Journal of Applied Analysis and Computation
Volume 13, Number 6, December 2023, 3195–3202
Website:http://www.jaac-online.com
DOI:10.11948/20220543

SPREADING SPEED OF A NONLOCAL DIFFUSIVE LOGISTIC MODEL WITH FREE BOUNDARIES IN TIME PERIODIC ENVIRONMENT*

Tong Wang[1] and Binxiang Dai[1,†]

Abstract In Zhang, Liu and Zhou [11], a nonlocal diffusion model with double free boundaries in time periodic environment was introduced and studied. A spreading-vanishing dichotomy is shown to govern the long time dynamical behavior. However, when spreading happens, the spreading speed was left open in [11]. In this paper, we answer this question. We obtain the spreading speed by solving the associated time periodic semi-wave problems and constructing new upper and lower solutions.

Keywords Nonlocal diffusion, time periodic, free boundary, spreading speed.

MSC(2010) 35R09, 35R35, 35K51.

Consider the following nonlocal diffusion model with free boundaries in the periodic environment:

$$\begin{cases} u_t = d\displaystyle\int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - du(t,x) & \\ \qquad + a(t,x)u - b(t,x)u^2, & t>0, g(t)<x<h(t), \\ h'(t) = \mu\displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{+\infty} J(x-y)u(t,x)dydx, & t>0, \\ g'(t) = -\mu\displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J(x-y)u(t,x)dydx, & t>0, \\ h(0)=-g(0)=h_0,\ u(0,x)=u_0(x), & -h_0\le x\le h_0, \\ u(t,x)=0, & t\ge 0, x\le g(t) \text{ or } x\ge h(t), \end{cases} \tag{0.1}$$

where $g(t)$, $h(t)$ are free boundaries to be determined with the population density $u(t,x)$, d, μ and h_0 are positive constants. The initial function u_0 satisfies

$$u_0(x)\in C([-h_0,h_0]),\quad u_0(-h_0)=u_0(h_0)=0,\quad u_0(x)>0 \text{ in } (-h_0,h_0). \tag{0.2}$$

†The corresponding author.
[1]School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, China
*The authors were supported by National Natural Science Foundation of China(No. 12271525).
Email: tongwang004@csu.edu.cn(T. Wang), bxdai@csu.edu.cn(B. Dai)

Assume that the kernel function $J(x)$ satisfies,

(J): $J(x) \in C^1(\mathbb{R})$ is nonnegative and symmetric, supported on the interval $[-r_0, r_0]$, where $0 < r_0 < +\infty$, and $J(0) > 0$, $\int_{\mathbb{R}} J(x)dx = 1$, $\sup_{\mathbb{R}} J < \infty$.
And the coefficient functions $a(t,x)$, $b(t,x)$ satisfy the following conditions:

$$\begin{cases} \text{(A) } b \in C(\mathbb{R}\times\mathbb{R}) \text{ and is T} - \text{periodic in } t \text{ for some } T>0,\ a(t,x) = \alpha(t) + \beta(x), \\ \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \text{where } \alpha : \mathbb{R} \to \mathbb{R} \text{ is a continuous T} - \text{periodic function and} \\ \quad \beta : \mathbb{R}^N \to \mathbb{R} \text{ is a continuous function;} \\ \text{(B) there are positive constants } C_1,\ C_2, \text{ such that } C_1 \le a(t,x),\ b(t,x) \le C_2, \\ \quad \forall\ (t,x) \in \mathbb{R}\times\mathbb{R}. \end{cases}$$

In [11], authors gave the existence and uniqueness of the global solution to model (0.1), then established a spreading-vanishing dichotomy (see Theorem 1.1 in [11]): either

