_k) = \left\{\mathcal{D}_{0+}^{\alpha}u \in V_{[0,T]}^{\alpha} : \|u - w_k\|_T = \sup_{t\in[0,T]} \left|\mathcal{D}_{0+}^{\alpha}u(t) - \mathcal{D}_{0+}^{\alpha}w_k(t)\right| < \varepsilon\right\}.$$

Denote

$$\begin{aligned} V_{ijk} = \{&u(t) \in V : u_{[0,T]} \in B_\varepsilon(u_i), \mathcal{D}_{0+}^{\alpha-1}u_{[0,T]} \in B_\varepsilon(\mathcal{D}_{0+}^{\alpha-1}v_j), \\ &\mathcal{D}_{0+}^{\alpha}u_{[0,T]} \in B_\varepsilon(\mathcal{D}_{0+}^{\alpha}w_k)\}\,. \end{aligned}$$

Clearly, $V_{[0,T]} \subset \bigcup_{1\le i\le n, 1\le j\le m, 1\le k\le l} V_{ijk}$. Now let us take $u_{ijk} \in V_{ijk}$, then V can be covered by the balls $B_{4\varepsilon}(u_{ijk}), 1 \le i \le n, 1 \le j \le m, 1 \le k \le l$.

In fact, for $u(t) \in V$, the above discussion implies that there exist i, j, k such that $u_{[0,T]} \in B_\varepsilon(u_i), \mathcal{D}_{0+}^{\alpha-1}u_{[0,T]} \in B_\varepsilon(\mathcal{D}_{0+}^{\alpha-1}v_j), \mathcal{D}_{0+}^{\alpha}u_{[0,T]} \in B_\varepsilon(\mathcal{D}_{0+}^{\alpha}w_k)$. Therefore, for $t \in [0,T]$, we can get

$$\begin{aligned} &\left|\frac{u(t)}{1+t^{\alpha-1}} - \frac{u_{ijk}(t)}{1+t^{\alpha-1}}\right| \\ &\le \left|\frac{u(t)}{1+t^{\alpha-1}} - \frac{u_i(t)}{1+t^{\alpha-1}}\right| + \left|\frac{u_i(t)}{1+t^{\alpha-1}} - \frac{u_{ijk}(t)}{1+t^{\alpha-1}}\right| < 2\varepsilon, \qquad (2.6) \\ &\left|\mathcal{D}_{0+}^{\alpha-1}u(t) - \mathcal{D}_{0+}^{\alpha-1}u_{ijk}(t)\right| \\ &\le \left|\mathcal{D}_{0+}^{\alpha-1}u(t) - \mathcal{D}_{0+}^{\alpha-1}v_j(t)\right| + \left|\mathcal{D}_{0+}^{\alpha-1}v_j(t) - \mathcal{D}_{0+}^{\alpha-1}u_{ijk}(t)\right| < 2\varepsilon, \qquad (2.7) \\ &\left|\mathcal{D}_{0+}^{\alpha}u(t) - \mathcal{D}_{0+}^{\alpha}u_{ijk}(t)\right| \\ &\le \left|\mathcal{D}_{0+}^{\alpha}u(t) - \mathcal{D}_{0+}^{\alpha}w_k(t)\right| + \left|\mathcal{D}_{0+}^{\alpha}w_k(t) - \mathcal{D}_{0+}^{\alpha}u_{ijk}(t)\right| < 2\varepsilon, \qquad (2.8) \end{aligned}$$

For $t \in [T, +\infty)$, by (2.5) and (2.6), we have

$$\left|\frac{u(t)}{1+t^{\alpha-1}} - \frac{u_{ijk}(t)}{1+t^{\alpha-1}}\right| \le \left|\frac{u(t)}{1+t^{\alpha-1}} - \frac{u(T)}{1+t^{\alpha-1}}\right| + \left|\frac{u(T)}{1+t^{\alpha-1}} - \frac{u_{ijk}(T)}{1+t^{\alpha-1}}\right|$$

$$+\left|\frac{u_{ijk}(T)}{1+t^{\alpha-1}}-\frac{u_{ijk}(t)}{1+t^{\alpha-1}}\right|<\varepsilon+2\varepsilon+\varepsilon=4\varepsilon. \tag{2.9}$$

In the same way, for $t\in[T,+\infty)$, by (2.5), (2.7) and (2.8), we can obtain

$$\left|\mathcal{D}_{0+}^{\alpha-1}u(t)-\mathcal{D}_{0+}^{\alpha-1}u_{ijk}(t)\right|<4\varepsilon, \tag{2.10}$$
$$\left|\mathcal{D}_{0+}^{\alpha}u(t)-\mathcal{D}_{0+}^{\alpha}u_{ijk}(t)\right|<4\varepsilon. \tag{2.11}$$

The inequalities (2.5)−(2.11) prove that $\|u(t)-u_{ijk}(t)\|<4\varepsilon$. Thus, V is totally bounded and Lemma 2.4 is complete. □

The following inequality on the p-Laplacian is needed in proving Theorem 3.1.

