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ANALYTICAL AND NUMERICAL DISCUSSION FOR THE PHASE-LAG VOLTERRA-FREDHOLM INTEGRAL EQUATION WITH SINGULAR KERNEL

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Abstract In this paper, we studied the existence and unique solution of the Volterra-Fredholm integral equation of the second kind (V-FIESK). The general singular kernel is considered to be in position with the Fredholm integral term. Singular kernel will tend to a logarithmic function under exceptional conditions and new discussions. The Volterra-Fredholm integral equation with the logarithmic form will be solved using Legendre polynomials, where the kernel of Volterra integral term is a positive continuous function in time. A system of infinite linear algebraic equations is obtained by solving the problem in series, where the convergence of this system is discussed. Finally, The error is calculated using Maple software after the numerical results have been acquired.

Keywords Volterra-Fredholm integral equation, Banach space, fixed point theorem, phase-lag term, logarithmic function, Legendre polynomials.

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1. Introduction

There are currently one and dual phases, and phase-lag plays an important role in our scientific and applied fields of science. The third phase has been defined with the development of modern applied science, and each phase has different significance and applications. The rapid transient heat transfer processes associated with micro-reactions have recently started to be significantly impacted by the three-phase-lag mode. Through the phase of time delay in heat transfer processes through bodies, it establishes the length of time necessary for various micro-reactions to happen, including reactions resulting in metals like the interaction of the phonon and the electron and the resulting reactions in insulating crystals like phonon scattering, as well as activating the movement of molecules at very low temperatures. For more information, see [5, 13]. The mathematical model of many evolutionary problems in mathematical physics, optimal control systems, biology, engineering, quantum mechanics, chemistry, and other fields is the Volterra-Fredholm integral equations

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with phase-lag term. For example, the dual phase-lag model of heat transfer uses integral equations.

According to [1, 6, 23, 24, 28], there are various types of integral equations that can be used as mathematical tools to represent the models of knowledge that are present in a variety of applied sciences fields. Volterra-Friedholm integral equations appear in many applications that have an impact on basic science, and because it can be difficult to find an exact solution in many cases, researchers have been interested in developing numerical methods to approximate the solution of these equations. Such these methods, we refer to [1, 2, 4, 11, 12, 17, 20, 27, 30]. The method presented in the article has high accuracy in approximating the solution of these equations because the solution is represented by a linear combination of polynomials with the help of non-orthogonal or orthogonal basis functions, for example in the case [9, 10, 32].

Volterra-Fredholm integral equations frequently appear in mathematical physics and chemistry problems, including kinetic theories of gases, theories of radiative transfer, queuing theories, kinetic theories of gases, theories of neutron transport, traffic theories, and many other applications. Previous works have examined existing solutions and numerical approaches to solve these kinds of integral equations, see [15, 16, 25].

Volterra-Fredholm integral equation with a singular form is discussed in this work, according to certain conditions. Using Picard's approach, it is shown that the integral equation's solution exists and is unique. In space $L_2[-1,1] \times C[0,T],\ 0 \leq T < 1$, the solution is expressed in the form of a series of Legendre polynomials, where an infinite system of linear algebraic equations is obtained. Numerical examples are presented in addition to a discussion of the convergence of this system.

There are seven sections in this article. The existence and unique solution of Eq. (2.1) are proved and discussed in section 3. We discussed a theory in section 4 that explains why the bad kernel has a logarithmic form. While, in section 5, we provide some integral and algebraic formulas for the Legendre polynomials, and also the Legendre polynomials approach is used to find the solution of the singular integral equation. Section 6 contains the computation of estimated errors and numerical results. Final remarks are deduced in section 7.

2. Volterra-Fredholm integral equations with phase-lag

Assume the Volterra-Fredholm integral equation with phase-lag term of the second kind

$$\begin{aligned}
&\gamma\Psi(u,t+\delta t) = g(u,t) + \int_0^t J(t,\tau)\Psi(u,\tau)\mathrm{d}\tau + \int_{-1}^1 \xi\left(\left|\frac{v-u}{\lambda}\right|\right)\Psi(v,t)\mathrm{d}v,\\
&g(u,t) = \frac{\pi}{\alpha_1+\alpha_2}[\sigma(t)+\eta(t)x-\theta_1(x)-\theta_2(x)],\\
&[|u| \leq 1,\ t \in [0,T],\ \lambda \in (0,\infty),\ \gamma \in (0,\infty),\ 0 < \delta t << 1],
\end{aligned} \tag{2.1}$$

we have

$$\xi\left(\left|\frac{v-u}{\lambda}\right|\right) = \int_0^\infty \left(\frac{L(w)}{w}\right)\cos(\frac{v-u}{\lambda}w)\mathrm{d}w,$$

$$L(w) = \frac{w+q}{1+w}; \quad q \geq 1. \tag{2.2}$$

Under the conditions

$$\int_{-1}^{1} \Psi(u,t)\mathrm{d}u = P(t), \quad \int_{-1}^{1} u\Psi(u,t)\mathrm{d}u = M(t), \tag{2.3}$$

$\psi(u,t)$ is unknown function in the space $L_2[-1,1] \times C[0,T],\ 0 \leq T < 1$, where δt is the phase-lag constant. The domain of integration with regard to the position is $[-1,1]$ and the time $t \in [0,T]$. The known function $g(u,t)$ is continuous in the space $L_2[-1,1] \times C[0,T], 0 \leq t \leq T$, and the kernel $J(t,\tau)$ is continuous in $C[0,T]$. Additionally, the kernel in position $\xi\left(\left|\frac{v-u}{\lambda}\right|\right)$ is discontinuous.

Taylor expansion is used when the second derivative in Eq. (2.1) is neglected, we obtain

$$\gamma\left[\Psi(u,t) + \delta t \frac{\partial \Psi(u,t)}{\partial t}\right] = g(u,t) + \int_0^t J(t,\tau)\Psi(u,\tau)\mathrm{d}\tau + \int_{-1}^{1} \xi\left(\left|\frac{v-u}{\lambda}\right|\right)\Psi(v,t)\mathrm{d}v, \tag{2.4}$$

with initial condition

$$\Psi(u,0) = f(u). \tag{2.5}$$

Integro-differential equation is the name given of equation (2.4) with initial condition (2.5). Integro-differential equations (IDEs) are a type of functional equations that have associated derivatives and integrals of an unknown function, as shown in [26,31]. These equations bear the names of the famous mathematicians who first explored them, including Volterra and Fredholm. The most numerous kinds are Volterra and Fredholm equations.

