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SUCCESSIVE ITERATIONS FOR POSITIVE EXTREMAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS ON THE HALF-LINE

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Abstract The authors study the existence of positive extremal solutions to the differential equation

$$-u'' + \lambda u = a\,(t)\, f(t, u(t)), \quad t \in I,$$

subject to the boundary conditions

$$u\,(0) = u\,(\infty) = 0,$$

where $I = (0, \infty)$, $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $a : I \to \mathbb{R}^+$, and $\lambda > 0$ is a parameter. Their results are obtained by using the monotone iterative method and are illustrated with an example.

Keywords Iterative methods, Green's function, fixed point, boundary value problems, positive solution, operators on a cone.

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1. Introduction

In 1837, Liouville introduced a fundamental approximation scheme in fixed point theory known as the fixed point iterative method. It was further developed by several mathematicians including Picard [22] in 1890. A number of mathematicians have used this method over the years, for example, see [12, 17, 20, 21, 24, 25].

In this work, we give sufficient conditions for the existence of a maximal and minimal positive solution to the second order boundary value problem posed on the half-line

$$\begin{cases} -u'' + \lambda u = a\,(t)\, f(t, u(t)), & t \in I, \\ u\,(0) = u\,(\infty) = 0, \end{cases} \tag{1.1}$$

where $I = (0, \infty)$, $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $a : I \to \mathbb{R}^+$, $\lambda > 0$ is a parameter, and there exists $t_0 \in I$ such that $f(t_0, 0) \neq 0$.

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Investigations of second order boundary value problems on the half-line have been conducted in a number of settings. For example, there have been studies of problems in which the equation under consideration is linear [5], nonlinear [3, 10, 24, 25], singular [2], contains the derivative of the unknown function [19, 23], contains a parameter [19], involves the Laplacian [9, 14, 15], be of the Kirchhoff type [13], or there may be impulse conditions involved [1,6,15,17]. In addition, the boundary conditions themselves may be of the Dirichlet type [9], the multi-point type [14, 18, 20], contain a functional [23], or the problem may be at resonance [14–16, 18]. The techniques used to prove the results have involved variational methods [7,9,11], critical point theory [4], or fixed point methods.

In our paper we choose to use a successive approximation approach to prove the existence of maximal and minimal positive solutions to problem (1.1). After presenting some preliminary concepts in the next section of the paper, our main result, its proof, and an example appear in Section 3.

2. Preliminaries

Let $G(t,s)$ the Green's function associated to our problem (1.1) which is given by

$$G(t,s) = \frac{1}{2\sqrt{\lambda}} \begin{cases} e^{-\sqrt{\lambda}s}\left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right), & t \le s, \\ e^{-\sqrt{\lambda}t}\left(e^{\sqrt{\lambda}s} - e^{-\sqrt{\lambda}s}\right), & s \le t. \end{cases}$$

Notice that for $t \neq s$, we have

$$\tilde{G}(t,s) = \frac{\partial G(t,s)}{\partial t} = \frac{1}{2} \begin{cases} e^{-\sqrt{\lambda}s}\left(e^{\sqrt{\lambda}t} + e^{-\sqrt{\lambda}t}\right), & t < s, \\ -e^{-\sqrt{\lambda}t}\left(e^{\sqrt{\lambda}s} - e^{-\sqrt{\lambda}s}\right), & s < t. \end{cases}$$

We will need the following lemmas.

Lemma 2.1. ([10, Lemma 5.1]) *We have*

$$G(t,s) \le \frac{1}{2\sqrt{\lambda}}, \quad |\tilde{G}(t,s)| < 1,$$

and

$$G(t,s)e^{-\mu t} \le G(s,s)e^{-\sqrt{\lambda}s} \quad \text{for } t, s \in I \text{ and all } \mu \ge \sqrt{\lambda}.$$

Lemma 2.2. *For all t_1, $t_2 \ge 0$ and all $s \in I$, we have*

$$|G(t_2,s) - G(t_1,s)| \le |t_2 - t_1|.$$

Proof. This follows easily from the Mean Value Theorem. □

For our construction, we let

$$E = \left\{ u \in C([0,+\infty),\mathbb{R}) : \sup_{t\in\mathbb{R}^+} |u(t)| < \infty \right\}$$

with the norm $\|u\| = \sup_{t\in\mathbb{R}^+} |u(t)|$.

We next have a lemma that gives an integral representation for solutions of our original problem.

Lemma 2.3. *If a function $u \in E$ is a solution of the integral equation $u(t) = \int_0^{+\infty} G(t,s)g(s)ds$ then $u \in E$ is a solution of the boundary value problem*

$$\begin{cases} -u'' + \lambda u = g(t), & t \in I, \\ u(0) = u(\infty) = 0, \end{cases}$$

where $g \in L^1(0,+\infty)$.

