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DECAY PROPERTIES AND ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS FOR THE NONLINEAR FRACTIONAL SCHRÖDINGER-POISSON SYSTEM

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Abstract In this paper, we study the following nonlinear fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^s u + \lambda V(x)u + \mu\phi u = |u|^{p-2}u, \text{ in } \mathbb{R}^3, \\ (-\Delta)^s \phi = u^2, \qquad \text{in } \mathbb{R}^3, \end{cases}$$

where $s \in (\frac{3}{4}, 1)$, $2 < p < 4$, λ, μ are positive parameters and the potential $V(x)$ is a nonnegative continuous function with a potential well $\Omega = intV^{-1}(0)$. By establishing truncation technique and the parameter-dependent compactness lemma, the existence, decay rate and asymptotic behavior of positive solutions are established. Moreover, we prove some nonexistence results in the case of $2 < p \le 3$.

Keywords Fractional Schrödinger-Poisson system, positive solutions, truncation technique, steep potential well.

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1. Introduction

In this paper, we are concerned with the following nonlinear fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^s u + \lambda V(x)u + \mu\phi u = |u|^{p-2}u, \text{ in } \mathbb{R}^3, \\ (-\Delta)^s \phi = u^2, \qquad \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $s \in (\frac{3}{4}, 1)$, $2 < p < 4$, λ, μ are positive parameters and the potential $V(x)$ satisfies the following conditions:
(V_1) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ and $V(x) \ge 0$ on $\mathbb{R}^3$.
(V_2) There exists $b > 0$ such that $\mathcal{V}_b := \{x \in \mathbb{R}^3 : V(x) < b\}$ is nonempty and has

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finite measure.
(V_3) $\Omega = intV^{-1}(0)$ is a nonempty open set with locally Lipschitz boundary and $\overline{\Omega} = V^{-1}(0)$.

It's well known that the fractional Laplacian $(-\Delta)^s(s \in (0,1))$ can be defined by

$$(-\Delta)^s v(x) = C_{3,s} P.V. \int_{\mathbb{R}^3} \frac{v(x)-v(y)}{|x-y|^{3+2s}} dy = C_{3,s} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3 \setminus B_\varepsilon(x)} \frac{v(x)-v(y)}{|x-y|^{3+2s}} dy$$

for $v \in \mathcal{S}(\mathbb{R}^3)$, where $P.V.$ denotes a Principal Value, $\mathcal{S}(\mathbb{R}^3)$ is the Schwartz space of rapidly decaying $\mathcal{C}^\infty$ function, $B_\varepsilon(x)$ denotes an open ball of radius ε centered at x and the normalization constant $C_{3,s} = \left(\int_{\mathbb{R}^3} \frac{1-\cos(\zeta_1)}{|\zeta|^{3+2s}}\right)^{-1}$(see e.g. [12,23,30,32,40] and the references therein). For $u \in \mathcal{S}(\mathbb{R}^3)$, the fractional Laplaction $(-\Delta)^s(s \in (0,1))$ can be defined by the Fourier transform $(-\Delta)^s u = \mathfrak{F}^{-1}(|\xi|^{2s}\mathfrak{F}u)$, $\mathfrak{F}$ being the usual Fourier transform. The applications of operator $(-\Delta)^s$ can be founded in several areas such as fractional quantum mechanics [25,26], physics and chemistry [31], obstacle problems [34], optimization and finance [6], conformal geometry and minimal surfaces [7] and so on.

When $s = \lambda = \mu = 1$, system (1.1) reduces to the classical Schrödinger-Poisson system written by a more general form

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u), \text{ in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, \qquad\qquad\qquad\quad \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

This kind of problem has been widely studied by many scholars in recent years. In the case of $V = K \equiv 1$ and $f = |u|^{p-2}u$, system (1.2) has been studied sufficiently as p varies. The readers may see [13] for the case $p \le 2$ or $p \ge 6$ and see [1,8,15,33] for the case $2 < p < 6$. Moreover, Azzollini and Pomponio [2] proved the existence of ground state solutions for the subcritical $3 < p < 6$ and the critical case $f = |u|^{p-2}u + u^5$ with $4 < p < 6$. In the case of V is non-radial, $K \equiv 1$ and $f = |u|^{p-2}u$, the existence of ground state solution for system (1.2) was obtained in [2] and [43] for $4 < p < 6$ and $3 < p \le 4$ respectively. When $V \equiv 1$, Cerami and Vaira [9] proved the existence of ground states and bound states of system (1.2), with $f = a(x)|u|^{p-2}u$ and $4 < p < 6$. If V is not constant, $K = 1$, $2 < p < 5$ and $f = \mu|u|^{p-1}u + u^5$, Liu and Guo [27] obtained the existence of ground state solution of system (1.2). For other results on the existence of solutions for system (1.2), the readers may see [1,3,5,10,14,21,24,35] and the references therein.

In addition, there are some results studied when V in (1.2) is a nonnegative continuous function with a potential well $\Omega = intV^{-1}(0)$. Du, Tian, Wang and Zhang [17] proved the existence, nonexistence and asymptotic behavior of solutions of system (1.2), with $K(x) \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$, $f(x,u) = a(x)f(u)$, $a(x)$ is a positive bounded function and $f(s)$ is either asymptotically linear or asymptotically 3-linear in s at infinity. Zhang, Tang and Zhang [44] obtained the existence and concentration of nontrivial solutions of system (1.2). When $V = \lambda a(x) + b(x)$, $a(x) \in C(\mathbb{R}^3)$ is nonnegative and has a potential well Ω_a and $a(x) \in C(\mathbb{R}^3)$ is unbounded below, Sun, Wu and Wu [36] studied the existence and concentration of nontrivial solutions of system (1.2), with $f = |u|^{p-2}u$ and $3 < p < 4$. For other related works, we refer the readers to [41,45] and the references therein.

As far as we know, there are few papers in the literature which considered the fractional Laplacian equations or systems. In particular, fractional Laplacian equation or system with steep potential well. In [37], Teng studied the following fractional Schrödinger-Poisson system

$$\begin{cases}(-\Delta)^s u+V(x)u+\phi u=\mu|u|^{q-1}u+|u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi=u^2, & \text{in } \mathbb{R}^3,\end{cases}$$

where $\mu\in\mathbb{R}^+$ is a parameter, $1<q<2_s^*-1=\frac{3+2s}{3-2s}$, $s,t\in(0,1)$ and $2s+2t<3$. Under certain assumptions on $V(x)$, by using the method of Pohozaev-Nehari manifold and the arguments of Brezis-Nirenberg, the monotonic trick and global compactness Lemma, the author proved the existence of a nontrivial ground state solution.

In [29], Yang and Liu investigated the following fractional Schrödinger equation with sublinear perturbation and steep potential well

$$\begin{cases}(-\Delta)^s u+\lambda V(x)u=f(x,u)+a(x)|u|^{v-2}u, & \text{in } \mathbb{R}^N,\\ u\in H^s(\mathbb{R}^N),\end{cases}$$

where $0<s<1$, $2s<N$, $\lambda>0$, $1<v<2$, $f\in C(\mathbb{R}^N\times\mathbb{R})$ is of subcritical growth. By using variational methods, they proved the existence of at least two nontrivial solutions. Moreover, the phenomenon of concentration of solutions was explored as well.

In [39], Torres and Cesar studied the nonlinear fractional Schrödinger equation with steep potential well

$$\begin{cases}(-\Delta)^s u+\lambda V(x)u=f(x,u), & \text{in } \mathbb{R}^N,\\ u\in H^s(\mathbb{R}^N),\end{cases}$$

where $0<s<1$, λ is a parameter, $V\in C(\mathbb{R}^N)$ and $V^{-1}(0)$ has nonempty interior. Under some suitable conditions, the existence of nontrivial solutions were obtained by using variational methods. Furthermore, the phenomenon of concentration of solutions was also explored. For more research on fractional elliptic equations, we refer the readers to [4, 11, 20, 22, 35, 42] and the references therein.

To our best knowledge, there is no result for system (1.1) with a steep potential well and $2<p<4$. In this paper we need to overcome the following difficulties. Comparing with the case of $4\le p<2_s^*=\frac{6}{3-2s}$, there are some new difficulties. One of the main difficulty is that the nonlinear term $u\mapsto f(u):=|u|^{p-2}u$ with $2<p<4$ does not satisfy the Ambrosetti-Rabinowitz condition

$$0<\rho\int_0^u f(s)ds\le f(u)u \quad \text{for all } u\neq 0 \text{ with some } \rho>4,$$

which would make it very difficult to prove the boundedness of Palais-Smale sequence or Cerami sequence. Another difficulty is given by the fact that the function f does not satisfy the Nehari-type monotonicity condition

$$\frac{f(s)}{|s|^3} \quad \text{is increasing on } (-\infty,0) \text{ and } (0,+\infty),$$

hence we can't apply the Nehari manifold and fibering methods and so on. Comparing with the case of $s = 1$, our major difficulty lies in the decay estimates of the sequences of solution to the nonlocal problem at infinity are different from those in the case of the classical local problem, we must build decay estimates for nonlocal operators.

Motivated by the above works, in this paper, we will consider the fractional Schrödinger-Poisson system (1.1) with $2 < p < 4$. As shown in Section 2, for every $u \in H^s(\mathbb{R}^3)$, there exists a unique $\phi_u^s \in D^{s,2}(\mathbb{R}^3)$ satisfying $(-\Delta)^s \phi_u^s = u^2$. Thus, we can rewrite (1.1) as follows

$$(-\Delta)^s u + \lambda V(x)u + \mu \phi_u^s u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^3.$$

The functional associated with (1.1) is

$$J_{\lambda,\mu}(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda V(x)u^2)dx + \frac{\mu}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |u^+|^p dx$$

defined in the space

$$E_\lambda = \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < \infty \right\}$$

with the norm

$$\|u\|_\lambda^2 = \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda V(x)u^2)dx,$$

where $u^+ = \max\{u, 0\}$. Our first main result can be stated as follows.