Lemma 2.5 ([5]). *Let $x,y\geq 0$. Then*

$$\varphi_p(x+y)\leq\varphi_p(x)+\varphi_p(y),\ \ if\ p<2.$$
$$\varphi_p(x+y)\leq 2^{p-2}\left(\varphi_p(x)+\varphi_p(y)\right),\ \ if\ p\geq 2.$$

The proof of our main result is based on an application of the Schauder's fixed point theorem which we list in the following.

Lemma 2.6 ([3]). *Let K be a bounded closed convex set in a Banach space E and $T:K\to K$ be a completely continuous operator. Then T has at least one fixed point in K.*

3. The main result

In this section we will use the Schauder's fixed point theorem to prove the existence of solution to BVP (1.1).

For $u\in Y$, we define the operator A by

$$Au(t)=f(t,u(t),\mathcal{D}_{0+}^{\alpha-1}u(t),\mathcal{D}_{0+}^{\alpha}u(t))$$

and the operator T by

$$\begin{aligned}Tu(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi_q\left(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au(\tau)d\tau\right)ds\\&+\frac{t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^m a_i\xi_i^\alpha\right)}\\&\times\sum_{i=1}^m a_i\int_0^{\xi_i}(\xi_i-s)^\alpha\varphi_q\left(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au(\tau)d\tau\right)ds,\end{aligned}$$

for all $t\in[0,+\infty)$. By the condition (H_3), we have

$$\begin{aligned}&\int_0^\infty|f(t,u(t),\mathcal{D}_{0+}^{\alpha-1}u(t),\mathcal{D}_{0+}^{\alpha}u(t))|dt\\\leq&\int_0^\infty\left[a(t)+b(t)\left|\frac{u(t)}{1+t^{\alpha-1}}\right|^{p-1}+c(t)|\mathcal{D}_{0+}^{\alpha-1}u(t)|^{p-1}+d(t)|\mathcal{D}_{0+}^{\alpha}u(t)|^{p-1}\right]dt\\\leq&\|a\|_1+\|u\|^{p-1}\int_0^\infty\left(b(t)+c(t)+d(t)\right)dt<+\infty.\end{aligned}$$

Hence, the operator T is well defined. Lemma 2.2 indicates that the solutions of BVP (1.1) are the fixed points of T.

Theorem 3.1. *Assume that $f : [0,\infty) \times R^3 \to R$ is continuous and $(H_1) - (H_3)$ hold. Then the BVP (1.1) has at least one solution.*

Proof. From the define of T, we have

$$\begin{aligned}\mathcal{D}_{0+}^{\alpha-1}Tu(t) = &\int_0^t \varphi_q\left(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)Au(\tau)d\tau\right)ds\\ &+ \frac{1}{\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^m a_i\xi_i^\alpha\right)}\\ &\times \sum_{i=1}^m a_i \int_0^{\xi_i}(\xi_i-s)^\alpha \varphi_q\left(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au(\tau)d\tau\right)ds,\\ \mathcal{D}_{0+}^{\alpha}Tu(t) =&\varphi_q\left(\frac{1}{p(t)}\int_0^{+\infty} G(t,s)Au(s)ds\right).\end{aligned}$$

By (H_2) and (H_3), we can get $Tu(t), \mathcal{D}_{0+}^{\alpha-1}Tu(t)$ and $\mathcal{D}_{0+}^{\alpha}Tu(t)$ are continuous on $[0,+\infty)$.

If $q < 2$, set

$$r \geq \max\left\{\frac{C\alpha\varphi_q(\|a\|_1)}{\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^m a_i\xi_i^\alpha\right) - C\alpha\varphi_q(B)}, \right.$$
$$\left.\frac{C\Gamma(\alpha+1)\varphi_q(\|a\|_1)}{\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^m a_i\xi_i^\alpha\right) - C\Gamma(\alpha+1)\varphi_q(B)}, \quad \frac{C\varphi_q(\|a\|_1)}{\varphi_q(\Gamma(\beta)) - C\varphi_q(B)}\right\}$$

and $P = \{u(t) \in Y : \|u(t)\| \leq r\}$.

