Journal of Applied Analysis and Computation
Volume 13, Number 6, December 2023, 3166–3194
Website:http://www.jaac-online.com
DOI:10.11948/20220532

A STUDY ON STABILITY, BIFURCATION ANALYSIS AND CHAOS CONTROL OF A DISCRETE-TIME PREY-PREDATOR SYSTEM INVOLVING ALLEE EFFECT

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Abstract This paper examines the stability and bifurcation of a discrete-time prey-predator system that is modified by the Allee effect on the prey population. The system undergoes flip and Neimark-Sacker bifurcations in a small neighborhood of the unique positive fixed point depending on the densities of prey-predator. The OGY method and hybrid control method are used to control the chaotic behavior that results from Neimark-Sacker bifurcation. In addition, numerical simulations are performed to illustrate the theoretical results. To keep the ecosystem stable, it is crucial to research how populations of prey and predator interact. The Allee effect is a significant evolutionary force that alters population size by affecting both prey and predator behavior. It would be more realistic to look into population behavior in light of this effect, which results from population density (number of individuals per unit area). The increase in the density of predator in the model with the Allee effect pushes the prey to extinction. When the density of predator is suppressed, the stability continues for a certain time before undergoing bifurcation.

Keywords Prey-predator system, local asymptotic stability, bifurcation theory, Neimark-Sacker bifurcation, flip bifurcation, chaos control.

MSC(2010) 37G35, 39A30, 39A33, 91B76, 92B05, 92D25.

1. Introduction

In population dynamics, mathematical models are usually formulated using continuous models that are created with differential equations and discrete-time models that are created with difference equations. There has been numerous research studies in the literature on the stability and bifurcation analysis of the discrete-time models (see [16, 21, 22, 27, 30–32, 45, 46, 52, 62] and references therein). Because models created with difference equations are effective in examining nonlinear populations with non-overlapping generations [2, 43, 44, 47]. Discrete-time models also exhibit rich dynamic behavior, and enable efficient calculations and numerical simulations [11, 22, 40, 54]. The dynamic results obtained from the analysis of discrete-time systems used in solving real-world problems can lead to great advances in ecology, biology, physics, economics and engineering [2, 5, 23, 47, 59].

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The interaction between predators and their prey has always been an important issue in mathematical models of ecological processes. Therefore, studies on the well-known Lotka-Volterra prey-predator models continue to play a remarkable role. Especially, studies on discrete-time prey-predator systems have become the focus of great interest in recent years. Most of these consist of research on the existence and stability of fixed points, complex and chaotic behaviors that require bifurcation analysis [3,7,10,17,24,28–30,37,50]. In prey-predator populations, the growth rate of the prey species may be less or more than that of the predator species, or they may be equal. This varies according to the potential characteristics of the prey and the predator [60]. The prey has the ability to survive without the predator. To maintain the balanced functioning between the populations, it is important to have information about the population of the prey-predator with an appropriate growth rate of prey that allows it to sustain its life. Therefore, examining the behavior of the population by considering the changes in the survival rate of the prey is an important study subject. Food, disease, migration, immigration, parasites, harvesting, and Allee factor can have a stabilizing effect on populations as well as cause fluctuations that bring instability. The factors regulating fluctuations have been studied by ecologists for a long time. Allee factor can dramatically change the number of a population from one time period to the next. It is noteworthy to see this change by incorporating the Allee effect into a prey-predator model. The positive relationship between population density and per capita growth rate at low density is defined as Allee effect. In single species populations, this effect is a factor that reduces reproduction (more specifically, individual fitness) and survival of individuals in small populations, and often saturates and disappears as the population increases (see [1,8,45]). Inbreeding depressions, difficulty finding mates, social dysfunction in small populations, food exploitation, environment conditioning, predator avoidance, and defense mechanisms against predators are among the various causes of this. Numerous native species, including plants [26], insects [38], marine invertebrates [56], birds and mammals [9] and others have shown empirical evidence of Allee effects. Both theoretical and applied ecologists [18,53,61] have paid great attention to the Allee effect. It is feasible to suggest more realistic approaches when taking this effect into account. Examination of population behavior considering these effects that change the intrinsic growth rate of prey allows us to obtain remarkable results.

In prey-predator models, besides the biological reasons that affect the intrinsic growth rate of the prey population, the biological reasons that affect the number of preys consumed by the predator per unit time are also important. It is not always possible to reach sufficient prey. As observed in nature, a prey community attacked by a predator can use interaction range distance, age and even physical structure, as well as cooperative interactions to escape the predator. In this case, with the assumption of the Allee effect, the number of preys consumed by the predator may vary depending on such effects. Many researchers have analyzed population behavior in discrete-time prey-predator models, taking into account the Allee factor effect, which modifies the intrinsic growth rate of the prey population (see [6, 14, 49, 65] and the references therein). While research on the effect of the Allee factor that takes into account the number of prey consumed is relatively rare, the lack of emphasis on additional model assumptions that prevent negative population levels is often a disadvantage in terms of biological significance [33, 34]. In this study, we introduce a discrete-time prey-predator system with the Allee effect that takes into account density of the prey consumed by the predator under additional model assumption. Difference modeling approaches derived directly from assumptions to avoid the existence of negative solutions; and even the approaches considered to determine all the global dynamics of the model are presented as different alternatives on prey-predator model studies, see for example, [57,58].

In such models, the growth rate of the predator population generally depends on the growth rate of the prey population. We can see that the prey population is a limiting effect on the population dynamics, because the predator population decreases with the decreased prey population. Therefore, small numerical changes in the prey population cause large changes in the dynamics of models. To see the effect of the Allee factor affecting the amount of prey consumed, taking the growth rate of prey as a bifurcation parameter allows us to better understand the dynamic processes involved and make practical predictions.

In this paper, we consider the following discrete-time prey-predator system describing interaction between two populations of non-overlapping generations subjected to the Allee effect:

$$\begin{aligned} x_{n+1} &= ax_n(1-x_n) - bx_ny_n\left(\frac{x_n}{x_n+c}\right), \\ y_{n+1} &= dx_ny_n \end{aligned} \tag{1.1}$$

where x_n and y_n denote the densities of prey and predator in year (generation) n, respectively and the parameters a, b, c, d are all positive parameters with $0 < c < 1$. In this model, bx_n represents the density of prey individuals consumed per unit area and per unit time by an individual predator without the Allee effect and dx_ny_n is the predator response. We can summarize the model parameters in the following table:

Parameter	Meaning
a	Intrinsic growth rate of the prey population
b	Predation rate
c	Allee constant
d	The growth rate of the predator limited by the density of prey

In [11], the authors present the dynamics of discrete-time prey-predator model (1.1) without the Allee effect. The model has rich dynamic behaviors including Neimark- Sacker and flip (period-doubling) bifurcation as well as semi-periodic, stable, chaotic, hyperchaotic attractors with different parameter values. The present study aims to analyze the stability and bifurcation of the system (1.1) incorporating the Allee effect and to observe the dynamics of the system. We refer to studies [2,12,20,40] for some basic concepts that we have used throughout the paper.

The paper is arranged as follows: Section 2 investigates the existence and local asymptotic stability of fixed points of the system (1.1) in $\mathbb{R}^2_+$, with graphs showing system behavior. Section 3 discusses the theory and dynamics of the system (1.1) which undergoes a Neimark-Sacker and flip bifurcation by choosing a as a bifurcation parameter. The chaos emerging with Neimark-Sacker is controlled by the OGY and hybrid methods. Section 4 includes numerical simulations exhibiting the dynamical properties of the system (1.1) by means of trajectories, bifurcation diagrams and phase portraits. The last section provides the results and various dynamics of the system (1.1).

2. The existence and local stability of fixed points of the system (1.1)

This section provides the existence and the local stability analysis of fixed points of the system (1.1) in the closed first quadrant $\mathbb{R}^2_+$. The magnitude of the eigenvalues of the Jacobian matrix determines the local stability conditions of the fixed points of discrete-time systems.

To keep the solutions in the closed first quadrant, we can make the following evaluation:

Let

$$\begin{aligned} f_{n+1}(x_n, y_n) &= ax_n(1-x_n) - bx_ny_n\left(\frac{x_n}{x_n+c}\right), \\ g_{n+1}(x_n, y_n) &= dx_ny_n \end{aligned}$$

such that $x_0 > 0$ and $y_0 > 0$. From $x_0 > 0$ and $y_0 > 0$, it is clear that $g_{n+1}(x_n, y_n) \geq 0$ for $n = 0, 1, 2, ...$ The set of ordered pairs (x_n, y_n) that makes $f_{n+1}(x_n, y_n) \geq 0$ can be obtained as

$$\Omega = \left\{(x_n, y_n) : 0 \leq y_n \leq \frac{a(1-x_n)(x_n+c)}{bx_n}, x_n < 1\right\}.$$

From an ecological point of view, for $(x_n, y_n) \in \Omega$, if $(f_{n+1}(x_n, y_n), g_{n+1}(x_n, y_n)) \notin \Omega$, then it means that the population collapses.

2.1. The existence of fixed points of the system (1.1)

When we examine the existence of all available fixed points of the prey-predator system (1.1), we obtain the following Lemma.

Lemma 2.1. *For the system (1.1), the following cases hold:*

(i.a1) *The system (1.1) has a single trivial (extinction) fixed point $E_0 = (0, 0)$ for all positive parameters;*

(i.a2) *If $a > 1$, then the system (1.1) has two fixed points. These are an extinction $E_0 = (0, 0)$ and an exclusion fixed point $E_1 = (\frac{a-1}{a}, 0)$;*

(i.a3) *If $a > \frac{d}{d-1}$ such that $d > 1$, then the system (1.1) has a unique positive coexistence fixed point $E_2 = (\frac{1}{d}, \frac{(-a-d+ad)(1+cd)}{bd})$.*

2.2. The local stability of the extinction and exclusion fixed points of the system (1.1)

We examine the locally asymptotic stability of the fixed points $E_0 = (0, 0)$ and $E_1 = (\frac{a-1}{a}, 0)$ by using the eigenvalues of the Jacobian matrix. Firstly, the Jacobian matrix of system (1.1) is

$$J_{E_0} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \tag{2.1}$$

evaluated at $(0,0)$. Therefore, we obtain $\lambda_1 = a$ and $\lambda_2 = 0$.

