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THE NON-EXISTENCE AND EXISTENCE OF NON-CONSTANT POSITIVE SOLUTIONS FOR A DIFFUSIVE AUTOCATALYSIS MODEL WITH SATURATION

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Abstract This paper deals with a diffusive autocatalysis model with saturation under Neumann boundary conditions. Firstly, some stability and Turing instability results are obtained. Then by the maximum principle, Hölder inequality and Poincaré inequality, a priori estimates and some basic characterizations of non-constant positive solutions are given. Moreover, some non-existence results are presented for three different situations. In particular, we find that the model does not have any non-constant positive solution when the parameter which represents the saturation rate is large enough. In addition, we use the theories of Leray-Schauder degree and bifurcation to get the existence of non-constant positive solutions, respectively. The steady-state bifurcations at both simple and double eigenvalues are intensively studied and we establish some specific condition to determine the bifurcation direction. Finally, a few of numerical simulations are provided to illustrate theoretical results.

Keywords Autocatalysis, saturation, Turing instability, non-constant positive steady-state solution, bifurcation.

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1. Introduction

A chemical reaction is said to have undergone autocatalysis if the reaction products have an accelerating effect on the reaction rate [34]. In the past decades, autocatalytic models have received extensive attentions in the study of morphogenesis, population dynamics and autocatalytic oxidation reactions [16, 30].

Assume that the initial concentration of reactants remains unchanged and the reaction rates are the same, then an autocatalysis model with arbitrary order can

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be presented in the following non-dimensional form

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - d_1\Delta u(x,t) = a - u(x,t)v^p(x,t), & x \in \Omega, t > 0, \\ \dfrac{\partial v(x,t)}{\partial t} - d_2\Delta v(x,t) = u(x,t)v^p(x,t) - v(x,t), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, \not\equiv 0,\ v(x,0) = v_0(x) \geq 0, \not\equiv 0,\ x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

in which Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary and ν is outward unit normal vector on $\partial\Omega$. The variables $u(x,t)$ and $v(x,t)$ respectively represent the dimensionless concentrations of reactants and autocatalyst, which are generally considered to be non-negative. The parameter a is the initial concentration of the reaction precursor, p represents the reaction order of autocatalytic species, d_1, d_2 are the diffusion coefficients, a, p, d_1 and d_2 are all positive. Here, we refer to [8,26] and the references therein for a more detailed description on the derivation of this model.

Taking a as a bifurcation parameter, the Hopf bifurcation and the steady-state bifurcation of system (1.1) were studied in [11], including the steady-state bifurcation at double eigenvalues and the techniques of space decomposition and the implicit function theorem were adopted to deal with the case of double eigenvalues. However, the direction of steady-state bifurcation was not mentioned in [11]. Without loss of generality, Guo et al. chose $p = 7$ and carried out a detailed steady-state bifurcation analysis for (1.1), some specific conditions to determine the direction of steady-state bifurcation given in [12]. The stability of the steady-state bifurcation solutions of (1.1) was investigated in [45]. The non-existence and existence of positive steady-state solutions for (1.1) with $p > 2$ were discussed in [17], and there was also obtained the steady-state bifurcation arising from the unique positive constant equilibria. The stability and pattern formations in a two-cell coupled autocatalysis system with arbitrary order were studied in [44], where Turing bifurcation solutions were obtained by weakly nonlinear theory. A general reaction-diffusion system modelling glycolysis was investigated in [46], where the parameter regions for the stability and instability of the unique constant steady-state solution were derived, and the existence of time-periodic orbits and non-constant steady-state solutions was proved by the bifurcation method and Leray-Schauder degree theory.

Biological and chemical applications of model equations often involve the effect of saturation laws. So in this paper, we mainly deal with the autocatalysis model (1.1) with saturation effects. Recall the reaction process proposed by Engelhardt [7],

$$\begin{aligned} &X + Y \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} XY, \quad XY \xrightarrow{k_2} P + X \quad \text{(autocatalysis)}, \\ &X + Y \xrightarrow{S(k_3,k_4)} P + X \quad \text{(saturation law)}, \end{aligned}$$

in which one substrate X reacts with an enzyme Y forming a complex XY through a reversible process, which then is converted into a product P plus the enzyme, and k_1, k_{-1}, k_2 are reaction rates. It is assumed that the concentrations of P is independent of time and spatial variables. Here, $S(k_3, k_4)$ accounts for the Michaelis-Menten law in enzyme-controlled processes, or the Langmuir-Hinshelwood law in heterogeneous catalysis and adsorption, the Holling law in ecology.