Proof. Notice that here we need that the function g satisfies $g \in L^1(0,+\infty)$ to assure that the integral $\int_0^{+\infty} G(t,s)g(s)ds$ is defined. From Lemma 2.1 we have that

$$\int_0^{+\infty} G(t,s)g(s)ds \leq \frac{1}{2\sqrt{\lambda}} \int_0^{+\infty} g(s)ds < +\infty.$$

We also have

$$\begin{aligned} u(t) &= \int_0^t G(t,s)g(s)ds + \int_t^{+\infty} G(t,s)g(s)ds \\ &= \frac{e^{-\sqrt{\lambda}t}}{2\sqrt{\lambda}} \int_0^t \left(e^{\sqrt{\lambda}s} - e^{-\sqrt{\lambda}s}\right) g(s)ds + \frac{\left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right)}{2\sqrt{\lambda}} \int_t^{+\infty} e^{-\sqrt{\lambda}s} g(s)ds. \end{aligned}$$

Then,

$$\begin{aligned} u'(t) = &-\frac{e^{-\sqrt{\lambda}t}}{2} \int_0^t \left(e^{\sqrt{\lambda}s} - e^{-\sqrt{\lambda}s}\right) g(s)ds + \frac{e^{-\sqrt{\lambda}t}\left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right)}{2\sqrt{\lambda}} g(t) \\ &+ \frac{\left(e^{\sqrt{\lambda}t} + e^{-\sqrt{\lambda}t}\right)}{2} \int_t^{+\infty} e^{-\sqrt{\lambda}s} g(s)ds + \frac{-e^{-\sqrt{\lambda}t}\left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right)}{2\sqrt{\lambda}} g(t), \end{aligned}$$

and so

$$\begin{aligned} u''(t) &= \frac{\sqrt{\lambda}e^{-\sqrt{\lambda}t}}{2} \int_0^t \left(e^{\sqrt{\lambda}s} - e^{-\sqrt{\lambda}s}\right) g(s)ds - \frac{e^{-\sqrt{\lambda}t}}{2} \left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right) g(t) \\ &\quad + \frac{\sqrt{\lambda}\left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right)}{2} \int_t^{+\infty} e^{-\sqrt{\lambda}s} g(s)ds + \frac{-e^{-\sqrt{\lambda}t}\left(e^{\sqrt{\lambda}t} + e^{-\sqrt{\lambda}t}\right)}{2} g(t) \\ &= \frac{\sqrt{\lambda}e^{-\sqrt{\lambda}t}}{2} \int_0^t \left(e^{\sqrt{\lambda}s} - e^{-\sqrt{\lambda}s}\right) g(s)ds \\ &\quad + \frac{\sqrt{\lambda}\left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right)}{2} \int_t^{+\infty} e^{-\sqrt{\lambda}s} g(s)ds - g(t). \end{aligned}$$

Clearly, we have $-u''(t) + \lambda u(t) = g(t)$.

Finally, $u(0) = \int_0^{+\infty} G(0,s)g(s)ds = \int_0^{+\infty} 0ds = 0$; and from the Lebesgue Dominated Convergence Theorem,

$$u(\infty) = \lim_{t\to+\infty} \int_0^{+\infty} G(t,s)g(s)ds = \int_0^{+\infty} \lim_{t\to+\infty} G(t,s)g(s)ds = 0.$$

□

Next, we define a cone P in E by

$$P = \left\{u \in E : u(t) \geq 0,\ t \in \mathbb{R}^+\right\}.$$

On P, we define the operator T by

$$Tu(t) = \int_0^{+\infty} a(s)G(t,s)f(s,u(s))ds.$$

The following compactness criteria is based on [8, p. 62].

Lemma 2.4. *Let $D \subset E$ be a bounded set. Then D is relatively compact in E if the following conditions hold:*

1. *D is equicontinuous on any compact sub-interval of $[0,+\infty)$, i.e., for any compact set $K \subset [0,+\infty)$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all t_1, $t_2 \in K$, $|t_1 - t_2| < \delta$ implies $|u(t_2) - u(t_1)| \leq \varepsilon$, for all $u \in D$;*
2. *D is equiconvergent at $+\infty$, i.e., for every $\varepsilon > 0$ there exists $T = T(\varepsilon)$ such that, $t \geq T(\varepsilon)$ implies $|u(t) - u(+\infty)| \leq \varepsilon$ for all $t \geq T(\varepsilon)$ and $u \in D$.*

3. Main result

Our main existence result in this paper is contained in the following theorem.