Hence, the following Lemma for the condition of locally asymptotic stability of the extinction fixed point E_0 is clearly obtained.

Lemma 2.2. *If $a < 1$, then the extinction fixed point E_0 is a sink point for all $b, c, d \in \mathbb{R}_+$. Furthermore, the fixed point E_0 can not be a source point since $\lambda_2 = 0 < 1$.*

Let us consider an example where $a < 1$ in the system (1.1). Throughout the study, the parameter values b and d are taken from [11] as $b = 0.2$, $d = 3.5$.

Example 2.1. Let us consider the following population model to exhibit the appearance of trajectories and phase portrait of system (1.1). For $a = 0.9$ and $c = 0.5$, the system has the following form:

$$\begin{aligned} x_{n+1} &= 0.9x_n(1-x_n) - 0.2x_n y_n \left(\frac{x_n}{x_n+0.5}\right), \\ y_{t+1} &= 3.5x_n y_n \end{aligned} \tag{2.2}$$

where the initial conditions $x_0 = 0.5$ and $y_0 = 0.2$.

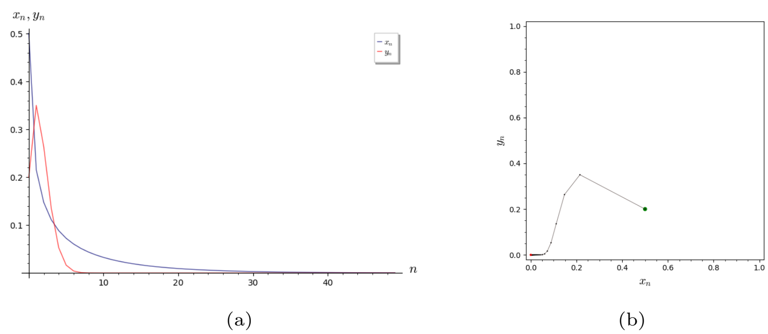


Figure 1. (a) The trajectories of prey and predator densities in the system (2.2) when $a = 0.9$, $b = 0.2$, $c = 0.5$ and $d = 3.5$. (b) The phase portrait of the system (2.2) when $a = 0.9$, $b = 0.2$, $c = 0.5$ and $d = 3.5$.

From Lemma 2.2, the fixed point $(0,0)$ is locally asymptotically stable.

Secondly, for $E_1 = (\frac{a-1}{a}, 0)$, the Jacobian matrix of system (1.1) is

$$J_{E_1} = \begin{pmatrix} 2-a & -\frac{(-1+a)^2 b}{a(-1+a+ac)} \\ 0 & \frac{(a-1)d}{a} \end{pmatrix}. \tag{2.3}$$

Therefore, the eigenvalues are $\lambda_1 = 2 - a$, $\lambda_2 = \frac{(a-1)d}{a}$.

The following Lemma presents the conditions of locally asymptotic stability of the exclusion fixed point E_1.

Lemma 2.3. *The fixed point E_1 is locally asymptotically stable if any of the following cases is provided:*

(i.b1) $1 < a < 3$ *and* $d < \frac{3}{2}, d \neq 1$

(i.b2) $1 < a < \frac{d}{d-1}$ *and* $d > \frac{3}{2}$

for all $b, c \in \mathbb{R}_+$.

Let us give an example to confirm the results obtained in Lemma 2.3.

Example 2.2. We consider the following population model to exhibit the appearance of trajectories and phase portrait of system (1.1). For $a = 1.39$ and $c = 0.5$, the system has the following form:

$$\begin{aligned} x_{n+1} &= 1.39x_n(1-x_n) - 0.2x_ny_n\left(\frac{x_n}{x_n+0.5}\right), \\ y_{t+1} &= 3.5x_ny_n \end{aligned} \tag{2.4}$$

where the initial conditions $x_0 = 0.5$ and $y_0 = 0.2$. The fixed point $(0.280576, 0)$ of this system that validates the condition **(i.b2)** with the selected values $a = 1.39$ and $c = 0.5$ is locally asymptotically stable. For this, the trajectories and phase portrait of prey and predator densities are exhibited in Figure 2 (a)-(b).

Furthermore, if we select the parameter $a = 3.6$, $b = 0.2$, $c = 0.5$ and $d = 1.4$, then system (1.1) can be written as

$$\begin{aligned} x_{n+1} &= 3.6x_n(1-x_n) - 0.2x_ny_n\left(\frac{x_n}{x_n+0.5}\right), \\ y_{t+1} &= 1.4x_ny_n. \end{aligned} \tag{2.5}$$

In this case, the fixed point is not locally asymptotically stable since the conditions given in **(i.b1)** and **(i.b2)** are not provided. For this, the trajectories and phase portrait of prey and predator densities are exhibited in Figure 2 (c)-(d). For, $a > \frac{d}{d-1}, d > \frac{3}{2}$, the fixed point E_1 disappears and the fixed point E_2 appears.

2.3. The local stability of the positive coexistence fixed point

We investigate the locally asymptotically stability of the coexistence fixed points as follows:

$$E_2 = (\overline{x}, \overline{y}) = (\frac{1}{d}, \frac{(-a-d+ad)(1+cd)}{bd}). \tag{2.6}$$

The Jacobian matrix of system (1.1) is

$$J_{E_2} = \begin{pmatrix} -\frac{a-d-2cd^2+acd^2}{d+cd^2} & \frac{-b}{d+cd^2} \\ \frac{[a(-1+d)-d][1+cd]}{b} & 1 \end{pmatrix}$$

evaluated at E_2, and the characteristic polynomial of the Jacobian matrix is

$$F(\lambda) = \lambda^2 + [\frac{a-2d-3cd^2+acd^2}{d+cd^2}]\lambda + [\frac{cd^2+a(-2+d-cd)}{d+cd^2}].$$

Then we have the following Lemma by using the characteristic polynomial of the Jacobian matrix, i.e. J_{E_2} (see [40]).

Lemma 2.4. ***A***. *Suppose that* $\frac{d}{d-1} < a < \frac{3d+5cd^2}{3-d+cd+cd^2}$. *Then the positive coexistence fixed point* E_2 *is locally asymptotically stable, if any of the following cases is provided:*

(i.b3) $\frac{3}{2} < d \leq \frac{9}{4}$ *and* $c < 1$;
(i.b4) $\frac{9}{4} < d \leq 3$ *and* $\frac{ad-2a-d}{ad} < c < 1(\frac{ad-2a-d}{ad} < \frac{-9+4d}{5d})$;
(i.b5) $d > 3$ *and* $\frac{-9+4d}{5d} < c < 1$.

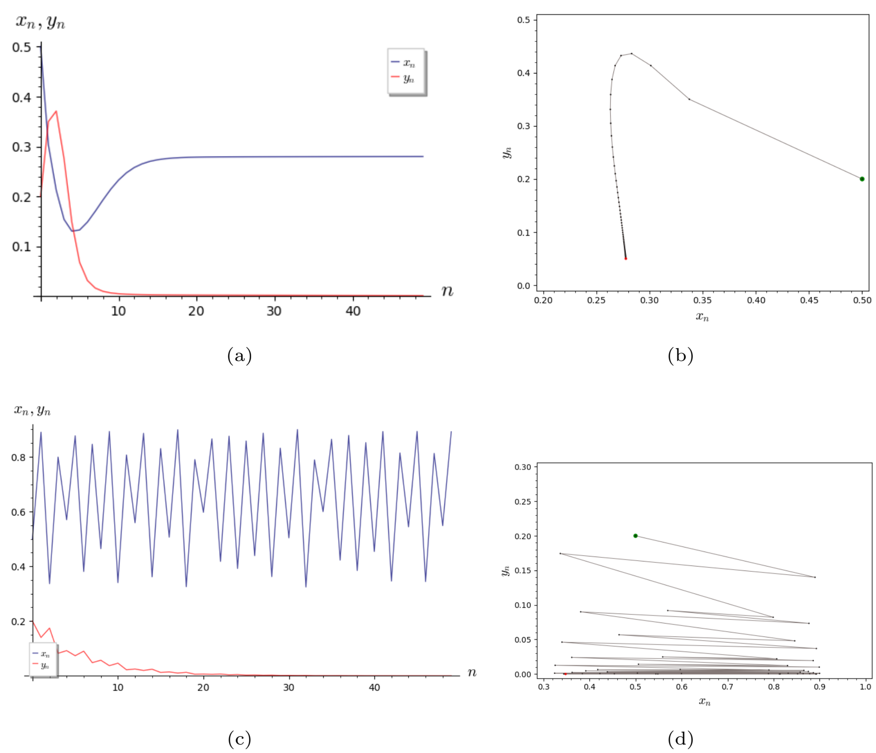


Figure 2. (a) The trajectories of prey and predator densities in the system (2.4) when $a = 1.39$, $b = 0.2$, $c = 0.5$ and $d = 3.5$. (b) The phase portrait of system (2.4) when $a = 1.39$, $b = 0.2$, $c = 0.5$ and $d = 3.5$. (c) The trajectories of prey and predator densities in the system (2.5) when $a = 3.6$, $b = 0.2$, $c = 0.5$ and $d = 1.4$. (d) The phase portrait of system (2.5) when $a = 3.6$, $b = 0.2$, $c = 0.5$ and $d = 1.4$.

B. *Suppose that* $\frac{d}{d-1} < a < \frac{-d}{2-d+cd}$. *Then the coexistence fixed point* E_2 *is locally asymptotically stable, if the following case is provided:*

(i.b6) $d > 3$ *and* $c \leq \frac{-3+d}{d+d^2}$.

Let us give an example to confirm the results obtained in Lemma 2.4.