Integrating Eq. (2.4) and using initial condition (2.5), we obtain

$$\begin{aligned} \Psi(u,t) =& f(u) + \frac{1}{\gamma\delta t}\int_0^t g(u,z)dz - \frac{1}{\delta t}\int_0^t \Psi(u,z)dz \\ &+ \frac{1}{\gamma\delta t}\int_0^t \int_0^z J(z,\tau)\Psi(u,\tau)d\tau dz \\ &+ \frac{1}{\gamma\delta t}\int_0^t \int_{-1}^{1} \xi\left(\left|\frac{v-u}{\lambda}\right|\right)\Psi(v,z)dvdz, \end{aligned} \tag{2.6}$$

interchanging the order of integration over the triangular domain in the τz-plane reveals that the equation (2.6) becomes,

$$\begin{aligned} \Psi(u,t) =& f(u) + \frac{1}{\gamma\delta t}\int_0^t g(u,\tau)d\tau + \frac{1}{\gamma\delta t}\int_0^t [G(t,\tau) - \gamma]\Psi(u,\tau)d\tau \\ &+ \frac{1}{\gamma\delta t}\int_0^t \int_{-1}^{1} \xi\left(\left|\frac{v-u}{\lambda}\right|\right)\Psi(v,\tau)dvd\tau. \end{aligned} \tag{2.7}$$

Where,

$$G(t,\tau) = \int_\tau^t J(z,\tau)dz.$$

Equation (2.7) is named Volterra-Fredholm integral equation with phase-lag term in time.

3. The existence and uniqueness of solution of the Volterra-Fredholm integral equation (2.7)

We establish the following assumptions to study the existence and uniqueness of the solution of Eq. (2.7):

(i) $G(t,\tau) \in C([0,T])$ and satisfies $\|G(t,\tau)\| \leq A$, $s.t\, A$ is a constant, $\forall\, t,\tau \in [0,T]$.

(ii) $f(u)$ is continuous function and satisfies $\|f(u)\| \leq B$, $s.t\, B$ is a constant.

(iii) The Bad kernel $\xi\left(\left|\frac{v-u}{\lambda}\right|\right)$ satisfies the condition

$$\left\{\int_{-1}^{1}\int_{-1}^{1}\left|\xi\left(\left|\frac{v-u}{\lambda}\right|\right)\right|^2 \mathrm{d}u\mathrm{d}v\right\}^{\frac{1}{2}} = H,$$

where H is a finite constant.

(iv) In space $L_2[-1,1]\times C[0,T]$, $0 \leq T < 1$, $g(u,\tau)$ is given continuous function with its partial derivatives with respect to the position and time, its norm is defined as,

$$\|g(u,\tau)\| = \max_{0<\tau\leq T}\int_0^{\tau}\left(\int_{-1}^{1} g^2(u,z)du\right)^{\frac{1}{2}} dz = Q, \quad Q \text{ is a constant.}$$

Theorem 3.1. *Let the conditions (i-iv) be satisfied. If the condition*

$$\left[\frac{2H + A - \gamma}{\gamma\delta t}\right] < 1 \tag{3.1}$$

is satisfied, then the equation (2.7) *has a unique solution* $\psi(u,t)$ *in the space* $L_2[-1,1]\times C[0,T]$.

Proof. We use the successive approximation method, often known as Picard's method, to prove the existence and uniqueness of the solution of equation (2.7).

The following form is the solution that approaches close to the exact solution of the equation (2.7):

$$\begin{aligned}\Psi_k(u,t) &= f(u) + \frac{1}{\gamma\delta t}\int_0^t g(u,\tau)d\tau + \frac{1}{\gamma\delta t}\int_0^t [G(t,\tau)-\gamma]\Psi_{k-1}(u,\tau)d\tau \\ &\quad + \frac{1}{\gamma\delta t}\int_0^t\int_{-1}^{1}\xi\left(\left|\frac{v-u}{\lambda}\right|\right)\Psi_{k-1}(v,\tau)dvd\tau, \qquad (3.2)\\ \psi_0(u,t) &= f(u) + \frac{1}{\gamma\delta t}\int_0^t g(u,\tau)d\tau.\end{aligned}$$

Since all of the functions $\psi_k(u,t)$ are continuous, $\psi_k(u,t)$ can be expressed as the sum of successive differences:

$$\psi_k(u,t) = \psi_0(u,t) + \sum_{j=1}^{k}(\psi_j(u,t) - \psi_{j-1}(u,t)).$$

It follows that the convergence of the sequence $\psi_k(u,t)$ is equivalent to the convergence of the finite series $\sum_{j=1}^{k}(\psi_j(u,t) - \psi_{j-1}(u,t))$, the solution will be

$$\psi(u,t) = \lim_{k\to\infty}\psi_k(u,t),$$

i.e. if the finite series $\sum_{j=1}^{k}(\psi_j(u,t)-\psi_{j-1}(u,t))$ converges, then the sequence $\psi_k(u,t)$ will converge to $\psi(u,t)$. For this reason, the following lemmas are currently proven: □

Lemma 3.1. *A sequence* $\{\psi_k(u,t)\}$ *is uniformly convergent to a continuous solution function* $\{\psi(u,t)\}$.