Firstly, we prove that $T : P \to P$. For any $u(t) \in P$, by (H_3), we can get

$$\begin{aligned}&\int_0^\infty |Au(t)|dt\\ &= \int_0^\infty |f(t,u(t),\mathcal{D}_{0+}^{\alpha-1}u(t),\mathcal{D}_{0+}^{\alpha}u(t))|dt\\ &\leq \int_0^\infty \left[a(t)+b(t)\left|\frac{u(t)}{1+t^{\alpha-1}}\right|^{p-1} + c(t)|\mathcal{D}_{0+}^{\alpha-1}u(t)|^{p-1} + d(t)|\mathcal{D}_{0+}^{\alpha}u(t)|^{p-1}\right]dt\\ &\leq \|a\|_1 + \|u\|^{p-1}\int_0^\infty (b(t)+c(t)+d(t))\,dt = \|a\|_1 + \|u\|^{p-1}B. \qquad (3.1)\end{aligned}$$

Hence by (2.4), (H_2), (H_3), and Lemma 2.5, we have

$$\frac{|Tu(t)|}{1+t^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}\varphi_q\left(\frac{1}{p(s)}\int_0^\infty G(s,\tau)|Au(\tau)|d\tau\right)ds$$

$$
\begin{aligned}
&+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\frac{1}{\Gamma(\alpha)\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}\\
&\times\sum_{i=1}^{m}a_i\int_0^{\xi_i}(\xi_i-s)^{\alpha}\varphi_q\left(\frac{1}{p(s)}\int_0^{\infty}G(s,\tau)|Au(\tau)|d\tau\right)ds\\
\leq&\frac{1}{\Gamma(\alpha)\varphi_q(\Gamma(\beta))}\int_0^{+\infty}\varphi_q\left(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au(\tau)|d\tau\right)ds\\
&+\frac{\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}}{\Gamma(\alpha)\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}\\
&\times\int_0^{+\infty}\varphi_q\left(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au(\tau)|d\tau\right)ds\\
\leq&\frac{C}{\Gamma(\alpha)\varphi_q(\Gamma(\beta))}\varphi_q(\|a\|_1+\|u\|^{p-1}B)\\
&+\frac{C\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}}{\Gamma(\alpha)\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}\varphi_q(\|a\|_1+\|u\|^{p-1}B)\\
\leq&\frac{C\alpha}{\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}(\varphi_q(\|a\|_1)+r\varphi_q(B))\leq r,\\
&|\mathcal{D}_{0+}^{\alpha-1}Tu(t)|\\
\leq&\int_0^{t}\varphi_q\left(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)|Au(\tau)|d\tau\right)ds\\
&+\frac{1}{\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}\\
&\times\sum_{i=1}^{m}a_i\int_0^{\xi_i}(\xi_i-s)^{\alpha}\varphi_q\left(\frac{1}{p(s)}\int_0^{\infty}G(s,\tau)|Au(\tau)|d\tau\right)ds\\
\leq&\frac{C}{\varphi_q(\Gamma(\beta))}\varphi_q(\|a\|_1+\|u\|^{p-1}B)\\
&+\frac{C\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}}{\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}\varphi_q(\|a\|_1+\|u\|^{p-1}B)\\
\leq&\frac{C\Gamma(\alpha+1)}{\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}(\varphi_q(\|a\|_1)+r\varphi_q(B))\leq r,
\end{aligned}
$$

and

$$
|\mathcal{D}_{0+}^{\alpha}Tu(t)|\leq\varphi_q\left(\frac{1}{p(t)}\int_0^{+\infty}G(t,s)|Au(s)|ds\right)
$$

$$\leq \frac{C}{\varphi_q(\Gamma(\beta))}(\varphi_q(\|a\|_1) + r\varphi_q(B)) \leq r.$$

Thus, $\|Tu(t)\| \leq r$. This implies that $T : P \to P$.

If $q \geq 2$, set

$$r \geq \max\Bigg\{\frac{2^{q-2}C\alpha\varphi_q(\|a\|_1)}{\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1) - \sum_{i=1}^{m} a_i\xi_i^\alpha\right) - 2^{q-2}C\alpha\varphi_q(B)},$$
$$\frac{2^{q-2}C\Gamma(\alpha+1)\varphi_q(\|a\|_1)}{\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1) - \sum_{i=1}^{m} a_i\xi_i^\alpha\right) - 2^{q-2}C\Gamma(\alpha+1)\varphi_q(B)},$$
$$\frac{2^{q-2}C\varphi_q(\|a\|_1)}{\varphi_q(\Gamma(\beta)) - 2^{q-2}\varphi_q(B)}\Bigg\}.$$

In the same way, we can prove that $T : P \to P$.

Secondly, let V be a subset of P. We will use Lemma 2.4 to show that TV is relatively compact.