Example 2.3. Let us consider the following population model to exhibit the appearance of trajectories and phase portrait of system (1.1). For $a = 2.33$ and $c = 0.031$, the system has the following form:

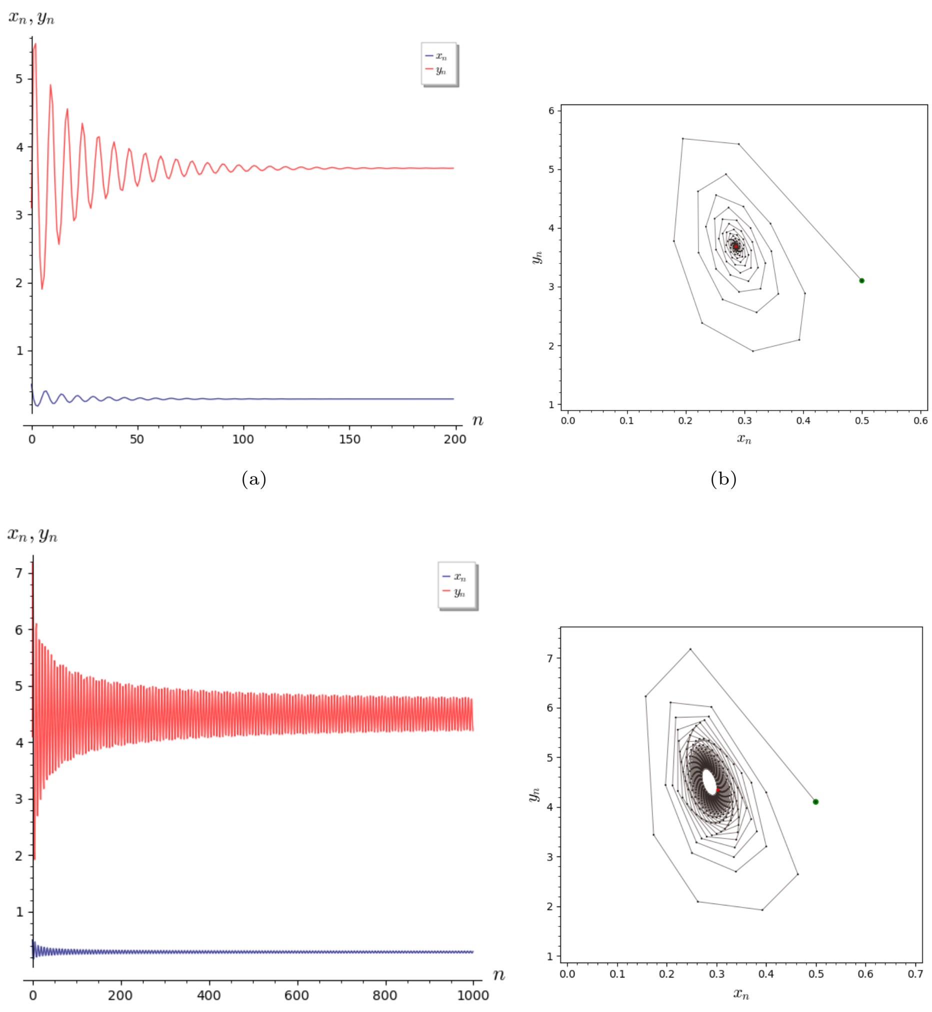
$$\begin{aligned} x_{n+1} &= 2.33x_n(1-x_n) - 0.2x_ny_n\left(\frac{x_n}{x_n+0.031}\right), \\ y_{t+1} &= 3.5x_ny_n \end{aligned} \tag{2.7}$$

where the initial conditions $x_0 = 0.5$ and $y_0 = 3.1$. The fixed point $(0.285714, 3.6818)$ of the system that validates the condition **B-(i.b6)** with the selected values $d = 3.5$ and $c = 0.031$ $(c \leq 0.031746)$ is locally asymptotically stable such that $1.4 < a < 2.51527$. For this, Figure 3(a)-(b) shows the trajectories and phase portrait of prey and predator densities.

Futhermore, if the parameter $a = 2.53$, $b = 0.2$, $c = 0.033$ and $d = 3.5$ are selected, then system (1.1) can be written as

$$x_{n+1} = 2.53x_n(1 - x_n) - 0.2x_n y_n \left(\frac{x_n}{x_n + 0.033} \right), \tag{2.8}$$
$$y_{t+1} = 3.5x_n y_n$$

where the initial conditions $x_0 = 0.5$ and $y_0 = 4.1$. In this case, the fixed point $(0.285714, 4.50184)$ of the system is unstable. For this situation, the trajectories and phase portrait of prey and predator densities are exhibited in Figure 3(c)-(d).



(a) (b) (c) (d)

Figure 3. (a)The trajectories of prey and predator densities in the system (2.7) when $a = 2.33$, $b = 0.2$, $c = 0.031$ and $d = 3.5$. (b) The phase portrait of system (2.7) when $a = 2.33$, $b = 0.2$, $c = 0.031$ and $d = 3.5$. (c) The trajectories of prey and predator densities in the system (2.8) when $a = 2.53$, $b = 0.2$, $c = 0.033$ and $d = 3.5$. (d) The phase portrait of system (2.8) when $a = 2.53$, $b = 0.2$, $c = 0.033$ and $d = 3.5$.

3. Bifurcation analysis and chaos control

This section discusses that positive coexistence fixed point E_2 of system (1.1) undergoes flip and Neimark-Sacker bifurcation in the interior of $\mathbb{R}_+^2$ by using bifurcation theory [39,63]. a is taken as bifurcation parameter to get the conditions of the flip and Neimark–Sacker bifurcations.

3.1. Flip bifurcation

First, the flip (period-doubling) bifurcation of the system (1.1) is investigated as the intrinsic growth rate of prey changes. The conditions that cause flip bifurcation occurring at the positive coexistence fixed point E_2 are determined. If

$$a = a_F = \frac{d(3+5cd)}{3+d(-1+c+cd)}$$

then $\lambda_1 = -1$ and $\lambda_2 = \frac{6-[3+c(-4+d)]d}{3+d(-1+c+cd)}$ with

$$|\lambda_2| \neq 1. \tag{3.1}$$

These conditions can be presented by the following set

$$FB_{E_2} = \left\{ a,b,c,d \in \mathbb{R}^+ : a = a_F = \frac{d(3+5cd)}{3+d(-1+c+cd)} \ , |\lambda_2| \neq 1 \right\}.$$

Using the transformation $u = x - \frac{1}{d}$, $v = y - \frac{(-a-d+ad)(1+cd)}{bd}$, the fixed point E_2 is shifted to the origin. Therefore, we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} \to J_{E_2} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F_1(u,v) \\ F_2(u,v) \end{pmatrix} \tag{3.2}$$

where

$$F_1(u,v) = \frac{c^2d^3 - a(1+2cd+c^2d^3)}{(1+cd)^2}u^2 - \frac{b(1+2cd)}{(1+cd)^2}uv \tag{3.3}$$

$$+\frac{c^2d^3[a(d-1)-d]}{(1+cd)^3}u^3 - \frac{bc^2d^2}{(1+cd)^3}u^2v + O(\|U\|^4), \tag{3.4}$$

$$F_2(u,v) = duv + O(\|U\|^4) \tag{3.5}$$

such that $U = (u,v)^T$. From there, the system (1.1) can be written as

$$(U_{n+1}) \to J_{E_2}\,(U_n) + \frac{1}{2}B(u_n,u_n) + \frac{1}{6}C(u_n,u_n,u_n) + O(\|u_n\|^4), \tag{3.6}$$

with the multilinear vector functions of $u,v,w \in \mathbb{R}^2$:

$$B(u,v) = \begin{pmatrix} B_1(u,v) \\ B_2(u,v) \end{pmatrix}$$

and

$$C(u,v,w)=\begin{pmatrix} C_1(u,v,w)\\ C_2(u,v,w)\end{pmatrix}.$$

These vectors are expressed by

$$\begin{aligned}
B_1(u,v) &= \sum_{j,k=1}^{2}\frac{\partial^2 F_1}{\partial\xi_j\partial\xi_k}\,|_{\xi=0}\; u_j v_k = \frac{2[c^2d^3-a(1+2cd+c^2d^3)]}{(1+cd)^2}u_1v_1\\
&\quad -\frac{b(1+2cd)}{(1+cd)^2}(u_2v_1+u_1v_2),\\
B_2(u,v) &= \sum_{j,k=1}^{2}\frac{\partial^2 F_2}{\partial\xi_j\partial\xi_k}\,|_{\xi=0}\; u_j v_k = d(u_2v_1+u_1v_2),\\
C_1(u,v,w) &= \sum_{j,k=1}^{2}\frac{\partial^3 F_1}{\partial\xi_j\partial\xi_k\xi_l}\,|_{\xi=0}\; u_j v_k w_l = \frac{6c^2d^3[a(d-1)-d]}{(1+cd)^3}u_1v_1w_1\\
&\quad -\frac{2bc^2d^3}{(1+cd)^3}(u_1v_1w_2+u_1v_2w_1),\\
C_2(u,v,w) &= \sum_{j,k=1}^{2}\frac{\partial^3 F_2}{\partial\xi_j\partial\xi_k\xi_l}\,|_{\xi=0}\; u_j v_k w_l = 0
\end{aligned}$$

and $a=a_F$. Let $q,p\in\mathbb{R}^2$ be eigenvectors of $J_{E_2}(a_F)$ and transposed matrix $J^T_{E_2}(a_F)$ respectively for $\lambda_1(a_F)=-1$. Then, we have $A(a_F)q=-q$ and $A^T(a_F)p=-p$. These eigenvectors calculated using Mathematica software are

$$q\sim\left(\frac{-b(3-d+cd+cd^2)}{(1+cd)^2(-3d+2d^2)},1\right)^T$$

and

$$p\sim\left(-1,\frac{-b}{2d+2cd^2}\right)^T.$$

We use standard scalar product in $\mathbb{R}^2$ in order to normalize p with respect to q, such that $\langle p,\,q\rangle=p_1q_1+p_2q_2$. Therefore, we can obtain

$$p\sim\left(\frac{-2d(-3+2d)(1+cd)^2}{b[9+(-4+5c)d]},\frac{-(-3+2d)(1+cd)}{9+(-4+5c)d}\right)^T.$$

It is clear that $\langle p,\,q\rangle=1$. To determine the direction of the flip bifurcation, we need to get the sign of the coefficient $c(a_F)$ as follows:

$$c\,(a_F)=\frac{1}{6}\langle p,C(q,q,q)\rangle-\frac{1}{2}\langle p,B(q,(A-I)^{-1}B(q,q)\rangle. \tag{3.7}$$

The following theorem gives the result on flip bifurcation regarding the coefficient of the critical normal form.

