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MODIFIED COLLOCATION METHODS FOR SECOND KIND OF VOLTERRA INTEGRAL EQUATIONS WITH WEAKLY SINGULAR HIGHLY OSCILLATORY BESSEL KERNELS*

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Abstract In this paper, we investigate the second kind of Volterra integral equations with weakly sinular highly oscillatory Bessel kernels by using two collocation methods: direct high-order interpolationorder (DO) and direct Hermite interpolation (DH). Based on hypergeometric and Gamma functions, we obtain a method for solving the modified moments $\int_0^1 x^\alpha (1-x)^\beta J_v(\omega x)dx$. Compared with the Filon-type (Q_N^F) method, piecewise constant collocation ($Q_N^{L,0}$) method and linear collocation ($Q_N^{L,1}$) method, we verified the efficiency of the method through error analysis and numerical examples.

Keywords Bessel transform, weakly singular kernel, highly oscillatory kernel, modified collocation methods.

MSC(2010) 65D30, 65D32.

1. Introduction

In this paper, we consider the following integral equation with weakly singular highly oscillatory Bessel kernel

$$u(x) - \int_0^x t^\alpha (x-t)^\beta J_m(\omega(x-t))u(t)dt = f(x),\ x \in [0,t],\ \alpha, \beta > -1, \tag{1.1}$$

where $f(x)$ is the known smooth function, ω is the frequency, and $u(x)$ is the unknown function. The second kind of volterra integral equations with weakly singular highly oscillatory kernels occur prominently in many fields such as quantum mechanics, optics, astronomy, seismology image processing, elec-tromagnetic scattering. In 1995 Berrone [1] proposed a model for heat conduction in the study of

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materials subjected to state change at specific temperatures

$$u(x) = f(x) + \int_0^x (x-t)^{-\alpha} k(x,t) u(t) dt.$$

For the study of the numerical solution of a scalar retarded potential integral equation posed on an infinite flat surface,

$$\int_L \frac{u(x', t - |x' - x|)}{|x' - x|} dx' = a(x,t), \ (x,t) \in L \times (0,T),$$

where u is the unknown function, if $L = \mathbb{R}^2$, u and a satisfy $u \equiv 0$, $a \equiv 0$ for any $t \leq 0$. Davis and Ducan [6] using the Fourier transform technology, the equations can be transformed into a class of first kind Volterra integral equations with Bessel kernel,

$$2\pi \int_0^t J_0(\omega s) \hat{u}(\omega, t-s) ds = \hat{a}(\omega, t), \ t \in I := [0,T], \ T < \infty,$$

where $J_0(\cdot)$ denotes the Bessel function of the first-kind and of order zero.

There are always difficulties in solving the integral Eq. (1.1), as shown by the fact that the Bessel function has parameter ω. It is obvious that the Bessel function becomes highly oscillatory when $\omega \gg 1$. While solving Eq. (1.1), the calculation of the integral problem on the Bessel function is crucial. However, the classical quadrature rules, such as Newton-Cotes rule, Clenshaw-Curtis rule or Gauss rule, are failed to calculate this kind of integral, because the cost increases steeply with ω. Furthermore, Eq. (1.1) also contains parameters α and β, that is, the kernel function in Eq. (1.1) contains not only highly oscillatory, but also different kinds of singular points, which makes the solution of Eq. (1.1) into a very challenging problem. Consequently, it is very important to us for finding an efficient method for solving Eq. (1.1).

In the last decades, there are lots of methods to solve highly oscillatory problems, such as Collocation methods [2,18], Filon-Clenshaw-Curtis quadrature [9,23], Levin method [15], fast multipole methods [26], Clenshaw-Curtis algorithms [13], Clenshaw–Curtis–Filon-type methods [22], BBFM-collocation [19] and so on. There are some methods for solving the special case in Eq. (1.1) with respect to α, β. If $\alpha = \beta = 0$, then Eq. (1.1) becomes

$$u(x) - \int_0^x J_m(\omega(x-t)) u(t) dt = f(x), \ x \in [0,t]. \tag{1.2}$$

As for Eq. (1.2), Fang, He and Xiang [7] proposed two kinds of hermite-type method, that is, dierct Hermite collocation method and piecewise Hermite collocation method. Fang, Ma and Xiang [8] proposed the Filon method. The given error analysis and numerical experiments show that these methods can effectively compute the highly oscillatory Bessel problem. Base on benefiteal from some ideas of Refs [7,8] taht provides us with experience in dealing with unknown functions in solving highly oscillatory problems. Moreover, Ma, Xiang and Kang [16] give the rate of convergence for the Filon method.

If $\alpha = 0, \beta \neq 0$, then Eq. (1.1) becomes

$$u(x) - \int_0^x (x-t)^{\beta} J_m(\omega(x-t)) u(t) dt = f(x), \ x \in [0,t]. \tag{1.3}$$

More details are presented in Ref. [4]. There are some methods to solve the equation (1.3). Xiang and Hermann [21] proposed three methods, that is, direct Filon method, piecewise constant method and linear collocation method. The corresponding convergence rates and the numerical experimental results of these three methods show that the numerical solutions become more accurate as the ω increases. Ma and Kang [14] gaven a frequency-explicit convergence analysis of these three methods in Ref. [21]. In addition, Xiang and He [24] gave the discontinuous Galerkin methods, which cost the same operations independent of large values of ω, and numerical experiments demonstrate the effectiveness of the method.

For the solution of Eq. (1.1), it is crucial to calculate $\int_0^x t^\alpha (x-t)^\beta J_m(\omega(x-t))f(t)dt$. Therefore, we are going to discuss highly oscillatory Bessel transforms

$$I[f] = \int_0^x t^\alpha (x-t)^\beta J_m(\omega(x-t))f(t)dt,\ m \geq 1. \tag{1.4}$$

As regards (1.4), Xu and Xiang [25] gave the Clenshaw-Curtis-Filon method that based on the Fast Fourier Transform (FFT), and gave the recurrence relation of the modified moments and the effective method to evaluate the modified moments by recurrence relation. Xiang [20] gave some lemmas for weakly singular with Bessel highly oscillatory integrals. Furthermore, Kang, Xiang, Xu and Wang [12] gave the effective quadrature rules for the singularly oscillatory Bessel transforms and the error analysis. Based on some ideas in Ref. [12,20,25] we get the idea and experience for solving equation (1.1).

However, there is few literature available for solving Eq. (1.1), which makes the problem of solving this kind of equation extremely challenging. Therefore, we are interested in studying Eq. (1.1) and giving two methods to solve this kind of equations efficiently.

This paper are composed of the following parts: In Sect. 2, we show that the direct order interpolation (DO) for solving Eq. (1.1). In Sect. 3, we introduce the direct Hermite interpolation (DH) method to solve Eq. (1.1). Then, we show that the convergence analysis of these two methods in Sect. 4. In Sect. 5, we give some numerical experiments to compare with Filon-type (Q_N^F) method, piecewise constant collocation ($Q_N^{L,0}$) method and linear collocation ($Q_N^{L,1}$) method in the Ref. [21], which for proving effectiveness of our methods.

2. Direct high-order interpolation (DO)

We suppose that $k(x,t)$ is a continuous function on $D = \{(x,t) : 0 \leq x, t \leq T\}$, and $C(I)$ denotes the space of all continuous functions on $I = [0,T]$. Then the linear Volterra integral equation operator $\mathcal{V}_\alpha u(x) : C(I) \to C(I)$ is expressed as

$$\mathcal{V}_\alpha u(x) := \int_0^x H_\alpha(x,t)u(t)dt, x = [0,T],\ u(x) \in C(I),$$

where $H_\alpha(x,t) = \frac{k(x,t)}{(x-t)^\alpha}$, $0 < \alpha < 1$.