Theorem 1.1. *Suppose that conditions* $(V_1) - (V_3)$ *hold and* $2 < p < 4$. *Then there exist* $\lambda^* > 1$ *and* $\mu_* > 0$ *such that for each* $\lambda \in (\lambda^*, \infty)$ *and* $\mu \in (0, \mu_*)$, *problem* (1.1) *has at least a positive solution* $u_{\lambda,\mu} \in E_\lambda$. *Moreover, there exist constants* $\tau, T > 0$ *(independent of* λ, μ *and* s*) such that*

$$\tau \le \|u_{\lambda,\mu}\|_\lambda \le T. \tag{1.3}$$

Remark 1.1. Now, we give the main idea of the proof of Theorem 1.1. In fact, it is easy to check that the functional $J_{\lambda,\mu}$ possesses the mountain pass geometry when $\mu > 0$ small. Then we will get a Cerami sequence of the functional $J_{\lambda,\mu}$ when $\mu > 0$ small. But we note that due to the lack of the Ambrosetti-Rabinowitz condition for system (1.1), it is hard to obtain the boundedness of the Cerami sequence. To overcome this obstacle, we use the truncation technique as e.g. in [28]. Indeed, for any $T > 0$, we consider the truncated functional $J_{\lambda,\mu}^T : E_\lambda \to \mathbb{R}$ defined by

$$\begin{aligned} J_{\lambda,\mu}^T(u) = &\frac{1}{2}\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda V(x)u^2)dx \\ &+ \frac{\mu}{4}\psi\left(\frac{\|u\|_\lambda^2}{T^2}\right)\int_{\mathbb{R}^3} \phi_u^s u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |u^+|^p dx, \end{aligned}$$

where ψ is a smooth cut-off function such that

$$\psi\left(\frac{\|u\|_\lambda^2}{T^2}\right) = \begin{cases} 1, & \|u\|_\lambda \le T, \\ 0, & \|u\|_\lambda > \sqrt{2}T. \end{cases}$$

Firstly, we can prove that the functional $J^T_{\lambda,\mu}$ possesses the mountain pass geometry when $\mu > 0$ small, then the Mountain Pass Theorem shows that there exists a Cerami sequence $\{u_n\}$ of $J^T_{\lambda,\mu}$ at the mountain pass level $c^T_{\lambda,\mu}$. Secondly, we will show that $c^T_{\lambda,\mu}$ has an upper bounded, after passing to a subsequence, such that $\|u_n\|_\lambda \le T$ for all n when $\mu > 0$ small, then we get that $\{u_n\}$ is a bounded Cerami sequence of $J^T_{\lambda,\mu}$ at the mountain pass level $c^T_{\lambda,\mu}$, that is

$$\sup_{n\in\mathbb{N}} \|u_n\|_\lambda \le T, \quad J_{\lambda,\mu}(u_n) \to c^T_{\lambda,\mu} \quad \text{and} \quad (1 + \|u_n\|_\lambda)\|J'_{\lambda,\mu}(u_n)\|_{E'_\lambda} \to 0,$$

where E'_λ is the dual space of E_λ. Finally, for $\lambda > 0$ large, we will prove that $u_n \to u_{\lambda,\mu}$ in E_λ through using the parameter-dependent compactness lemma, hence, $u_{\lambda,\mu}$ is a solution of problem (1.1).

Another aim of the paper is to prove the nonexistence of nontrivial solutions to problem (1.1) when λ and μ are large enough.

Theorem 1.2. *Suppose that conditions $(V_1) - (V_3)$ hold.*

(i) If $2 < p < 3$ and $|\mathcal{V}_b| \le S_s^{\frac{2_s^*}{2_s^*-2}}$, then problem (1.1) has no nontrivial solution in E_λ for all $\lambda > \frac{1}{b}$ and $\mu \ge \frac{1}{4(1-|\mathcal{V}_b|^{\frac{2_s^*-2}{2_s^*}} S_s^{-1})}$. Here S_s is the best constant for the embedding $D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$.
(ii) If $p = 3$, then problem (1.1) has no nontrivial solution in E_λ for all $\lambda > 0$ and $\mu > \frac{1}{4}$.

Next, we are concerned with the decay rate of the positive solutions at infinity. Clearly, it is possible that $\liminf_{|x|\to\infty} V(x) = 0$ since $(V_1) - (V_3)$, hence we need to replace (V_2) by the following condition.
(V_2') there exists $b > 0$ such that $\mathcal{V}_b := \{x \in \mathbb{R}^3 : V(x) < b\}$ is nonempty and bounded.

Theorem 1.3. *Suppose that conditions (V_1), (V_2') and (V_3) hold and $2 < p < 4$. In addition, suppose that $V(x) \in L^\infty(\mathbb{R}^3)$. Let $u_{\lambda,\mu}$ be the positive solution of (1.1) for each $\lambda \in (\lambda^*, \infty)$ and $\mu \in (0, \mu_*)$ satisfying (1.3). Then there exists $\Lambda^* > \lambda^*$ such that for each $\lambda \in (\Lambda^*, \infty)$ and $\mu \in (0, \mu_*)$, we have*

$$0 < u_{\lambda,\mu}(x) < \frac{C}{1 + |x|^{3+2s}},$$

where constants $C > 0$ independent of λ and μ.

Remark 1.2. (1) As far as we know, Theorem 1.3 is a new result for fractional Schrödinger-Poisson problem with steep potential well.
(2) It is easy to see that (V_2') is stronger than (V_2). Thus, the conclusions of Theorem 1.1 still hold when (V_2) replaced by (V_2').
(3) We know that the potential function $V(x)$ satisfying condition (V_1), (V_2') and (V_3) may be bounded or unbounded. For example, the bounded potential function:

$$V(x) = \begin{cases} 0, & |x| \le 1, \\ (|x| - 1)^2, & 1 < |x| \le 2, \\ 1, & |x| > 2, \end{cases}$$

and the unbounded potential function:

$$V(x)=\begin{cases}0, & |x|\le 1,\\ (|x|-1)^2, & |x|>1.\end{cases}$$

However, Theorem 1.3 only obtain that the decay rate of positive solution $u_{\lambda,\mu}$ at infinity when $V(x)$ is bounded. It is still an open question that the decay rate of positive solution $u_{\lambda,\mu}$ at infinity when $V(x)$ is unbounded, which is under consideration in my following work.

Finally, we give the asymptotic behavior of the positive solutions as $\lambda\to\infty$ and $\mu\to 0$.

Theorem 1.4. *Let $u_{\lambda,\mu}$ be the positive solution of* (1.1) *obtained by Theorem 1.1. Then for each $\mu\in(0,\mu_*)$ be fixed, $u_{\lambda,\mu}\to u_\mu$ in $H^s(\mathbb{R}^3)$ as $\lambda\to\infty$ up to a subsequence, where $u_\mu\in H_0^s(\Omega)$ is a positive solution of*

$$\begin{cases}(-\Delta)^s u+\mu\phi_u^s u=|u|^{p-2}u, & \textit{in } \Omega,\\ u=0, & \textit{in } \partial\Omega.\end{cases}\tag{1.4}$$

Theorem 1.5. *Let $u_{\lambda,\mu}$ be the positive solution of* (1.1) *obtained by Theorem 1.1. Then for each $\lambda\in(\lambda^*,\infty)$ be fixed, $u_{\lambda,\mu}\to u_\lambda$ in E_λ as $\mu\to 0$ up to a subsequence, where $u_\lambda\in E_\lambda$ is a positive solution of*

$$\begin{cases}(-\Delta)^s u+\lambda V(x)u=|u|^{p-2}u, & \textit{in } \mathbb{R}^3,\\ u\in H^s(\mathbb{R}^3).\end{cases}\tag{1.5}$$

Theorem 1.6. *Let $u_{\lambda,\mu}$ be the positive solution of* (1.1) *obtained by Theorem 1.1. Then $u_{\lambda,\mu}\to u_0$ in $H^s(\mathbb{R}^3)$ as $\mu\to 0$ and $\lambda\to\infty$ up to a subsequence, where $u_0\in H_0^s(\Omega)$ is a positive solution of*

$$\begin{cases}(-\Delta)^s u=|u|^{p-2}u, & \textit{in } \Omega,\\ u=0, & \textit{in } \partial\Omega.\end{cases}\tag{1.6}$$

In the sequel, we use the following notations:

- X' denotes the dual space of X.
- $\widehat{u}$ denotes the Fourier transform of u.
- $|M|$ is the Lebesgue measure of the set M.
- For $\rho>0$ and $z\in\mathbb{R}^3$, $B_\rho(z)$ denotes the ball of radius ρ centered at z.
- $L^s(\mathbb{R}^3)$ denotes the usual Lebesgue space with norm $|u|_s:=(\int_{\mathbb{R}^3}|u|^s dx)^{\frac{1}{s}}$, $1\le s\le\infty$.
- C, C_i $(i=1,2,3\cdots)$ denotes various positive constants which may vary from one line to another and which is not important for the analysis of the problem.

The paper is organized as follows. In Section 2, we set up the variational framework and present some preliminaries results. In Section 3, we will prove Theorems 1.1 and 1.2. In Section 4, we will prove Theorem 1.3. Section 5 is devoted to proving Theorems 1.4, 1.5 and 1.6.