Theorem 3.1. *If (3.1) becomes valid, $c\,(a_F)\neq 0$, and the parameter a changes its value around a_F, then the system (1.1) undergoes a flip bifurcation at positive coexistence fixed point E_2. Furthermore, if $c\,(a_F)>0$ ($c\,(a_F)<0$), then the period 2 orbits that bifurcate from E_2 are stable (unstable).*

3.2. Neimark-Sacker bifurcation

This section provides the direction and the existence of Neimark–Sacker bifurcation for the system (1.1). In addition, if the system (1.1) provides eigenvalue assignment, transversality and nonresonance conditions, then the Neimark–Sacker bifurcation occurs at a bifurcation point. To work Neimark–Sacker bifurcation in the system (1.1), we define the parameters providing non-hyperbolic conditions as follows:

$$NSB_{E_2} = \{a, b, c, d \in \mathbb{R}_+ : a_1 < a < a_2 \text{ and } a = a_{NS}\}$$

where

$$a_1 = \frac{3 + (2 - 9c + 4cd - 2c^2d)}{(c + c^2d^2)} + \frac{-2 + 2c(3 + (-3 + c)d)}{c(1 + cd^2)^2} - 2\sqrt{\frac{d^3(1 + cd)^3(-2 + d + c(1 + (-3 + d)d))}{(1 + cd^2)^4}},$$

$$a_2 = \frac{3 + (2 - 9c + 4cd - 2c^2d)}{(c + c^2d^2)} + \frac{-2 + 2c(3 + (-3 + c)d)}{c(1 + cd^2)^2} + 2\sqrt{\frac{d^3(1 + cd)^3(-2 + d + c(1 + (-3 + d)d))}{(1 + cd^2)^4}},$$

and

$$a_{NS} = \frac{d}{d(1 - c) - 2}, \; d(1 - c) > 2.$$

For $a = a_{NS}$, the eigenvalues of the matrix J_{E_2} associated with the linearization in the map (3.2) are conjugate complex numbers whose modules are one. These eigenvalues are

$$\lambda, \overline{\lambda} \mid_{a=a_{NS}} = \frac{5 + (-2 + 3c)d \pm i\sqrt{(-9 + (4 - 5c)d)(1 + cd)}}{(4 + 2(-1 + c)d)}$$

with

$$|\lambda(a_{NS})| = 1.$$

For $a \in NSB_{E_2}$, we get

$$\frac{\partial |\lambda_i(a)|}{\partial a} \mid_{a=a_{NS}} \neq 0 \; , \; i = 1, 2. \tag{3.8}$$

Also, if

$$trJ(a_{NS}) \mid_{a=a_{NS}} \neq 0, -1, \tag{3.9}$$

then, we reach

$$\lambda^k(a_{NS}) \neq 1 \; , \; k = 1, 2, 3, 4. \tag{3.10}$$

Let $q, p \in \mathbb{C}^2$ be two eigenvectors that correspond to the eigenvalues λ of the matrix $J(NSB_{E_2})$ and the eigenvalues $\overline{\lambda}$ of the matrix $J(NSB_{E_2})^T$, respectively. If these eigenvectors are calculated with Mathematica program, then we get

$$q \sim \left(\frac{-b\sqrt{1 + cd} - bi\sqrt{-9 + (4 - 5c)d}}{2d(1 + cd)^{\frac{3}{2}}}, 1 \right)^T$$

and

$$p \sim \left(1, \frac{b\sqrt{1+cd} + bi\sqrt{-9+(4-5c)d}}{2d(1+cd)^{\frac{3}{2}}}\right).$$

By using the scalar product in $\mathbb{C}^2 : \langle p, q\rangle = \overline{p_1}q_1 + \overline{p_2}q_2$, we get the following vector in order to normalize p according to q

$$p \sim \left(\frac{-id(1+cd)^{\frac{3}{2}}}{b\sqrt{-9+(4-5c)d}}, \frac{-1}{2} - \frac{i\sqrt{1+cd}}{2\sqrt{-9+(4-5c)d}}\right)$$

where $\langle p, q\rangle = 1$. $\forall\, U \in \mathbb{R}^2$ can be uniquely represented as

$$U = zq + \overline{zq} \tag{3.11}$$

for some $z \in \mathbb{C}$. Here, $\overline{z}$ is the conjugate of that complex number z, and $z = \langle p, U\rangle$. For all sufficiently small $|a|$ about a_{NS}, we can transform the system (1.1) as follows:

$$z \to \lambda(a)z + g(z, \overline{z}, a), \tag{3.12}$$

where $\lambda(a) = (1 + \omega(a))e^{i\theta(a)}$ with $\omega(a_{NS}) = 0$ and $g(z, \overline{z}, a)$ is a complex valued smooth function of z and $\overline{z}$. Taylor expression of g with respect to $g(z, \overline{z})$ is

$$g(z, \overline{z}, a) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl}(a) z^k \overline{z}^l, \tag{3.13}$$

and the Taylor coefficients g_{kl} calculated through multilinear vector functions are expressed by the following formulas:

$$\begin{aligned}
g_{20}(a_{NS}) =& \langle p, B(q, q)\rangle,\\
g_{11}(a_{NS}) =& \langle p, B(q, \overline{q})\rangle,\\
g_{02}(a_{NS}) =& \langle p, B(\overline{q}, \overline{q})\rangle,\\
g_{21}(a_{NS}) =& \langle p, C(q, q, \overline{q})\rangle.
\end{aligned}$$

For the system (3.2) that exhibits the Neimark-Sacker bifurcation, the coefficient $\varphi(a_{NS})$ determining the direction of the appearance of the invariant curve can be calculated as:

$$\begin{aligned}
\varphi(a_{NS}) =& Re(\frac{e^{-i\theta(a_{NS})}g_{21}}{2}) - Re\left(\frac{(1-2e^{i\theta(a_{NS})})e^{-2i\theta(a_{NS})}}{2(1-e^{i\theta(a_{NS})})} g_{20}g_{11}\right)\\
& -\frac{1}{2}\left|g_{11}\right|^2 - \frac{1}{4}\left|g_{02}\right|^2
\end{aligned} \tag{3.14}$$

where $e^{i\theta(a_{NS})} = \lambda(a_{NS})$. Consequently, we have the following theorem on Neimark-Sacker bifurcation:

Theorem 3.2. *If (3.9) holds, $\varphi(b_1) \neq 0$ and the parameter a changes its value in small vicinity of NSB_{E_2}, then the system (1.1) passes through a Neimark-Sacker bifurcation at the only fixed point E_2. Furthermore, if $\varphi(a_{NS}) < 0$ ($\varphi(a_{NS}) > 0$), then there exists a unique attracting (repelling) invariant closed curve which bifurcates from E_2.*

3.3. Chaos control

Recently, the control of chaos in discrete-time systems has become the focal point for many researchers. It is possible to optimize the system and avoid chaos with some chaos strategies applied to the systems. These practical methods can be used in many fields such as communication, physics laboratories, cardiology and turbulence, as well as providing control of dynamical systems [42]. Various methods such as state feedback method, pole placement technique, OGY method and hybrid control method are useful for controlling chaos in discrete-time models (see [13–17, 19, 20, 25, 29, 30, 34, 36, 41, 48, 49, 51, 64]). The first feedback control strategy known as the OGY method was proposed by Ott et al. [48], which can be used to control not only fixed points but also periodic trajectories. The core of the OGY theory relies on stabilizing one (or more) of many unstable periodic orbitals embedded in a chaotic attractor by applying small perturbations. One of the system parameters must be accessible to apply these perturbations. The instability is controlled by the perturbation of this parameter, which serves as the input of the system. The solution of a chaotic system is difficult to predict, which requires a way to control it. The OGY algorithm, which is useful in discrete-time systems, has disadvantages in that control can be limited to one of the buried trajectories and the time required to obtain control can be very long and unpredictable. For discrete-time models, which are discrete counterparts of continuous systems with an application of the Euler approximation, the OGY technique may be particularly ineffective [13, 15]. Additional information on the mathematics of the OGY algorithm can be found in [25,51]. A hybrid control strategy is an alternative method combining parameter perturbation and state feedback. This is an effective method. Advantageously, bifurcations in the discrete nonlinear dynamical system can be delayed or even completely eliminated. Therefore, the system exhibits stable dynamic behavior over a wide range of parameter values. With this method, besides stabilizing the desired unstable periodic orbit embedded in a chaotic attractor, it is possible to control the lower stable periodic orbital towards the highly stable periodic orbital. We refer to the studies [19, 41, 64] for further details related to hybrid control method. In addition, the studies [4,55] are referred to review the logic of deriving chaos control techniques and the biological meanings of the parameters used in these techniques.

We endeavor to control the chaos via chaos controlling strategy based on OGY method and hybrid control feedback methodology. We first apply the OGY control strategy to move the trajectory towards the desired stabilizing orbit. Let us control the chaos occurring in the system (1.1) by taking a as control parameter. Besides this, a is restricted in some small interval $|a - a_0| < \delta$ with $\delta > 0$, and a_0 denotes the nominal value belonging to chaotic region. Assume that $(\overline{x}, \overline{y})$ be unstable fixed point of system (1.1) in chaotic region produced by Neimark-Sacker bifurcation, then the system (1.1) can be approximated in the neighborhood of the unstable fixed point $(\overline{x}, \overline{y})$ by the following linear map:

$$\begin{bmatrix} x_{n+1} - \overline{x} \\ y_{n+1} - \overline{y} \end{bmatrix} \approx A \begin{bmatrix} x_n - \overline{x} \\ y_n - \overline{y} \end{bmatrix} + B[a - a_0] \tag{3.15}$$

where

$$A = \begin{bmatrix} \frac{\partial f(\overline{x},\overline{y},a_0)}{\partial x_n} & \frac{\partial f(\overline{x},\overline{y},a_0)}{\partial y_n} \\ \frac{\partial g(\overline{x},\overline{y},a_0)}{\partial x_n} & \frac{\partial g(\overline{x},\overline{y},a_0)}{\partial y_n} \end{bmatrix} = \begin{bmatrix} \frac{-a_0+d-(-2+a_0)cd^2}{d(1+cd)} & \frac{-b}{d+cd^2} \\ \frac{(a_0(-1+d)-d)(1+cd)}{b} & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} \frac{\partial f(\overline{x},\overline{y},a_0)}{\partial a} \\ \frac{\partial g(\overline{x},\overline{y},a_0)}{\partial a} \end{bmatrix} = \begin{bmatrix} \frac{-1+d}{d^2} \\ 0 \end{bmatrix}.$$

It is clear that the system (1.1) is controlled by the following matrix

$$C = [B : AB] = \begin{bmatrix} \frac{-1+d}{d^2} & \frac{(-1+d)(-a+d-(-2+a)cd^2)}{d^3(1+cd)} \\ 0 & \frac{a(-1+d)-d)(-1+d)(1+cd)}{bd^2} \end{bmatrix} \tag{3.16}$$

such that the rank of C is 2.