(i) Vanishing: $\lim_{t\to+\infty}(g(t), h(t)) = (g_\infty, h_\infty)$ is a finite interval and

$$\lim_{t\to+\infty} \|u(t,x)\|_{C([g(t),h(t)])} = 0 \ \text{ or}$$

(ii) Spreading: $\lim_{t\to+\infty}(g(t), h(t)) = \mathbb{R}$ and $\lim_{t\to+\infty} u(t+nT, x) = w(t,x)$ in $C_{loc}([0,T]\times\mathbb{R})$, where $w(t,x)$ is the unique positive time-periodic solution of the following equation

$$w_t = d\int_{\mathbb{R}} J(x-y)w(t,y)dy - dw(t,x) + a(t,x)w - b(t,x)w^2, \quad t \in \mathbb{R},\ x \in \mathbb{R}. \quad (0.3)$$

Besides, the sharp criteria for spreading and vanishing were also obtained. However, when spreading happens, the question of spreading speed was not considered in [11]. The main purpose of this paper is to determine the spreading speed.

When the nonlocal diffusion term is replaced by the local diffusion operator du_{xx}, model (0.1) has been studied by Du et al. [1]. They not only established a spreading-vanishing dichotomy, but also determined the asymptotic spreading speed by studying the corresponding semi-wave problem. Many works studied the spreading speed for nonlocal diffusion model [4, 5, 7, 10, 12]. For the free boundary problem of nonlocal diffusive model in homogeneous environment, the semi-wave problems and spreading speed have been fully studied in [2, 3]. Inspired by their works, when spreading occurs, we study the spreading speed of (0.1) by constructing suitable upper and lower solutions based on the time periodic semi-wave problem.

First, we consider an associated semi-wave problem. From [6], we have the following results:

Lemma 0.1. *Let $d > 0$ be a given constant. Suppose $J(x) \in C^1(\mathbb{R})$ is nonnegative, symmetric, $J(0) > 0$, $\int_{\mathbb{R}} J(x)dx = 1$ and satisfies a "thin-tailed" condition, namely, there exists $\hat{\eta} > 0$ such that*

$$\int_{-\infty}^{+\infty} J(x)e^{\eta x}dx < +\infty, \quad \forall\ \eta \in [0, \hat{\eta}).$$

Assume $p(t)$, $q(t) \in C([0,T])$ are positive T-periodic functions and $k(t) \in C([0,T])$ is a nonnegative T-periodic function with $0 \le k(t) < c^*$ for $t \in [0,T]$, where

$$c^* := \inf_{0<\eta<\hat{\eta}} \frac{\frac{1}{T}\int_0^T p(t)dt + d(\int_{\mathbb{R}} J(y)e^{\eta y}dy - 1)}{\eta}.$$

Then the time periodic semi-wave problem

$$\begin{cases} \phi_t = d\displaystyle\int_{-\infty}^{0} J(\xi - y)\phi(t, y)dy - d\phi(t, \xi) + k(t)\phi_\xi \\ \qquad + p(t)\phi - q(t)\phi^2, & 0 \le t \le T, \xi \in (-\infty, 0), \\ \phi(t, 0) = 0, & 0 \le t \le T, \end{cases} \tag{0.4}$$

has a unique positive T-periodic solution $\phi^k(t,\xi) \in C([0,T] \times (-\infty, 0])$. Furthermore, the following conclusions hold:

(i) $\phi^k_\xi(t,\xi) < 0$ and $\phi^k(t,\xi) \to v(t)$ uniformly on $[0,T]$ as $\xi \to -\infty$, where $v(t)$ is the unique T-periodic solution of

$$\frac{dv}{dt} = p(t)v - q(t)v^2, \; 0 \le t \le T, \quad v(0) = v(T).$$

(ii) For any given nonnegative T-periodic function $m(t) \in C([0,T])$ satisfying $0 \le m(t) < c^*$ for $t \in [0,T]$, if $m(t) \le, \not\equiv k(t)$, then $\phi^m(t,\xi) > \phi^k(t,\xi)$ for $t \in [0,T]$, $\xi < 0$.