2. Preliminaries

We define the homogeneous fractional Sobolev space $D^{s,2}(\mathbb{R}^3)$ as follows

$$D^{s,2}(\mathbb{R}^3) = \{u \in L^{2_s^*}(\mathbb{R}^3) : |\xi|^s \widehat{u}(\xi) \in L^2(\mathbb{R}^3)\}$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{D^{s,2}}^2 = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx = \int_{\mathbb{R}^3} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi.$$

The fractional Sobolev space $H^s(\mathbb{R}^3)$ can be described by means of the Fourier transform, i.e.

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2s} |\widehat{u}(\xi)|^2 + |\widehat{u}(\xi)|^2) d\xi < \infty \right\}.$$

In this case, the norm is defined as

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^3} (|\xi|^{2s} |\widehat{u}(\xi)|^2 + |\widehat{u}(\xi)|^2) d\xi.$$

From Plancherel's theorem we have $|u|_2 = |\widehat{u}|_2$ and $||\xi|^s \widehat{u}|_2 = |(-\Delta)^{\frac{s}{2}} u|_2$. Hence

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + u^2) dx.$$

Now, let

$$E = \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x) u^2 dx < \infty \right\}$$

with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x) uv) dx$$

and the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + V(x) u^2) dx.$$

It is clear that $E \hookrightarrow H^s(\mathbb{R}^3)$. In fact, by virtue of $(V_1) - (V_2)$, Hölder's inequality and Sobolev inequality, it is easy to deduce that

$$\begin{aligned}
\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + u^2) dx &\le \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + |\mathcal{V}_b|^{\frac{2_s^*-2}{2_s^*}} \left(\int_{\mathcal{V}_b} |u|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \\
&\quad + \frac{1}{b} \int_{\mathbb{R}^3 \setminus \mathcal{V}_b} V(x) u^2 dx \\
&\le \max\{1 + |\mathcal{V}_b|^{\frac{2_s^*-2}{2_s^*}} S_s^{-1}, b^{-1}\} \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + V(x) u^2) dx.
\end{aligned}$$

Now, we set

$$E_\lambda = (E, \|u\|_\lambda),$$

where

$$\|u\|_\lambda^2 = \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda V(x) u^2) dx.$$

Thus, there exists $L_s > 0$ (independent of $\lambda \ge 1$) such that

$$|u|_s \le L_s \|u\| \le L_s \|u\|_\lambda \quad \text{for} \quad s \in [2, 2_s^*]. \tag{2.1}$$

It is easy to show that problem (1.1) can be reduced to a single fractional Schrödinger equation with a nonlocal term. More precisely, for $u \in H^s(\mathbb{R}^3)$, consider the linear functional $f_u(v)$ defined in $D^{s,2}(\mathbb{R}^3)$ by

$$f_u(v) = \int_{\mathbb{R}^3} u^2 v dx.$$

By Hölder's inequality and Sobolev inequality, we can get that

$$\begin{aligned} |f_u(v)| &\le \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2s}} dx\right)^{\frac{3+2s}{6}} \left(\int_{\mathbb{R}^3} |v|^{2_s^*} dx\right)^{\frac{1}{2_s^*}} \\ &\le S_s^{-\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2s}} dx\right)^{\frac{3+2s}{6}} \|v\|_{D^{s,2}} \\ &\le S_s^{-\frac{1}{2}} C\|u\|_{H^s}^2 \|v\|_{D^{s,2}}, \end{aligned}$$

where using the following fact that $H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2s}}$ if $s > \frac{1}{2}$. By the Lax-Milgram theorem, there exists a unique $\phi_u^s \in D^{s,2}(\mathbb{R}^3)$ such that $(-\Delta)^s \phi_u^s = u^2$ and possesses an explicit formula

$$\phi_u^s(x) = c_s \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2s}} dy, \quad x \in \mathbb{R}^3, \quad c_s = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(\frac{3-2s}{2})}{\Gamma(s)}.$$

Let us summarize some properties of the function ϕ_u^s, their proof can be found in [38].

Lemma 2.1. *If $s \in (\frac{1}{2}, 1)$, then for any $u \in H^s(\mathbb{R}^3)$, we have*
(i) $\phi_u^s \ge 0$*;*
(ii) $\phi_{\rho u}^s = \rho^2 \phi_u^s$, $\forall \rho > 0$*;*
(iii) $\int_{\mathbb{R}^3} \phi_u^s u^2 dx \le C|u|^4_{\frac{12}{3+2s}} \le C\|u\|^4_{H^s}$*;*
(iv) If $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$*, then* $\phi_{u_n}^s \rightharpoonup \phi_u^s$ *in* $D^{s,2}(\mathbb{R}^3)$*;*
(v) If $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$ *and* $u_n \to u$ *in* $L^r(\mathbb{R}^3)$ *for* $2 \le r < 2_s^*$*, then*

$$\int_{\mathbb{R}^3} \phi_{u_n}^s u_n v dx \to \int_{\mathbb{R}^3} \phi_u^s u v dx \quad \textit{for all } v \in H^s(\mathbb{R}^3),$$

and

$$\int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx \to \int_{\mathbb{R}^3} \phi_u^s u^2 dx.$$

Substituting ϕ_u^s in (1.1), we can rewrite (1.1) as follows

$$(-\Delta)^s u + \lambda V(x) u + \mu \phi_u^s u = |u|^{p-2} u. \tag{2.2}$$

In order to find weak solutions to (2.2), we look for critical points of the functional $J_{\lambda,\mu}(u) : E_\lambda \to \mathbb{R}$ associated with (2.2) which is defined by

$$J_{\lambda,\mu}(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda V(x) u^2) dx + \frac{\mu}{4} \int_{\mathbb{R}^3} \phi_u^s u^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} |u^+|^p dx.$$

This is a well defined C^1-functional with derivative given by

$$\langle J'_{\lambda,\mu}(u), v\rangle = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} \lambda V(x) u v dx + \mu \int_{\mathbb{R}^3} \phi_u^s u v dx - \int_{\mathbb{R}^3} |u^+|^{p-2} u^+ v dx,$$

for all $u, v \in H^s(\mathbb{R}^3)$.

Next, we give a stronger version of the Mountain Pass Theorem.

Lemma 2.2 ([18]). *Let X be a real Banach space with its dual space X', and suppose that $J \in C^1(X, \mathbb{R})$ satisfies*

$$\max\{J(0), J(e)\} \leq \mu < \eta \leq \inf_{\|u\|_X = \rho} J(u)$$

for some $\mu < \eta$, $\rho > 0$ and $e \in X$ with $\|e\|_X > \rho$. Let $c \geq \eta$ be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e. Then there exists a sequence $\{u_n\} \subset X$ such that

$$J(u_n) \to c \geq \eta \quad \text{and} \quad (1 + \|u_n\|_X)\|J'(u_n)\|_{X'} \to 0,$$

as $n \to \infty$.

The vanishing Lemma for fractional Sobolev space is stated as follows.

Lemma 2.3 ([37]). *Assume that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and it satisfies*

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_\rho(y)} |u_n(x)|^2 dx = 0,$$

where $\rho > 0$. Then $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2_s^*$.

Lemma 2.4. *Suppose that $(V_1) - (V_2)$ hold with $2 < p < 4$. Then every nontrivial critical point of $J_{\lambda,\mu}$ is a positive solution of problem (1.1).*

Proof. Let $u \in E_\lambda$ is a nontrivial critical point of $J_{\lambda,\mu}$, we have

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} \lambda V(x) uv dx + \mu \int_{\mathbb{R}^3} \phi_u^s uv dx - \int_{\mathbb{R}^3} |u^+|^{p-2} u^+ v dx = 0,$$

for every $v \in E_\lambda$. Taking $v = u^- = \max\{-u, 0\}$, we deduce that

$$\int\int_{\mathbb{R}^6} \frac{(u(x) - u(y))(u^-(x) - u^-(y))}{|x-y|^{3+2s}} dxdy - \int_{\mathbb{R}^3} \lambda V(x)|u^-|^2 dx - \mu \int_{\mathbb{R}^3} \phi_u^s |u^-|^2 dx = 0,$$

which implies that

$$\int\int_{\mathbb{R}^6} \frac{(u(x) - u(y))(u^-(x) - u^-(y))}{|x-y|^{3+2s}} dxdy \geq 0. \tag{2.3}$$

On the other hand, by direct computation, it follows that

$$\begin{aligned} &\int\int_{\mathbb{R}^6} \frac{(u(x) - u(y))(u^-(x) - u^-(y))}{|x-y|^{3+2s}} dxdy \\ &= \int_{\{u(x)\geq 0\}\times\{u(y)<0\}} \frac{(u(x) - u(y))u(y)}{|x-y|^{3+2s}} dxdy \\ &\quad + \int_{\{u(x)<0\}\times\{u(y)\geq 0\}} \frac{(u(y) - u(x))u(x)}{|x-y|^{3+2s}} dxdy \\ &\quad - \int_{\{u(x)<0\}\times\{u(y)<0\}} \frac{(u(x) - u(y))^2}{|x-y|^{3+2s}} dxdy \leq 0. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we have

$$\int\int_{\mathbb{R}^6} \frac{(u(x) - u(y))(u^-(x) - u^-(y))}{|x-y|^{3+2s}} dxdy = 0,$$

which leads to $u^- = 0$, so, $u \geq 0$ and $u \not\equiv 0$. Similar argument to the proof of Proposition 4.4 in [38], we know that $u \in C^{1,\alpha}$ for some $\alpha \in (0,1)$. Then, by Lemma 3.2 in [16], we have

$$(-\Delta)^s u(x) = -\frac{1}{2}C(3,s)\int_{\mathbb{R}^3}\frac{u(x+y)+u(x-y)-2u(x)}{|x-y|^{3+2s}}dy, \quad \forall x \in \mathbb{R}^3.$$