Furthermore, we suppose that $[a - a_0] = -K \begin{bmatrix} x_n - \overline{x} \\ y_n - \overline{y} \end{bmatrix}$, where $K = [k_1 \;\; k_2]$, then the system (3.15) can be written as

$$\begin{bmatrix} x_{n+1} - \overline{x} \\ y_{n+1} - \overline{y} \end{bmatrix} \approx [A - BK] \begin{bmatrix} x_n - \overline{x} \\ y_n - \overline{y} \end{bmatrix}. \tag{3.17}$$

The corresponding controller system can be given by

$$\begin{aligned} x_{n+1} &= [a_0 - k_1(x_n - \overline{x}) - k_2(y_n - \overline{y})]x_n(1 - x_n) - bx_ny_n\left(\frac{x_n}{x_n + c}\right), \\ y_{t+1} &= dx_ny_n. \end{aligned} \tag{3.18}$$

In addition, the fixed point $(\overline{x}, \overline{y})$ is locally asymptotically stable if and only if eigenvalues μ_1 and μ_2 of the matrix $A - BK$ lie in an open unit disk. Therefore, the Jacobian matrix $A - BK$ of the controlled system (3.18) is found as

$$A - BK = \begin{bmatrix} \frac{d^2(1+2cd)-(d+cd^3)a_0-(-1+d)(1+cd)k_1}{d^2(1+cd)} & -\frac{bd+(-1+d)(1+cd)k_2}{d^2(1+cd)} \\ \frac{(1+cd)(-d+(-1+d)a_0)}{b} & 1 \end{bmatrix}. \tag{3.19}$$

The characteristic equation of the Jacobian matrix $A - BK$ can be given as

$$\begin{aligned} &\mu^2 + \frac{-d^2(2+3cd) + (d+cd^3)a_0 + (-1+d)(1+cd)k_1}{d^2(1+cd)}\mu \\ &+ \frac{-b(-1+d)(1+cd)k_1 + d(bcd^2 - (-1+d)(1+cd)^2k_2)}{d^2(b+bcd)} \\ &+ \frac{a_0(bd(-2+d-cd) + (-1+d)^2(1+cd)^2k_2)}{d^2(b+bcd)} = 0. \end{aligned}$$

Let μ_1 and μ_2 be the roots of characteristic equation of system (3.18), then we have

$$\mu_1 + \mu_2 = \frac{-d^2(2+3cd) + (d+cd^3)a_0 + (-1+d)(1+cd)k_1}{d^2(1+cd)}, \tag{3.20}$$

$$\mu_1\mu_2 = \frac{-b(-1+d)(1+cd)k_1 + d(bcd^2 - (-1+d)(1+cd)^2k_2)}{d^2(b+bcd)} \tag{3.21}$$

$$+\frac{a_0(bd(-2+d-cd)+(-1+d)^2(1+cd)^2k_2)}{d^2(b+bcd)}. \tag{3.22}$$

To determine the marginal stability lines, we examine with conditions $\mu_1=\pm1$ and $\mu_1\mu_2=1$. If $\mu_1\mu_2=1$, then from Equation (3.21), we get

$$\begin{aligned} l_1 :=&\frac{-b(-1+d)(1+cd)k_1+d(-bd-(-1+d)(1+cd)^2k_2)}{d^2(b+bcd)} \\ &+\frac{a_0(bd(-2+d-cd)+(-1+d)^2(1+cd)^2k_2)}{d^2(b+bcd)}=0. \end{aligned} \tag{3.23}$$

Now, we suppose that $\mu_1=1$, then Equation (3.20) and Equation (3.21) imply

$$\begin{aligned} l_2 :=&\frac{bd^2(3+5cd)-bd(3+d(-1+c+cd))a_0-2b(-1+d)(1+cd)k_1}{d^2(b+bcd)} \\ &+\frac{(-1+d)(1+cd)^2(-d+(-1+d)a_0)k_2}{d^2(b+bcd)}=0. \end{aligned} \tag{3.24}$$

If $\mu_1=-1$, by using Equation (3.20) and Equation (3.21), we obtain

$$l_3 := -\frac{(-d+(-1+d)a_0)(bd+(-1+d)(1+cd)k_2)}{bd^2}=0. \tag{3.25}$$

The triangular region determined by the lines l_1, l_2 and l_3 in k_1k_2 plane is the region of the values that make the eigenvalues less than 1.

Secondly, we use hybrid control method to control the chaos in the system (1.1). Let us suppose that system (1.1) undergoes Neimark-Sacker bifurcation at fixed point $(\overline{x},\overline{y})$, then the corresponding controlled system can be taken as follows:

$$\begin{aligned} x_{n+1} =&\gamma[ax_n(1-x_n)-bx_ny_n\left(\frac{x_n}{x_n+c}\right)]+(1-\gamma)x_n, \\ y_{n+1} =&\gamma dx_ny_n+(1-\gamma)y_n \end{aligned} \tag{3.26}$$

where γ is control parameter with $0<\gamma<1$. The Jacobian matrix of controlled system (3.26) is given by

$$\begin{bmatrix} 1+(-1+a-2a\overline{x}-\frac{bx(2c+\overline{x})\overline{y}}{(c+\overline{x})^2})\gamma & \frac{-\gamma b\overline{x}^2}{c+\overline{x}} \\ \gamma d\overline{y} & 1-\gamma+d\overline{xy} \end{bmatrix}.$$

When

$$\begin{aligned} &\left|\frac{-2d+a\gamma+cd^2(-2+(-1+a)\gamma)}{d(1+cd)}\right| \\ &< 1+\frac{-a\gamma(1+\gamma)+cd^2(1+\gamma-a\gamma+(-1+a)\gamma^2)-d(-1+(1+a(-1+c))\gamma^2))}{d(1+cd)} \\ &< 2 \end{aligned}$$

is provided then the positive equilibrium point $(\overline{x},\overline{y})$ of the controlled system (3.26) is locally asymptotically stable.

4. Numerical simulations

This section provides some numerical simulations to demonstrate the existence of flip and Neimark-Sacker bifurcation for the system (1.1). Theoretical analysis is verified with suitable examples by taking some special cases for system (1.1). Numerical simulations clearly manifest interesting complex dynamics behaviors. The dynamic nature of the system (1.1) near the positive coexistence fixed point is displayed under different sets of parameter values. Here, trajectories, bifurcation diagrams and phase portraits are illustrated via SageMath programming [35] by taking a as bifurcation parameter.

The following Example 4.1 and 4.2 illustrates the emergence of flip and Neimark-Sacker bifurcation based on our theoretical results, respectively.

Example 4.1. By considering the parameter values $b = 0.2$, $c = 0.5$, $d = 3.5$, we have the following system

$$\begin{aligned} x_{n+1} &= 5.57627 x_n(1 - x_n) - 0.2 x_n y_n \left(\frac{x_n}{x_n + 0.5} \right), \\ y_{t+1} &= 3.5 x_n y_n. \end{aligned} \tag{4.1}$$

$a_F = 5.57627$ is flip bifurcation point. The computation yields $(\overline{x}, \overline{y}) = (0.285714, 41.0169)$. The Jacobian matrix is $J = \begin{bmatrix} -2.49153 & -0.0207792 \\ 143.559 & 1 \end{bmatrix}$. The eigenvalues are $\lambda_1 = -1$, and $\lambda_2 = -0.491525$ such that $|\lambda_2| \neq 1$. The flip bifurcation diagram are displayed in Figure 4. The system (1.1) undergoes a flip bifurcation at positive fixed point E_2 when the parameter changes in a small neighborhood of a_F. This defines that the fixed point E_2 is stable for $a < 5.57627$, loses its stability around $a = 5.57627$ and there exists a period doubling phenomena for $a > 5.57627$. Upon the necessary calculations, we obtain $c\,(a_F) = 36.0037 > 0$ (see Appendix A for detailed steps on obtaining $c\,(a_F)$). The period-2 orbits that bifurcate from E_2 are stable. Figure 4 shows flip bifurcation diagram of the system (4.1) with the initial conditions $x_0 = 0.3$ and $y_0 = 40.1$.

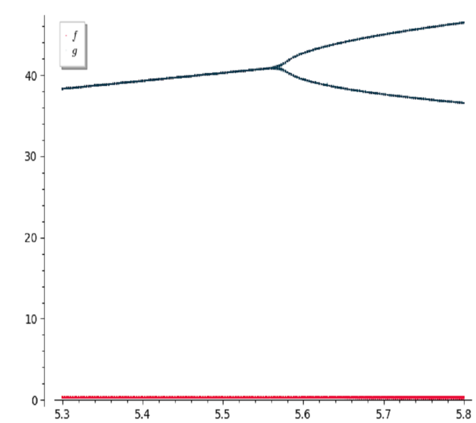


Figure 4. Bifurcations diagram of the prey-predator system (4.1) with the parameter values $a \in (5.3, 5.8)$, $b = 0.2$, $d = 3.5$, $c = 0, 5$.