Let $I \subset [0, +\infty)$ be a compact interval, $t_1, t_2 \in I$ and $t_1 < t_2$, then for any $u(t) \in V$, we obtain

$$\begin{aligned}
&\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}} - \frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right| \\
\leq&\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\varphi_q\left(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au(\tau)d\tau\right)ds\right. \\
&\left.-\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\varphi_q\left(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au(\tau)d\tau\right)ds\right| \\
&+\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\varphi_q\left(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au(\tau)d\tau\right)ds\right. \\
&\left.-\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\varphi_q\left(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au(\tau)d\tau\right)ds\right| \\
&+\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\frac{1}{\Gamma(\alpha)\left(\Gamma(\alpha+1) - \sum_{i=1}^{m} a_i\xi_i^\alpha\right)} \\
&\times\sum_{i=1}^{m} a_i\left|\int_0^{\xi_i}(\xi_i-s)^\alpha\varphi_q\left(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au(\tau)d\tau\right)ds\right| \\
\leq&\frac{1}{\Gamma(\alpha)\varphi_p(\Gamma(\beta))}\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\varphi_q\left(\frac{s^{\beta-1}}{p(s)}\int_0^\infty |Au(\tau)|d\tau\right)ds \\
&+\frac{1}{\Gamma(\alpha)\varphi_p(\Gamma(\beta))}\int_0^{t_1}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\varphi_q\left(\frac{s^{\beta-1}}{p(s)}\int_0^\infty |Au(\tau)|d\tau\right)ds \\
&+\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\frac{1}{\Gamma(\alpha)\varphi_p(\Gamma(\beta))\left(\Gamma(\alpha+1) - \sum_{i=1}^{m} a_i\xi_i^\alpha\right)} \\
&\times\sum_{i=1}^{m} a_i\xi_i^\alpha\int_0^{\xi_i}\varphi_q\left(\frac{s^{\beta-1}}{p(s)}\int_0^\infty |Au(\tau)|d\tau\right)ds,
\end{aligned}$$

$$
\begin{aligned}
&|\mathcal{D}_{0+}^{\alpha-1}Tu(t_2)-\mathcal{D}_{0+}^{\alpha-1}Tu(t_1)|\\
\leq&\frac{1}{\varphi_p(\Gamma(\beta))}\int_{t_1}^{t_2}\varphi_q\left(\frac{s^{\beta-1}}{p(s)}\int_0^{+\infty}|Au(\tau)|d\tau\right)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&|\varphi_p(\mathcal{D}_{0+}^{\alpha}Tu(t_2))-\varphi_p(\mathcal{D}_{0+}^{\alpha}Tu(t_1))|\\
\leq&\frac{1}{\Gamma(\beta)}\left|\int_0^{t_2}\frac{(t_2-s)^{\beta-1}}{p(t_2)}Au(s)ds-\int_0^{t_1}\frac{(t_2-s)^{\beta-1}}{p(t_2)}Au(s)ds\right|\\
&+\frac{1}{\Gamma(\beta)}\left|\int_0^{t_1}\left(\frac{(t_2-s)^{\beta-1}}{p(t_2)}-\frac{(t_1-s)^{\beta-1}}{p(t_1)}\right)Au(s)ds\right|\\
&+\frac{1}{\Gamma(\beta)}\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|\int_0^{+\infty}|Au(s)|ds\\
\leq&\frac{1}{\Gamma(\beta)}\left|\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{p(t_2)}Au(s)ds\right|\\
&+\frac{1}{\Gamma(\beta)}\int_0^{t_1}\left(\frac{(t_2-s)^{\beta-1}}{p(t_2)}-\frac{(t_1-s)^{\beta-1}}{p(t_1)}\right)|Au(s)|ds\\
&+\frac{1}{\Gamma(\beta)}\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|\int_0^{+\infty}|Au(s)|ds.
\end{aligned}
$$

Combining (H_2) and (H_3), it is easy to show that $\dfrac{Tu(t)}{1+t^{\alpha-1}}$, $\mathcal{D}_{0+}^{\alpha-1}Tu(t)$ and $\mathcal{D}_{0+}^{\alpha}Tu(t)$ are equicontinuous on any interval I.

Next we will prove that for any $u(t)\in V$, $\dfrac{Tu(t)}{1+t^{\alpha-1}}$, $\mathcal{D}_{0+}^{\alpha-1}Tu(t)$, and $\mathcal{D}_{0+}^{\alpha}Tu(t)$ satisfy the condition (ii) of Lemma 2.4.

By (3.1), we know that for any $u(t)\in V$, $\int_0^\infty|Au(t)|dt$ is bounded. From (H_2), for any given $\varepsilon>0$, there exists a constant $N>0$ such that for any $t_1,t_2\geq N$,

$$
\int_{t_1}^{t_2}\varphi_q\left(\frac{t^{\beta-1}}{p(t)}\right)dt<\varepsilon,\qquad\int_{t_1}^{t_2}|Au(t)|dt<\varepsilon. \tag{3.2}
$$

In addition, due to $\lim\limits_{t\to+\infty}\dfrac{t^{\alpha-1}}{1+t^{\alpha-1}}=1$ and $\lim\limits_{t\to+\infty}\dfrac{t^{\beta-1}}{p(t)}=0$, there exists a constant $T_1>N>0$ such that for any $t_1,t_2\geq T_1$,

$$
\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\leq\left|1-\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right|+\left|1-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|<\varepsilon, \tag{3.3}
$$

$$
\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|\leq\left|\frac{t_2^{\beta-1}}{p(t_2)}\right|+\left|\frac{t_1^{\beta-1}}{p(t_1)}\right|<\varepsilon. \tag{3.4}
$$