(iii) For each $\mu > 0$, there exists a unique positive T-periodic function $k_0(t) = k_0(\mu, p, q)(t) \in C([0,T])$ and $0 < k_0(t) < c^*$ for $t \in [0,T]$, such that

$$k_0(t) = \mu \int_{-\infty}^{0}\int_{0}^{+\infty} J(\xi - y)\phi^{k_0}(t,\xi)dyd\xi, \quad 0 \le t \le T. \tag{0.5}$$

By assumption (A) and (B), we have that

$$\begin{aligned} a^\infty(t) &:= \limsup_{|x|\to\infty} a(t,x) \le C_2, \quad a_\infty(t) := \liminf_{|x|\to\infty} a(t,x) \ge C_1, \\ b^\infty(t) &:= \limsup_{|x|\to\infty} a(t,x) \le C_2, \quad b_\infty(t) := \liminf_{|x|\to\infty} a(t,x) \ge C_1, \end{aligned} \tag{0.6}$$

where $a^\infty(t)$, $a_\infty(t)$, $b^\infty(t)$ and $b_\infty(t)$ are T-periodic continuous functions.

Lemma 0.2. *Let (u, g, h) be the unique solution of* (0.1) *and suppose that spreading happens. We have*

$$\limsup_{t\to+\infty} \frac{h(t)}{t} \le \frac{1}{T}\int_0^T k_0(\mu, a^\infty, b_\infty)(t)dt \tag{0.7}$$

and

$$\liminf_{t\to+\infty} \frac{h(t)}{t} \ge \frac{1}{T}\int_0^T k_0(\mu, a_\infty, b^\infty)(t)dt, \tag{0.8}$$

where $k_0(\mu, a^\infty, b_\infty)(t)$ and $k_0(\mu, a_\infty, b^\infty)(t)$ are given in (0.5) *with $p(t) = a^\infty(t)$, $q(t) = b_\infty(t)$ and $p(t) = a_\infty(t)$, $q(t) = b^\infty(t)$, respectively.*

Proof. From (0.6), for any small $\epsilon > 0$, there is $R_* := R(\epsilon) > 1$, such that for $x \ge R_*$,

$$\begin{aligned} a(t,x) &\le a^\infty_\epsilon(t) := a^\infty(t) + \epsilon, \qquad & a(t,x) &\ge a^\epsilon_\infty(t) := a_\infty(t) - \epsilon, \\ b(t,x) &\le b^\infty_\epsilon(t) := b^\infty(t) + \epsilon, \qquad & b(t,x) &\ge b^\epsilon_\infty(t) := b_\infty(t) - \epsilon. \end{aligned}$$

Firstly, we prove that the unique solution of (0.3) satisfies

$$\limsup_{x\to\infty} w(t,x) \le \bar{v}(t), \quad \liminf_{x\to\infty} w(t,x) \ge \underline{v}(t) \quad \text{for } t\in[0,T], \tag{0.9}$$

where $\bar{v}(t)$, $\underline{v}(t)$ are respectively the unique positive T-periodic solutions of

$$\frac{d\bar{v}(t)}{dt} = a^{\infty}(t)\bar{v} - b_{\infty}(t)\bar{v}^2 \text{ in } [0,T], \quad \bar{v}(0)=\bar{v}(T),$$

and

$$\frac{d\underline{v}(t)}{dt} = a_{\infty}(t)\underline{v} - b^{\infty}(t)\underline{v}^2 \text{ in } [0,T], \quad \underline{v}(0)=\underline{v}(T).$$

For large $R > R_*$, according to [9, Theorem B], the following problem

$$\begin{cases} v_t = d\displaystyle\int_{\mathbb{R}} J(x-y)v(t,y)dy - dv + a(t,x)v - b(t,x)v^2, & t\in[0,T], x\in[-R,R],\\ v(0,x)=v(T,x), & x\in[-R,R] \end{cases}$$

admits a unique T-periodic solution $v_R(t,x)$. Then by [8, Theorem E, Proposition 3.1], we can deduce that the following problem

$$\begin{cases} z_t = d\displaystyle\int_{\mathbb{R}} J(x-y)z(t,y)dy - dz + a_{\epsilon}^{\infty}(t)z - b_{\infty}^{\epsilon}(t)z^2, & t\in(0,T), x\in(R_*,R),\\ z(t,R_*) = v_R(t,R_*), z(t,R)=0, & t\in[0,T],\\ z(0,x)=z(T,x), & x\in[R_*,R] \end{cases}$$

admits a unique T-periodic solution $z_R^{\epsilon}(t,x)$ and

$$v_R(t,x) \le z_R^{\epsilon}(t,x) \le \frac{C_2}{C_1} \quad \text{for } t\in[0,T], x\in[R^*,R].$$