Example 4.2. Let us consider the following system for the parameter values $b = 0.2$, $d = 3.5$, $c = 0.05$,

$$\begin{aligned} x_{n+1} &= 2.64151x_n(1-x_n) - 0.2x_ny_n\left(\frac{x_n}{x_n+0.05}\right), \\ y_{t+1} &= 3.5x_ny_n. \end{aligned} \tag{4.2}$$

$a_{NS} = 2.64151$ is Neimark-Sacker bifurcation point. The computation yields $(\overline{x}, \overline{y}) = (0.285714, 5.20991)$, and the Jacobian matrix evaluated at $(\overline{x}, \overline{y})$ is

$$J_{(\overline{x},\overline{y})} = \begin{pmatrix} 0.113208 & -0.0486322 \\ 18.2347 & 1 \end{pmatrix}.$$

The eigenvalues are $\lambda_{1,2} = 0.556604 \mp 0.830778i$ such that $|\lambda_{1,2}| = 1$. Here $\theta = 0.980504$; and from (3.14), we get $\varphi(a_{NS}) = -0.843535 < 0$. (see Appendix B for detailed steps on obtaining $\varphi(a_{NS})$). Consequently, the Neimark-Sacker bifurcation emerges at $a_{NS} = 2.64151$. The following graphs give the bifurcation and phase portraits of the system (4.2) with the initial conditions $x_0 = 0.3$ and $y_0 = 5.1$. Figure 5(a) shows Neimark-Sacker bifurcation diagram of the system (4.2). The phase portraits of the system (4.2) are presented Figure 5(b)-(d).

The phase portraits of bifurcation diagram in Figure 5(a) shown in Figure 5(b)-(d) for different values of a demonstrate the process how smooth invariant curve bifurcates from the stable fixed point. Furthermore, at the value $a = 2.64151$, the positive fixed point $(\overline{x}, \overline{y})$ becomes unstable and closed invariant curve enclosing the unique positive unstable fixed point $(\overline{x}, \overline{y})$ also is generated. Therefore, it is confirmed that Neimark Sacker bifurcation emerges at $a = 2.64151$.

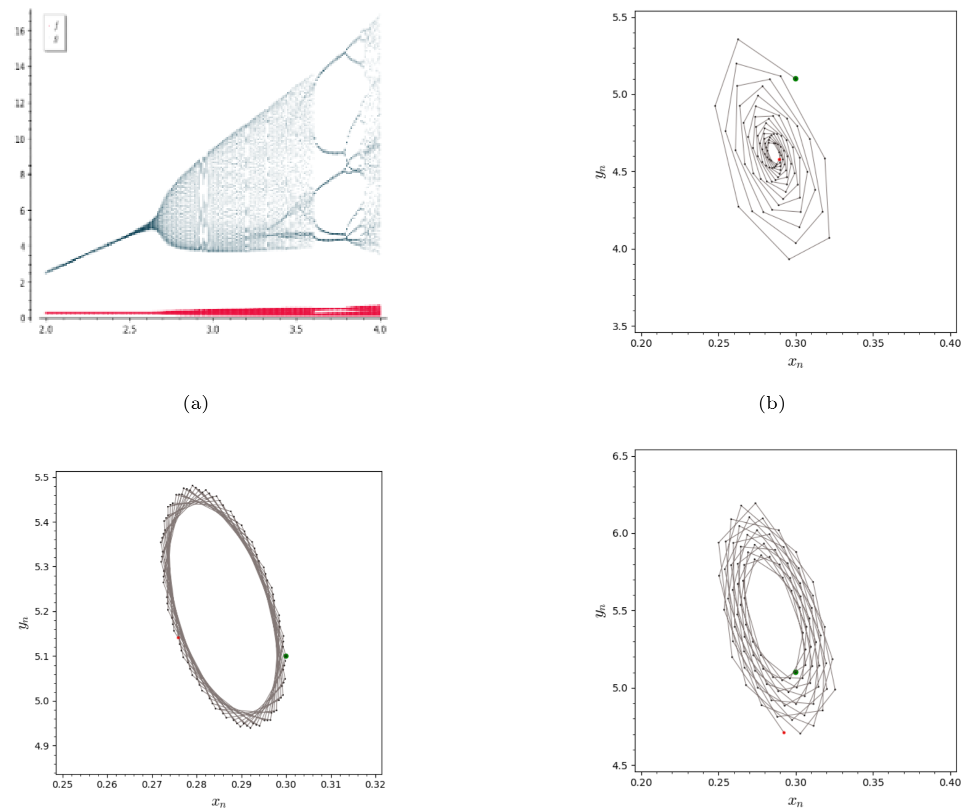
The following example 4.3-4.4 is used to control the chaotic behavior of the system (1.1).

Example 4.3. Let us take the parameters $b = 0.2$, $d = 3.5$, $c = 0.05$, $a = 2.64151$ and the initial conditions $x_0 = 0.3$ and $y_0 = 5.1$. In this case, we know that the system (1.1) undergoes Neimark-Sacker bifurcations.

Let us take $a = 2.7$ in order to control the system (1.1) with OGY control method. Then corresponding controlled system is given as

$$\begin{aligned} x_{n+1} &= [2.7 - k_1(x_n - 0.285714) - k_2(y_n - 5.45536)]x_n(1-x_n) \\ &\quad -0.2x_ny_n\left(\frac{x_n}{x_n+0.05}\right), \\ y_{t+1} &= 3.5x_ny_n \end{aligned} \tag{4.3}$$

for $K = [k_1\ k_2]$. We have $A = \begin{bmatrix} 0.0902736 & -0.0486322 \\ 19.0938 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0.204082 \\ 0 \end{bmatrix}$ and $C = [B : AB] = \begin{bmatrix} 0.204082 & 0.0184232 \\ 0 & 3.89668 \end{bmatrix}$. Since the rank of C is 2, the system is controllable. Therefore, the Jacobian matrix $A - BK$ of the controlled system (4.3) can be given as $A - BK = \begin{bmatrix} 0.0902736 - 0.204082k_1 & -0.0486322 - 0.204082k_2 \\ 19.0938 & 1 \end{bmatrix}$.



(a) (b)

(c) (d)

Figure 5. (a) Bifurcations diagram of the prey-predator system (4.2) with the parameter values $a \in (2, 4)$, $b = 0.2$, $d = 3.5$, $c = 0,05$. (b) The phase portrait of system (4.2) when $a = 2.5$, $b = 0.2$, $d = 3.5$, $c = 0,05$. (c) The phase portrait of system (4.2) when $a = 2.64151$, $b = 0.2$, $d = 3.5$, $c = 0,05$. (d) The phase portrait of system (4.2) when $a = 2.7$, $b = 0.2$, $d = 3.5$, $c = 0,05$.

The marginal lines l_1, l_2 and l_3 in Equation (3.23)-(3.25) are obtained as

$$l1 = 0.018845 - 0.204082k_1 + 3.896685k_2 = 0,$$
$$l2 = 3.10912 - 0.408163k_1 + 3.89668k_2 = 0,$$

and

$$l3 = -0.928571 - 3.89668k_2 = 0.$$

The marginal lines l_1, l_2 and l_3 determine the stable triangular region in the k_1k_2 plane. Figure 6 shows the region bounded by these lines of the controlled system (4.3).

Example 4.4. To implement the hybrid control method, once again, we consider the parameters $b = 0.2$, $d = 3.5$, $c = 0.05$, $a = 2.64151$ with the initial conditions $x_0 = 0.3$ and $y_0 = 5.1$. For these parametric values, the controlled system is

$$x_{n+1} = \gamma[2.64151x_n(1 - x_n) - 0.2x_ny_n\left(\frac{x_n}{x_n + 0.05}\right)] + (1 - \gamma)x_n, \qquad (4.4)$$

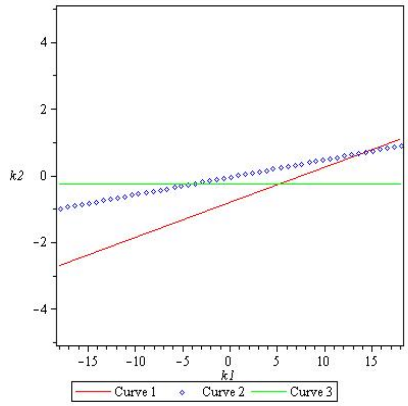


Figure 6. Stability region of the controlled system (4.3) in k_1k_2 plane.

$$y_{n+1} = 3.5\gamma x_n y_n + (1-\gamma)y_n$$

and the system (4.4) has unique positive coexistence fixed point $(\overline{x}, \overline{y}) = (0.285714, 5.20991)$. In addition, the Jacobian matrix evaluated at $(0.285714, 5.20991)$ is

$$\begin{bmatrix} 1-0.861724\gamma & -0.0486322\gamma \\ 18.2347\gamma & 1-1.10^{-6}\gamma \end{bmatrix} \tag{4.5}$$

and the characteristic equation of (4.5) is obtained as

$$\lambda^2 - (2-0.861724\gamma)\lambda + 1 - 0.861724\gamma + 0.886792\gamma^2 = 0. \tag{4.6}$$

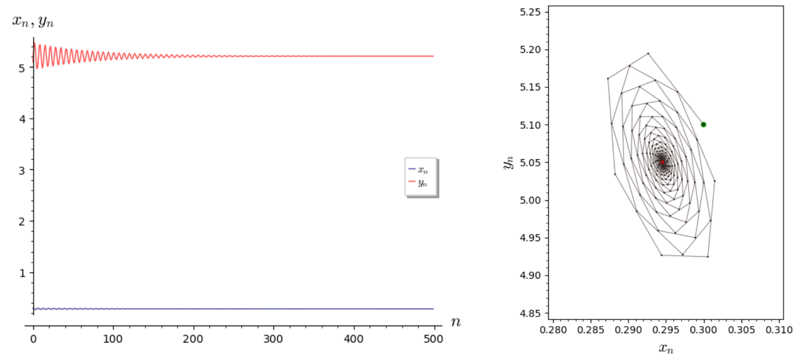
Based on the Jury conditions, we conclude that if $0 < \gamma < 0.97173$, then the roots of (4.6) lie in a unit open disk. Therefore, the Neimark-Sacker bifurcation is fully controlled for values γ in the obtained range. The unique positive coexistence fixed point $(\overline{x}, \overline{y}) = (0.285714, 5.20991)$ of (4.4) is locally asymptotically stable.

For $\gamma = 0.97$, Figure 7-(a)-(b) shows plots of the controlled system (4.4).