Next, we show that $u > 0$. Assume by contradiction that there exists $x_0 \in \mathbb{R}^3$ such that $u(x_0) = 0$, then we can see that

$$(-\Delta)^s u(x_0) = -\frac{1}{2}C(3,s)\int_{\mathbb{R}^3}\frac{u(x_0+y)+u(x_0-y)}{|x_0-y|^{3+2s}}dy < 0,$$

since $u \geq 0$ and $u \not\equiv 0$. However, it is easy to see that

$$(-\Delta)^s u(x_0) = -\lambda V(x_0)u(x_0) - \mu(\phi_u^s)(x_0) + u(x_0)^{p-2}u(x_0) = 0,$$

which gives a contradiction. Hence $u > 0$ for all $x \in \mathbb{R}^3$. □

3. Proof of Theorems 1.1 and 1.2

In this section, we prove the existence and nonexistence of solutions to problem (1.1) when $V(x)$ satisfies the conditions $(V_1) - (V_3)$. Since we do not impose the 4-superlinear Ambrosetti-Rabinowtiz condition, the boundedness of the Cerami sequence becomes not easy to obtain. A penalization problem is introduced to overcome this difficulty. More precisely, we define a cut-off function $\psi \in C^1([0,\infty),\mathbb{R})$ satisfying $0 \leq \psi \leq 1$, $\psi(t) = 1$ if $0 \leq t \leq 1$, $\psi(t) = 0$ if $t \geq 2$, $\max_{t>0}|\psi'(t)| \leq 2$ and $\psi'(t) \leq 0$ for each $t > 0$.

Next, for $T > 0$, we consider the truncated functional $J_{\lambda,\mu}^T : E_\lambda \to \mathbb{R}$ defined by

$$J_{\lambda,\mu}^T(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{\mu}{4}\psi\left(\frac{\|u\|_\lambda^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_u^s u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3}|u^+|^p dx. \tag{3.1}$$

It is easy to check that $J_{\lambda,\mu}^T$ is well-defined, $J_{\lambda,\mu}^T \in C^1(E_\lambda,\mathbb{R})$ and its differential is given by

$$\begin{aligned}\langle (J_{\lambda,\mu}^T)'(u), v\rangle &= \int_{\mathbb{R}^3}((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + \lambda V(x)uv)dx + \mu\psi\left(\frac{\|u\|_\lambda^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_u^s uv dx \\ &\quad + \frac{\mu}{2T^2}\psi'\left(\frac{\|u\|_\lambda^2}{T^2}\right)\int_{\mathbb{R}^3}((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + \lambda V(x)uv)dx\int_{\mathbb{R}^3}\phi_u^s u^2 dx \\ &\quad - \int_{\mathbb{R}^3}|u^+|^{p-2}u^+ v dx,\end{aligned} \tag{3.2}$$

for all $u, v \in E_\lambda$. Clearly, if a Cerami sequence $\{u_n\}$ of $J_{\lambda,\mu}^T$ satisfying $\|u_n\|_\lambda \leq T$, then $\{u_n\}$ is also a Cerami sequence of $J_{\lambda,\mu}$ satisfying $\|u_n\|_\lambda \leq T$.

Now we show that the functional $J_{\lambda,\mu}^T$ possesses a mountain pass geometry.

Lemma 3.1. *Suppose that $2 < p < 4$ and $(V_1) - (V_3)$ hold. Then the functional $J_{\lambda,\mu}^T$ satisfies the following conditions:*
(i) for each $T, \mu > 0$ and $\lambda \geq 1$, there exists $\alpha, \rho > 0$ (independent of T, λ and μ) such that $J_{\lambda,\mu}^T(u) \geq \alpha$ for all $u \in E_\lambda$ with $\|u\|_\lambda = \rho$;
(ii) there exists $\mu^* > 0$ such that for each $T, \lambda > 0$ and $\mu \in (0,\mu^*)$, we have $J_{\lambda,\mu}^T(e_0) < 0$ for some $e_0 \in C_0^\infty(\Omega)$ with $\|e_0\|_\lambda > \rho$.

Proof. (i) For all $u \in E_\lambda$, by using (2.1), we get

$$J^T_{\lambda,\mu}(u) \geq \frac{1}{2}\|u\|^2_\lambda - \frac{1}{p}L^p_p\|u\|^p_\lambda,$$

where $L_p > 0$ is independent of T, μ and λ. Note that $p > 2$, the conclusion (i) follows by choosing $\rho > 0$ sufficiently small.

(ii) Define the functional $J_\lambda : E_\lambda \to \mathbb{R}$ by

$$J_\lambda(u) = \frac{1}{2}\|u\|^2_\lambda - \frac{1}{p}\int_{\mathbb{R}^3} |u^+|^p dx.$$

Since $2 < p < 4$, then (2.1) shows that J_λ is well-defined. Let $e \in C^\infty_0(\Omega)$ be a positive smooth function, it is easy to see that

$$J_\lambda(te) = \frac{t^2}{2}\int_\Omega |(-\Delta)^{\frac{s}{2}}e|^2 dx - \frac{t^p}{p}\int_\Omega |e|^p dx.$$

Since $p > 2$, we have $J_\lambda(te) \to -\infty$ as $t \to \infty$. Hence, there exists $e_0 \in C^\infty_0(\Omega)$ with $\|e_0\|_\lambda > \rho$ such that $J_\lambda(e_0) \leq -1$. By Lemma 2.1, we can see that

$$J^T_{\lambda,\mu}(e_0) = J_\lambda(e_0) + \frac{\mu}{4}\psi\left(\frac{\|e_0\|^2_\lambda}{T^2}\right)\int_{\mathbb{R}^3} \phi^s_{e_0} e_0^2 dx \leq -1 + \frac{\mu}{4}C|e_0|^4_{\frac{12}{3+2s}}.$$

Then there exists $\mu^* > 0$ such that for each $T, \lambda > 0$ and $\mu \in (0, \mu^*)$, we have $J^T_{\lambda,\mu}(e_0) < 0$ for some $e_0 \in C^\infty_0(\Omega)$ with $\|e_0\|_\lambda > \rho$. □

Now, we define the mountain pass value

$$c^T_{\lambda,\mu} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J^T_{\lambda,\mu}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0, \gamma(1) = e_0\}$. From Lemma 2.2 and Lemma 3.1, we know that for each $T > 0$, $\lambda \geq 1$ and $\mu \in (0, \mu^*)$, there exists a Cerami sequence $\{u_n\} \subset E_\lambda$ such that

$$J^T_{\lambda,\mu}(u_n) \to c^T_{\lambda,\mu} \quad \text{and} \quad (1 + \|u_n\|_\lambda)\|(J^T_{\lambda,\mu})'(u_n)\|_{E'_\lambda} \to 0. \tag{3.3}$$

Moreover, $c^T_{\lambda,\mu} \geq \alpha > 0$.

Next, we give an estimate on the upper bound of $c^T_{\lambda,\mu}$.

Lemma 3.2. *Suppose that $2 < p < 4$ and $(V_1) - (V_3)$ hold. Then for each $T > 0$, $\lambda \geq 1$ and $\mu \in (0, \mu^*)$, there exists $M > 0$ (independent of T, μ and λ) such that $c^T_{\lambda,\mu} \leq M$.*

Proof. For $e_0 \in C^\infty_0(\Omega)$, by Lemma 2.1, it is easy to check that

$$J^T_{\lambda,\mu}(te_0) \leq \frac{t^2}{2}\int_\Omega |(-\Delta)^{\frac{s}{2}}e_0|^2 dx + \frac{\mu^* t^4}{4}C\left(\int_\Omega |e_0|^{\frac{12}{3+2s}} dx\right)^{\frac{3+2s}{3}} - \frac{t^p}{p}\int_\Omega |e_0|^p dx,$$

which implies that there exists $M > 0$ (independent of T, μ and λ) such that

$$c^T_{\lambda,\mu} \leq \max_{t\in[0,1]} J^T_{\lambda,\mu}(te_0) \leq M.$$

□

Now, we show that for a given $T > 0$ properly, after passing to a subsequence, the sequence $\{u_n\}$ given by (3.3) is also a bounded Cerami sequence of $J_{\lambda,\mu}$ satisfying $\|u_n\|_\lambda \leq T$.

Lemma 3.3. *Suppose that $2 < p < 4$ and $(V_1) - (V_3)$ hold, and let $T = \sqrt{\frac{2p(M+1)}{p-2}}$. Then there exists $\mu_* \in (0, \mu^*)$ such that for each $\lambda \geq 1$ and $\mu \in (0, \mu_*)$, if $\{u_n\} \subset E_\lambda$ is a sequence satisfying (3.3), then we have, up to a subsequence, $\|u_n\|_\lambda \leq T$ that is $\{u_n\}$ is also a Cerami sequence at level $c^T_{b,\lambda}$ for $J_{\lambda,\mu}$.*

Proof. Firstly, we prove that $\|u_n\|_\lambda \le \sqrt{2}T$ for n large enough. Assume by contradiction that there exists a subsequence, still denoted by $\{u_n\}$, such that $\|u_n\|_\lambda > \sqrt{2}T$. From (3.1), (3.2) and the definition of ψ, it is easy to see that

$$\begin{aligned} c_{\lambda,\mu}^T &= \lim_{n\to\infty}\left(J_{\lambda,\mu}^T(u_n) - \frac{1}{p}\langle (J_{\lambda,\mu}^T)'(u_n), u_n\rangle\right) \\ &= \lim_{n\to\infty}[(\frac{1}{2}-\frac{1}{p})\|u_n\|_\lambda^2 - (\frac{\mu}{p}-\frac{\mu}{4})\psi\left(\frac{\|u_n\|_\lambda^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2 dx \\ &\quad - \frac{\mu}{2pT^2}\psi'\left(\frac{\|u_n\|_\lambda^2}{T^2}\right)\|u_n\|_\lambda^2\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2 dx] \\ &\ge \frac{p-2}{2p}2T^2 = 2\frac{p-2}{2p}\frac{2p(M+1)}{p-2} = 2(M+1), \end{aligned} \tag{3.4}$$

which gives a contradiction by Lemma 3.2. Therefore $\|u_n\|_\lambda \le \sqrt{2}T$ for n large enough.