Moreover, using [11, Theorem 3.8], we have $v_R(t,x) \to w(t,x)$ and $z_R^{\epsilon}(t,x) \to \hat{z}^{\epsilon}(t,x)$ locally uniformly in $[0,T]$ as $R\to\infty$, where $\hat{z}^{\epsilon}(t,x)$ is the unique T-periodic solution of

$$\begin{cases} z_t = d\displaystyle\int_{\mathbb{R}} J(x-y)z(t,y)dy - dz + a_{\epsilon}^{\infty}(t)z - b_{\infty}^{\epsilon}(t)z^2, & t\in(0,T), x\in(R_*,\infty),\\ z(t,R_*) = w(t,R_*), & t\in[0,T],\\ z(0,x)=z(T,x), & x\in[R_*,\infty). \end{cases}$$

And by [9, Theorem C], $\hat{z}^{\epsilon}(t,x) \to \bar{v}_{\epsilon}(t)$ uniformly in $[0,T]$ as $x\to\infty$, where $\bar{v}_{\epsilon}(t)$ is the unique T-periodic solution of

$$\frac{dv(t)}{dt} = a_{\epsilon}^{\infty}(t)v - b_{\infty}^{\epsilon}(t)v^2 \text{ in } [0,T], \quad v(0)=v(T).$$

Then we have

$$\limsup_{x\to\infty} w(t,x) \le \bar{v}_{\epsilon}(t) \quad \text{for } t\in[0,T].$$

Letting $\epsilon\to 0$, the first inequality in (0.9) is proved. Similarly, we have

$$\liminf_{x\to\infty} w(t,x) \ge \underline{v}_{\epsilon}(t) \quad \text{for } t\in[0,T],$$

where $\underline{v}_\epsilon(t)$ is the unique T-periodic solution of

$$\frac{dv(t)}{dt} = a^\epsilon_\infty(t)v - b^\infty_\epsilon(t)v^2 \text{ in } [0,T], \quad v(0) = v(T).$$

Letting $\epsilon \to 0$, the second inequality in (0.9) is also proved.

According to (0.9), there exists $R^* > R_* > 1$ such that $\underline{v}_{\frac{\epsilon}{2}}(t) \le w(t,x) \le \bar{v}_{\frac{\epsilon}{2}}(t)$ for $t \in [0,T]$, $x \in [R^*, +\infty)$. Since $\lim_{t\to+\infty} u(t+nT, x) = w(t,x)$, combined the fact that $-g_\infty = h_\infty = +\infty$, there exists a large $N > 0$, such that

$$g(NT) < -3R^*, \; h(NT) > 3R^* \text{ and } u(t+NT, 2R^*) < \bar{v}_\epsilon(t), \quad \forall t \ge 0.$$

Setting $\tilde{u}(t,x) = u(t+NT, x+2R^*)$, $\tilde{h}(t) = h(t+NT) - 2R^*$, we have