Let $\{x_j\}_{j=0}^N$ be the collocation point, satisfying $0 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_N = 1$. By linear interpolation at the points $x = 0$ and $x = x_j$, we have

$$u_d(x) = d_1(x)u(0) + d_2(x)u(x_j), \tag{2.1}$$

where $d_1(x) = (1 + \frac{2x}{x_j})(\frac{x-x_j}{x_j})^2$, $d_2(x) = (1 + 2\frac{x-x_j}{-x_j})(\frac{x}{x_j})^2$.

Since Eq. (1.1) holds at any collocation point, we get

$$u(x_j) - \int_0^{x_j} t^\alpha (x_j - t)^\beta J_m(x_j - t)u(t)dt = f(x_j). \tag{2.2}$$

By transforming the variables $\tau = x_j - t$, we obtain

$$u(x_j) - \int_0^{x_j} (x_j - \tau)^\alpha \tau^\beta J_m(\omega\tau)u(x_j - \tau)d\tau = f(x_j). \tag{2.3}$$

Approximating $u(x_j)$ with u_j, and substituting Eq. (2.1) into Eq. (2.3), so the collocation equation is obtained

$$u_j - \int_0^{x_j} (x_j - \tau)^\alpha \tau^\beta J_m(\omega\tau)(d_1(x)u(0) + d_2(x)u_j)d\tau = f(x_j). \tag{2.4}$$

In particular, when $x_j = 0$, we have $u(0) = f(0)$.

Solving Eq. (2.4), we conclude that

$$u_j = \frac{f(x_j) + f(0)[\frac{2}{x_j^3}I(\alpha+3,\beta,m,\omega) - \frac{3}{x_j^2}I(\alpha+2,\beta,m,\omega) + I(\alpha,\beta,m,\omega)]}{1 - \frac{3}{x_j^2}I(\alpha+2,\beta,m,\omega) + \frac{2}{x_j^3}I(\alpha+3,\beta,m,\omega)}. \tag{2.5}$$

Based on some ideas of Ref ([10], p681), we arrive at

$$\begin{aligned} I(\alpha,\beta,m,\omega) &= \int_0^b x^\alpha (b-x)^\beta J_m(\omega x)dx \\ &= b^{\alpha+\beta+1}\int_0^1 t^\alpha (1-t)^\beta J_m(b\omega t)dt \\ &= b^{\alpha+\beta+1} M(\alpha,\beta,m,b\omega), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &M(\alpha,\beta,m,\omega) \\ =&\int_0^1 x^\alpha (1-x)^\beta J_m(\omega x)dx \\ =&\frac{\Gamma(\beta+1)\Gamma(\alpha+m+1)}{2^m\omega^{-m}\Gamma(m+1)\Gamma(\alpha+\beta+m+2)} \\ &\times {}_2\mathrm{F}_3(\frac{\alpha+m+1}{2}, \frac{\alpha+m+2}{2}; m+1, \frac{\alpha+\beta+m+2}{2}, \frac{\alpha+\beta+m+3}{2}; \frac{-\omega^2}{4}), \end{aligned}$$

$\Gamma(x)$ is the Gamma function, defined as follows:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,$$

and ${}_2\mathrm{F}_3(\alpha_1,\alpha_2;\beta_1,\beta_2,\beta_3;x)$ is the hypergeometric function that can be efficiently computed for smaller $|x|$ by truncating the power series with the appropriate number of terms ([17], p404),

$${}_2\mathrm{F}_3(\alpha_1,\alpha_2;\beta_1,\beta_2,\beta_3;x) = \sum_{n=0}^\infty \frac{(\alpha_1)_n(\alpha_2)_n}{(\beta_1)_n(\beta_2)_n(\beta_3)_n}\frac{x^n}{n!}.$$

For larger $|x|$, there exists an asymptotic expansion [11],

$$
\begin{aligned}
&{}_2\mathrm{F}_3(\alpha_1,\alpha_2;\beta_1,\beta_2,\beta_3;-x)\\
=&\frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\left\{\frac{\Gamma(\alpha_1)\Gamma(\alpha_2-\alpha_1)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_2-\alpha_1)\Gamma(\beta_3-\alpha_1)}[1+O(\frac{1}{x})]\right.\\
&+\frac{\Gamma(\alpha_2)\Gamma(\alpha_1-\alpha_2)}{\Gamma(\beta_1-\alpha_2)\Gamma(\beta_2-\alpha_2)\Gamma(\beta_3-\alpha_2)}[1+O(\frac{1}{x})]\\
&+\frac{(-x)^{\chi}}{\sqrt{\pi}}[\cos(\pi\chi+2\sqrt{-x})(1+O(\frac{1}{x}))\\
&+\frac{\sin(\pi\chi+2\sqrt{-x})}{8\sqrt{-x}}((3\alpha_1+3\alpha_2+\beta_1+\beta_2+\beta_3-2)(4\chi-1)\\
&\left.+8\beta_1\beta_2+8\beta_1\beta_3+8\beta_2\beta_3-8\alpha_1\alpha_2-\frac{3}{2})[1+O(\frac{1}{x})]]\right\},
\end{aligned}
$$

where $\chi=\frac{1}{2}(\alpha_1+\alpha_2-\beta_1-\beta_2-\beta_3+\frac{1}{2})$. When $\omega\to\infty$, we have

$$
\begin{aligned}
&\left|{}_2\mathrm{F}_3(\frac{\alpha+m+1}{2},\frac{\alpha+m+2}{2};m+1,\frac{\alpha+\beta+m+2}{2},\frac{\alpha+\beta+m+3}{2};\frac{-\omega^2}{4})\right|\\
\le&C[\frac{\Gamma(\frac{\alpha+m+1}{2})}{\Gamma(\frac{m-\alpha+1}{2})\Gamma(\frac{\beta+1}{2})\Gamma(\frac{\beta+2}{2})}(\frac{\omega}{2})^{-(\alpha+m+1)}\\
&+\frac{\Gamma(\frac{\alpha+m+2}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{\beta}{2})\Gamma(\frac{\beta+1}{2})}(\frac{\omega}{2})^{-(\alpha+m+2)}+(\frac{\omega}{2})^{-(\beta+m+\frac{3}{2})}].
\end{aligned}
$$

Here the operation of $(\alpha)_j$ obeys $(\alpha)_j=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+j-1), j\ge 1, (j\in N)$.

3. Direct Hermite interpolation (DH)

In this section, we will provide a detailed procedure for solving equation (1.1) using the direct Hermite-type interpolation method. Before that, we need to prove the differentiability of the solution to equation (1.1).

Lemma 3.1 ([2,21]). *Assume that the functions $f=f(x)$ and $K_\alpha=t^\alpha J_m(\omega(x-t))$ are cuntinuous on their respective domains $[0,1]$ and $D=\{0\le t\le x\le 1\}$. Then the equation (1.1) possesses a unique continuous solution $u=u(x)$.*

Lemma 3.2 ([2, 21]). *Suppose $K_\alpha=t^\alpha J_m(\omega(x-t))\in C[0,1]$ and $-1<\beta<0$, the solution $u(x)$ of equation (1.1) is uniformly bounded for $\omega\ge 0$, that is, $\sup\limits_{\omega\in[0,\infty)}\max\limits_{x\in[0,1]}|u(x)|<\infty$.*

Theorem 3.1. *Let $f\in C^q[0,1](q\ge 1), (x-\tau)^\alpha\in C^q[0,1](\alpha>0), J_m(\omega\tau)\in C[0,1]$ and $-1<\beta<0$, then the solution $u(x)$ of equation (1.1) satisfies $u(x)\in C^1[0,1]$.*

Proof. The solution of the integral equation (1.1) is given by

$$u(x)=f(x)+\int_0^x R_\beta(x,t)J_m(\omega(x-t))t^\alpha f(t)dt,$$

where $R_\beta(x,t)=(x-t)^{-\beta}\sum_{n=1}^{\infty}\frac{(\Gamma(\beta))^{(2n)}}{\Gamma(n\beta)}(x-t)^{(n-1)(1-\beta)}:=(x-t)^{-\beta}Q(x,t,\beta)$ [3].