5. Conclusions

In this article, we discuss the analysis of the complex dynamic behavior of a discrete-time prey-predator system (1.1). We investigate the existence of the fixed points of the system (1.1) and the stability conditions of these points. We also show that the system (1.1) exhibits flip and Neimark-Sacker bifurcation at the positive coexistence fixed point. We apply OGY method and hybrid control feedback methodology to prevent the chaos exhibited by the dynamic system (1.1).

We obtain that the system (1.1) has a trivial (extinction) fixed point E_0, an exclusion fixed point E_1 and a coexistence fixed point E_2. The asymptotic stability conditions of these fixed points are investigated by using the linearization method. It can seen that there is a unique positive coexistence fixed point $E_2 = (\frac{1}{d}, \frac{(-a-d+ad)(1+cd)}{bd})$ of the system (1.1) with $a > \frac{d}{d-1}$, $d > 1$. To examine flip



(a) (b)

Figure 7. (a) The trajectories of the controlled system (4.4) when $b = 0.2$, $d = 3.5$, $c = 0.05$, $a = 2.64151$ and $\gamma = 0.97$. (b) Phase portrait of the controlled system (4.4) when $b = 0.2$, $d = 3.5$, $c = 0.05$, $a = 2.64151$ and $\gamma = 0.97$.

and Neimark-Sacker bifurcation, prey growth rate a is taken as bifurcation parameter. By using mathematical techniques of bifurcation theory, we show that the system (1.1) undergoes flip bifurcation under the condition $a = a_F = \frac{d(3+5cd)}{3+d(-1+c+cd)}$, and the system (1.1) undergoes Neimark-Sacker bifurcation under the condition $a_{NS} = \frac{d}{d(1-c)-2}$. The dynamic properties of system (1.1) are presented by some figures. By choosing a value as bifurcation parameter, the effects of Allee factor on prey were observed. Chaos caused by Neimark-Sacker bifurcation is successfully controlled by OGY control method by considering the parameter values $a = 2.7, b = 0.2$, $d = 3.5$, $c = 0.05$. The stabilization of the unstable fixed point of the system (1.1) with $b = 0.2$, $d = 3.5$, $c = 0.05$, $a = 2.64151$ is provided by the hybrid control method. The hybrid control strategy allows us to successfully the stable behavior by suppressing the unstable fixed point.

Also, a flip bifurcation occurs when the intrinsic growth rate of the prey increases despite the low initial condition of the prey. In response to high levels of predators, the density of the prey population is greatly reduced, but not completely extinct. In [11], the researchers numerically reach that for some parameter values, y_n can be lost and x_n remains chaotic. An anti-control algorithm is implemented to prevent extinction. In our study, with the addition of the Allee effect to the model, y_n continues its chaotic behavior and does not disappear. When the density of predator is dominant, the density of prey continues to exist with a very small number (see Figures 4 and 5). Neimark-Sacker bifurcation occurs with parameter values $d = 3.5$ and $b = 0.2, a = 2.333...$ in the system without Allee effect. When the Allee effect is added (with Allee constant $c = 0.05$), Neimark-Sacker bifurcation occurs at value $a = 2.64151$. The Allee effect appears to delay Neimark-Sacker bifurcation. This effect keeps the stability going for a while. Namely, the stability decreases as the Allee effect decreases. In addition, the system undergoes flip bifurcation at the point $a = 47.1739$ with parameter values $d = 3.5$, $b = 0.2$ and $c = 0.05$. The chaos control techniques were used to avoid the earlier Neimark-Sacker bifurcation.

As species become extinct, they are removed from the food chain. Animals that eat endangered species must find new food sources to avoid starvation. This can harm other plant or animal populations. In other words, each extinct species triggers the extinction of other species in the ecosystem. In particular, if a predator population goes extinct, the population of its prey multiplies and can destabilize local ecosystems. The result obtained is ecologically important since the Allee effect supports the survival of the predator and contributes to the survival of the species in a certain balance.

Furthermore, for some parameter values, figures (by using SageMath and Matlab programming) exhibit the trajectories, bifurcation diagrams, phase portraits and maximal Lyapunov exponent of the prey-predator system (1.1) in an enriched way:

Example 5.1. The prey-predator system (1.1) exhibits dynamical behaviors according to different parameter values. The initial conditions are taken $x_0 = 0.5$ and $y_0 = 0.2$ in these simulations.

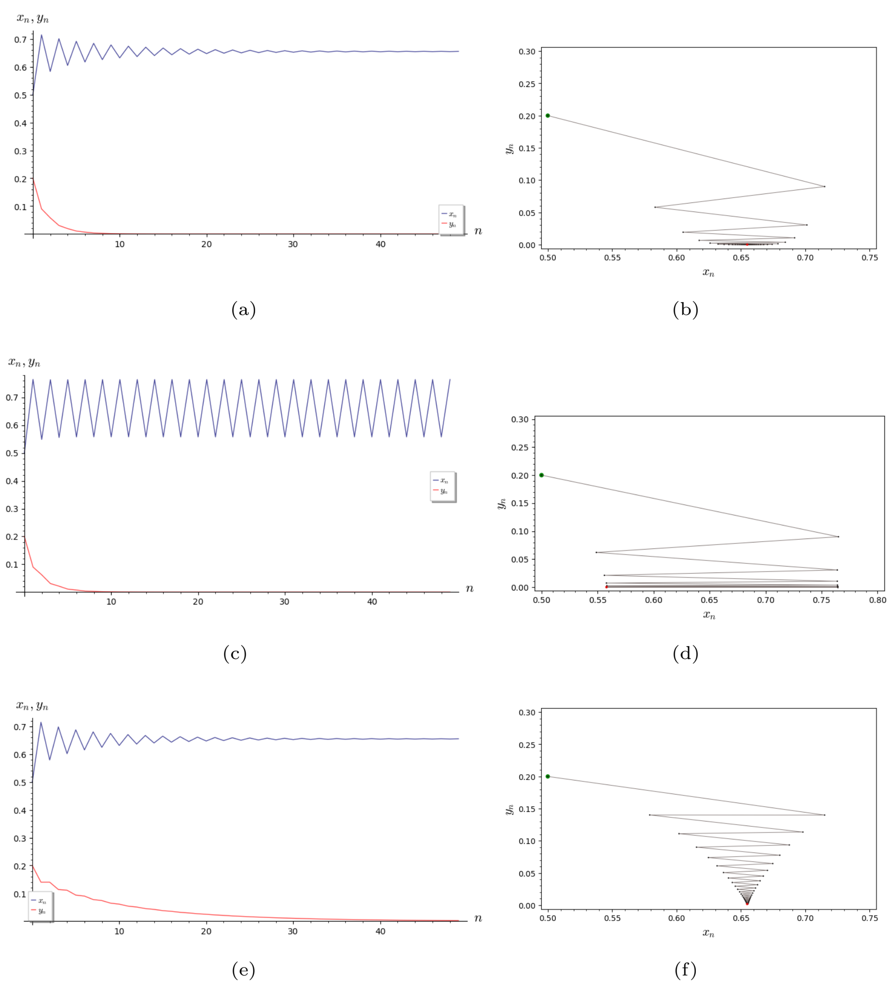


Figure 8. (a) The trajectories of prey-predator densities when $a = 2.9$, $b = 0.2$, $c = 0.5$ and $d = 0.9$. (b) The phase portrait prey-predator system when $a = 2.9$, $b = 0.2$, $c = 0.5$ and $d = 0.9$. (c) The trajectories of prey-predator densities when $a = 3.1$, $b = 0.2$, $c = 0.5$ and $d = 0.9$. (d) The phase portrait prey-predator system when $a = 3.1$, $b = 0.2$, $c = 0.5$ and $d = 0.9$. (e) The trajectories of prey-predator densities when $a = 2.9$, $b = 0.2$, $c = 0.5$ and $d = 1.4$. (f) The phase portrait prey-predator system when $a = 2.9$, $b = 0.2$, $c = 0.5$ and $d = 1.4$.

The phase portraits corresponding to the densities of prey-predator in **(a), (c)** and **(e)** are given with **(b), (d)** and **(f),** in Figure 8 respectively.

From Lemma 2.3- **(i.b1)**, the fixed point $(0.655172, 0)$ is locally asymptotically stable when $a = 2.9$, $b = 0.2$, $c = 0.5$ and $d = 0.9$. Also, Lemma 2.3- **(i.b1)** is not provided when $a = 3.1, b = 0.2$, $c = 0.5$ and $d = 0.9$, the fixed point $(0.677419, 0)$ is not locally asymptotically stable. From Lemma 2.3-**(i.b1)**, the fixed point $(0.655172, 0)$ is locally asymptotically stable when $a = 2.9$, $b = 0.2$, $c = 0.5$ and $d = 1.4$. For, $1 < a < 3$ and $d > \frac{3}{2}$, we can see that the fixed point E_1 disappears and the fixed point E_2 appears.

Example 5.2. The prey-predator system (1.1) exhibits dynamical behaviors according to different parameter values and initial conditions (See Figure 9).

When $a = 2.53$, $b = 0.2$, $c = 0.30$ and $d = 3.5$, these parameter values satisfy the case in **A. (i.b5)**- $(2.51527 < a < 5.57627$ and $0.285714 < c < 1)$. The fixed point $(0.285714, 8.27321)$ is locally asymptotically stable.

When $a = 3.381, b = 0.2$, $c = 0.30$ and $d = 3$, these parameter values satisfy the case in **A. (i.b4)**. The fixed point $(0.333333, 11.913)$ is locally asymptotically stable.

When $a = 3.381, b = 0.2$, $c = 0.033$ and $d = 2.99$, these parameter values do not satisfy the case in **A. (i.b4)**. The fixed point $(0.334448, 6.86796)$ is unstable.

Furthermore, when $a = 4, b = 0.2$, $c = 0.5$ and $d = 1.4$, these parameter values do not satisfy the case in **A. (i.b3)**. The fixed point $(0.714286, 1.21429)$ is unstable.

Example 5.3. Finally, let us take the parameters $b = 0.0001$; $d = 9.5$; $c = 0.0001$. For the system (1.1) with these values, Figure 10(a)-(c) presents the Neimark-Sacker bifurcation and the Lyapunov exponent graph supporting the chaotic behavior.

Conflict of interest

The author declares that they do not have any known competing financial interests or personal relationships that could have influence the work reported in this paper.