Proof. In order to prove the uniform convergence of $\{\psi_k(u,t)\}$, we take into consideration the related series

$$\sum_{k=1}^{\infty}(\psi_k(u,t)-\psi_{k-1}(u,t)).$$

From Eq. (3.2), for $k=1$, we get

$$\begin{aligned}\psi_1(u,t)-\psi_0(u,t)=&\frac{1}{\gamma\delta t}\int_0^t[G(t,\tau)-\gamma]\Psi_0(u,\tau)d\tau\\&+\frac{1}{\gamma\delta t}\int_0^t\int_{-1}^{1}\xi\left(\left|\frac{v-u}{\lambda}\right|\right)\Psi_0(v,\tau)dvd\tau,\end{aligned}$$

using conditions (i–iv), we obtain

$$\begin{aligned}\|\psi_1(u,t)-\psi_0(u,t)\|&\le\frac{1}{\gamma\delta t}\left\|\int_0^t[A-\gamma]\Psi_0(u,\tau)d\tau\right\|+\frac{H}{\gamma\delta t}\left\|\int_0^t\int_{-1}^{1}\Psi_0(v,\tau)dvd\tau\right\|\\&\le\frac{1}{\gamma\delta t}[A-\gamma](B+\frac{Q}{\gamma\delta t})+\frac{2H}{\gamma\delta t}(B+\frac{Q}{\gamma\delta t})\\&\le\left(B+\frac{Q}{\gamma\delta t}\right)\left[\frac{2H+A-\gamma}{\gamma\delta t}\right].\end{aligned}\tag{3.3}$$

By using this method repeatedly, we are able to obtain the following general estimate for the terms of the series:

$$\|\psi_k(x,t)-\psi_{k-1}(x,t)\|\le\eta^k\left(B+\frac{Q}{\gamma\delta t}\right);\ \eta=\left[\frac{2H+A-\gamma}{\gamma\delta t}\right];\ k=1,2,3,\dots.$$

Since $\left[\frac{2H+A-\gamma}{\gamma\delta t}\right]<1$, then the uniform convergence of

$$\sum_{k=1}^{\infty}(\psi_k(u,t)-\psi_{k-1}(u,t)),$$

is proved and so the sequence $\{\psi_k(u,t)\}$ is uniformly convergent.

$$\begin{aligned}\psi(u,t)=&\lim_{k\to\infty}(f(u)+\frac{1}{\gamma\delta t}\int_0^t g(u,\tau)d\tau+\frac{1}{\gamma\delta t}\int_0^t[G(t,\tau)-\gamma]\Psi_k(u,\tau)d\tau\\&+\frac{1}{\gamma\delta t}\int_0^t\int_{-1}^{1}\xi\left(\left|\frac{v-u}{\lambda}\right|\right)\Psi_k(v,\tau)dvd\tau)\\=&f(u)+\frac{1}{\gamma\delta t}\int_0^t g(u,\tau)d\tau+\frac{1}{\gamma\delta t}\int_0^t[G(t,\tau)-\gamma]\Psi(u,\tau)d\tau\end{aligned}$$

$$+\frac{1}{\gamma\delta t}\int_0^t\int_{-1}^1 \xi\left(\left|\frac{v-u}{\lambda}\right|\right)\Psi(v,\tau)dvd\tau.$$

Thus, the existence of a solution of equation (2.7) is proved. □

Lemma 3.2. *The function $\psi(u,t)$ represents a unique solution of integral equation (2.7).*

Proof. To prove the uniqueness of Eq. (2.7), let $\Phi(u,t)$ be a different continuous solution of Eq. (2.7). We obtain

$$\begin{aligned}\Phi(u,t) =& f(u)+\frac{1}{\gamma\delta t}\int_0^t g(u,\tau)d\tau+\frac{1}{\gamma\delta t}\int_0^t[G(t,\tau)-\gamma]\Phi(u,\tau)d\tau\\ &+\frac{1}{\gamma\delta t}\int_0^t\int_{-1}^1 \xi\left(\left|\frac{v-u}{\lambda}\right|\right)\Phi(v,\tau)dvd\tau,\end{aligned}$$

and

$$\begin{aligned}\Psi(u,t)-\Phi(u,t) =& \frac{1}{\gamma\delta t}\int_0^t[G(t,\tau)-\gamma][\Psi(u,\tau)-\Phi(u,\tau)]d\tau\\ &+\frac{1}{\gamma\delta t}\int_0^t\int_{-1}^1 \xi\left(\left|\frac{v-u}{\lambda}\right|\right)[\Psi(v,\tau)-\Phi(v,\tau)]dvd\tau.\end{aligned}$$

Using conditions (i–iv) and the properties of the norm, we get

$$\begin{aligned}\|\Psi(u,t)-\Phi(u,t)\| \leq& \frac{1}{\gamma\delta t}\int_0^t[A-\gamma]\|\Psi(u,\tau)-\Phi(u,\tau)\|d\tau\\ &+\frac{H}{\gamma\delta t}\int_0^t\int_{-1}^1\|\Psi(v,\tau)-\Phi(v,\tau)\|dvd\tau\\ \leq& \left[\frac{2H+A-\gamma}{\gamma\delta t}\right]\|\Psi(u,t)-\Phi(u,t)\|.\end{aligned}$$

But

$$\|\Psi(u,t)-\Phi(u,t)\|\leq\eta\|\Psi(u,t)-\Phi(u,t)\|;\ \eta=\left[\frac{2H+A-\gamma}{\gamma\delta t}\right]. \tag{3.4}$$

The equation (3.4) can be written as,

$$(1-\eta)\|\Psi(u,t)-\Phi(u,t)\|\leq 0,$$

since $\eta<1$, so that $\Psi(u,t)=\Phi(u,t)$, that is the solution is a unique. Which ends the proof. □

4. The kernel of the Fredholm integral term

Theorem 4.1. *The bad kernel of equation (2.2) takes the logarithmic form.*

Proof. For $w\in(0,\infty)$, the function $L(w)$ is continuous and positive. The asymptotic equalities can therefore be satisfied:

$$L(w) =q-(q-1)w+O(w^2),\quad w\to 0, \tag{4.1}$$

$$L(w) = 1 + \frac{q-1}{w} + O(w^{-2}), \quad w \to \infty, \quad q \geq 1. \tag{4.2}$$

Most of the previous authors have, when $w \to 0$, i.e. $L(w) = q$, solved the Fredholm integral equations of the second and first kinds in the continuum mechanics problems.