Similarly, from $\lim\limits_{t\to+\infty}\dfrac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}=1$ and $\lim\limits_{t\to+\infty}\dfrac{(t-s)^{\beta-1}}{p(t)}=0$, there exists a constant $T_2>N>0$ such that for any $t_1,t_2\geq T_2$ and $0\leq s\leq N$,

$$
\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\leq\left|1-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|+\left|1-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|<\varepsilon, \tag{3.5}
$$

$$\left|\frac{(t_2-s)^{\beta-1}}{p(t_2)}-\frac{(t_1-s)^{\beta-1}}{p(t_1)}\right|\le\left|\frac{(t_2-s)^{\beta-1}}{p(t_2)}\right|+\left|\frac{(t_1-s)^{\beta-1}}{p(t_1)}\right|<\varepsilon. \tag{3.6}$$

Set $T>\max\{T_1,T_2\}$, then for $t_1,t_2\ge T$, by (3.1)−(3.6), we can get

$$\begin{aligned}
&\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}}-\frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right|\\
\le&\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\varphi_q\left(\frac{1}{p(s)}\int_0^{\infty}G(s,\tau)Au(\tau)d\tau\right)ds\right.\\
&\left.-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\varphi_q\left(\frac{1}{p(s)}\int_0^{\infty}G(s,\tau)Au(\tau)d\tau\right)ds\right|\\
&+\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\frac{1}{\Gamma(\alpha)\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}\\
&\times\sum_{i=1}^{m}a_i\left|\int_0^{\xi_i}(\xi_i-s)^{\alpha}\varphi_q\left(\frac{1}{p(s)}\int_0^{\infty}G(s,\tau)Au(\tau)d\tau\right)ds\right|\\
\le&\frac{1}{\Gamma(\alpha)\varphi_q(\Gamma(\beta))}\int_0^{N}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\varphi_q\left(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au(\tau)|d\tau\right)ds\\
&+\frac{1}{\Gamma(\alpha)\varphi_q(\Gamma(\beta))}\int_N^{t_2}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|\varphi_q\left(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au(\tau)|d\tau\right)ds\\
&+\frac{1}{\Gamma(\alpha)\varphi_q(\Gamma(\beta))}\int_N^{t_1}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\varphi_q\left(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au(\tau)|d\tau\right)ds\\
&+\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\frac{1}{\Gamma(\alpha)\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}\\
&\times\sum_{i=1}^{m}a_i\xi_i^{\alpha}\int_0^{\xi_i}\varphi_q\left(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au(\tau)|d\tau\right)ds\\
\le&\frac{C}{\Gamma(\alpha)\varphi_q(\Gamma(\beta))}\varphi_q\left(\int_0^{\infty}|Au(\tau)|d\tau\right)\varepsilon+\frac{2}{\Gamma(\alpha)\varphi_q(\Gamma(\beta))}\varphi_q\left(\int_0^{\infty}|Au(\tau)|d\tau\right)\varepsilon\\
&+\frac{C\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}}{\Gamma(\alpha)\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}\right)}\varphi_q\left(\int_0^{\infty}|Au(\tau)|d\tau\right)\varepsilon,\\
&|\mathcal{D}_{0+}^{\alpha-1}Tu(t_2)-\mathcal{D}_{0+}^{\alpha-1}Tu(t_1)|\\
\le&\frac{1}{\varphi_q(\Gamma(\beta))}\int_{t_1}^{t_2}\varphi_q\left(\frac{s^{\beta-1}}{p(s)}\int_0^{+\infty}|Au(\tau)|d\tau\right)ds\\
\le&\frac{1}{\varphi_q(\Gamma(\beta))}\varphi_q\left(\int_0^{\infty}|Au(\tau)|d\tau\right)\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
&|\varphi_p(\mathcal{D}_{0+}^{\alpha}Tu(t_2))-\varphi_p(\mathcal{D}_{0+}^{\alpha}Tu(t_1))|\\
\le&\frac{1}{\Gamma(\beta)}\left|\int_0^{t_2}\frac{(t_2-s)^{\beta-1}}{p(t_2)}Au(s)ds-\int_0^{t_1}\frac{(t_1-s)^{\beta-1}}{p(t_1)}Au(s)ds\right|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\beta)}\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|\int_0^{+\infty}|Au(s)|ds\\
\leq&\frac{1}{\Gamma(\beta)}\int_0^{N}\left|\frac{(t_2-s)^{\beta-1}}{p(t_2)}-\frac{(t_1-s)^{\beta-1}}{p(t_1)}\right||Au(s)|ds\\
&+\frac{1}{\Gamma(\beta)}\int_N^{t_2}\frac{(t_2-s)^{\beta-1}}{p(t_2)}|Au(s)|ds\\
&+\frac{1}{\Gamma(\beta)}\int_N^{t_1}\frac{(t_1-s)^{\beta-1}}{p(t_1)}|Au(s)|ds+\frac{1}{\Gamma(\beta)}\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|\int_0^{+\infty}|Au(s)|ds\\
\leq&\frac{2}{\Gamma(\beta)}\int_0^{+\infty}|Au(s)|ds\cdot\varepsilon+\frac{2C}{\Gamma(\beta)}\varepsilon.
\end{aligned}
$$