$$\begin{cases} \tilde{u}_t = d\displaystyle\int_{\tilde{g}(t)}^{\tilde{h}(t)} J(x-y)\tilde{u}(t,y)dy - d\tilde{u}(t,x) & \\ \qquad + a(t,x+2R^*)\tilde{u} - b(t,x+2R^*)\tilde{u}^2, & t>0, \tilde{g}(t) < x < \tilde{h}(t), \\ \tilde{h}'(t) = \mu \displaystyle\int_{\tilde{g}(t)}^{\tilde{h}(t)} \int_{\tilde{h}(t)}^{+\infty} J(x-y)\tilde{u}(t,x)dydx, & t > 0, \\ \tilde{g}'(t) = -\mu \displaystyle\int_{\tilde{g}(t)}^{\tilde{h}(t)} \int_{-\infty}^{\tilde{g}(t)} J(x-y)\tilde{u}(t,x)dydx, & t>0, \\ \tilde{u}(0,x) = u(NT, x+2R^*), & -\tilde{h}_0 \le x \le \tilde{h}_0, \\ \tilde{u}(t,x) = 0, & t \ge 0, x \le \tilde{g}(t) \text{ or } x \ge \tilde{h}(t). \end{cases}$$

Let $U(t)$ be the unique solution of the problem:

$$\frac{dU}{dt} = a^\infty_\epsilon(t)U - b^\epsilon_\infty(t)U^2 \text{ for } t>0, \quad \|U(0)\| = \max\{\bar{v}_\epsilon, \|\tilde{u}(t,\cdot)\|_\infty\}.$$

Then $U(t) \ge \bar{v}_\epsilon$ for $t > 0$ and $\lim_{n\to\infty} U(t+nT) = \bar{v}_\epsilon(t)$. Since $a(t, x+2R^*) \le a^\infty_\epsilon(t)$, $b(t, x+2R^*) \ge b^\epsilon_\infty(t)$ for $x > 0$ by the choice of R^*, we can use comparison principle to deduce that $\tilde{u}(t,x) \le U(t)$ for $t > 0$, $\tilde{g}(t) < x < \tilde{h}(t)$. Hence, there exists $\tilde{N} > N$ such that

$$\tilde{u}(t,x) \le (1+\epsilon/2)\bar{v}(t), \quad \forall t \ge \tilde{N}T, \; \tilde{g}(t) < x < \tilde{h}(t).$$

Let ϕ^{k^ϵ} denote the unique solution in Lemma 0.1 with $p(t) = a^\infty_\epsilon(t)$, $q(t) = b^\epsilon_\infty(t)$ and $k^\epsilon(t) := k_0(\mu, a^\infty_\epsilon, b^\epsilon_\infty)(t)$. Recall that $\phi^{k^\epsilon}(t, -\infty) = \bar{v}_\epsilon(t)$, then there exists $R^*_0 > 2R^*$ such that

$$\phi^{k^\epsilon}(t,x) \ge (1+\epsilon/2)^{-1}\bar{v}(t), \quad t \in [0,T], \; x \in (-\infty, R^*_0].$$

Define

$$\bar{h}(t) = (1+2\epsilon)\int_0^t k^\epsilon(s)ds + R^*_0 + \tilde{h}(\tilde{N}T),$$
$$\bar{u}(t,x) = (1+\epsilon)\phi^{k^\epsilon}(t, x - \bar{h}(t)).$$

By direct calculation (see [2, 6]), we have

$$\tilde{h}(t+\tilde{N}T) < \bar{h}(t) \text{ and } \tilde{u}(t+\tilde{N}T, x) < \bar{u}(t,x) \text{ for } t>0, \; x \in [\tilde{g}(t+\tilde{N}T), \tilde{h}(t+\tilde{N}T)],$$

which implies

$$\begin{aligned}\limsup_{t\to+\infty}\frac{h(t)}{t}&=\limsup_{t\to+\infty}\frac{\tilde{h}(t-NT)+2R^*}{t}\\&\le\lim_{t\to+\infty}\frac{\bar{h}(t-NT-\tilde{N}T)+2R^*}{t}\\&=(1+2\epsilon)\frac{1}{T}\int_0^T k^\epsilon(t)dt.\end{aligned}$$

Therefore, letting $\epsilon\to 0$, $k^\epsilon(t)\to k_0(\mu,a^\infty,b_\infty)(t)$, (0.7) holds.