Here, we consider the case when $w \to \infty$. i.e. $L(w) = 1$, and then for the second and first approximations of $L(w)$ after applying the two well-known relations

$$\begin{aligned}
&\int_0^\infty \frac{\cos(\frac{v-u}{\lambda}w)}{w}\mathrm{d}w = -\ln|v-u| + d; \quad d = \ln\frac{4\lambda}{\pi}; \quad \lambda \in (0,\infty), \\
&\int_0^\infty \cos(\frac{v-u}{\lambda}w)\mathrm{d}w = \delta(v-u); \quad \delta(v-u) \text{ is the Dirac function.}
\end{aligned} \tag{4.3}$$

We can arrive

$$\xi\left(\left|\frac{v-u}{\lambda}\right|\right) = -\ln|v-u| + d. \tag{4.4}$$

Substituting (4.4) into (2.7), we get

$$\begin{aligned}
\Psi(u,t) = & f(u) + \frac{1}{\gamma\delta t}\int_0^t g(u,\tau)d\tau + \frac{1}{\gamma\delta t}\int_0^t [G(t,\tau) - \gamma]\Psi(u,\tau)d\tau \\
& + \frac{1}{\gamma\delta t}\int_0^t\int_{-1}^1 (-\ln|v-u| + d)\Psi(v,\tau)dvd\tau.
\end{aligned} \tag{4.5}$$

□

5. The solution algorithm of the Volterra-Fredholm integral equation (2.7)

5.1. Quadratic numerical method

The importance of Quadratic numerical approach comes from its numerous applications in mathematical physics problems, wherever the eigenfunctions and eigenvalues of the integral equations are often discussed and studied. Additionally, this approach has numerous applications in the applied sciences, especially in the theory of elasticity, mixed problems in the field of mechanics, and contact problems.

Here, we often apply this numerical technique to convert the Volterra-Fredholm integral equation (2.7) to a linear Fredholm integral equations of second type. We divide $[0,T]$, $0 \leq T < 1$, as $0 = t_0 < t_1 < ... < t_m < ... < t_L = T$, where $t = t_m$, $m = 0,1,...,L$, to get

$$\begin{aligned}
\Psi(u,t_m) = & f(u) + \frac{1}{\gamma\delta t}\int_0^{t_m} g(u,\tau)d\tau + \frac{1}{\gamma\delta t}\int_0^{t_m} [G(t_m,\tau) - \gamma]\Psi(u,\tau)d\tau \\
& + \frac{1}{\gamma\delta t}\int_0^{t_m}\int_{-1}^1 (-\ln|v-u| + d)\Psi(v,\tau)dvd\tau.
\end{aligned} \tag{5.1}$$

For the Volterra integral terms, we have the following using the quadrature formula [14]:

$$\int_0^{t_m}\int_{-1}^1 (-\ln|v-u| + d)\Psi(v,\tau)dvd\tau$$

$$
\begin{aligned}
&= \sum_{n=0}^{m} \mu_n \int_{-1}^{1} (-\ln|v-u| + d)\Psi(v,t_n)dv + O(\hbar_m^{\wp_1+1}),\\
&\int_0^{t_m} [G(t_m,\tau)-\gamma]\Psi(u,\tau)d\tau = \sum_{n=0}^{m} \nu_n [G(t_m,t_n)-\gamma]\Psi(u,t_n) + O(\hbar_m^{\wp_2+1}),\\
&\int_0^{t_m} g(u,\tau)d\tau = \sum_{n=0}^{m} \omega_n g(u,t_n) + O(\hbar_m^{\wp_3+1}),\\
&(\hbar_m^{\wp_1+1} \to 0,\ \hbar_m^{\wp_2+1} \to 0,\ \hbar_m^{\wp_3+1} \to 0;\ \wp_1>0,\ \wp_2>0,\ \wp_3>0,
\end{aligned} \tag{5.2}
$$

where, $\Im$ denotes the step size of the partition,

$$\hbar_m = \max_{0\le n\le m} \Im_n \qquad \text{and} \qquad \Im_n = t_{n+1} - t_n.$$

In [19], more details regarding the quadrature coefficients and characteristic points are presented.

Equation (5.2) is applied in equation (5.1), yielding

$$
\begin{aligned}
\Psi(u,t_m) =& f(u) + \frac{1}{\gamma\delta t}\sum_{n=0}^{m}\omega_n g(u,t_n) + \frac{1}{\gamma\delta t}\sum_{n=0}^{m}\nu_n[G(t_m,t_n)-\gamma]\Psi(u,t_n)\\
&+ \frac{1}{\gamma\delta t}\sum_{n=0}^{m}\mu_n\int_{-1}^{1}(-\ln|v-u|+d)\Psi(v,t_n)dv.
\end{aligned} \tag{5.3}
$$

Utilizing the notations shown below:

$$g(u,t_n) = g_n(u), \quad \Psi(u,t_m) = \Psi_m(u), \quad G(t_m,t_n) = G_{m,n}.$$

In the following format, we can write (5.3):

$$
\begin{aligned}
\Psi_m(u) =& f(u) + \frac{1}{\gamma\delta t}\sum_{n=0}^{m}\omega_n g_n(u) + \frac{1}{\gamma\delta t}\sum_{n=0}^{m}\nu_n[G_{m,n}-\gamma]\Psi_n(u)\\
&+ \frac{1}{\gamma\delta t}\sum_{n=0}^{m}\mu_n\int_{-1}^{1}(-\ln|v-u|+d)\Psi_n(v)dv.
\end{aligned} \tag{5.4}
$$

Equation (5.4) can be rewritten as follows:

$$
\begin{aligned}
\aleph_m\Psi_m(u) =& f(u) + \frac{1}{\gamma\delta t}\sum_{n=0}^{m}\omega_n g_n(u) + \frac{1}{\gamma\delta t}\sum_{n=0}^{m-1}\nu_n[G_{m,n}-\gamma]\Psi_n(u)\\
&+ \frac{1}{\gamma\delta t}\sum_{n=0}^{m}\mu_n\int_{-1}^{1}(-\ln|v-u|+d)\Psi_n(v)dv,
\end{aligned} \tag{5.5}
$$

where $\aleph_m = [1 - (\nu_m/\gamma\delta t)(G_{m,m}-\gamma)]$.

Also the boundary condition (2.3) becomes

$$\int_{-1}^{1}\Psi_m(u)\mathrm{d}u = P_m, \quad \int_{-1}^{1}u\Psi_m(u)\mathrm{d}u = M_m. \tag{5.6}$$

According to equation (5.5), $\Psi_m(u)$ represents the number of finite unknown functions, with values of $m = 0, 1, ..., L$, which correspond to $0 = t_0 < t_1 < ... <$

$t_m < ... < t_L = T$. In this case, equation (5.5) describes a finite system of Fredholm integral equations of the second type with a logarithmic form in position, whereas, for $\aleph_m = 0$, we get the first type of a finite system of Fredholm integral equations.