According to Lemma 2.4, it is clear that TV is relatively compact.

Finally, we prove that $T:P\to P$ is a continuous operator.

Let $u_n,u\in P,\ n=1,2,\cdots$, and $u_n\to u(n\to+\infty)$. Then, by (H_2), (H_3) and (3.1), we have

$$
\begin{aligned}
&\left|\frac{Tu_n(t)}{1+t^{\alpha-1}}-\frac{Tu(t)}{1+t^{\alpha-1}}\right|\\
\leq&\frac{1}{\Gamma(\alpha)}\left|\int_0^t\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}\left(\varphi_q(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au_n(\tau)d\tau)\right.\right.\\
&\left.\left.-\varphi_q(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au(\tau)d\tau)\right)ds\right|+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\cdot\frac{1}{\Gamma(\alpha)\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^m a_i\xi_i^\alpha\right)}\\
&\times\sum_{i=1}^m a_i\left|\int_0^{\xi_i}(\xi_i-s)^\alpha\left(\varphi_q(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au_n(\tau)d\tau)\right.\right.\\
&\left.\left.-\varphi_q(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au(\tau)d\tau)\right)ds\right|\\
\leq&\frac{1}{\Gamma(\alpha)}\int_0^t\left(\left|\varphi_q(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au_n(\tau)d\tau)\right|+\left|\varphi_q(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au(\tau)d\tau)\right|\right)ds\\
&+\frac{1}{\Gamma(\alpha)\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^m a_i\xi_i^\alpha\right)}\sum_{i=1}^m a_i\xi_i^\alpha\\
&\times\int_0^{\xi_i}\left(\left|\varphi_q(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au_n(\tau)d\tau)\right|+\left|\varphi_q(\frac{1}{p(s)}\int_0^\infty G(s,\tau)Au(\tau)d\tau)\right|\right)ds\\
\leq&\frac{1}{\Gamma(\alpha)\varphi_q(\Gamma(\beta))}\int_0^t\left(\varphi_q(\frac{s^{\beta-1}}{p(s)}\int_0^\infty|Au_n(\tau)|\,d\tau)+\varphi_q(\frac{s^{\beta-1}}{p(s)}\int_0^\infty|Au(\tau)|\,d\tau)\right)ds\\
&+\frac{1}{\Gamma(\alpha)\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^m a_i\xi_i^\alpha\right)}\\
&\times\sum_{i=1}^m a_i\xi_i^\alpha\int_0^{\xi_i}\left(\varphi_q(\frac{s^{\beta-1}}{p(s)}\int_0^\infty|Au_n(\tau)|\,d\tau)+\varphi_q(\frac{s^{\beta-1}}{p(s)}\int_0^\infty|Au(\tau)|\,d\tau)\right)ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\alpha}{\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m} a_i\xi_i^{\alpha}\right)} \\
&\quad \times \int_0^{+\infty}\left(\varphi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au_n(\tau)|\,d\tau)+\varphi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au(\tau)|\,d\tau)\right)ds \\
&\le \frac{\alpha C}{\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m} a_i\xi_i^{\alpha}\right)} \\
&\quad \times \left(\varphi_q(\int_0^{\infty}a(\tau)d\tau+\|u_n\|^{p-1}\int_0^{\infty}(b(\tau)+c(\tau)+d(\tau))d\tau)\right. \\
&\quad \left.+\ \varphi_q(\int_0^{\infty}a(\tau)d\tau+\|u\|^{p-1}\int_0^{\infty}(b(\tau)+c(\tau)+d(\tau))d\tau)\right) \\
&\le \frac{2\alpha C}{\varphi_q(\Gamma(\beta))\left(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m} a_i\xi_i^{\alpha}\right)}\varphi_q\left(\int_0^{\infty}a(\tau)d\tau+r^{p-1}\int_0^{\infty}(b(\tau)+c(\tau)+d(\tau))d\tau\right),
\end{aligned}
$$