Now, we show that $\|u_n\|_\lambda \le T$. Assume by contradiction that there exists a subsequence, still denoted by $\{u_n\}$, such that $T < \|u_n\|_\lambda \le \sqrt{2}T$ for n large enough. According to the definition of ψ and Lemma 2.1, we can see that

$$\begin{aligned} c_{\lambda,\mu}^T &= \lim_{n\to\infty}\left(J_{\lambda,\mu}^T(u_n) - \frac{1}{p}\langle (J_{\lambda,\mu}^T)'(u_n), u_n\rangle\right) \\ &\ge \liminf_{n\to\infty}\left((\frac{1}{2}-\frac{1}{p})\|u_n\|_\lambda^2 - (\frac{\mu}{p}-\frac{\mu}{4})CL^4_{\frac{12}{3+2s}}\|u_n\|_\lambda^4\right) \\ &\ge \frac{p-2}{2p}\frac{2p(M+1)}{p-2} - \frac{4-p}{4p}\mu CL^4_{\frac{12}{3+2s}}\frac{16p^2(M+1)^2}{(p-2)^2} \\ &= M+1 - \frac{4p(4-p)}{(p-2)^2}\mu CL^4_{\frac{12}{3+2s}}(M+1)^2, \end{aligned} \tag{3.5}$$

which implies a contradiction by choosing $\mu_* > 0$ small. □

Next, we need to establish the following parameter-dependent compactness lemma, which is crucial to prove our main result.

Lemma 3.4. *Suppose that $2 < p < 4$ and $(V_1) - (V_3)$ hold, and let $T = \sqrt{\frac{2p(M+1)}{p-2}}$. Then there exists $\lambda^* > 1$ such that for each $\mu \in (0, \mu_*)$ and $\lambda \in (\lambda^*, \infty)$, if $\{u_n\} \subset E_\lambda$ is a sequence satisfying* (3.3)*, then $\{u_n\}$ has a convergent subsequence in E_λ.*

Proof. From Lemma 3.3, we know that $\|u_n\|_\lambda \le T$. Up to a subsequence again, we may assume that there exists $u \in E_\lambda$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad E_\lambda, \quad \text{and} \quad \int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2 dx \ge \int_{\mathbb{R}^3}\phi_u^s u^2 dx.$$

In view of $J'_{\lambda,\mu}(u_n) \to 0$ and Lemma 2.1, we can see that

$$\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v dx + \int_{\mathbb{R}^3}\lambda V(x)uv dx + \mu\int_{\mathbb{R}^3}\phi_u^s uv dx - \int_{\mathbb{R}^3}|u^+|^{p-2}u^+ v dx = 0, \tag{3.6}$$

for all $v \in E_\lambda$. Taking $v = u$ as test function in (3.6), we get

$$\|u\|_\lambda^2 + \mu\int_{\mathbb{R}^3}\phi_u^s u^2 dx - \int_{\mathbb{R}^3}|u^+|^p dx = 0. \tag{3.7}$$

Let $v_n := u_n - u$. By (V_2), we can obtain that

$$|v_n|_2^2 = \int_{\mathbb{R}^3\setminus\mathcal{V}_b} v_n^2 dx + \int_{\mathcal{V}_b} v_n^2 dx \le \frac{1}{\lambda b}\|v_n\|_\lambda^2 + o(1). \tag{3.8}$$

From (3.8), Hölder's inequality and Sobolev inequality, we deduce that

$$\begin{aligned}|v_n|_p &\le |v_n|_2^{\theta}|v_n|_{2_s^*}^{1-\theta} \le d_0|v_n|_2^{\theta}|(-\Delta)^{\frac{s}{2}}v_n|_2^{1-\theta}\\ &\le d_0(\lambda b)^{-\frac{\theta}{2}}\|v_n\|_\lambda^{\theta}|(-\Delta)^{\frac{s}{2}}v_n|_2^{1-\theta}+o(1)\\ &\le d_0(\lambda b)^{-\frac{\theta}{2}}\|v_n\|_\lambda^{\theta}\|v_n\|_\lambda^{1-\theta}+o(1)\\ &= d_0(\lambda b)^{-\frac{\theta}{2}}\|v_n\|_\lambda+o(1),\end{aligned} \tag{3.9}$$

where θ satisfies that $\frac{1}{p}=\frac{\theta}{2}+\frac{1-\theta}{2_s^*}$, $\theta\in(0,1)$ and the constant $d_0>0$ is independent of μ and λ. By (2.1), (3.7) and (3.9), we have

$$\begin{aligned}o(1) &= \langle J'_{\lambda,\mu}(u_n),u_n\rangle-\left[\|u\|_\lambda^2+\mu\int_{\mathbb{R}^3}\phi_u^s u^2dx-\int_{\mathbb{R}^3}|u^+|^pdx\right]\\ &= \|u_n\|_\lambda^2+\mu\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx-\int_{\mathbb{R}^3}|u_n^+|^pdx-\|u\|_\lambda^2-\mu\int_{\mathbb{R}^3}\phi_u^s u^2dx+\int_{\mathbb{R}^3}|u^+|^pdx\\ &\ge \|v_n\|_\lambda^2-|v_n^+|_p^p+o(1)\\ &\ge \|v_n\|_\lambda^2-|v_n|_p^{p-2}|v_n|_p^2+o(1)\\ &\ge \|v_n\|_\lambda^2-(2L_pT)^{p-2}d_0^2(\lambda b)^{-\theta}\|v_n\|_\lambda^2+o(1)\\ &\ge [1-(2L_pT)^{p-2}d_0^2(\lambda b)^{-\theta}]\|v_n\|_\lambda^2+o(1).\end{aligned}$$

Therefore, there exists $\lambda^*>1$ such that $v_n\to0$ in E_λ for all $\lambda>\lambda^*$. □

Proof of Theorem 1.1. Let T be Defined as in Lemma 3.3. Form Lemma 2.2 and Lemma 3.1, we know that there exists $\mu^*>0$ such that for $\lambda\ge1$ and $\mu\in(0,\mu^*)$, $J_{\lambda,\mu}^T$ possesses a Cerami sequence $\{u_n\}\subset E_\lambda$ at mountain pass level $c_{\lambda,\mu}^T$. By using Lemma 3.2 and Lemma 3.3, it is easy to see that there exists $\mu_*\in(0,\mu^*)$ such that for $\lambda\ge1$ and $\mu\in(0,\mu_*)$, $\{u_n\}$ is also a Cerami sequence at level $c_{\lambda,\mu}^T$ for $J_{\lambda,\mu}$ satisfying $\|u_n\|_\lambda\le T$, that is

$$\sup_{n\in\mathbb{N}}\|u_n\|_\lambda\le T,\quad J_{\lambda,\mu}(u_n)\to c_{\lambda,\mu}^T\quad\text{and}\quad(1+\|u_n\|_\lambda)\|J'_{\lambda,\mu}(u_n)\|_{E'_\lambda}\to0.$$

By applying Lemma 3.4, we get that there exists $\lambda^*>1$ such that for each $\mu\in(0,\mu_*)$ and $\lambda\in(\lambda^*,\infty)$, then $\{u_n\}$ has a convergent subsequence in E_λ. We assume that $u_n\to u_{\lambda,\mu}$ as $n\to\infty$, then we have

$$\|u_{\lambda,\mu}\|_\lambda\le T,\quad J_{\lambda,\mu}(u_{\lambda,\mu})=c_{\lambda,\mu}^T\quad\text{and}\quad J'_{\lambda,\mu}(u_{\lambda,\mu})=0.$$

Consequently, Lemma 2.4 shows that $u_{\lambda,\mu}$ is a positive solution of problem (1.1). Finally, by using $\langle J'_{\lambda,\mu}(u_{\lambda,\mu}),u_{\lambda,\mu}\rangle$ and $u_{\lambda,\mu}\ne0$, we can see that

$$\|u_{\lambda,\mu}\|_\lambda^2\le|u_{\lambda,\mu}^+|_p^p\le L_p^p\|u_{\lambda,\mu}\|_\lambda^p,$$

which implies that there exists $\tau>0$ (independent of μ and λ) such that $\|u_{\lambda,\mu}\|_\lambda\ge\tau$. □

Now, we will prove the nonexistence of nontrivial solutions to problem (1.1) when λ and μ are large enough.