There are various approaches to getting the solution of the system (5.5), for $\aleph_m \neq 0$. For example, by using a variation of Nyström method [14] and the collocation method [21]. Also, Galerkin method is used in [8] to solve the system (5.5). If we obtain, firstly, the value of $\Psi_0(u)$, and let $m = 0$ in (5.5), we obtain

$$\begin{aligned} &\aleph_0 \Psi_0(u) = f(u) + \frac{1}{\gamma\delta t}\omega_0 g_0(u) + \frac{1}{\gamma\delta t}\mu_0 \int_{-1}^{1} (-\ln|v-u| + d)\Psi_0(v)dv, \\ &\aleph_0 = [1 - (\nu_0/\gamma\delta t)(G_{0,0} - \gamma)]. \end{aligned} \tag{5.7}$$

After finding the solution of equation (5.7), we may apply mathematical induction to find the general solution of (5.5).

Using the famous relation, found in [29], it is possible to get the Fredholm integral equation with Carleman kernel from Eq. (5.7)

$$\ln|v-u| = U(u,v)|v-u|^{-\vartheta}; \qquad \vartheta \in]0,1[\ , \tag{5.8}$$

where $U(u,v) = |v-u|^{\vartheta} \ln|v-u| \in C[-1,1]$ for all $(-1 \leq u, v \leq 1)$. The work of Artiunian, who established that the first approximation of the nonlinear integral equation in the theory of plasticity represents a Fredholm integral equation of the second type with the Carleman form, provided the reason for the importance of the Carleman kernel. For further information, see [7].

Differentiating the integral equation (5.7) with respect to u, we arrive at

$$\aleph_0 \frac{\mathrm{d}\Psi_0(u)}{\mathrm{d}u} = \frac{\mathrm{d}f(u)}{\mathrm{d}u} + \frac{1}{\gamma\delta t}\omega_0 \frac{\mathrm{d}g_0(u)}{\mathrm{d}u} + \frac{1}{\gamma\delta t}\mu_0 \int_{-1}^{1} \frac{\Psi_0(v)dv}{v-u}, \tag{5.9}$$

here $\int_{-1}^{1}$ represents integration in the sense of Cauchy principal value, the unknown function $\Psi_0(u)$ with its derivatives are continuous in $L_2[-1,1]$, $u \in [-1,1]$.

5.2. Legendre polynomials

Now, in order to obtain the solution of problem (5.9), we suppose the unknown function $\Psi_0(u)$ in the Legendre polynomials form:

$$\Psi_0(u) = \sum_{k=0}^{\infty} C_k^0 P_k(u), \tag{5.10}$$

where $P_k(u)$ is a Legendre polynomial that satisfies the orthogonal relation, see [22] and C_k^0 are constants

$$\int_{-1}^{1} P_k(u)P_s(u)\mathrm{d}u = \begin{cases} 0; & s \neq k, \\ \frac{2}{2k+1}; & s = k. \end{cases} \tag{5.11}$$

The polynomial series (5.10) exhibits the following behavior at the two end points of contact, $u = \pm 1$

$$\Psi_0(1) = \sum_{k=0}^{\infty} C_k^0, \qquad \Psi_0(-1) = \sum_{k=0}^{\infty} (-1)^k C_k^0.$$

Also, we say that, if $\Psi_0(u) \in L_2[-1,1]$, then the polynomial series (5.10) belongs to $L_2[-1,1]$ (see [22]).

Differentiating (5.10) with respect to u, we obtain

$$\Psi_0'(u) = \sum_{k=0}^{\infty} C_k^0 P_k^1(u).(1-x^2)^{-\frac{1}{2}}, \tag{5.12}$$

where $P_k^{\ell}(u),\ k,\ \ell \geq 0$, are the first type of the associated Legendre polynomials that satisfy the following relation that satisfy the following relation (see [[18], p. 808]):

$$\int_{-1}^{1} P_k^{\ell}(u) P_s^{\ell}(u)\mathrm{d}u = \begin{cases} 0; & s \neq k, \\ \frac{2(k+\ell)!}{(k-\ell)!(2k+1)}; & s = k. \end{cases} \tag{5.13}$$

The known term of (5.9) can be formed as follows in view of (5.10)

$$f'(u) = \sum_{k=0}^{\infty} f_k P_k^1(u).(1-x^2)^{-\frac{1}{2}}, \quad g_0'(u) = \sum_{k=0}^{\infty} g_k^0 P_k^1(u).(1-x^2)^{-\frac{1}{2}}, \tag{5.14}$$

where Eq. (5.13) can be used to calculate the constant coefficients $f_k, g_k^0, k \geq 0$. The polynomial series (5.14) belongs to $L_2[-1,1]$ if the known functions $f'(u)$, $g_0'(u) \in L_2[-1,1]$ (see [22]). Applying the following famous relation (see [[18], p. 835])

$$Q_k(u) = \frac{1}{2}\int_{-1}^{1} \frac{P_k(v)}{u-v}\mathrm{d}v, \tag{5.15}$$

by using (5.10), the integral term of (5.9) becomes

$$\int_{-1}^{1} \frac{\Psi_0(v)}{u-v}\mathrm{d}v = 2\sum_{k=0}^{\infty} C_k^0 Q_k(u), \tag{5.16}$$

where $Q_k^{\ell}(u),\ \ell,\ k \geq 0$, are the second type of the associated Legendre polynomials that satisfy the following relation, (see [[18], p. 808])

$$\int_{-1}^{1} Q_k^{\ell}(u) P_s^{\ell}(u)\mathrm{d}u = (-1)^{\ell} \frac{1-(-1)^{(k+s)}(k+\ell)!}{(s-k)(s+k+1)(k-\ell)!};\ \ell \geq 0. \tag{5.17}$$

Using equations (5.12), (5.14) and (5.16) in form (5.9), we obtain

$$\aleph_0 \sum_{k=0}^{\infty} C_k^0 P_k^1(u) = \sum_{k=0}^{\infty}(f_k + \frac{\omega_0}{\gamma\delta t} g_k^0) P_k^1(u) - \frac{2\mu_0\sqrt{1-x^2}}{\gamma\delta t}\sum_{k=0}^{\infty} C_k^0 Q_k(u). \tag{5.18}$$