Finally, let $v(t,x)$ be the unique T-periodic solution of the following problem:

$$\begin{cases}v_t=d\displaystyle\int_{\tilde{g}(t)}^{\tilde{h}(t)}J(x-y)v(t,y)dy-dv(t,x)\\\qquad+a_\infty^\epsilon(t)v-b_\epsilon^\infty(t)v^2, & t>0,\tilde{g}(t)<x<\tilde{h}(t),\\v(t,0)=\tilde{u}(t,0),\ v(t,\tilde{g}(t))=v(t,\tilde{h}(t))=0, & t>0,\\v(0,x)=\tilde{u}(0,x), & -\tilde{h}_0\le x\le\tilde{h}_0.\end{cases}$$

Since $\tilde{u}(t+nT,0)\to w(t,2R^*)>\underline{v}_{\epsilon/2}(t)$ as $n\to\infty$, $\liminf_{n\to\infty}v(t+nT,x)\ge\underline{v}_\epsilon(t)$ locally uniformly for $t\in[0,T]$, $x\in\mathbb{R}$. Because $a(t,x+2R^*)\ge a_\infty^\epsilon(t)$, $b(t,x+2R^*)\le b_\epsilon^\infty(t)$ for $x>0$, we can use comparison principle to deduce that $\tilde{u}(t,x)\le v(t,x)$ for $t>0$, $-\tilde{h}(t)<x<\tilde{h}(t)$. Hence

$$\liminf_{n\to\infty}\tilde{u}(t+nT,x)\ge\underline{v}_\epsilon(t),\quad \forall t\in[0,T],\ \tilde{g}(t)<x<\tilde{h}(t).$$

Let ϕ^{k_ϵ} denote the unique solution in Lemma 0.1 with $p(t)=a_\infty^\epsilon(t)$, $q(t)=b_\epsilon^\infty(t)$ and $k_\epsilon(t):=k_0(\mu,a_\infty^\epsilon,b_\epsilon^\infty)(t)$. Define

$$\begin{aligned}&\underline{h}(t)=(1-2\epsilon)\int_0^t k_\epsilon(s)ds+\tilde{h}(0)+2R^*,\\&\underline{u}(t,x)=(1-\epsilon)[\phi^{k_\epsilon}(t,x-\underline{h}(t))+\phi^{k_\epsilon}(t,-x-\underline{h}(t))-\underline{v}_\epsilon(t)].\end{aligned}$$

By direct calculation (see [2, 6]), we can prove $(\underline{u},-\underline{h},\underline{h})$ is a lower solution of the problem (0.1), which implies

$$\liminf_{t\to+\infty}\frac{h(t)}{t}\ge(1-2\epsilon)\frac{1}{T}\int_0^T k_\epsilon(t)dt.$$

Therefore, letting $\epsilon\to 0$, $k_\epsilon(t)\to k_0(\mu,a_\infty,b^\infty)(t)$, (0.8) holds. Proof completed. □

Now we are ready to prove the main result of this paper.

Theorem 0.1. *Suppose that*

$$\lim_{|x|\to+\infty}a(t,x)=a^*(t),\qquad\lim_{|x|\to+\infty}b(t,x)=b^*(t)$$

uniformly in $[0,T]$. *Then when spreading happens to the solution* (u,g,h) *of problem* (0.1), *we have*

$$\lim_{t\to+\infty}\frac{h(t)}{t}=-\lim_{t\to+\infty}\frac{g(t)}{t}=\frac{1}{T}\int_0^T k_0(t)dt,$$

where $k_0(t)=k_0(\mu,a^*,b^*)(t)$ *is given in* (0.5) *with* $p(t)=a^*(t)$, $q(t)=b^*(t)$.