Proof of Theorem 1.2. Suppose that $(u,\phi)\in E_\lambda\times D^{s,2}(\mathbb{R}^3)$ is a solution of (1.1). Multiplying the first equation of (1.1) by u and integrate over $\mathbb{R}^3$, we can obtain that

$$\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u|^2+\lambda V(x)u^2)dx+\mu\int_{\mathbb{R}^3}\phi u^2dx-\int_{\mathbb{R}^3}|u|^pdx=0. \tag{3.10}$$

Multiplying the first equation of (1.1) by ϕ and integrate over $\mathbb{R}^3$, by using Plancherel Theorem and $\phi=\overline{\phi}$, we get

$$\int_{\mathbb{R}^3}\phi u^2dx=\int_{\mathbb{R}^3}(-\Delta)^s\phi\phi dx=\int_{\mathbb{R}^3}(|\xi|^{2s}\widehat{\phi})\overline{\widehat{\phi}}d\xi=\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\phi|^2dx. \tag{3.11}$$

Multiplying the first equation of (1.1) by $|u|$, and using the element fact that $z_1\overline{z_2}+\overline{z_1}z_2 \leq |z_1|^2+|z_2|^2$ if $z_1\overline{z_2}=\overline{z_1}z_2$ for any $z_1, z_2 \in \mathbb{C}$, we have

$$\begin{aligned}\int_{\mathbb{R}^3}|u|^3dx &= \int_{\mathbb{R}^3}(-\Delta)^s\phi|u|dx = \int_{\mathbb{R}^3}|\xi|^{2s}\widehat{\phi}\overline{\widehat{|u|}}d\xi\\ &= \int_{\mathbb{R}^3}(\sqrt{2\mu}|\xi|^s\widehat{\phi})(\frac{1}{\sqrt{2\mu}}|\xi|^s\overline{\widehat{|u|}})d\xi\\ &= \frac{1}{2}\int_{\mathbb{R}^3}[(\sqrt{2\mu}|\xi|^s\widehat{\phi})(\frac{1}{\sqrt{2\mu}}|\xi|^s\overline{\widehat{|u|}})+(\sqrt{2\mu}|\xi|^s\overline{\widehat{|\phi|}})(\frac{1}{\sqrt{2\mu}}|\xi|^s\widehat{|u|})]d\xi\\ &\leq \frac{1}{2}\left(\int_{\mathbb{R}^3}(\sqrt{2\mu}|\xi|^s\widehat{\phi})^2d\xi+\int_{\mathbb{R}^3}(\frac{1}{\sqrt{2\mu}}|\xi|^s\widehat{|u|})^2d\xi\right)\\ &= \frac{1}{4\mu}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx+\mu\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\phi|^2dx\\ &= \frac{1}{4\mu}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx+\mu\int_{\mathbb{R}^3}\phi u^2dx.\end{aligned} \tag{3.12}$$

If $p=3$, for $\lambda>0$ and $\mu>\frac{1}{4}$, from (3.10) and (3.12), we have

$$\begin{aligned}0 &= \int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u|^2+\lambda V(x)u^2)dx+\mu\int_{\mathbb{R}^3}\phi u^2dx-\int_{\mathbb{R}^3}|u|^3dx\\ &\geq (1-\frac{1}{4\mu})\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx,\end{aligned}$$

which implies that $u\equiv 0$.

If $2<p<3$, for $\lambda>\frac{1}{b}$ and $\mu \geq \frac{1}{4\left(1-|\mathcal{V}_b|^{\frac{2_s^*-2}{2_s^*}}S_s^{-1}\right)}$, from (3.10) and (3.12), we have

$$\begin{aligned}0 &= \int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u|^2+\lambda V(x)u^2)dx+\mu\int_{\mathbb{R}^3}\phi u^2dx-\int_{\mathbb{R}^3}|u|^pdx\\ &\geq (1-\frac{1}{4\mu})\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx+\int_{\mathbb{R}^3\setminus\mathcal{V}_b}u^2dx+\int_{\mathbb{R}^3}|u|^3dx-\int_{\mathbb{R}^3}|u|^pdx\\ &\geq \left(1-\frac{1}{4\mu}-|\mathcal{V}_b|^{\frac{2_s^*-2}{2_s^*}}S_s^{-1}\right)\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx+\int_{\mathbb{R}^3}u^2dx+\int_{\mathbb{R}^3}|u|^3dx-\int_{\mathbb{R}^3}|u|^pdx\\ &\geq \int_{\mathbb{R}^3}(|u|^3+u^2-|u|^p)dx.\end{aligned}$$

Since the function $t^2+t^3-t^p$ for $2<p<3$ is nonnegative for all $t>0$. Hence, $u\equiv 0$. □

4. Proof of Theorem 1.3

In this section, we study the decay rate of the positive solutions for (1.1) at infinity. For this purpose, we always assume that for each $\lambda\in(\lambda^*,\infty)$ and $\mu\in(0,\mu_*)$, $u_{\lambda,\mu}$ is the positive solution of (1.1) obtained by Theorem 1.1. Firstly, let us give an important estimate involving the L^∞-norm of $u_{\lambda,\mu}$ under the assumptions of Theorem 1.3.

Lemma 4.1. *Under the assumptions of Theorem 1.3, the positive solution* $u_{\lambda,\mu}\in L^\infty(\mathbb{R}^3)\cap C^{1,\alpha}(\mathbb{R}^3)$ *for some* $\alpha<2s-1$, *and there exists* $C_0>0$ *such that*

$$|u_{\lambda,\mu}|_\infty \leq C_0, \quad \text{for all } \mu \text{ and } \lambda.$$

Moreover,

$$\lim_{|x|\to\infty}u_{\lambda,\mu}(x)=0.$$

Proof. For $\beta \geq 1$ and $\widehat{T} > 0$, we define

$$\varphi(t) = \begin{cases} 0, & t \leq 0, \\ t^{\beta}, & 0 < t < \widehat{T}, \\ \beta \widehat{T}^{\beta-1}(t-\widehat{T}) + \widehat{T}^{\beta}, & t \geq \widehat{T}. \end{cases}$$

Clearly, φ is convex and Lipschitz continuous, then we have

$$(-\Delta)^s \varphi(u_{\lambda,\mu}) \leq \varphi'(u_{\lambda,\mu})(-\Delta)^s u_{\lambda,\mu}, \tag{4.1}$$

in the weak sense. By direct computation, we have

$$\begin{aligned} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \varphi(u_{\lambda,\mu})|^2 dx &= \int\int_{\mathbb{R}^6} \frac{|\varphi(u_{\lambda,\mu})(x) - \varphi(u_{\lambda,\mu})(y)|^2}{|x-y|^{3+2s}} dxdy \\ &\leq \beta \widehat{T}^{\beta-1} \int\int_{\mathbb{R}^6} \frac{|u_{\lambda,\mu}(x) - u_{\lambda,\mu}(y)|^2}{|x-y|^{3+2s}} dxdy \\ &< +\infty, \end{aligned}$$

which implies that $\varphi(u_{\lambda,\mu}) \in D^{s,2}(\mathbb{R}^3)$. By using Sobolev inequality, (V_1), (4.1) and integrating by parts, we can deduce that

$$\begin{aligned} |\varphi(u_{\lambda,\mu})|_{2_s^*}^2 &\leq S_s^{-1} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \varphi(u_{\lambda,\mu})|^2 dx \\ &= S_s^{-1} \int_{\mathbb{R}^3} \varphi(u_{\lambda,\mu})(-\Delta)^s \varphi(u_{\lambda,\mu}) dx \\ &\leq S_s^{-1} \int_{\mathbb{R}^3} \varphi(u_{\lambda,\mu})\varphi'(u_{\lambda,\mu})(-\Delta)^s u_{\lambda,\mu} dx \\ &= S_s^{-1} \int_{\mathbb{R}^3} \varphi(u_{\lambda,\mu})\varphi'(u_{\lambda,\mu})[-\lambda V(x) u_{\lambda,\mu} - \mu \phi_{u_{\lambda,\mu}} u_{\lambda,\mu} + |u_{\lambda,\mu}^+|^{p-2} u_{\lambda,\mu}^+] dx \\ &\leq C_1 \int_{\mathbb{R}^3} \varphi(u_{\lambda,\mu})\varphi'(u_{\lambda,\mu})(1 + u_{\lambda,\mu}^{2_s^*-1}) dx \\ &= C_1 \left(\int_{\mathbb{R}^3} \varphi(u_{\lambda,\mu})\varphi'(u_{\lambda,\mu}) dx + \int_{\mathbb{R}^3} \varphi(u_{\lambda,\mu})\varphi'(u_{\lambda,\mu}) u_{\lambda,\mu}^{2_s^*-1} dx \right), \end{aligned}$$

where $C_1 > 0$ independent of μ, λ and β. Now by using the fact that

$$\varphi'(u_{\lambda,\mu})\varphi(u_{\lambda,\mu}) \leq \beta u_{\lambda,\mu}^{2\beta-1} \quad \text{and} \quad u_{\lambda,\mu}\varphi'(u_{\lambda,\mu}) \leq \beta \varphi(u_{\lambda,\mu}),$$

we can see that

$$|\varphi(u_{\lambda,\mu})|_{2_s^*}^2 \leq C_1 \beta \left(\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2\beta-1} dx + \int_{\mathbb{R}^3} (\varphi(u_{\lambda,\mu}))^2 u_{\lambda,\mu}^{2_s^*-2} dx \right). \tag{4.2}$$

By direct computation, we can obtain that

$$\begin{aligned} \int_{\mathbb{R}^3} (\varphi(u_{\lambda,\mu}))^2 u_{\lambda,\mu}^{2_s^*-2} dx &= \int_{\{u_{\lambda,\mu} \leq \widehat{T}\}} (\varphi(u_{\lambda,\mu}))^2 u_{\lambda,\mu}^{2_s^*-2} dx + \int_{\{u_{\lambda,\mu} > \widehat{T}\}} (\varphi(u_{\lambda,\mu}))^2 u_{\lambda,\mu}^{2_s^*-2} dx \\ &\leq \widehat{T}^{2\beta-2} \int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*} dx + C \int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*} dx < +\infty, \end{aligned}$$

where we have used that $\beta \geq 1$ and that $\varphi(u_{\lambda,\mu})$ is linear when $u_{\lambda,\mu} \geq \widehat{T}$. Then, the above inequality shows that $\int_{\mathbb{R}^3} (\varphi(u_{\lambda,\mu}))^2 u_{\lambda,\mu}^{2_s^*-2} dx$ is well defined for every $\widehat{T}$.