Using the following famous integral relation, after multiplying both sides of (5.18) by the term $P_s^1(u)\mathrm{d}u$, then integrating the result from -1 to 1, (see [[18], p. 807])

$$\int_{-1}^{1} \sqrt{1-x^2} Q_k(u) P_s^1(u)\mathrm{d}u = \begin{cases} 0; & (k = s \pm 1), \\ \frac{-2s(s+1)[1+(-1)^{k+s}]}{(s-k-1)(s-k+1)(s+k)(s+k+2)}; & (k \neq s \pm 1), \end{cases} \tag{5.19}$$

the formula (5.18) becomes

$$\begin{aligned}\aleph_0 C_s^0 =& f_s + \frac{\omega_0}{\gamma\delta t} g_s^0 + \frac{2\mu_0}{\gamma\delta t}\sum_{k=0}^{\infty}\frac{(2s+1)[1+(-1)^{k+s}]C_k^0}{(s-k-1)(s-k+1)(s+k)(s+k+2)},\\ &(C_0^0 = \frac{P_0}{2},\ C_1^0 = \frac{3}{2}M_0,\ s \geq 2).\end{aligned} \tag{5.20}$$

The following relation can be obtained by applying the same previous method and mathematical induction:

$$\begin{aligned}\aleph_m C_s^m =& f_m + \frac{1}{\gamma\delta t}\sum_{n=0}^{m}\omega_n g_s^n + \frac{1}{\gamma\delta t}\sum_{n=0}^{m-1}\nu_n[G_{m,n}-\gamma]C_k^n\\ &+ \frac{2}{\gamma\delta t}\sum_{n=0}^{m}\sum_{k=0}^{\infty}\mu_n\frac{(2s+1)[1+(-1)^{k+s}]C_k^n}{(s-k-1)(s-k+1)(s+k)(s+k+2)},\\ &(C_0^m = \frac{P_m}{2},\ C_1^m = \frac{3}{2}M_m,\ m = 0,1,2,\cdots,L),\end{aligned} \tag{5.21}$$

where, we assume

$$\Psi_m(u) = \sum_{k=0}^{\infty} C_k^m P_k(u), \quad g'_m(u) = \sum_{k=0}^{\infty} g_k^m P_k^1(u).(1-x^2)^{-\frac{1}{2}}. \tag{5.22}$$

Lemma 5.1. *The infinite series* $\frac{(2s+1)[1+(-1)^{k+s}]}{(s-k-1)(s-k+1)(s+k)(s+k+2)}$, *is bounded for all values of* $s,\ k \geq 1$.

Proof. We consider that in order to prove the lemma

$$\sum_{s=1}^{\infty}\left|\frac{(2s+1)[1+(-1)^{k+s}]}{(s-k-1)(s-k+1)(s+k)(s+k+2)}\right| < \Lambda, \tag{5.23}$$

where

$$\Lambda = \frac{3}{2} + 4\sum_{s=3}^{\infty}\frac{2s+1}{(s-2)s(s+1)(s+3)}.$$

The value of Λ can be defined as

$$\Lambda = \frac{3}{2} + 4(\Gamma_1 + \Gamma_2),$$

where

$$\begin{aligned}\Gamma_1 =& \sum_{s=0}^{\infty}\frac{1}{(s+1)(s+4)(s+6)} = -\frac{1}{15}\Theta(1) + \frac{1}{6}\Theta(4) - \frac{1}{10}\Theta(6),\\ \Gamma_2 =& \sum_{s=0}^{\infty}\frac{1}{(s+1)(s+3)(s+6)} = -\frac{1}{10}\Theta(1) + \frac{1}{6}\Theta(4) - \frac{1}{15}\Theta(6).\end{aligned}$$

Where $\Theta(s)$ is called the Euler function and Gredshtein tables [18] can be used to determine its value for various values of s. In the end, we get $\Lambda = 11/5$. This proves the lemma. □

Table 1. Absolute error of solution in some arbitrary points by using Legendre polynomials with $L = 3$ and $0 \leq T \leq 0.6$

u_i	$\lvert\Psi(u,t_0) - \Psi_0(u)\rvert$	$\lvert\Psi(u,t_1) - \Psi_1(u)\rvert$	$\lvert\Psi(u,t_2) - \Psi_2(u)\rvert$	$\lvert\Psi(u,t_3) - \Psi_3(u)\rvert$
0.8	9.50026×10^{-13}	7.02314×10^{-8}	2.66215×10^{-7}	1.00198×10^{-6}
0.6	9.20214×10^{-14}	3.23021×10^{-9}	6.30862×10^{-8}	5.65481×10^{-7}
0.4	7.40035×10^{-14}	3.02155×10^{-9}	5.30287×10^{-8}	3.32002×10^{-7}
0.2	7.00026×10^{-15}	1.02139×10^{-9}	3.02547×10^{-8}	2.52014×10^{-7}
0.0	0	1.00071×10^{-10}	2.02154×10^{-9}	3.58420×10^{-8}
-0.2	5.44028×10^{-15}	2.00215×10^{-9}	2.02781×10^{-8}	4.29813×10^{-7}
-0.4	5.00211×10^{-14}	$4.3.251\times10^{-9}$	4.63220×10^{-8}	5.21170×10^{-7}
-0.6	6.20302×10^{-14}	7.62314×10^{-9}	5.32358×10^{-8}	6.45287×10^{-7}
-0.8	7.21005×10^{-13}	5.35269×10^{-8}	1.50000×10^{-7}	1.00214×10^{-6}

6. Problems and numerical results

Example 6.1. In this section, we applied the method that was described in this study for solve the integral equation (5.5).

We determine the constant C_k^m of Eq. (5.21) to get the numerical solution of integral equation (5.5). Then, using the main relation results, we can compute the unknown function $\Psi_m(u)$; $-1 \leq u \leq 1$, when $k = 20$, $\gamma = 25$, $\delta t = 0.001$, $\lambda = 0.1$, $g(u,t) = ut$, $J(t,\tau) = t^2\tau$, $f(u) = u$, $P_0 = 1$, $M_0 = 1$.

If we divide the interval $[0,T]$, $0 \leq T < 1$, as $0 = t_0 < t_1 < t_2 < t_3 = T$, where, $t = t_m$, $m = 0,1,2,3$. Applying the presented numerical technique, where $L = 3$ and $T = 0.6$ in the interval $[0,0.6]$.