By setting $\tau = x - t$, then the solution $u(x)$ can be written as

$$u(x) = f(x) + \int_0^x R_\beta(\tau) J_m(\omega\tau)(x-\tau)^\alpha f(x-\tau) d\tau. \tag{3.1}$$

Differentiating both sides of equation (3.1) with respect to variable x, we obtain

$$\begin{aligned} u'(x) =& f'(x) + \alpha \int_0^x R_\beta(\tau) J_m(\omega\tau)(x-\tau)^{\alpha-1} f(x-\tau) d\tau \\ &+ \int_0^x R_\beta(\tau) J_m(\omega\tau)(x-\tau)^\alpha f'(x-\tau) d\tau, x \in [0,1]. \end{aligned} \tag{3.2}$$

Then the theorem is proved. □

If the interpolation condition satisfies $H(x_j) = f(x_j), H'(x_j) = f'(x_j)$, then it is called Hermite interpolation. Choosing two points $x = 0$ and $x = x_j$ and Hermite interpolation, we have

$$u_h(x) = h_1(x)u(0) + h_2(x)u(x_j) + h_3(x)u'(0) + h_4(x)u'(x_j), \tag{3.3}$$

where

$$\begin{aligned} h_1(x) &= (1 + \frac{2x}{x_j})(\frac{x - x_j}{x_j})^2, \quad h_3(x) = x(\frac{x - x_j}{-x_j})^2, \\ h_2(x) &= (1 + 2\frac{x - x_j}{-x_j})(\frac{x}{x_j})^2, \ h_4(x) = (x - x_j)(\frac{x}{x_j})^2, \end{aligned}$$

denote the basic polynomial with respect to $x = 0$ and $x = x_j$.

Combining with Theorem 3.1, and applying Hermite interpolation, we consider the case $\alpha > 0$ in equation (1.1). Differentiating both sides of Eq. (1.1) with respect to variable x, one has

$$\begin{aligned} &u'(x) - \alpha \int_0^x (x-\tau)^{\alpha-1} \tau^\beta J_m(\omega\tau) u(x-\tau) d\tau \\ &- \int_0^x (x-\tau)^\alpha \tau^\beta J_m(\omega\tau) u'(x-\tau) d\tau = f'(x). \end{aligned} \tag{3.4}$$

The Eq. (3.4) holds at every collocation point, then we get

$$\begin{aligned} &u'(x_j) - \alpha \int_0^{x_j} (x_j-\tau)^{\alpha-1} \tau^\beta J_m(\omega\tau) u(x_j-\tau) d\tau \\ &- \int_0^{x_j} (x_j-\tau)^\alpha \tau^\beta J_m(\omega\tau) u'(x_j-\tau) d\tau = f'(x_j). \end{aligned} \tag{3.5}$$

Approximating $u(x_j)$ by u_j^d, and form Eq. (3.3), Eq. (2.3) and Eq. (3.5), we have

$$\begin{aligned} &u_j'^d - \alpha \int_0^{x_j} (x_j-\tau)^{\alpha-1} \tau^\beta J_m(\omega\tau) u_h(x_j-\tau) d\tau \\ &- \int_0^{x_j} (x_j-\tau)^\alpha \tau^\beta J_m(\omega\tau) u_h'(x_j-\tau) d\tau = f'(x_j). \end{aligned} \tag{3.6}$$

$$u_j^d - \int_0^{x_j} (x_j-\tau)^\alpha \tau^\beta J_m(\omega\tau) u_h(x_j-\tau) d\tau = f(x_j). \tag{3.7}$$

It follows from Eq. (3.3), Eq. (3.6) and Eq. (3.7) that

$$\begin{aligned} & u_j'^d + \alpha\int_0^{x_j}(x_j-\tau)^{\alpha-1}\tau^\beta J_m(\omega\tau)(h_1(x_j-\tau)u(0)+h_2(x_j-\tau)u_j \\ & + h_3(x_j-\tau)u'(0)+h_4(x_j-\tau)u_j')d\tau \\ & - \int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta-1} J_m(\omega\tau)(h_1'(x_j-\tau)u(0) \\ & + h_2'(x_j-\tau)u_j+h_3'(x_j-\tau)u'(0)+h_4'(x_j-\tau)u_j')d\tau \\ =& f'(x_j), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} & u_j^d - \int_0^{x_j}(x_j-\tau)^{\alpha}\tau^\beta J_m(\omega\tau)(h_1(x_j-\tau)u(0)+h_2(x_j-\tau)u_j \\ & + h_3(x_j-\tau)u'(0)+h_4(x_j-\tau)u_j')d\tau = f(x_j). \end{aligned} \tag{3.9}$$

From Eq. (3.8) and Eq. (3.9), we get

$$\begin{aligned} u_j^d =& \frac{f(x_j)+(1/x_j^2 I(\alpha+3,\beta,m,\omega)-2/x_j I(\alpha+2,\beta,m,\omega)+I(\alpha+1,\beta,m,\omega))u'(0)}{1-(3/x_j^2 I(\alpha+2,\beta,m,\omega)-2/x_j^3 I(\alpha+3,\beta,m,\omega))} \\ & + \frac{(1/x_j^2 I(\alpha+3,\beta,m,\omega)-1/x_j I(\alpha+2,\beta,m,\omega))u_j'}{1-(3/x_j^2 I(\alpha+2,\beta,m,\omega)-2/x_j^3 I(\alpha+3,\beta,m,\omega))} \\ & + \frac{(2/x_j^3 I(\alpha+3,\beta,m,\omega)-3/x_j^2 I(\alpha+2,\beta,m,\omega)+I(\alpha,\beta,m,\omega))u(0)}{1-(3/x_j^2 I(\alpha+2,\beta,m,\omega)-2/x_j^3 I(\alpha+3,\beta,m,\omega))}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} u_j'^d =& \frac{f'(x_j)+\alpha(3/x_j^2 I(\alpha+1,\beta,m,\omega)-2/x_j^3 I(\alpha+2,\beta,m,\omega))u_j^d}{1-A-B} \\ & + \frac{(6/x_j^2 I(\alpha+1,\beta,m,\omega)-6/x_j^3 I(\alpha+2,\beta,m,\omega))u_j^d}{1+A-B} \\ & + \frac{\alpha(1/x_j^2 I(\alpha+2,\beta,m,\omega)-2/x_j I(\alpha+1,\beta,m,\omega)+I(\alpha,\beta,m,\omega))u'(0)}{1-A-B} \\ & + \frac{\alpha(2/x_j^3 I(\alpha+2,\beta,m,\omega)-3/x_j^2 I(\alpha+1,\beta,m,\omega)+I(\alpha-1,\beta,m,\omega))u(0)}{1-A-B} \\ & + \frac{(3/x_j^2 I(\alpha+2,\beta,m,\omega)-4/x_j I(\alpha+1,\beta,m,\omega)+I(\alpha,\beta,m,\omega))u'(0)}{1+A-B} \\ & + \frac{(6/x_j^3 I(\alpha+2,\beta,m,\omega)-6/x_j^2 I(\alpha+1,\beta,m,\omega)u(0)}{1+A-B}, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} A &= \alpha(1/x_j^2 I(\alpha+2,\beta,m,\omega)-1/x_j I(\alpha+1,\beta,m,\omega)), \\ B &= (3/x_j^2 I(\alpha+2,\beta,m,\omega)-2/x_j I(\alpha+1,\beta,m,\omega)). \end{aligned}$$