Engelhardt [7] not only used a balanced reaction equation to express the stoichiometric relationship between reactants and products, but also proposed a sel'kov model with saturation effects,

$$\begin{cases} \dfrac{\partial x}{\partial t} - D_1 \nabla^2 x = \nu_1 - \dfrac{k_1 x y^\gamma}{1 + K_1 y^\gamma}, \\ \dfrac{\partial y}{\partial t} - D_2 \nabla^2 y = \dfrac{k_1 x y^\gamma}{1 + K_1 y^\gamma} - k_2 y, \end{cases}$$

where x and y represent two different concentrations of either chemical species or morphogenes in a reaction-diffusion model, D_1 and D_2 represent diffusive coefficients, ν_1 is a constant uniform rate, $\gamma > 1$ is the Hill coefficient, and K_1 is the saturation coefficient. For the model above, Hopf bifurcation was considered, but the effect of the saturation coefficient K_1 on the existence and nonexistence of non-constant positive solutions was ignored. One can also seen [15] for details. For the Sel'kov model with saturation effects, Du et al. [6] studied the existence and non-existence conditions of non-constant positive solutions, and Wang and Gao [36] derived a formula in terms of the diffusion rates to determine the Turing instability of the spatially homogeneous Hopf bifurcating periodic solutions. The works on the bimolecular model with saturation can be seen in [25, 29, 39, 40, 42], where detailed qualitative analyses were carried out, including the non-existence and existence of non-constant positive solutions, Hopf bifurcation and steady-state bifurcation, and many pattern formation dynamics were presented. In addition, there are many works on other autocatalysis models. For example, see [2, 4, 24, 41, 43] for the Lengyel-Epstein model, see [14, 21, 23, 27] for the Brusselator model, see [1, 5, 18, 37] for the Degn-Harrison model.

Motivated by above works, we consider the following diffusive autocatalysis model with saturation

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = a - \dfrac{u v^p}{1 + k v^p}, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial v}{\partial t} - d_2 \Delta v = \dfrac{u v^p}{1 + k v^p} - v, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega, \ t > 0, \end{cases} \tag{1.2}$$

where $k > 0$ represents the saturation coefficient, and the initial conditions are nonnegative and not idential to 0. The steady-state problem corresponding to (1.2) is given by

$$\begin{cases} -d_1 \Delta u = a - \dfrac{u v^p}{1 + k v^p}, \ x \in \Omega, \\ -d_2 \Delta v = \dfrac{u v^p}{1 + k v^p} - v, \ x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega. \end{cases} \tag{1.3}$$

System (1.3) has a unique positive equilibrium $E^* := (u^*, v^*) = (a^{1-p} + ka, a)$.

In this paper, we shall focus on the non-existence and existence of non-constant positive solutions for system (1.3). We first discuss the stability and Turing instability of the equilibrium E^* and then a priori estimates and some related properties of non-constant positive solutions for system (1.3) are established. The effects of the diffusion and saturation are intensively investigated and three non-existence results are given. Moreover, we derive the existence of non-constant positive solutions based on the Leray-Schauder degree. Using d_1 as a bifurcation parameter, we obtain the local steady-state bifurcation at simple and double eigenvalues, respectively. In particular, we extend the local bifurcation to the global one and present a formula to determine the local bifurcation direction.

It must be pointed out that the steady-state bifurcation at double eigenvalues is difficult to tackle because the classic Crandall-Rabinowitz theorem does not work and so we need to propose an effective method in this case. Here, we fortunately use the techniques of space decomposition and implicit function theorem to solve this problem, ever if only for two special cases. In addition, the diffusion and saturation effects are fully taken into accounts in our arguments, and we find out that there is no non-constant positive solution of (1.3) when d_1 is small, d_2 is large or k is large.

The outline of this paper is arranged as follows. In Section 2, the stability and Turing instability of the unique positive equilibrium are discussed. In Section 3, a priori estimates and some basic properties of positive solutions are given. In Section 4, the non-existence and existence of non-constant positive solutions are established from different perspectives. The fixed-point index theory in Banach space are used in this section. In Section 5, taking d_1 as a bifurcation parameter, a detailed steady-state bifurcation analysis is carried out, where the local bifurcation, the global one and the direction of local bifurcation are involved. Some numerical simulations are given to illustrate some theoretical results in Section 6.