In Table 1, we showed the absolute error $|\Psi(u,t_m) - \Psi_m(u)|, m = 0,1,2,3$, from the Legendre polynomials with $L = 3$ in the interval $[0,0.6]$.

In Figs. 1, 2, 3, and 4, we showed a comparison between the exact solution, the approximate solution, and absolute error of solution by using the presented numerical approach (Legendre polynomials) with different values of t_m, $m = 0,1,2,3$ with $L = 3$.

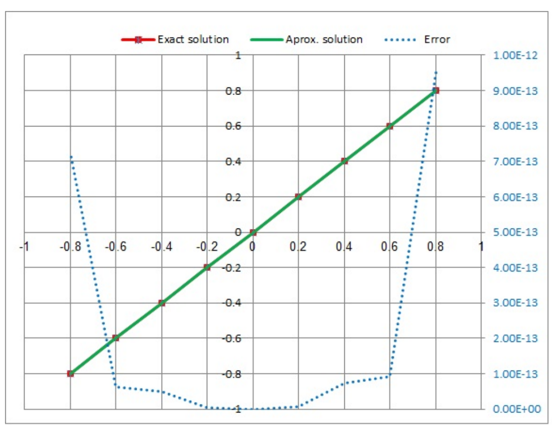

Figure 1. Exact and approximate solution of Legendre polynomials for $t_0 = 0$.

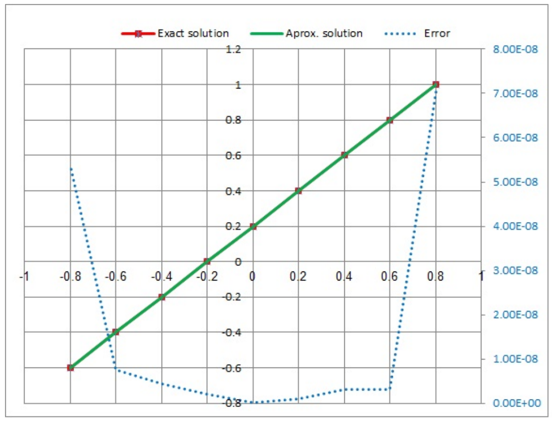

Figure 2. Exact and approximate solution of Legendre polynomials for $t_1 = 0.2$.

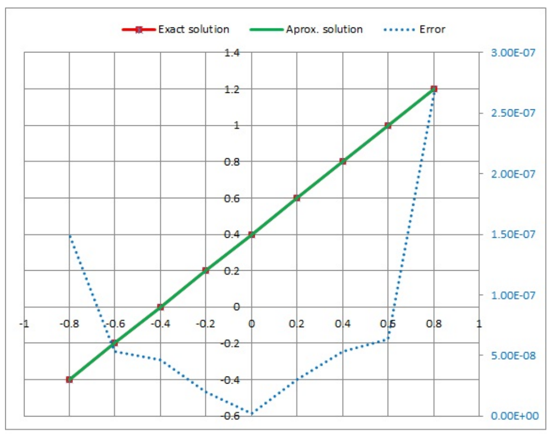

Figure 3. Exact and approximate solution of Legendre polynomials for $t_1 = 0.4$.

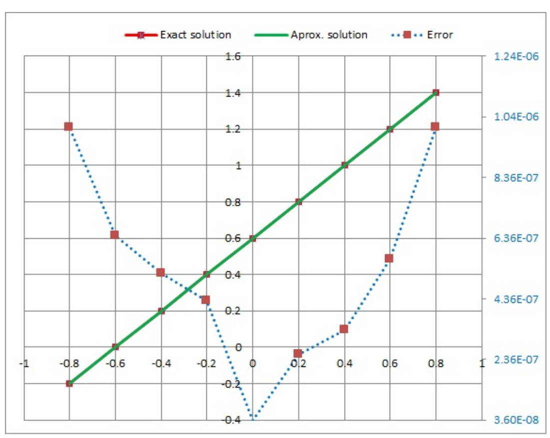

Figure 4. Exact and approximate solution of Legendre polynomials for $t_1 = 0.6$.

Table 2. Absolute error of solution in some arbitrary points by applying Legendre polynomials with $L = 3$ and $0 \leq T \leq 0.6$

u_i	$\lvert\Psi(u,t_0) - \Psi_0(u)\rvert$	$\lvert\Psi(u,t_1) - \Psi_1(u)\rvert$	$\lvert\Psi(u,t_2) - \Psi_2(u)\rvert$	$\lvert\Psi(u,t_3) - \Psi_3(u)\rvert$
0.9	8.85697×10^{-13}	9.85264×10^{-9}	7.00369×10^{-8}	1.90874×10^{-7}
0.7	8.23588×10^{-14}	2.25874×10^{-9}	5.65287×10^{-8}	1.23251×10^{-7}
0.5	4.02175×10^{-14}	7.00254×10^{-10}	3.32980×10^{-8}	6.21005×10^{-8}
0.3	1.00214×10^{-15}	1.00487×10^{-10}	5.65312×10^{-9}	3.56974×10^{-8}
0.1	3.00251×10^{-16}	3.92584×10^{-11}	3.05879×10^{-10}	5.00022×10^{-9}
-0.1	1.00214×10^{-16}	5.21479×10^{-11}	3.65874×10^{-10}	6.05877×10^{-9}
-0.3	1.85241×10^{-15}	1.56215×10^{-10}	3.00033×10^{-9}	5.32841×10^{-8}
-0.5	2.00021×10^{-14}	8.89521×10^{-10}	2.02580×10^{-8}	7.25682×10^{-8}
-0.7	3.66587×10^{-14}	6.25849×10^{-9}	5.69856×10^{-8}	1.00268×10^{-7}
-0.9	3.69852×10^{-13}	9.98654×10^{-9}	7.30002×10^{-8}	2.02858×10^{-7}

Example 6.2. Consider the following Volterra-Fredholm integral equations with phase-lag term:

$$\begin{aligned}20\Psi(u,t+0.0002) =&(u^2+t^2) + \int_0^t t^2\tau^2\Psi(u,\tau)\mathrm{d}\tau \\ &+ \int_{-1}^{1}\left(-\ln|v-u| + \ln\frac{4}{\pi}\right)\Psi(v,t)\mathrm{d}v, \qquad (6.1)\end{aligned}$$

the unknown function $\Psi_m(u)$; $-1 \leq u \leq 1$, when $k = 50$, $\lambda = 1$, $f(u) = u^2$, $P_0 = 1$, $M_0 = 1$.