Combining Eq. (3.10) and Eq. (3.11) yields

$$u_j^d = \frac{a_2 s + b_2 c}{a_2 b_1 + b_2 a_1}, \quad u_j'^d = \frac{a_1 s - b_1 c}{a_1 b_2 + b_1 a_2},$$

where

$$
\begin{aligned}
a_1 &= 1 - 3/x_j^2 I(\alpha+2,\beta,m,\omega) + 2/x_j^3 I(\alpha+3,\beta,m,\omega),\\
a_2 &= 1/x_j I(\alpha+2,\beta,m,\omega) - 1/x_j^2 I(\alpha+3,\beta,m,\omega),\\
b_1 &= -\alpha(3/x_j^2 I(\alpha+1,\beta,m,\omega) - 2/x_j^3 I(\alpha+2,\beta,m,\omega))\\
&\quad - (6/x_j^2 I(\alpha+1,\beta,m,\omega) - 6/x_j^3 I(\alpha+2,\beta,m,\omega)),\\
b_2 &= 1 - \alpha(3/x_j^2 I(\alpha+2,\beta,m,\omega) - 2/x_j I(\alpha+1,\beta,m,\omega))\\
&\quad - (3/x_j^2 I(\alpha+2,\beta,m,\omega) - 2/x_j I(\alpha+1,\beta,m,\omega)),\\
c &= f(x_j) + (1/x_j^2 I(\alpha+3,\beta,m,\omega) - 2/x_j I(\alpha+2,\beta,m,\omega) + I(\alpha+1,\beta,m,\omega))u'(0)\\
&\quad + (2/x_j^3 I(\alpha+3,\beta,m,\omega) - 3/x_j^2 I(\alpha+2,\beta,m,\omega) + I(\alpha,\beta,m,\omega))u(0),\\
s &= f'(x_j) + \alpha(1/x_j^2 I(\alpha+2,\beta,m,\omega) - 2/x_j I(\alpha+1,\beta,m,\omega) + I(\alpha,\beta,m,\omega))u'(0)\\
&\quad + (3/x_j^2 I(\alpha+2,\beta,m,\omega) - 4/x_j I(\alpha+1,\beta,m,\omega) + I(\alpha,\beta,m,\omega))u'(0)\\
&\quad + \alpha(2/x_j^3 I(\alpha+2,\beta,m,\omega) - 3/x_j^2 I(\alpha+1,\beta,m,\omega) + I(\alpha-1,\beta,m,\omega))u(0)\\
&\quad + (6/x_j^3 I(\alpha+2,\beta,m,\omega) - 6/x_j^2 I(\alpha+1,\beta,m,\omega)u(0).
\end{aligned}
$$

4. Error analysis

In this section, we are going to give an analysis of the convergence of our methods for solving Eq. (1.1). to further analysis, we introduce the two lemmas that will be used.

Lemma 4.1 ([5,20]). *Assuming that $\alpha+m>-1, \beta>-1, b>0$, for $\omega \gg 1$, then the following equation holds*

$$\int_0^b t^\alpha (b-t)^\beta J_m(\omega t)dt = O(\omega^{-\min\{\alpha+1,\beta+\frac{3}{2}\}}).$$

Lemma 4.2 ([20]). *Supposing that $f \in C^{v+1}[0,1]$, for $\omega \gg 1$, then the following equation holds*

$$\int_0^1 t^\alpha (1-t)^\beta J_m(\omega t) f(t)dt = O(\omega^{-\min\{\alpha+1,\beta+\frac{3}{2}\}}).$$

We are now position to state the error analysis of our methods, direct order interpolation (DO) method and direct Hermite interpolation (DH) method, and show the convergence rate.

Theorem 4.1. *The error estimate for solving Eq. (1.1) by using direct high-order interpolation is*

$$u_j - u(x_j) = O(\omega^{-1-\min\{\alpha+\frac{3}{2},\beta+1\}}).$$

Proof. By the direct high-order interpolation method in section 2, we know that Eq. (1.1) holds at any collocation point, giving

$$u(x_j) - \int_0^{x_j} t^\alpha (x_j-t)^\beta J_m(\omega(x_j-t))u(t)dt = f(x_j). \tag{4.1}$$

Let the variables $\tau = x_j - t$, then we get

$$u(x_j) - \int_0^{x_j} (x_j - \tau)^\alpha \tau^\beta J_m(\omega\tau) u(x_j - \tau) d\tau = f(x_j). \tag{4.2}$$

Next, using u_j to approximate $u(x_j)$, the following equation holds

$$u_j - \int_0^{x_j} (x_j - \tau)^\alpha \tau^\beta J_m(\omega\tau) u_h(x_j - \tau) d\tau = f(x_j). \tag{4.3}$$

It follows from Eq. (4.2) and Eq. (4.3) that

$$u(x_j) - u_j = \int_0^{x_j} (x_j - \tau)^\alpha \tau^\beta J_m(\omega\tau)(u(x_j - \tau) - u_h(x_j - \tau)) d\tau. \tag{4.4}$$

Assume that the error at the interpolation point $x = 0$ is $E(0) = 0$. Then the error equation can be expressed as follows

$$E(x) = u(x) - u_h(x) = d_2(x)E(x_j) + R(x), \tag{4.5}$$

where $R(x)$ is the direct high-order interpolation residual term. Substituting Eq. (4.5) into Eq. (4.4) yields

$$E(x_j) = \int_0^{x_j} (x_j - \tau)^\alpha \tau^\beta J_m(\omega\tau) E(x_j - \tau) d\tau. \tag{4.6}$$

Combining Eq. (4.5) with Eq. (4.6), we get

$$E(x_j) = \int_0^{x_j} (x_j - \tau)^\alpha \tau^\beta J_m(\omega\tau)(d_2(x_j - \tau)E(x_j) + R(x_j - \tau)) d\tau. \tag{4.7}$$

Thus, we know that the interpolation error at $x = x_j$ is presented as follows

$$E(x_j) = \frac{\int_0^{x_j} (x_j - \tau)^\alpha \tau^\beta J_m(\omega\tau) R(x_j - \tau) d\tau}{1 - \int_0^{x_j} (x_j - \tau)^\alpha \tau^\beta J_m(\omega\tau) d_2(x_j - \tau) d\tau} \triangleq \frac{I_1}{I_2}. \tag{4.8}$$

In what follows, we show estimates for I_1 and I_2 in Eq. (4.8). Applying Lemma 4.1 and Lemma 4.2, we get that $I_2 = O(1)$ when $\omega \to \infty$. According to the direct high order interpolation residual term property, we have

$$R(x) = u(x) - u_h(x) = C(x)x(x_j - x),$$

where $C(x) \in C[0, x_j]$. Applying to Lemma 4.2, we get

$$I_1 = \int_0^{x_j} (x_j - \tau)^{\alpha+1} \tau^{\beta+1} J_m(\omega\tau) C(x_j - \tau) d\tau = O(\omega^{-1-\min\left\{\alpha+\frac{3}{2},\beta+1\right\}}).$$

Therefore, the error convergence rate can be obtained

$$E(x_j) = \frac{I_1}{I_2} = O(\omega^{-1-\min\left\{\alpha+\frac{3}{2},\beta+1\right\}}).$$

$\square$

Theorem 4.2. *The error estimate obtained by solving Eq. (1.1) using the direct hermite interpolation method is*

$$u_j - u(x_j) = O(\omega^{-2-\min\left\{\alpha+\frac{3}{2},\beta+1\right\}}).$$

Proof. When $\alpha > 0$, by the direct Hermite interpolation method in section 3, we know that

$$u(x) - \int_0^x (x-\tau)^\alpha \tau^\beta J_m(\omega\tau) u(x-\tau) d\tau = f(x), \tag{4.9}$$

$$\begin{aligned} &u'(x) - \alpha \int_0^x (x-\tau)^{\alpha-1} \tau^\beta (J_m(\omega\tau))) u(x-\tau) d\tau \\ &- \int_0^x (x-\tau)^\alpha \tau^\beta J_m(\omega\tau) u'(x-\tau) d\tau = f'(x). \end{aligned} \tag{4.10}$$