Theorem 3.1. *Assume that $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying the following conditions:*

(H1) There exist nonnegative functions b, $c \in C(\mathbb{R}^+, \mathbb{R}^+)$ and constant $1 > p \geq 0$, such that

$$f(t,u) \leq b(t) + c(t)|u|^p \text{ for all } (t,u) \in \mathbb{R}^+ \times \mathbb{R}^+$$

with

$$\int_0^{+\infty} a(s)b(s)ds = \beta < \infty, \quad \int_0^{+\infty} a(s)c(s)ds = \delta < \infty.$$

(H2) f is nondecreasing with respect to its second variable.

Then there exists a positive constant R such that in $(0,R]$ the problem (1.1) has a minimal positive solution u^* and a maximal positive solution v^* with

$$u_0 \equiv 0, \quad u_{n+1} = \int_0^{+\infty} G(t,s)a(s)f(s,u_n(s))ds,$$

and

$$v_0 \equiv R, \quad v_{n+1} = \int_0^{+\infty} G(t,s)a(s)f(s,v_n(s))ds,$$

and for which we have

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq u^* \leq \cdots \leq v^* \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{3.1}$$

Proof. To show that the operator $T : E \to E$ is relatively compact, let Ω be any bounded subset of E. Then there exists a constant $M > 0$ such that, for any $u \in \Omega$ with $\|u\| \leq M$, we have

$$\|Tu\| = \sup_{t\in\mathbb{R}^+} |Tu(t)| \leq \frac{1}{2\sqrt{\lambda}}[\beta + \delta M^p].$$

This implies that $T(\Omega)$ is uniformly bounded.

To see that $S = \{Tu : u \in P \cap \Omega\}$ is almost equicontinuous on I, that is, S is equicontinuous on compact subsets of I, let $K \subset I$ be compact and let $t_2, t_1 \in K$. For all $u \in P \cap \Omega$, we have

$$\begin{aligned}|Tu(t_1) - Tu(t_2)| &\leq \int_0^{+\infty} |G(t_1, s) - G(t_2, s)|a(s)\left[b(s) + c(s)|u(s)|^p\right] ds \\ &\leq |t_1 - t_2|\left[\beta + \delta\|u\|^p\right] \\ &\leq |t_1 - t_2|\left[\beta + \delta M^p\right],\end{aligned}$$

proving the equicontinuity of $T(\Omega)$ on K by Lemma 2.4.

To show that the functions $\{Tu : u \in P \cap \Omega\}$ are equiconvergent at ∞, notice that for all $u \in \Omega$ and $t > 0$, we have

$$\begin{aligned}|Tu(t)| &\leq \int_0^{+\infty} a(s)G(t, s)\left[b(s) + c(s)\|u\|^p\right] ds \\ &= \int_0^{+\infty} a(s)G(t, s)\left[b(s) + c(s)M^p\right] ds.\end{aligned}$$

Thus, the equiconvergence of T follows from the fact that $\lim_{t\to\infty} G(t, s) = 0$.

Next, we need to show that T is continuous, so let $(u_n) \subset P \cap \Omega$ be such that $u_n \to u$ as $n \to +\infty$. Since f is continuous, we have that $f(t, u_n(t)) \to f(t, u(t))$ as $n \to +\infty$ and

$$\begin{aligned}|a(s)G(t, s)f(s, u_n(s))| &= a(s)G(t, s)f(s, u_n(s)) \\ &\leq a(s)G(t, s)\left[b(s) + c(s)\|u\|^p\right] \\ &\leq a(s)G(t, s)\left[b(s) + c(s)M^p\right] \\ &\leq \frac{a(s)}{2\sqrt{\lambda}}[b(s) + c(s)M^p].\end{aligned}$$

From condition (H1), we see that

$$\int_0^{+\infty} a(s)b(s)ds = \beta < \infty, \ \int_0^{+\infty} a(s)c(s)ds = \delta < \infty,$$

so we can apply the Lebesgue Dominated Convergence Theorem to obtain,

$$\lim_{n\to+\infty} \int_0^{+\infty} a(s)G(t, s)f(s, u_n(s))ds = \int_0^{+\infty} a(s)G(t, s)f(s, u(s))ds,$$

so $\|Tu_n - Tu\| \to 0$ as $n \to \infty$.