Appendix

A. Calculation of the formula in (3.7) for example 4.1

$$\begin{aligned}
F_1(u,v) &= -9.80431u^2 + 5.38114u^3 - 0.119008uv - 0.10308u^2v + O(\|U\|^4),\\
F_2(u,v) &= 3.5uv + O(\|U\|^4), \qquad \text{(A.1)}\\
B(q,q) &= \begin{pmatrix} 0.276464 \\ -0.09751 \end{pmatrix},\\
C(q,q,q) &= \begin{pmatrix} 0.00909692 \\ 0 \end{pmatrix},\\
B_1(u,v) &= -19.6086u_1v_1 - 0.119008(u_2v_1 + u_1v_2),
\end{aligned}$$

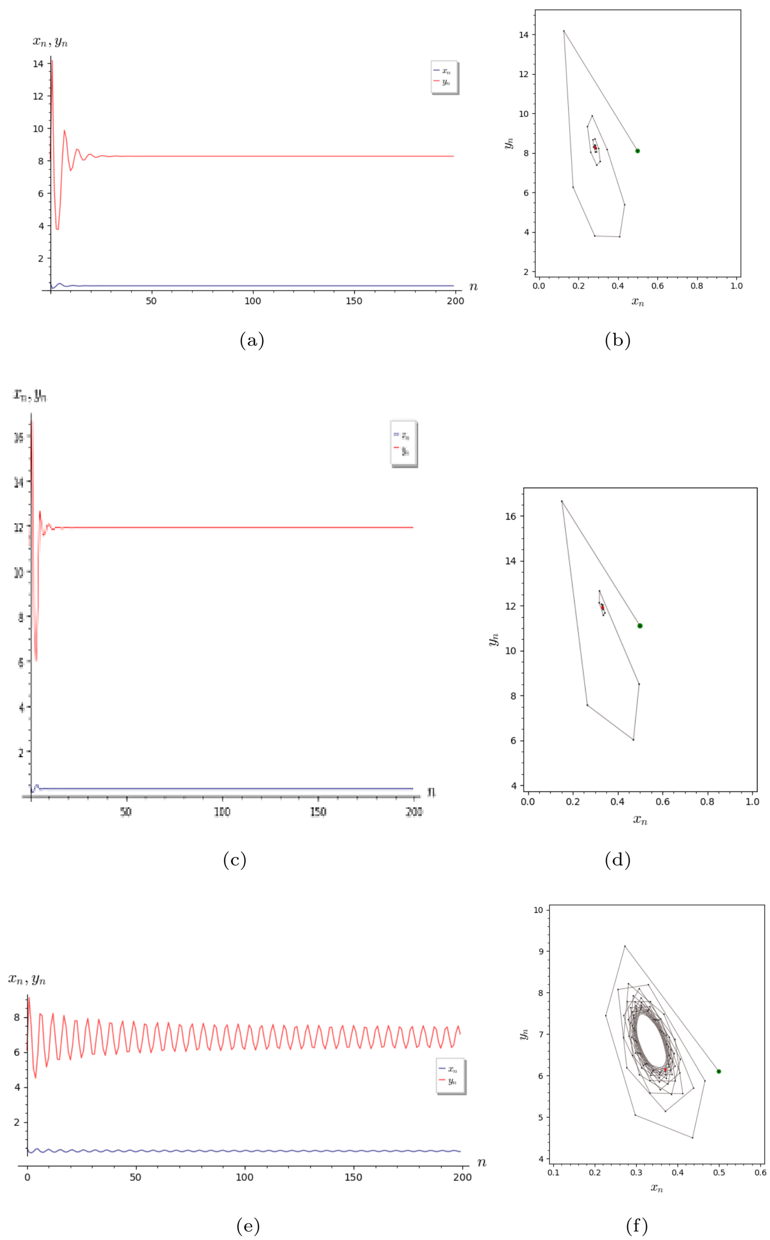
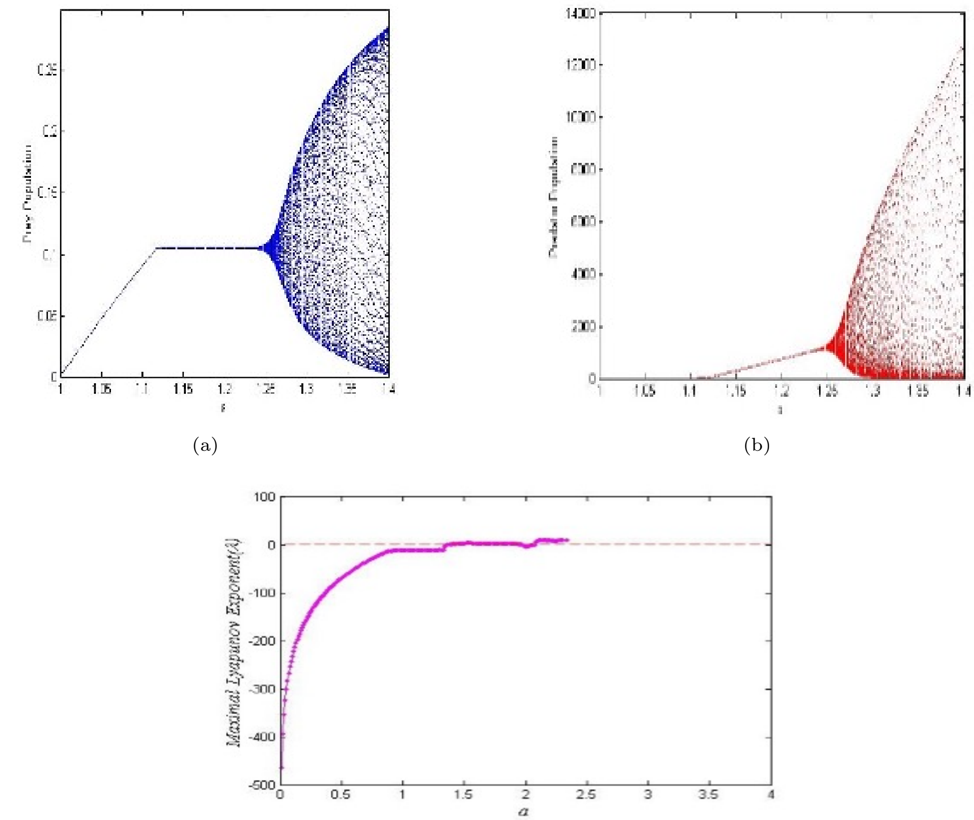


Figure 9. (a) The trajectories of prey-predator densities with the initial conditions $x_0 = 0.5$ and $y_0 = 8.1$ when $a = 2.53$, $b = 0.2$, $c = 0.30$ and $d = 3.5$. (b) The phase portrait prey-predator system when $a = 2.53$, $b = 0.2$, $c = 0.30$ and $d = 3.5$. (c) The trajectories of prey and predator densities with the initial conditions $x_0 = 0.5$ and $y_0 = 11.1$ when $a = 3.381$, $b = 0.2$, $c = 0.30$ and $d = 3$. (d) The phase portrait prey-predator system when $a = 3.381, b = 0.2$, $c = 0.30$ and $d = 3$. (e) The trajectories of prey and predator densities with the initial conditions $x_0 = 0.5$ and $y_0 = 6.1$ when $a = 3.381$, $b = 0.2$, $c = 0.033$ and $d = 2.99$. (f) The phase portrait prey-predator system when $a = 3.381$, $b = 0.2$, $c = 0.033$ and $d = 2.99$.

$$B_2(u,v) = 3.5(u_2 v_1 + u_1 v_2),$$

$$C_1(u,v,w) = 32.2869 u_1 v_1 w_1 - 0.206161(u_1 v_1 w_2 + u_1 v_2 w_1),$$



(a) (b)

(c)

Figure 10. Neimark-Sacker bifurcations of system (1.1) when $a \in (1, 1.4)$; $b = 0.0001$; $d = 9.5$; $c = 0.0001$; with corresponding Maximal Lyapunov Exponent. (a) Bifurcation diagram of prey population (b) Bifurcation diagram of predator population (c) Maximal Lyapunov Exponent.

$$C_2(u, v, w) = 0$$

and $p \sim (-282.4, -2.933)^T$, $q \sim (-0.01393, 1)^T$.

B. Calculation of the formula in (3.14) for example 4.2

Let $q, p \in \mathbb{C}^2$ be the complex eigenvectors corresponding to $\lambda_{1,2}$, respectively, $q \sim (-0.0243 - 0.0455i, 1)^T$ and $p \sim (1, 0.0243 + 0.0455i)^T$. We get the vector $p \sim (10.989i, -0.5 + 0.267i)^T$ to normalize p according to q, such that $\langle p, q \rangle = 1$. When the coefficient of the form (3.14) are calculated, we get

$$\begin{aligned} g_{20}(a_{NS}) &= -2.73583 + 1.66575i, \\ g_{11}(a_{NS}) &= -2.62531 + 1.50654i, \\ g_{02}(a_{NS}) &= -2.71034 + 1.34729i, \\ g_{21}(a_{NS}) &= -0.0132148 + 0.04226i \end{aligned}$$

where

$$
\begin{aligned}
F_1(u,v) &= -2.71036u^2 + 0.205079u^3 - 0.195564uv - 0.0132148u^2v + O(\|U\|^4),\\
F_2(u,v) &= 3.5uv + O(\|U\|^4), \qquad\qquad \text{(B.1)}\\
B(q,q) &= \begin{pmatrix} 0.141228 + 0.264439i \\ -0.1701 - 0.3185i \end{pmatrix},\\
C(q,q,q) &= \begin{pmatrix} 0.00113946 + 0.00386505i \\ 0 \end{pmatrix},\\
C(q,q,\overline{q}) &= \begin{pmatrix} 0.00384589 + 0.00120255i \\ 0 \end{pmatrix},\\
B_1(u,v) &= -5.42072u_1v_1 - 0.195564(u_2v_1 + u_1v_2),\\
B_2(u,v) &= 3.5(u_2v_1 + u_1v_2),\\
C_1(u,v,w) &= 1.23047u_1v_1w_1 - 0.0264296(u_1v_1w_2 + u_1v_2w_1),\\
C_2(u,v,w) &= 0.
\end{aligned}
$$

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