Equations (4.9) and (4.10) hold at any collocation point, we have

$$u(x_j) - \int_0^{x_j} (x_j-\tau)^\alpha \tau^\beta J_m(\omega\tau) u(x_j-\tau) d\tau = f(x_j), \tag{4.11}$$

$$\begin{aligned} &u'(x_j) - \alpha \int_0^{x_j} (x_j-\tau)^{\alpha-1} \tau^\beta (J_m(\omega\tau)) u((x_j-\tau)) d\tau \\ &- \int_0^{x_j} (x_j-\tau)^\alpha \tau^{\beta-1} J_m(\omega\tau) u'((x_j-\tau)) d\tau = f'(x_j). \end{aligned} \tag{4.12}$$

Approximate $u(x_j)$ by u_j, and we have

$$u_j - \int_0^{x_j} (x_j-\tau)^\alpha \tau^\beta J_m(\omega\tau) u_h(x_j-\tau) d\tau = f(x_j), \tag{4.13}$$

$$\begin{aligned} &u_j' - \alpha \int_0^{x_j} (x_j-\tau)^{\alpha-1} \tau^\beta (J_m(\omega\tau)) u_h((x_j-\tau)) d\tau \\ &- \int_0^{x_j} (x_j-\tau)^\alpha \tau^\beta J_m(\omega\tau) u_h'((x_j-\tau)) d\tau = f'(x_j). \end{aligned} \tag{4.14}$$

It follows from Eq. (4.11) - Eq. (4.14) that

$$u(x_j) - u_j = \int_0^{x_j} (x_j-\tau)^\alpha \tau^\beta J_m(\omega\tau)(u(x_j-\tau) - u_h(x_j-\tau)) d\tau = f(x_j), \tag{4.15}$$

$$\begin{aligned} u'(x_j) - u_j' = {} & \alpha \int_0^{x_j} (x_j-\tau)^{\alpha-1} \tau^\beta (J_m(\omega\tau))(u(x_j-\tau) - u_h((x_j-\tau))) d\tau \\ & + \int_0^{x_j} (x_j-\tau)^\alpha \tau^\beta J_m(\omega\tau)(u'(x_j-\tau) - u_h'((x_j-\tau))) d\tau. \end{aligned} \tag{4.16}$$

Based on Hermite interpolation, the error function is expressed as follows

$$E(x) = u(x) - u_h(x) = h_2(x)E(x_j) + h_4(x)E'(x_j) + R(x), \tag{4.17}$$

where $R(x)$ is the residual term of Hermite interpolation. And the errors at the collocation point $x = 0$ satisfy $E(0) = E'(0) = 0$. Substituting Eq. (4.17) into Eq.

(4.15) and Eq. (4.16), respectively, we have

$$\begin{aligned}
&(1-\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta}J_m(\omega\tau)h_2(x_j-\tau)d\tau)E(x_j)\\
&-\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta}J_m(\omega\tau)h_4(x_j-\tau)d\tau E'(x_j)\\
=&\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta}J_m(\omega\tau)R(x_j-\tau)d\tau,
\end{aligned}\tag{4.18}$$

$$\begin{aligned}
&(1-\alpha\int_0^{x_j}(x_j-\tau)^{\alpha-1}\tau^{\beta}(J_m(\omega\tau))h_4(x_j-\tau)d\tau\\
&-\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta-1}J_m(\omega\tau)h_4'(x_j-\tau)d\tau)E'(x_j)\\
&-(\alpha\int_0^{x_j}(x_j-\tau)^{\alpha-1}\tau^{\beta}J_m(\omega\tau)h_2(x_j-\tau)d\tau\\
&-\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta}J_m(\omega\tau)h_2'(x_j-\tau)d\tau)E(x_j)\\
=&\alpha\int_0^{x_j}(x_j-\tau)^{\alpha-1}\tau^{\beta}J_m(\omega\tau)R(x_j-\tau)d\tau\\
&+\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta-1}(J_m(\omega\tau))R'(x_j-\tau)d\tau.
\end{aligned}\tag{4.19}$$

Combining Eq. (4.18) and Eq. (4.19) yields

$$E(x_j)=\frac{Q_2}{Q_1},E^{'}(x_j)=\frac{Q_3}{Q_1},$$

where

$$\begin{aligned}
Q_1=&(1-\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta}J_m(\omega\tau)h_2(x_j-\tau)d\tau)\\
&\times(1-\alpha\int_0^{x_j}(x_j-\tau)^{\alpha-1}\tau^{\beta}(J_m(\omega\tau))h_4(x_j-\tau)d\tau\\
&-\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta-1}J_m(\omega\tau)h_4'(x_j-\tau)d\tau)\\
&-\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta}J_m(\omega\tau)h_4(x_j-\tau)d\tau\\
&\times(\alpha\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta}(J_m(\omega\tau))h_2(x_j-\tau)d\tau\\
&+\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta}J_m(\omega\tau)h_2'(x_j-\tau)d\tau)\\
Q_2=&\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta}J_m(\omega\tau)h_4(x_j-\tau)d\tau\\
&\times(\alpha\int_0^{x_j}(x_j-\tau)^{\alpha-1}\tau^{\beta}J_m(\omega\tau)R(x_j-\tau)d\tau\\
&+\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta}(J_m(\omega\tau))R'(x_j-\tau)d\tau)\\
&+\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^{\beta}J_m(\omega\tau)R(x_j-\tau)d\tau
\end{aligned}$$

$$
\begin{aligned}
&\times (1-\alpha\int_0^{x_j}(x_j-\tau)^{\alpha-1}\tau^\beta(J_m(\omega\tau)h_4(x_j-\tau)d\tau \\
&-\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^\beta J_m(\omega\tau)h_4'(x_j-\tau)d\tau), \\
Q_3 =&(1-\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^\beta J_m(\omega\tau)h_2(x_j-\tau)d\tau) \\
&\times (\alpha\int_0^{x_j}(x_j-\tau)^{\alpha-1}\tau^\beta J_m(\omega\tau)R(x_j-\tau)d\tau \\
&+\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^\beta (J_m(\omega\tau))R'(x_j-\tau)d\tau) \\
&+\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^\beta J_m(\omega\tau)R(x_j-\tau)d\tau \\
&\times (\alpha\int_0^{x_j}(x_j-\tau)^{\alpha-1}\tau^\beta (J_m(\omega\tau)h_2(x_j-\tau)d\tau \\
&+\int_0^{x_j}(x_j-\tau)^{\alpha}\tau^\beta J_m(\omega\tau)h_2'(x_j-\tau)d\tau).
\end{aligned}
$$

Applying Lemma 4.1 and 4.2, we can get that $Q_1 = O(1)$ when $\omega \to \infty$. According to the Hermite interpolation residual term property, we have

$$R(x) = u(x) - u_h(x) = Cx^2(x_j - x)^2,$$

where C is a constan. Then according to Lemma 4.1 and the expression for Q_2, we can easily get

$$Q_2 = O(\omega^{-2-\min\left\{\alpha+\frac{3}{2},\beta+1\right\}}).$$

Therefore, the error convergence rate can be obtained

$$E(x_j) = O(\omega^{-2-\min\left\{\alpha+\frac{3}{2},\beta+1\right\}}).$$

□

5. Numerical experiments

In this section, in order to verify the effectiveness of our methods, we demonstrate some numerical experiments by using direct high-order interpolation (DO) method and the direct Hermite interpolation (DH) method to solve Eq. (1.1). The case of $m = 0, \alpha = 0$ and $\beta = -0.5$ is chosen and our two methods are applied to compare with the Filon-type (Q_N^F) method, piecewise constant collocation ($Q_N^{L,0}$) method and linear collocation ($Q_N^{L,1}$) method in the literature [21].

Next, we give four numerical examples to demonstrate the effectiveness of our methods.

Example 5.1. We consider the following equation ($m = 0, \alpha = 0$),

$$u(x) + \int_0^x (x-t)^\beta J_0(\omega(x-t))u(t)dt = f(x),$$

then $Q_A^2 = f(x) - \int_0^x t^\beta J_0(\omega t)f(x-t)dt$, when $f(x) = \sin(x)$.