If we divide the interval $[0, T]$, $0 \leq T < 1$, as $0 = t_0 < t_1 < t_2 < t_3 = T$, where, $t = t_m$, $m = 0, 1, 2, 3$, the Volterra-Fredholm integral equations (6.1) have the following form:

$$\begin{aligned}\aleph_m\Psi_m(u) =&f(u) + \frac{1}{20(0.0002)}\sum_{n=0}^{m}\omega_n(u^2+t_n^2) \\ &+ \frac{1}{20(0.0002)}\sum_{n=0}^{m-1}\nu_n[\frac{1}{3}(t_m^3t_n^2 - t_n^5) - 20]\Psi_n(u) \\ &+ \frac{1}{20(0.0002)}\sum_{n=0}^{m}\mu_n\int_{-1}^{1}\left(-\ln|v-u| + \ln\frac{4}{\pi}\right)\Psi_n(v)dv,\end{aligned}$$

where $\aleph_m = [1 - (\nu_m/20(0.0002))[\frac{1}{3}(t_m^5 - t_m^5) - 20]]$.

Using the Legendre polynomials with $L = 3$ and $T = 0.6$ in the interval $[0, 0.6]$.

In Table 2, we used the Legendre polynomials with $L = 3$ in the interval $[0, 0.6]$ to present the absolute error $|\Psi(u, t_m) - \Psi_m(u)|, m = 0, 1, 2, 3$.

In Figs. 5, 6, 7, and 8, we introduced a comparison between the exact solution, the approximate solution, and absolute error of solution using the proposed numerical approach (Legendre polynomials) with various values of t_m, $m = 0, 1, 2, 3$ with $L = 3$.

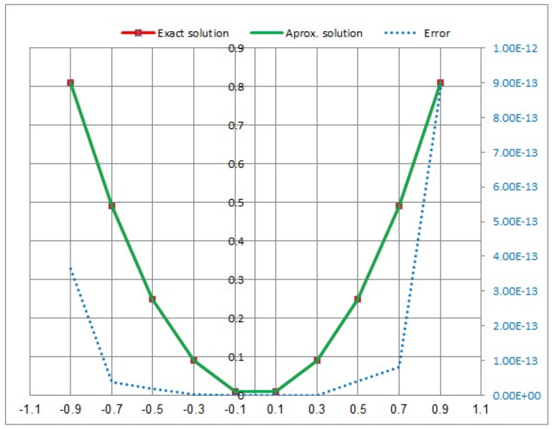

Figure 5. Exact and approximate solution of Legendre polynomials for $t_0 = 0$.

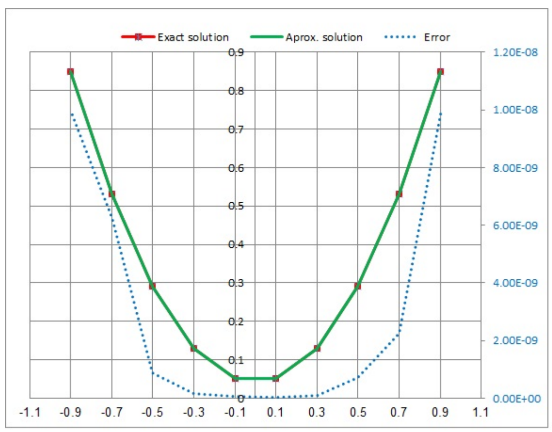

Figure 6. Exact and approximate solution of Legendre polynomials for $t_1 = 0.2$.

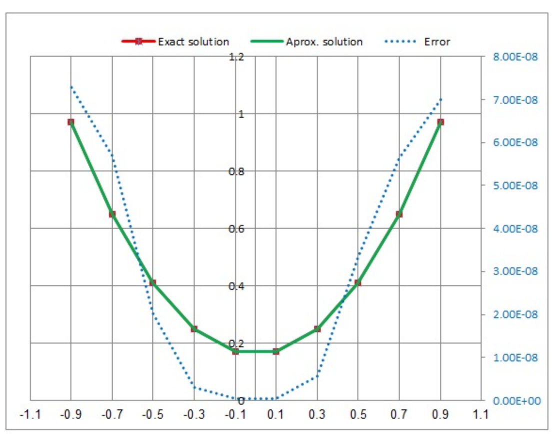

Figure 7. Exact and approximate solution of Legendre polynomials for $t_1 = 0.4$.

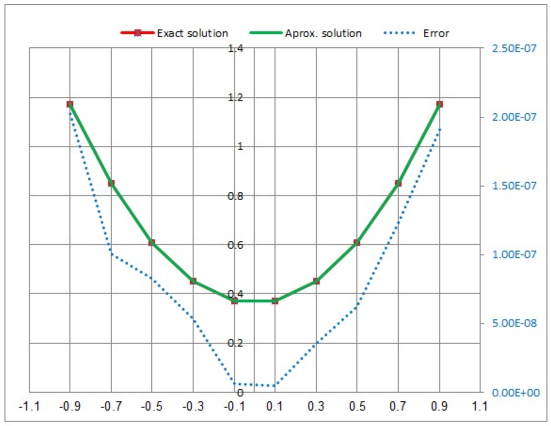

Figure 8. Exact and approximate solution of Legendre polynomials for $t_1 = 0.6$.

7. Conclusion and remarks

The following can be concluded from the results and discussion in this article:

In the space $L_2[-1,1] \times C[0,T]$, the equation (2.1) has a unique solution $\Psi(u,t)$ under some conditions. In many types of integral equations, it is usually difficult to obtain exact solutions, so it is necessary to find approximate solutions. From the Tables 1, 2, we note that the error takes maximum value at the ends when $u = 1$ and $u = -1$, while it is minimum at the middle when $u = 0$. If $\delta t \to 0$, we find that the numerical solution converges to the exact solution.

This work can be used to construct an integral equation with Carleman form by using Eq. (5.8). Currently, this work is considered a special case of the Fredholm integral equations of the second type with Carleman and logarithmic kernels. Numerous spectral relations are established from the problem these relations have various important applications in mathematical physics problems.

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