Now, since $0 \leq p < 1$, choose $R \geq \frac{1}{2\sqrt{\lambda}} \max\left\{2\beta, (2\delta)^{\frac{1}{1-p}}\right\}$ and set $B = \{u \in E : \|u\| \leq R\}$. In order to show that $T(B) \subset B$, let $u \in B$. By Lemma 2.1, we have

$$\begin{aligned}\sup_{t\in\mathbb{R}^+} |Tu(t)| &= \sup_{t\in\mathbb{R}^+} \int_0^{+\infty} a(s)G(t,s)f\left(s,u(s)\right)ds \\ &\leq \int_0^{+\infty} a(s)G(t,s)\left[b(s)+c(s)|u|^p\right]ds \\ &\leq \frac{1}{2\sqrt{\lambda}}\left[\beta+\delta\|u\|^p\right] \\ &\leq \frac{1}{2\sqrt{\lambda}}\left[\beta+\delta R^p\right] \leq R.\end{aligned}$$

We then have $\|Tu\| \leq R$, for all $u \in B$, and so $T(B) \subset B$ as needed.

From the definition of the operator T and condition $(H2)$, we see that T is nondecreasing. Define a sequence (u_n) as follows:

$$u_0 \equiv 0,\ u_{n+1} = Tu_n \text{ for } n = 0,1,2,\dots \text{ and for all } t \in \mathbb{R}^+.$$

Since $u_0 \equiv 0 \in B$ and $T : B \to B$, we have $(u_n)_{n\geq 1} \subset T(B) \subset B$. Notice that for all $t \in \mathbb{R}^+$,

$$u_j(t) = (Tu_{j-1})(t) \geq u_{j-1}(t) \text{ for } j = 1,2,\dots$$

By the complete continuity of the operator T, we have that $(u_n)_{n\geq 1}$ has a convergent subsequence $(u_{n_k})_{k\geq 1}$, and there exists a $u^* \in B$ such that $u_{n_k} \to u^*$ as $k \to +\infty$. This, together with (3.1), implies that $\lim_{n\to\infty} u_n = u^*$. Since T is continuous and $u_{n+1} = Tu_n$, we have $Tu^* = u^*$, that is, u^* is a fixed point of the operator T.

In a similar way, we define a sequence (v_n) by

$$v_0 \equiv R,\ v_{n+1} = Tv_n \text{ for } n = 0,1,2,\dots \text{ and for all } t \in \mathbb{R}^+.$$

Since $v_0 \equiv R \in B$ and $T : B \to B$, we have $v_j \in T(B) \subset B$ and

$$v_j(t) = (Tv_{j-1})(t) \leq u_{j-1} \text{ for } j = 1,2,\dots$$

Reasoning as above, we can prove the existence of $v^* \in B$ such that $\lim_{n\to+\infty} v_n = v^*$ and $Tv^* = v^*$, that is, v^* is a fixed point of T.

Next, we will prove that v^* and u^* are the maximal and minimal positive solutions of (1.1), respectively, in $[0,R]$. Let $w \in [0,R]$ be a solution of (1.1); then $u_0 \equiv 0 \leq w \leq R \equiv v_0$, and so $u_1(t) = Tu_0(t) \leq w(t) \leq Tv_0(t) = v_1(t)$ for all $t \in \mathbb{R}^+$. By induction, we obtain $u_n(t) \leq w(t) \leq v_n(t)$ for all $t \in I$ and $n = 0,1,2,\dots$ Passing to the limit, we see that

$$u_0 \leq u_1 \leq \cdots \leq u^* \leq w \leq v^* \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

This completes the proof of the theorem. □

We conclude this paper with an example of our main result.

Example 3.1. Consider the BVP

$$\begin{cases}-u'' + \lambda u = a\,(t)\,f(t,u(t)), & t \in I, \\ u\,(0) = u\,(\infty) = 0,\end{cases} \tag{3.2}$$

where
$$f(t,u) = 1 + e^{-t}u^{\frac{1}{2}}$$
and $a(t) = e^{-t}$.

Clearly, f is nondecreasing with respect to its second variable. Thus, $(H2)$ holds. We also have
$$f(t,u) \leq b(t) + c(t)|u|^p,$$
where
$$b(t) = 1,\ c(t) = e^{-t},\ p = \frac{1}{2}.$$
Since
$$\beta = \int_0^{+\infty} e^{-s}ds = 1 < \infty \text{ and } \delta = \int_0^{+\infty} e^{-2s}ds = \frac{1}{2} < \infty,$$
condition $(H1)$ holds. Therefore, by Theorem 3.1, the problem (3.2) has maximal and minimal positive solutions v^* and u^* for which (3.1) holds.

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References

[1] S. D. Akgol and A. Zafer, *Boundary value problems on half-line for second-order nonlinear impulsive differential equations*, Math. Methods Appl. Sci., 2018, 41, 5459–5465.