In Tables 1-2, we apply the direct high-order interpolation (DO) method to solve the errors of the Example 5.1 and 5.2, respectively, comparing them with the Filon-type (Q_N^F) method, the piecewise constant collocation ($Q_N^{L,0}$) method and the linear collocation ($Q_N^{L,1}$) in the literature [21]. In Tables 3-8, we give the errors of solving the equation at the point $x = 0.2, 0.4, 0.6$ and 0.8 when $\omega = 10, 100, 1000, 10000, 100000$ using the direct high-order interpolation (DO) method and the direct Hermite interpolation (DH) method, and we could find that the errors are decreasing as the ω increases. The curves of the change of error with ω at point $0.2, 0.5$ and 0.8 using the direct high-order interpolation (DO) method and the direct Hermite interpolation (DH) method are shown in Figs. 1, 3, 5, 7, 9 and 11. In Figs. 2, 4, 6, 8, 10 and 12, the errors of the two methods at the point $x = 0.2$ are compared.

Table 1. The errors of solving Example 5.1 by direct high-order interpolation (DO) method is compared with the Filon-type (Q_N^F) method, piecewise constant collocation ($Q_N^{L,0}$) method and linear collocation ($Q_N^{L,1}$) method in the literature [21] at point $x = 0.1, 0.5$ and 1 ($m = 0, \alpha = 0, \beta = -1/2, \omega = 10^4$).

$Method \backslash x$	0.1	0.5	1
Q_{10}^F	4.4032e-05	2.0520e-04	3.6216e-04
$Q_{10}^{L,0}$	4.3846e-05	2.0421e-04	3.6135e-04
$Q_{100}^{L,0}$	4.4492e-05	2.0571e-04	3.6235e-04
$Q_{10}^{L,1}$	4.4033e-05	2.0517e-04	3.6204e-04
$Q_{100}^{L,1}$	4.4032e-05	2.0516e-04	3.6202e-04
$Q_{1000}^{L,1}$	4.4032e-05	2.0516e-04	3.6202e-04
DO	4.8816e-07	4.2864e-07	2.6381e-07

Example 5.2. Considering the following equation ($m = 0.5, \alpha = 0$)

$$u(x) - \int_0^x (x-t)^\beta J_{0.5}(\omega(x-t))u(t)dt = f(x),$$

then $Q_A^2 = f(x) - \int_0^x t^\beta J_0(\omega t) f(x-t)dt$, when $f(x) = e^x$.

Table 2. The errors of solving Example 5.2 by direct high-order interpolation (DO) method is compared with the Filon-type (Q_N^F) method, piecewise constant collocation ($Q_N^{L,0}$) method and linear collocation ($Q_N^{L,1}$) method in the literature [21] at point $x = 1$ when $\beta = -0.1, -0.5, -0.8$ ($m = 0.5, \alpha = 0, \omega = 10^4$).

$Method \backslash \beta$	-0.1	-0.5	-0.8
Q_{10}^F	2.8459e-06	4.2702e-04	1.3440e-01
$Q_{10}^{L,0}$	2.7280e-06	4.2645e-04	1.3440e-01
$Q_{100}^{L,0}$	2.9371e-06	4.2705e-04	1.3440e-01
$Q_{10}^{L,1}$	2.8585e-06	4.2771e-04	1.3440e-01
$Q_{100}^{L,1}$	2.8601e-06	4.2780e-04	1.3441e-01
$Q_{1000}^{L,1}$	2.8604e-06	4.2781e-04	1.3441e-01
DO	3.9101e-08	2.1964e-06	5.2862e-05

Example 5.3. We consider the following equation

$$u(x) - \int_0^x t^\alpha (x-t)^\beta J_m(\omega(x-t))u(t)dt = f(x),$$

when $f(x) = e^x - \int_0^x t^\alpha (x-t)^\beta J_m(\omega(x-t))e^t dt$, then the exact solution of the equation is $u(x) = e^x$.

Table 3. Absolute error of the two collocation methods for solving Example 5.3 ($m = 0.5, \alpha = 3/2, \beta = -1/2$).

$\omega \backslash x$		0.2	0.4	0.6	0.8
10	DO	2.3709e-04	3.5164e-03	1.6230e-02	4.4382e-02
	DH	8.5931e-05	2.1769e-03	1.0828e-02	2.8508e-02
100	DO	8.5176e-05	3.0840e-04	7.1452e-04	1.3917e-03
	DH	7.3557e-06	3.0944e-05	7.3363e-05	1.3400e-04
1000	DO	2.7653e-06	9.6179e-06	2.1767e-05	4.1351e-05
	DH	6.4986e-08	2.7097e-07	6.1943e-07	1.0689e-06
10000	DO	8.7263e-08	3.0434e-07	6.7964e-07	1.4438e-06
	DH	6.3759e-10	4.9046e-09	5.9797e-09	1.7183e-07
100000	DO	2.7212e-09	9.1004e-09	8.9242e-08	4.0175e-08
	DH	2.9798e-11	4.0531e-10	6.7894e-08	1.9120e-11

$m = 0.5, \alpha = 3/2, \beta = -1/2, u(x) = e^x$

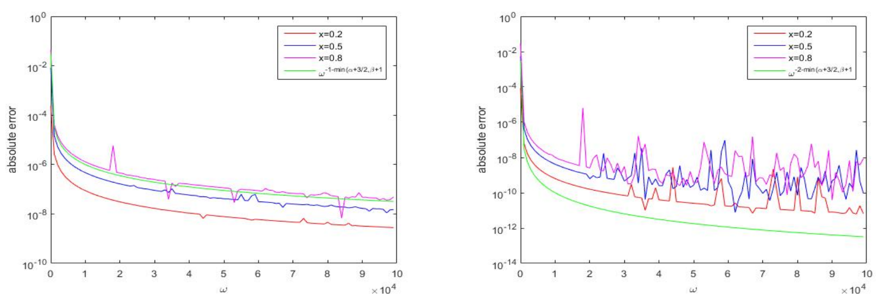

Figure 1. The error curves with ω at point $x = 0.2, 0.5$ and 0.8 for Example 5.3 are solved by the direct high-order interpolation (DO) method (left) and the direct Hermite interpolation (DH) method (right).

$m = 0.5, \alpha = 3/2, \beta = -1/2, u(x) = e^x$

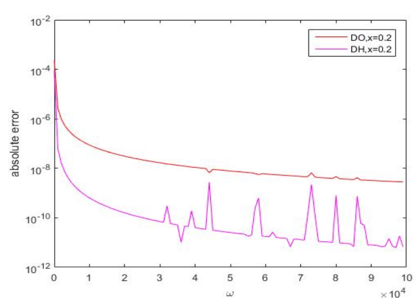

Figure 2. Comparison of the error at point $x = 0.2$ for solving Example 5.3 by using the direct high-order interpolation (DO) method and the direct Hermite interpolation (DH) method.

Table 4. Absolute error of the two collocation methods for solving Example 5.3 ($m = 0, \alpha = 2, \beta = -1/2$).