Now, we choose β in (4.2) such that $2\beta - 1 = 2_s^*$ and let

$$\beta_1 = \frac{2_s^* + 1}{2}.$$

Let $R > 0$ be fixed later, by using Hölder's inequality, we can get that

$$\begin{aligned}\int_{\mathbb{R}^3} (\varphi(u_{\lambda,\mu}))^2 u_{\lambda,\mu}^{2_s^*-2} dx &= \int_{\{u_{\lambda,\mu} \le R\}} (\varphi(u_{\lambda,\mu}))^2 u_{\lambda,\mu}^{2_s^*-2} dx + \int_{\{u_{\lambda,\mu} > R\}} (\varphi(u_{\lambda,\mu}))^2 u_{\lambda,\mu}^{2_s^*-2} dx \\ &\le R^{2_s^*-1} \int_{\{u_{\lambda,\mu} \le R\}} \frac{(\varphi(u_{\lambda,\mu}))^2}{u_{\lambda,\mu}} dx \\ &\quad + \left(\int_{\{u_{\lambda,\mu} > R\}} u_{\lambda,\mu}^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}} \left(\int_{\mathbb{R}^3} (\varphi(u_{\lambda,\mu}))^{2_s^*} dx \right)^{\frac{2}{2_s^*}}.\end{aligned} \tag{4.3}$$

Since $u_{\lambda,\mu}$ is bounded in E_λ, so we can choose R sufficiently large such that

$$\left(\int_{\{u_{\lambda,\mu} > R\}} u_{\lambda,\mu}^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}} \le \frac{1}{2C_1\beta_1}. \tag{4.4}$$

From (4.2)–(4.4), it is easy to check that

$$\left(\int_{\mathbb{R}^3} (\varphi(u_{\lambda,\mu}))^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \le 2C_1\beta_1 \left(\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*} dx + R^{2_s^*-1} \int_{\mathbb{R}^3} \frac{(\varphi(u_{\lambda,\mu}))^2}{u_{\lambda,\mu}} dx \right). \tag{4.5}$$

Then, by using $\varphi(u_{\lambda,\mu}) \le u_{\lambda,\mu}^{\beta_1}$ and let $\widehat{T} \to \infty$, we deduce that

$$\left(\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*\beta_1} dx \right)^{\frac{2}{2_s^*}} \le 2C_1\beta_1 \left(\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*} dx + R^{2_s^*-1} \int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*} dx \right) < \infty,$$

which implies that

$$u_{\lambda,\mu} \in L^{2_s^*\beta_1}(\mathbb{R}^3). \tag{4.6}$$

Now we suppose $\beta > \beta_1$. Thus, by using $\varphi(u_{\lambda,\mu}) \le u_{\lambda,\mu}^{\beta}$, (4.2) and taking $\widehat{T} \to \infty$, we have

$$\left(\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*\beta} dx \right)^{\frac{2}{2_s^*}} \le C_1\beta \left(\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2\beta-1} dx + \int_{\mathbb{R}^3} u_{\lambda,\mu}^{2\beta+2_s^*-2} dx \right). \tag{4.7}$$

Let

$$u_{\lambda,\mu}^{2\beta-1} = u_{\lambda,\mu}^{l} u_{\lambda,\mu}^{k},$$

where $l = \frac{2_s^*(2_s^*-1)}{2(\beta-1)}$ and $k = 2\beta - 1 - l$. Moreover, $\beta > \beta_1$ implies that $0 < l, k < 2_s^*$, by using Young's inequality with exponents

$$r = \frac{2_s^*}{l} \quad \text{and} \quad r' = \frac{2_s^*}{2_s^* - l},$$

we can deduce that

$$\begin{aligned}\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2\beta-1} dx &\le \frac{l}{2_s^*} \int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*} dx + \frac{2_s^* - l}{2_s^*} \int_{\mathbb{R}^3} u_{\lambda,\mu}^{\frac{2_s^* k}{2_s^* - l}} dx \\ &\le \int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*} dx + \int_{\mathbb{R}^3} u_{\lambda,\mu}^{2\beta+2_s^*-2} dx \\ &\le C \left(1 + \int_{\mathbb{R}^3} u_{\lambda,\mu}^{2\beta+2_s^*-2} dx \right).\end{aligned} \tag{4.8}$$

From (4.7) and (4.8), we can see that

$$\left(\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*\beta}dx\right)^{\frac{2}{2_s^*}} \le C\beta\left(1+\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2\beta+2_s^*-2}dx\right). \tag{4.9}$$

Consequently, we have

$$\left(1+\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*\beta}dx\right)^{\frac{1}{2_s^*(\beta-1)}} \le (C\beta)^{\frac{1}{2(\beta-1)}}\left(1+\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2\beta+2_s^*-2}dx\right)^{\frac{1}{2(\beta-1)}}. \tag{4.10}$$

Iterating this argument, we obtain

$$\left(1+\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*\beta_{i+1}}dx\right)^{\frac{1}{2_s^*(\beta_{i+1}-1)}} \le (C\beta_{i+1})^{\frac{1}{2(\beta_{i+1}-1)}}\left(1+\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*\beta_i}dx\right)^{\frac{1}{2(\beta_i-1)}}, \tag{4.11}$$

where

$$2\beta_{i+1}+2_s^*-2 = 2_s^*\beta_i \quad \text{and} \quad \beta_{i+1} = (\frac{2_s^*}{2})^i(\beta_i-1)+1.$$

Denoting $C_{i+1} = C\beta_{i+1}$ and

$$K_i = \left(1+\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*\beta_i}dx\right)^{\frac{1}{2_s^*(\beta_i-1)}}.$$

We can see that there exists a constant $C_2 > 0$ independent of i , such that

$$K_{i+1} \le \Pi_{i=2}^{i+1} C_i^{\frac{1}{2(\beta_i-1)}} K_1 \le C_2K_1.$$

Hence, we can obtain that

$$|u_{\lambda,\mu}|_\infty \le C_0, \quad \text{for all } \mu \text{ and } \lambda.$$

From (4.6) and (4.7), we have

$$\begin{aligned}|u_{\lambda,\mu}|_\infty &\le C\left(\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*}dx + \int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*\beta_1}dx\right)^{\frac{1}{2_s^*(\beta_1-1)}}\\ &\le C\left[\left(\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*}dx\right)^{\frac{1}{2_s^*(\beta_1-1)}} + \left(\int_{\mathbb{R}^3} u_{\lambda,\mu}^{2_s^*}dx\right)^{\frac{1}{2(\beta_1-1)}}\right],\end{aligned}$$

where $C > 0$ independent of $u_{\lambda,\mu}$ and $\beta_1 = \frac{2_s^*+1}{2}$. This yields $u_{b,\lambda} \in L^r(\mathbb{R}^3)$ for all $r \in [2,+\infty]$. Moreover, $V(x) \in L^\infty(\mathbb{R}^3)$ and $\phi_{u_{\lambda,\mu}}^s \in L^\infty(\mathbb{R}^3)$. Hence, according to Proposition 2.9 in [34], we can get $u_{\lambda,\mu} \in C^{1,\alpha}(\mathbb{R}^3)$ for all $\alpha < 2s-1$ when $\frac{3}{4} < s < 1$. Finally, the fact $u_{\lambda,\mu} \in L^r(\mathbb{R}^3)\cap C^{1,\alpha}$ for all $2 \le r \le \infty$ implies that $\lim_{|x|\to\infty} u_{\lambda,\mu}(x) = 0$. □

Next, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. By Lemma 4.3 in [19], there exists a function w such that

$$0 < w \le \frac{C}{1+|x|^{3+2s}}, \tag{4.12}$$

and

$$(-\Delta)^s w + \frac{1}{2}w = 0, \quad \text{in } \mathbb{R}^3\backslash B_{R_1}(0), \tag{4.13}$$

for some suitable $R_1 > 0$. By direct computation, we can see that

$$\begin{aligned}(-\Delta)^s u_{\lambda,\mu} + \frac{1}{2}u_{\lambda,\mu} &= -\lambda V(x)u_{\lambda,\mu} - \mu\phi_{u_{\lambda,\mu}}^s u_{\lambda,\mu} + |u_{\lambda,\mu}|^{p-2}u_{\lambda,\mu} + \frac{1}{2}u_{\lambda,\mu}\\ &\le |u_{\lambda,\mu}|^{p-2}u_{\lambda,\mu} - \lambda V(x)u_{\lambda,\mu} + \frac{1}{2}u_{\lambda,\mu}.\end{aligned} \tag{4.14}$$

From (V_2'), there exists $R_2 > 0$ such that $\mathcal{V}_b \subset B_{R_2}(0)$ and so

$$V(x) \geq b, \quad \text{for } |x| \geq R_2. \tag{4.15}$$

By (4.14), (4.15) and Lemma 4.1, we can obtain that

$$(-\Delta)^s u_{\lambda,\mu} + \frac{1}{2} u_{\lambda,\mu} \leq u_{b,\lambda}(C_0^{p-2} - \lambda b + \frac{1}{2}). \tag{4.16}$$

Hence, there exists $\Lambda^* > \lambda^* > 1$ large such that $\lambda \in (\Lambda^*, \infty)$, we have

$$(-\Delta)^s u_{\lambda,\mu} + \frac{1}{2} u_{\lambda,\mu} \leq 0, \tag{4.17}$$

for $|x| \geq R_2$.