$$
\begin{aligned}
&|\mathcal{D}_{0+}^{\alpha-1}Tu_n(t)-\mathcal{D}_{0+}^{\alpha-1}Tu(t)| \\
&\le \left|\int_0^t\left(\varphi_q(\frac{1}{p(s)}\int_0^{\infty}G(s,\tau)Au_n(\tau)d\tau)-\varphi_q(\frac{1}{p(s)}\int_0^{\infty}G(s,\tau)Au(\tau)d\tau)\right)ds\right| \\
&\quad +\frac{1}{\Gamma(\alpha+1)-\sum\limits_{i=1}^{m} a_i\xi_i^{\alpha}}\cdot\sum_{i=1}^{m}a_i\left|\int_0^{\xi_i}(\xi_i-s)^{\alpha}\left(\varphi_q(\frac{1}{p(s)}\int_0^{\infty}G(s,\tau)Au_n(\tau)d\tau)\right.\right. \\
&\quad \left.\left.-\varphi_q(\frac{1}{p(s)}\int_0^{\infty}G(s,\tau)Au(\tau)d\tau)\right)ds\right| \\
&\le \frac{1}{\varphi_q(\Gamma(\beta))}\int_0^t\left(\varphi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au_n(\tau)|\,d\tau)+\varphi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au(\tau)|\,d\tau)\right)ds \\
&\quad +\frac{1}{\varphi_q(\Gamma(\beta))(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m} a_i\xi_i^{\alpha})} \\
&\quad \times\sum_{i=1}^{m}a_i\xi_i^{\alpha}\int_0^{\xi_i}\left(\varphi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au_n(\tau)|\,d\tau)+\varphi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au(\tau)|\,d\tau)\right)ds \\
&\le \frac{\Gamma(\alpha+1)}{\varphi_q(\Gamma(\beta))(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m} a_i\xi_i^{\alpha})} \\
&\quad \times\int_0^{+\infty}\left(\varphi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au_n(\tau)|\,d\tau)+\varphi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au(\tau)|\,d\tau)\right)ds \\
&\le \frac{\Gamma(\alpha+1)C}{\varphi_q(\Gamma(\beta))(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m} a_i\xi_i^{\alpha})} \\
&\quad \times\left(\varphi_q(\int_0^{\infty}a(\tau)d\tau+\|u_n\|^{p-1}\int_0^{\infty}(b(\tau)+c(\tau)+d(\tau))d\tau)\right. \\
&\quad \left.+\ \varphi_q(\int_0^{\infty}a(\tau)d\tau+\|u\|^{p-1}\int_0^{\infty}(b(\tau)+c(\tau)+d(\tau))d\tau)\right)
\end{aligned}
$$

$$\leq \frac{2\Gamma(\alpha+1)C}{\varphi_q(\Gamma(\beta))(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m} a_i\xi_i^{\alpha})}\varphi_q\left(\int_0^{\infty} a(\tau)d\tau + r^{p-1}\int_0^{\infty}(b(\tau)+c(\tau)+d(\tau))d\tau\right),$$

and

$$\begin{aligned}
&|\mathcal{D}_{0+}^{\alpha}Tu_n(t)-\mathcal{D}_{0+}^{\alpha}Tu(t)| \\
=&\left|\varphi_q(\frac{1}{p(s)}\int_0^{\infty}G(s,\tau)Au_n(\tau)d\tau)-\varphi_q(\frac{1}{p(s)}\int_0^{\infty}G(s,\tau)Au(\tau)d\tau)\right| \\
\leq&\frac{1}{\varphi_q(\Gamma(\beta))}\varphi_q\left(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au_n(\tau)|\,d\tau\right)+\frac{1}{\varphi_q(\Gamma(\beta))}\varphi_q\left(\frac{s^{\beta-1}}{p(s)}\int_0^{\infty}|Au(\tau)|\,d\tau\right) \\
\leq&\frac{2C}{\varphi_q(\Gamma(\beta))}\varphi_q\left(\int_0^{\infty}a(\tau)d\tau+r^{p-1}\int_0^{\infty}(b(\tau)+c(\tau)+d(\tau))d\tau\right).
\end{aligned}$$

It follows from the Lebesgue's dominated convergence theorem that $T: P \to P$ is continuous. Therefore, $T: P \to P$ is completely continuous. So, by Lemma 2.6, BVP (1.1) has at least one solution in P. The proof is complete. □

4. Example

Now as an application, we give an example to illustrate our result.