$\omega\backslash x$		0.2	0.4	0.6	0.8
10	DO	2.2021e-04	3.0671e-03	1.4568e-02	4.6287e-02
	DH	7.1429e-05	2.0607e-03	1.3485e-02	5.6109e-02
100	DO	2.8492e-05	1.3168e-04	3.6535e-04	8.3125e-04
	DH	2.2051e-06	1.8479e-05	7.0467e-05	1.9608e-04
1000	DO	7.5960e-07	3.6927e-06	1.0241e-05	2.2617e-05
	DH	1.8129e-08	1.5817e-07	5.8390e-07	1.5031e-06
10000	DO	2.3434e-08	1.1462e-07	3.2646e-07	6.9185e-07
	DH	1.7662e-10	1.5403e-09	1.5876e-08	1.5427e-08
100000	DO	1.3835e-09	5.0064e-09	1.2821e-08	2.1485e-08
	DH	6.4636e-10	1.4095e-09	2.9367e-09	2.6690e-13

$m = 0, \alpha = 2, \beta = -1/2, u(x) = e^x$

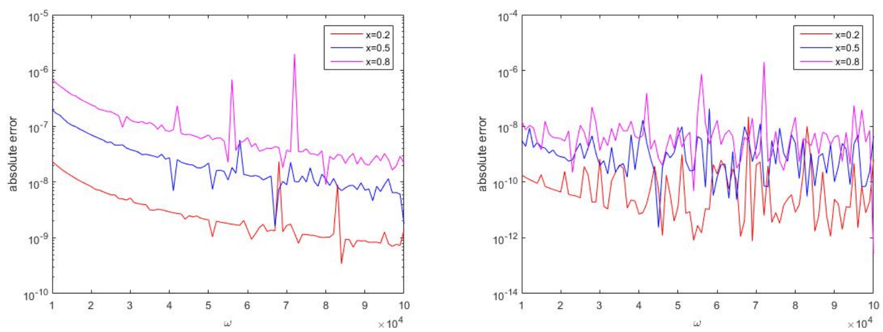

Figure 3. The error curves with ω at point $x = 0.2, 0.5$ and 0.8 for Example 5.3 are solved by the direct high-order interpolation (DO) method (left) and the direct Hermite interpolation (DH) method (right).

$m = 0, \alpha = 2, \beta = -1/2, u(x) = e^x$

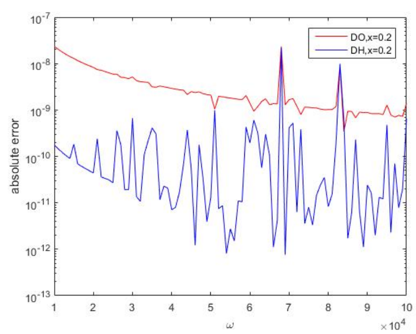

Figure 4. Comparison of the error at point $x = 0.2$ for solving Example 5.3 by using the direct high-order interpolation (DO) method and the direct Hermite interpolation (DH) method.

Table 5. Absolute error of the two collocation methods for solving Example 5.3 ($m = 1, \alpha = 2, \beta = -1/2$).

$\omega\backslash x$		0.2	0.4	0.6	0.8
10	DO	3.5970e-05	1.0715e-03	7.4853e-03	2.7667e-02
	DH	2.8131e-06	2.9724e-04	3.4400e-03	1.6011e-02
100	DO	4.1594e-05	2.3155e-04	6.7093e-04	1.5214e-03
	DH	1.7032e-06	1.6184e-05	6.1963e-05	1.6508e-04
1000	DO	1.5909e-06	7.8698e-06	2.1822e-05	4.7858e-05
	DH	1.7674e-08	1.5501e-07	5.6671e-07	1.4326e-06
10000	DO	5.1041e-08	2.4987e-07	6.8817e-07	1.5061e-06
	DH	1.7607e-10	1.5334e-09	5.5370e-09	2.1462e-08
100000	DO	1.6134e-09	7.8956e-09	2.3699e-08	4.7581e-08
	DH	8.9240e-13	1.2020e-11	2.0335e-09	5.1282e-10

$m = 1, \alpha = 2, \beta = -1/2, u(x) = e^x$

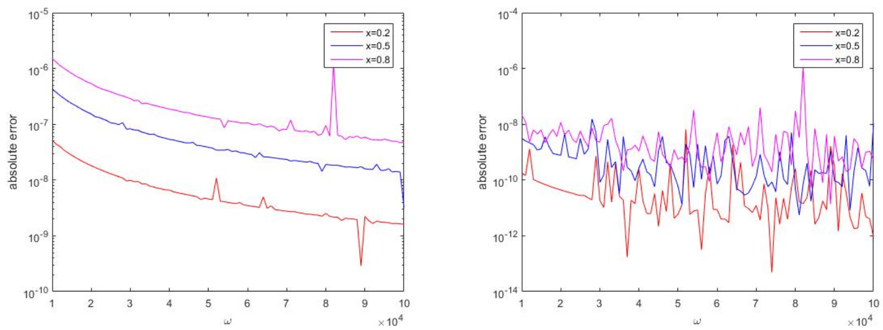

Figure 5. The error curves with ω at point $x = 0.2, 0.5$ and 0.8 for Example 5.3 are solved by the direct high-order interpolation (DO) method (left) and the direct Hermite interpolation (DH) method (right).

$m = 1, \alpha = 2, \beta = -1/2, u(x) = e^x$

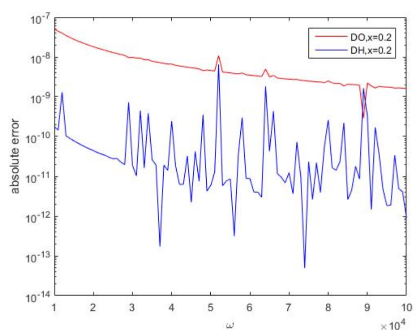

Figure 6. Comparison of the error at point $x = 0.2$ for solving Example 5.3 by using the direct high-order interpolation (DO) method and the direct Hermite interpolation (DH) method.

Example 5.4. We consider the following equation

$$u(x) - \int_0^x t^\alpha (x-t)^\beta J_m(\omega(x-t)) u(t) dt = f(x),$$

when $f(x) = \sin(x) - \int_0^x t^\alpha (x-t)^\beta J_m(\omega(x-t)) \sin(t) dt$, then the exact solution of this equation is $u(x) = \sin(x)$.

Table 6. Absolute error of the two collocation methods for solving Example 5.4 ($m = 0.5, \alpha = 3/2, \beta = -1/2$).

$\omega \backslash x$		0.2	0.4	0.6	0.8
10	DO	1.5490e-04	1.6807e-03	6.0735e-03	1.2568e-02
	DH	1.3980e-05	5.6850e-04	3.3575e-03	9.1942e-03
100	DO	6.8146e-05	1.9016e-04	3.2337e-04	4.3537e-04
	DH	1.1965e-06	8.0778e-06	2.2735e-05	4.3195e-05
1000	DO	2.2188e-06	5.9381e-06	9.8592e-06	1.2945e-05
	DH	1.0570e-08	7.0734e-08	1.9195e-07	3.4452e-07
10000	DO	7.0068e-08	1.8649e-07	3.0752e-07	4.1083e-07
	DH	1.5147e-10	6.7481e-10	1.5221e-09	3.7438e-09
100000	DO	2.2349e-09	5.8742e-09	9.9666e-09	1.2225e-08
	DH	2.3333e-11	4.0039e-12	2.8880e-10	3.5677e-10

$m = 0.5, \alpha = 3/2, \beta = -1/2, u(x) = \sin(x)$

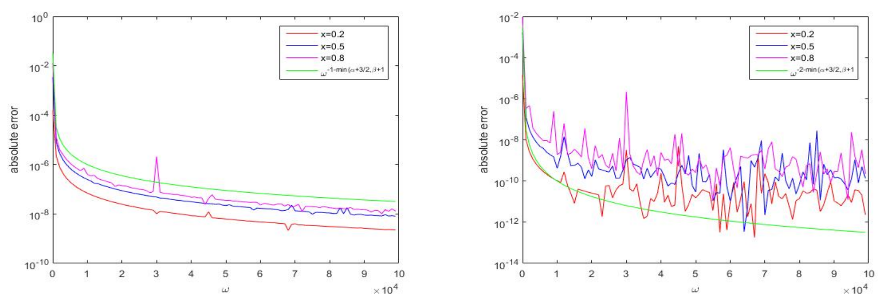

Figure 7. The error curves with ω at point $x = 0.2, 0.5$ and 0.8 for Example 5.4 are solved by the direct high-order interpolation (DO) method (left) and the direct Hermite interpolation (DH) method (right).