Let $R_3 = \max\{R_1, R_2\}$ and set

$$\gamma := \inf_{B_{R_3}(0)} w > 0 \quad \text{and} \quad w_{\lambda,\mu} = (k+1)w - \gamma u_{\lambda,\mu}, \tag{4.18}$$

where $k = \sup |u_{\lambda,\mu}|_\infty < \infty$. Now, we show that $w_{\lambda,\mu} \geq 0$ in $\mathbb{R}^3$. In fact, suppose by contradiction that there exists a sequence x_j such that

$$\inf_{x \in \mathbb{R}^3} w_{\lambda,\mu}(x) = \lim_{j \to \infty} w_{\lambda,\mu}(x_j) < 0. \tag{4.19}$$

By Lemma 4.1 and (4.12), we know that

$$\lim_{|x| \to \infty} w(x) = \lim_{|x| \to \infty} u_{\lambda,\mu}(x) = 0,$$

then (4.18) shows that

$$\lim_{|x| \to \infty} w_{\lambda,\mu}(x) = 0. \tag{4.20}$$

Putting together (4.19) and (4.20), we can see that $\{x_j\}$ is bounded and therefore, up to a subsequence, we may suppose that $x_j \to x_*$ for some $x_* \in \mathbb{R}^3$ as $j \to \infty$. By using (4.19), we get that

$$\inf_{x \in \mathbb{R}^3} w_{\lambda,\mu}(x) = w_{\lambda,\mu}(x_*) < 0. \tag{4.21}$$

Hence, from the minimality property of x_* and the integral representation of the fractional Laplace of $w_{\lambda,\mu}$ at the point x_*, we deduce that

$$(-\Delta)^s w_{\lambda,\mu}(x_*) = \frac{C_{3,s}}{2} \int_{\mathbb{R}^3} \frac{2w_{\lambda,\mu}(x_*) - w_{\lambda,\mu}(x_* + y) - w_{\lambda,\mu}(x_* - y)}{|y|^{3+2s}} dy \leq 0. \tag{4.22}$$

By (4.18), we can obtain that

$$w_{\lambda,\mu}(x) \geq k\gamma + w - k\gamma > 0, \quad \text{in } B_{R_3}(0),$$

then, (4.19) shows that

$$x_* \in \mathbb{R}^3 \backslash B_{R_3}(0). \tag{4.23}$$

Putting together (4.12), (4.13), (4.17) and (4.18), we have

$$(-\Delta)^s w_{\lambda,\mu} + \frac{1}{2} w_{\lambda,\mu} \geq 0, \quad \text{in } \mathbb{R}^3 \backslash B_{R_3}(0). \tag{4.24}$$

So, by (4.21)–(4.24), we can see that

$$0 \leq (-\Delta)^s w_{\lambda,\mu}(x_*) + \frac{1}{2} w_{\lambda,\mu}(x_*) < 0,$$

which is a contradiction, then $w_{\lambda,\mu} \geq 0$ in $\mathbb{R}^3$. Therefore, from (4.12), we have

$$u_{\lambda,\mu} \leq (k+1)\gamma^{-1} w \leq \frac{C}{1 + |x|^{3+2s}}.$$

□

5. Proofs of Theorems 1.4, 1.5 and 1.6

In the last section, we study the asymptotic behavior of positive solutions for (1.1) and give the proofs of Theorems 1.4, 1.5 and 1.6.

Proof of Theorem 1.4. Let $\mu \in (0, \mu_*)$ be fixed, then for any sequence $\lambda_n \to \infty$, let $u_n := u_{\lambda_n,\mu}$ be the positive solution of (1.1) obtained by Theorem 1.1. By (1.3), we know that

$$0 < \tau \le \|u_n\|_{\lambda_n} \le T \quad \text{for all } n. \tag{5.1}$$

Up to a subsequence, we may assume that there exists $u_\mu \in E$ such that

$$\begin{cases} u_n \rightharpoonup u_\mu, \text{ in } E, \\ u_n \to u_\mu, \text{ in } L^t_{loc}(\mathbb{R}^3) \quad \text{for } t \in [2, 2^*_s), \\ u_n \to u_\mu, \text{ a.e. on } \mathbb{R}^3. \end{cases} \tag{5.2}$$

From (5.1), (5.2) and Fatou's Lemma, it follows that

$$\int_{\mathbb{R}^3} V(x) u_\mu^2 dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^3} V(x) u_n^2 dx \le \liminf_{n\to\infty} \frac{\|u_n\|_{\lambda_n}}{\lambda_n} = 0,$$

which implies that $u_\mu = 0$ a.e. in $\mathbb{R}^3 \backslash V^{-1}(0)$. Hence, the condition (V_3) shows that $u_\mu \in H^s_0(\Omega)$.

Next, we show that $u_n \to u_\mu$ in $L^t(\mathbb{R}^3)$ for $t \in (2, 2^*_s)$. If not, by Lemma 2.3, we can get that there exist $\delta, r > 0$ and $x_n \in \mathbb{R}^3$ such that

$$\int_{B_r(x_n)} (u_n - u_\mu)^2 dx \ge \delta,$$

which implies that $|x_n| \to \infty$, then $|B_r(x_n) \cap \mathcal{V}_b| \to 0$. Hence, by using Hölder's inequality, it is easy to check that

$$\int_{B_r(x_n)\cap\mathcal{V}_b} (u_n - u_\mu)^2 dx \to 0.$$

Hence

$$\begin{aligned} \|u_n\|^2_{\lambda_n} &\ge \lambda_n b \int_{B_r(x_n)\cap\{V\ge b\}} u_n^2 dx = \lambda_n b \int_{B_r(x_n)\cap\{V\ge b\}} (u_n - u_\mu)^2 dx \\ &= \lambda_n b \left(\int_{B_r(x_n)} (u_n - u_\mu)^2 dx - \int_{B_r(x_n)\cap\mathcal{V}_b} (u_n - u_\mu)^2 dx \right) \\ &\to \infty, \end{aligned}$$

which contradicts (5.1).

Now, we show that $u_n \to u_\mu$ in E. Using $\langle J'_{\lambda_n,\mu}(u_n), u_n\rangle = \langle J'_{\lambda_n,\mu}(u_n), u_\mu\rangle = 0$, we can obtain that

$$\|u_n\|^2_{\lambda_n} + \mu \int_{\mathbb{R}^3} \phi^s_{u_n} u_n^2 dx = |u_n^+|^p_p, \tag{5.3}$$

and

$$\|u_\mu\|^2 + \mu \int_{\mathbb{R}^3} \phi^s_{u_\mu} u_\mu^2 dx = |u_n^+|^p_p + o(1). \tag{5.4}$$

By (5.2) and Fatou's Lemma, after passing to subsequence, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \phi^s_{u_n} u_n^2 dx \ge \int_{\mathbb{R}^3} \phi^s_{u_\mu} u_\mu^2 dx. \tag{5.5}$$

From (5.3)–(5.5), it follows that

$$\lim_{n\to\infty} \|u_n\|_{\lambda_n}^2 \le \|u_\mu\|^2.$$

It follows from the weakly lower semi-continuity of norm that

$$\|u_\mu\|^2 \le \liminf_{n\to\infty} \|u_n\|^2 \le \limsup_{n\to\infty} \|u_n\|^2 \le \lim_{n\to\infty} \|u_n\|_{\lambda_n}^2 \le \|u_\mu\|^2, \tag{5.6}$$

which implies that $u_n \to u_\mu$ in E.

Finally, we prove that u_μ is a positive solution of (1.4). For any $v \in C_0^\infty(\Omega)$, by using $\langle J'_{\lambda_n,\mu}(u_n), v\rangle = 0$, we can see that

$$\int_\Omega (-\Delta)^{\frac{s}{2}} u_\mu (-\Delta)^{\frac{s}{2}} v dx + \mu \int_\Omega \phi_{u_\mu}^s u_\mu v dx = \int_\Omega |u_\mu^+|^{p-2} u_\mu^+ v dx,$$

which implies that u_μ is a nonnegative solution of (1.4) by the density of $C_0^\infty(\Omega)$ in $H_0^s(\Omega)$. Moreover, by (5.1) and (5.6), we can see that

$$\|u_\mu\| = \lim_{n\to\infty} \|u_n\|_{\lambda_n} \ge \tau > 0,$$

which implies that $u_\mu \ne 0$. Similar to the proof Lemma 2.4, we obtain $u_\mu > 0$ in $\mathbb{R}^3$ and and this ends the proof of Theorem 1.4. □

Proof of Theorem 1.5. Let $\lambda \in (\lambda^*, \infty)$ be fixed, then for any sequence $\mu_n \to 0$, let $u_n := u_{\lambda,\mu_n}$ be the positive solution of (1.1) obtained by Theorem 1.1. By (1.3), we known that

$$0 < \tau \le \|u_n\|_\lambda \le T \quad \text{for all } n. \tag{5.7}$$

Up to a subsequence, we may assume that

$$u_n \rightharpoonup u_\lambda \quad \text{in } E_\lambda. \tag{5.8}$$

By $J'_{\lambda,\mu_n}(u_n) = 0$ and Lemma 3.4, we can obtain that $u_n \to u_\lambda$ in E_λ.

we prove that u_λ is a positive solution of (1.5). For any $v \in E_\lambda$, by using $\langle J'_{\lambda,\mu_n}(u_n), v\rangle = 0$, we can see that

$$\int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u_\lambda (-\Delta)^{\frac{s}{2}} v + \lambda V(x) u_\lambda v) dx = \int_{\mathbb{R}^3} |u_\lambda^+|^{p-2} u_\lambda^+ v dx,$$

which implies that u_λ is a nonnegative solution of (1.5). Moreover, (5.7) shows that $u_\lambda \ne 0$. Finally, similar to the proof Lemma 2.4, we obtain $u_\lambda > 0$ in $\mathbb{R}^3$ and this ends the proof of Theorem 1.5. □

Proof of Theorem 1.6. We can proceed exactly as in the proof of Theorem 1.4 and omit the details. □

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