Example 4.1. Consider the following BVP

$$\begin{cases}
\mathcal{D}_{0+}^{\frac{3}{2}}\Big((t+1)^{\frac{9}{2}}\varphi_3(\mathcal{D}_{0+}^{\frac{3}{2}}u(t))\Big) \\
+\Big(\frac{\sin t}{(t+3)^3}+\frac{u^2(t)}{(t+4)^3(2+\sqrt{t})^2}+\frac{(\mathcal{D}_{0+}^{\frac{1}{2}}u(t))^2}{(t+5)^3}+\frac{(\mathcal{D}_{0+}^{\frac{3}{2}}u(t))^2}{(t+6)^3}\Big)=0, \\
u(0)=0, \quad \mathcal{D}_{0+}^{\frac{1}{2}}u(0)=\frac{1}{8}\int_0^{\sqrt[3]{4}}u(t)dt+\frac{1}{9}\int_0^{\sqrt[3]{9}}u(t)dt+\frac{1}{10}\int_0^{\sqrt[3]{16}}u(t)dt, \\
\mathcal{D}_{0+}^{\alpha}u(0)=0, \quad \lim\limits_{t\to+\infty}\mathcal{D}_{0+}^{\frac{1}{2}}(p(t)\varphi_p(\mathcal{D}_{0+}^{\frac{3}{2}}u(t)))=0,
\end{cases} \tag{4.1}$$

where $\alpha=\beta=\frac{3}{2}$, $p=3$, $p(t)=(t+1)^{\frac{9}{2}}$, $a_1=\frac{1}{8}$, $a_2=\frac{1}{9}$, $a_3=\frac{1}{10}$, $\xi_1=\sqrt[3]{4}$, $\xi_2=\sqrt[3]{9}$, $\xi_3=\sqrt[3]{16}$,

$$f(t,u,v,w)=\frac{\sin t}{(t+3)^3}+\frac{u^2(t)}{(t+4)^3(2+\sqrt{t})^2}+\frac{v^2(t)}{(t+5)^3}+\frac{w^2(t)}{(t+6)^3}.$$

From $p=3$, we have $q=\frac{3}{2}<2$.

By simple computation, we get

$$\begin{aligned}
&\Gamma(\alpha+1)=\Gamma(\frac{5}{2})=\frac{3}{4}\sqrt{\pi}\approx 1.3293, \\
&\sum_{i=1}^{i=3}a_i\xi_i^{\alpha}=\frac{1}{8}(\sqrt[3]{4})^{\frac{3}{2}}+\frac{1}{9}(\sqrt[3]{9})^{\frac{3}{2}}+\frac{1}{10}(\sqrt[3]{16})^{\frac{3}{2}}=\frac{59}{60}<\Gamma(\alpha+1).
\end{aligned}$$

So the condition (H_1) holds.

Obviously, $p(t) \in C[0,+\infty), p(t) > 0$ and $\varphi_q\left(\frac{t^{\beta-1}}{p(t)}\right) = \frac{t^{\frac{1}{4}}}{(t+1)^{\frac{9}{4}}} \in L^1[0,+\infty)$, $\lim\limits_{t\to+\infty} \frac{t^{\beta-1}}{p(t)} = 0$, which implies that the condition (H_2) is satisfied and

$$\int_0^{+\infty} \varphi_q\left(\frac{t^{\beta-1}}{p(t)}\right) dt = \int_0^{+\infty} \frac{t^{\frac{1}{4}}}{(t+1)^{\frac{9}{4}}} dt = \frac{4}{5}, \quad \sup_{t\in[0,\infty)} \varphi_q\left(\frac{t^{\beta-1}}{p(t)}\right) = \frac{64}{81\sqrt[4]{9}} < \frac{4}{5},$$

thus, $C = \max\left\{ \int_0^{+\infty} \varphi_q\left(\frac{t^{\beta-1}}{p(t)}\right) dt, \sup\limits_{t\in[0,\infty)} \varphi_q\left(\frac{t^{\beta-1}}{p(t)}\right) \right\} = \frac{4}{5}$.

Let $a(t) = \frac{1}{(t+3)^3}$, $b(t) = \frac{1}{(t+4)^3}$, $c(t) = \frac{1}{(t+5)^3}$, $d(t) = \frac{1}{(t+6)^3}$. Then $a(t), b(t), c(t), d(t) \in L^1[0,+\infty)$,

$$|f(t,u,v,w)| \le a(t) + b(t)\left|\frac{u(t)}{1+t^{\frac{1}{2}}}\right|^2 + c(t)|v(t)|^2 + d(t)|w(t)|^2,$$

and

$$B = \int_0^{+\infty} (b(t)+c(t)+d(t))\, dt = \frac{1}{2}(\frac{1}{16} + \frac{1}{25} + \frac{1}{36}) \approx 0.0651,$$

$$\min\left\{\frac{\Gamma(\beta)\varphi_p\Big(\Gamma(\alpha+1) - \sum\limits_{i=1}^m a_i\xi_i^\alpha\Big)}{\varphi_p(C\alpha)}, \frac{\Gamma(\beta)\varphi_p\Big(\Gamma(\alpha+1) - \sum\limits_{i=1}^m a_i\xi_i^\alpha\Big)}{\varphi_p(C\Gamma(\alpha+1))}, \frac{\Gamma(\beta)}{\varphi_p(C)}\right\} \approx 0.0737.$$

So, the condition (H_3) holds. Hence by Theorem 3.1, the BVP (4.1) has at least one solution.

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