$m = 0.5, \alpha = 3/2, \beta = -1/2, u(x) = \sin(x)$

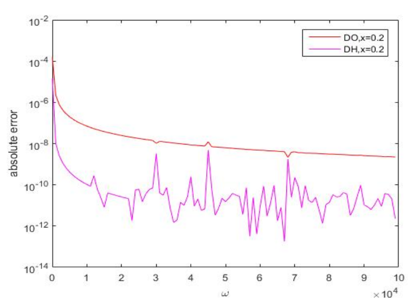

Figure 8. Comparison of the error at point $x = 0.2$ for solving Example 5.4 by using the direct high-order interpolation (DO) method and the direct Hermite interpolation (DH) method.

Table 7. Absolute error of the two collocation methods for solving Example 5.4 ($m = 0, \alpha = 2, \beta = -1/2$).

$\omega \backslash x$		0.2	0.4	0.6	0.8
10	DO	1.6011e-04	1.6748e-03	6.1170e-03	1.4424e-02
	DH	1.1620e-05	5.3803e-04	4.1800e-03	1.8088e-02
100	DO	2.2993e-05	8.1987e-05	1.6714e-04	2.6327e-04
	DH	3.5865e-07	4.8233e-06	2.1834e-05	6.3198e-05
1000	DO	6.1014e-07	2.2824e-06	4.6440e-06	7.0898e-06
	DH	2.9488e-09	4.1285e-08	1.8093e-07	4.8447e-07
10000	DO	1.8921e-08	7.0759e-08	1.4333e-07	2.1686e-07
	DH	1.4384e-10	3.8557e-10	1.8275e-09	5.2115e-09
100000	DO	4.5856e-10	2.4730e-09	3.4994e-09	7.4287e-09
	DH	1.3403e-10	2.4677e-10	9.8570e-10	7.0416e-10

$m = 0, \alpha = 2, \beta = -1/2, u(x) = \sin(x)$

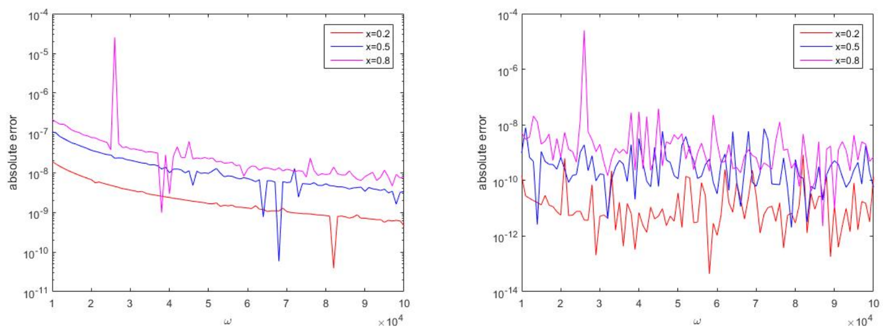

Figure 9. The error curves with ω at point $s = 0.2, 0.5$ and 0.8 for Example 5.4 are solved by the direct high-order interpolation (DO) method (left) and the direct Hermite interpolation (DH) method (right).

$m = 0, \alpha = 2, \beta = -1/2, u(x) = \sin(x)$

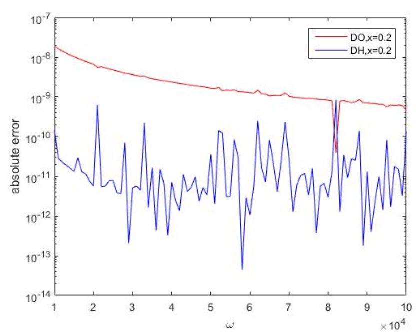

Figure 10. Comparison of the error at point $x = 0.2$ for solving Example 5.4 by using the direct high-order interpolation (DO) method and the direct Hermite interpolation (DH) method.

Table 8. Absolute error of the two collocation methods for solving Example 5.4 ($m = 1, \alpha = 2, \beta = -1/2$).

$\omega\backslash x$		0.2	0.4	0.6	0.8
10	DO	2.2199e-05	4.5756e-04	2.3779e-03	6.4867e-03
	DH	4.5804e-07	7.7688e-05	1.0674e-03	5.1660e-03
100	DO	3.3023e-05	1.4170e-04	3.0127e-04	4.7181e-04
	DH	2.7708e-07	4.2254e-06	1.9206e-05	5.3221e-05
1000	DO	1.2756e-06	4.8554e-06	9.8767e-06	1.4969e-05
	DH	2.8749e-09	4.0464e-08	1.7563e-07	4.6181e-07
10000	DO	4.0952e-08	1.5428e-07	3.1160e-07	4.6894e-07
	DH	2.7593e-11	4.2685e-10	1.6321e-09	4.3299e-09
100000	DO	1.2959e-09	4.8539e-09	9.9020e-09	1.4843e-08
	DH	5.1514e-13	1.8895e-11	8.0537e-11	1.1000e-10

$$m = 1, \alpha = 2, \beta = -1/2, u(x) = \sin(x)$$

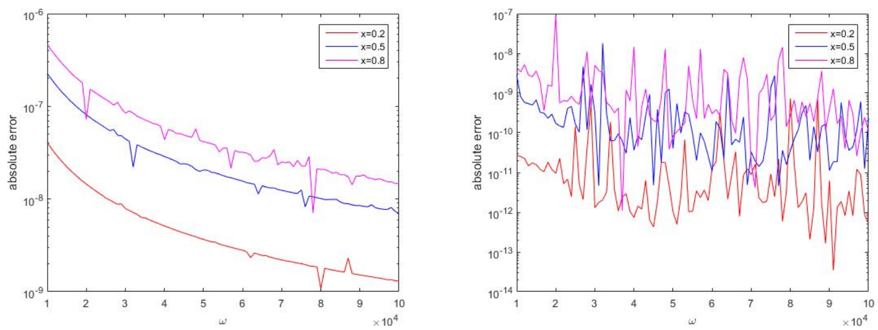

Figure 11. The error curves with ω at point $s = 0.2, 0.5$ and 0.8 for Example 5.4 are solved by the direct high-order interpolation (DO) method (left) and the direct Hermite interpolation (DH) method (right).

$$m = 1, \alpha = 2, \beta = -1/2, u(x) = \sin(x)$$

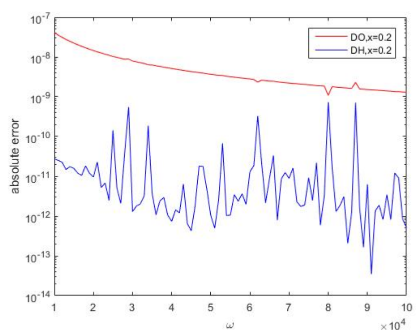

Figure 12. Comparison of the error at point $x = 0.2$ for solving Example 5.4 by using the direct high-order interpolation (DO) method and the direct Hermite interpolation (DH) method.

From the above examples, it is easy to see that efficiency of our methods for solving the second kind of Volterra integral equations with weakly singular highly oscillatory Bessel kernels. And from the Examples 5.1 and 5.2, we conclude that our method, the direct high-order interpolation (DO) method, is superior to the Filon-type (Q_N^F) method, the piecewise constant collocation ($Q_N^{L,0}$) method and the linear collocation ($Q_N^{L,1}$) method in the Ref. [21].

6. Conclusion

We focus on the second kind of Volterra integral equations with the weakly singular highly oscillatory Bessel kernel. For this type of equations, we propose two collocation methods: direct high-order interpolation and direct Hermite interpolation, based on the solution of the modified moments $\int_0^{x_j} t^\alpha (x_j - t)^\beta J_m(\omega(x_j - t))dt$. From the convergence analysis and numerical experiments, it is easy to see that the methods we propose are very efficient for solving weakly singular highly oscillatory problems. However, we only consider the kernel of the Bessel transform to be highly oscillatory combined with weakly singular. Future research should consider other highly oscillatory equations containing different weakly singular forms and try to apply different methods to solve the highly oscillatory weakly singular equations to achieve improved accuracy of the approximate solutions.

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