Proof. Since $-g_\infty = h_\infty = +\infty$ and

$$a(t,x) \to a^*(t), \quad b(t,x) \to b^*(t)$$

uniformly in $[0,T]$ as $|x| \to \infty$, we have

$$\liminf_{t\to+\infty} \frac{h(t)}{t} \geq \frac{1}{T}\int_0^T k_0(\mu, a^*(t), b^*(t))(t)dt$$

and

$$\limsup_{t\to+\infty} \frac{h(t)}{t} \leq \frac{1}{T}\int_0^T k_0(\mu, a^*(t), b^*(t))(t)dt,$$

i.e.

$$\lim_{t\to+\infty} \frac{h(t)}{t} = \frac{1}{T}\int_0^T k_0(\mu, a^*(t), b^*(t))(t)dt.$$

It remains to show that

$$-\lim_{t\to+\infty} \frac{g(t)}{t} = \frac{1}{T}\int_0^T k_0(\mu, a^*(t), b^*(t))(t)dt.$$

Let $\tilde{u}(t,x) := u(t,-x)$, $\tilde{h}(t) := -g(t)$, $\tilde{g}(t) := -h(t)$. Then $(\tilde{u}, \tilde{g}, \tilde{h})$ satisfies (0.1) with initial value function $\tilde{u}_0(x) := u_0(-x)$ and spreading happens. Hence we can conclude that

$$-\lim_{t\to+\infty} \frac{g(t)}{t} = \lim_{t\to+\infty} \frac{\tilde{h}(t)}{t} = \frac{1}{T}\int_0^T k_0(\mu, a^*(t), b^*(t))(t)dt.$$

Proof completed. □

Acknowledgements

The authors wish to thank the editor and the referees for their many valuable comments and suggestions which improve the contents of this work.

References

[1] Y. Du, Z. Guo and R. Peng, *A diffusive logistic model with a free boundary in time-periodic environment*, J. Funct. Anal., 2013, 265(9), 2089–2142.

[2] Y. Du, F. Li and M. Zhou, *Semi-wave and spreading speed of the nonlocal Fisher-KPP equation with free boundaries*, J. Math. Pures. Appl., 2021, 154, 30–66.

[3] Y. Du and W. Ni, *Spreading speed for some cooperative systems with nonlocal diffusion and free boundaries, part 1: Semi-wave and a threshold condition*, J. Diff. Eqs., 2022, 308, 369–420.

[4] S. Gan and Z. Yu, *The approach of solutions for the nonlocal diffusion equation to traveling fronts*, J. Nonlinear Model. Anal., 2020, 2(2), 205–226.

[5] D.-Y. Kong and B.-S. Han, *Long-time asymptotic behavior of Fisher-KPP equation for nonlocal dispersal in asymmetric kernel*, J. Appl. Anal. Comput., 2023. DOI: 10.11948/20220521.

[6] Z. Li and B. Dai, *The dynamics of a nonlocal dispersal logistic model with seasonal succession and free boundaries*, J. Dyn. Diff. Eqs., 2022. DOI:10.1007/s10884–022–10184–9.

[7] X. Liang and T. Zhou, *Spreading speeds of nonlocal KPP equations in almost periodic media*, J. Funct. Anal., 2020, 279(9), 108723.

[8] N. Rawal and W. Shen, *Criteria for the existence and lower bounds of principal eigenvalues of time periodic nonlocal dispersal operators and applications*, J. Dyn. Diff. Eqs., 2012, 24(4), 927–954.

[9] Z. Shen and H.-H. Vo, *Nonlocal dispersal equations in time-periodic media: Principal spectral theory, limiting properties and long-time dynamics*, J. Diff. Eqs., 2019, 267(2), 1423–1466.

[10] X. Yang, G. Lin and J. Yang, *Asymptotic spreading in a competition system with nonlocal dispersal*, J. Appl. Anal. Comput., 2021, 11(4), 1951–1962.

[11] W. Zhang, Z. Liu and L. Zhou, *Dynamics of a nonlocal diffusive logistic model with free boundaries in time periodic environment*, Discrete Contin. Dyn. Syst. B, 2021, 26(7), 3767–3784.

[12] M. Zhao, R. Yuan, Z. Ma and X. Zhao, *Spreading speeds for the predator-prey system with nonlocal dispersal*, J. Diff. Eqs., 2022, 316, 552–598.