[2] I. Bachar, *Combined effects in nonlinear singular second-order differential equations on the half-line*, Electron. J. Differential Equations, 2015, 2015, No. 235, 12 pp.

[3] I. Bachar and H. Mâagli, *Existence and global asymptotic behavior of positive solutions for combined second-order differential equations on the half-line*, Adv. Nonlinear Anal., 2016, 5, 205–222.

[4] A. Benmezai, *Existence of weak solutions for second-order BVPS on the half-line via critical point theory*, Electron. J. Math. Anal. Appl., 2022, 10, 154–166.

[5] A. Benmezai and S. Mellal, *Nodal solutions for asymptotically linear second-order BVPs on the half-line*, Bull. Iranian Math. Soc., 2019, 45, 315–335.

[6] D. Bouafia and T. Moussaoui, *Existence results for a sublinear second order Dirichlet boundary value problem on the half-line*, Opuscula Math., 2020, 40, 537–548.

[7] D. Bouafia, T. Moussaoui and D. O'Regan, *Existence of solutions for a second order problem on the half-line via Ekeland's variational principle*, Discuss. Math. Differ. Incl. Control Optim., 2016, 36, 131–140.

[8] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.

[9] S. Djebali and K. Mebarki, *Singular semi-positone ϕ-Laplacian BVP on infinite interval in Banach spaces*, Dynam. Systems Appl., 2016, 25, 197–217.

[10] S. Djebali and T. Moussaoui, *A class of second order BVPs on infinite intervals*, Electron. J. Qual. Theor. Differ. Eqs., 2006, 2006, No. 4, 1–19.

[11] M. Djibaoui and T. Moussaoui, *Variational methods to second-order Dirichlet boundary value problems with impulses on the half-line*, Cubo, 2022, 24, 227–237.

[12] P. Habets and L. Sanchez, *A monotone method for fourth order boundary value problems involving a factorizable linear operator*, Portugal. Math., 2007, 64, 255–279.

[13] S. Heidarkhani, G. A. Afrouzi and S. Moradi, *Existence results for a Kirchhoff-type second-order differential equation on the half-line with impulses*, Asymptot. Anal., 2017, 105, 137–158.

[14] O. F. Imaga and S. A. Iyase, *Existence of solution for a resonant p-Laplacian second-order m-point boundary value problem on the half-line with two dimensional kernel*, Bound. Value Probl., 2020, 2020, Paper No. 114, 11 pp.

[15] W. Jiang, *The existence of solutions for impulsive p-Laplacian boundary value problems at resonance on the half-line*, Bound. Value Probl., 2015, 2015, No. 39, 13 pp.

[16] R. Khaldi and A. Guezane-Lakoud, *Minimal and maximal solutions for a fractional boundary value problem at resonance on the half line*, Fract. Differ. Calc., 2018, 8, 299–307.

[17] E. Liz and J. J. Nieto, *The monotone iterative technique for periodic boundary value problems of second order impulsive differential equations*, Comment. Math. Univ. Carolin., 1993, 34, 405–411.

[18] L. López-Somoza and F. Minhós, *On multipoint resonant problems on the half-line*, Bound. Value Probl., 2019, 2019, Paper No. 38, 19 pp.

[19] S. Matucci, *A new approach for solving nonlinear BVP's on the half-line for second order equations and applications*, Math. Bohem., 2015, 140, 153–169.

[20] H. Pang, M. Q. Feng, and W. G. Ge, *Existence and monotone iteration of positive solutions for a three-point boundary value problem*, Appl. Math. Lett., 2008, 21, 656–661.

[21] M. Pei and S. K. Chang, *Monotone iterative technique and symmetric positive solutions for a fourth order boundary value problem*, Math. Comput. Modelling, 2010, 51, 1260–1267.

[22] E. Picard, *Mémoire sur la théorie des équations aux dérivitées partielles et la méthode d'approximations successives*, J. Math. Pures Appl., 1890, 6, 145–210.

[23] R. L. Pouso and J. Rodrigo López, *Second-order discontinuous problems with nonlinear functional boundary conditions on the half-line*, Electron. J. Qual. Theory Differ. Equ., 2018, 2018, Paper No. 79, 18 pp.

[24] P. J. Torres and M. Zhang, *A monotone iterative scheme for a nonlinear second order equation based on a generalized anti-maximum principle*, Math. Nachr., 2003, 251, 101–107.

[25] L. Zhang, B. Ahmed, and G. Wang, *Successive iterations for positive extremal solutions of nonlinear fractional differential equations on a half-line*, Bull. Austral. Math. Soc., 2015, 91, 116–128.