2. The stability and Turing instability

In this section, we discuss the saturation effect on the stability of the unique positive equilibrium and the effect of diffusion coefficients on Turing instability is also given. An equilibrium point of the reaction-diffusion system is said to be Turing unstable if it is stable in the absence of diffusion and it becomes unstable for the diffusive system.

Assume $p > 1$ throughout the whole arguments, since E^* is always locally asymptotically stable if $0 < p \leq 1$. The local system corresponding to (1.2) which is an ordinary differential equation takes in the following form

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = a - \dfrac{uv^p}{1+kv^p},\ t > 0, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{uv^p}{1+kv^p} - v,\ t > 0. \end{cases} \tag{2.1}$$

The Jacobian matrix of (2.1) at E^* is

$$J = \begin{pmatrix} -\dfrac{a^p}{1+ka^p} & -\dfrac{p}{1+ka^p} \\ \dfrac{a^p}{1+ka^p} & \dfrac{p}{1+ka^p} - 1 \end{pmatrix}.$$

The characteristic equation can be given by $\mu^2 - T\mu + D = 0$, where

$$T = \frac{p-1-(k+1)a^p}{1+ka^p}, \qquad D = \frac{a^p}{1+ka^p} > 0.$$

As we know, E^* is locally asymptotically stable if $T < 0$ and $D > 0$, and it is unstable if $T > 0$ or $D < 0$. It is easy to see E^* is locally asymptotically stable if $p > 1$ and $a \geq \sqrt[p]{p-1}$.

Let $k_0 = \dfrac{p-1-a^p}{a^p}$. Simple analysis leads to the following stability results.

Theorem 2.1. *Assume that $p > 1$.*

(i) *If $a \geq \sqrt[p]{p-1}$, the equilibrium E^* is asymptotically stable for system* (2.1)*;*

(ii) *If $0 < a < \sqrt[p]{p-1}$, the equilibrium E^* is asymptotically stable for system* (2.1) *when $k > k_0$ and is unstable when $0 < k < k_0$.*

Now we focus on the stability of E^* for system (1.2). Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots$ be the eigenvalues of the operator $-\Delta$ subject to the Neumman boundary condition on Ω, where λ_i has multiplicity $m_i \geq 1$. Set ϕ_{ij} $(1 \leq j \leq m_i)$ be the normalized eigenfunctions corresponding to λ_i and $\phi_i := \phi_{i1}$. Then $\{\phi_{ij}\}(i \geq 0, 1 \leq j \leq m_i)$ forms a complete orthogonal basis in $L^2(\Omega)$.

The linearization operator of (1.2) at E^* is

$$L = \begin{pmatrix} d_1\Delta - \dfrac{a^p}{1+ka^p} & -\dfrac{p}{1+ka^p} \\ \dfrac{a^p}{1+ka^p} & d_2\Delta + \dfrac{p}{1+ka^p} - 1 \end{pmatrix}. \tag{2.2}$$

Then, the characteristic equation of (2.2) can be denoted by

$$\mu^2 - T_n(k)\mu + D_n(k) = 0, \tag{2.3}$$

where

$$\begin{aligned} T_n(k) &= T - (d_1 + d_2)\lambda_n, \\ D_n(k) &= d_1 d_2 \lambda_n^2 + \frac{1}{1+ka^p}[(1-p+ka^p)d_1 + a^p d_2]\lambda_n + \frac{a^p}{1+ka^p}. \end{aligned}$$

If $p > 1$ and $k \geq k_0 + 1$, we have $T_n \leq T < 0, D_n > 0$ for $n \in \mathbb{N}_0$, and thus the equilibrium E^* is locally asymptotically stable.

Assume that $p > 1$ and $\max\{0, k_0\} < k < k_0 + 1$. Then we have $T_n \leq T < 0$ for all $n \in \mathbb{N}_0$. If $d_2/d_1 \geq k_0 + 1 - k$, then we have $D_n > 0$ and thus the equilibrium E^* is locally asymptotically stable.

Next we consider the stability of E^* if $d_2/d_1 < k_0 + 1 - k$. Let

$$\begin{aligned} \Delta &= \left[\frac{(1-p+ka^p)d_1 + a^p d_2}{1+ka^p}\right]^2 - \frac{4a^p d_1 d_2}{1+ka^p} \\ &= \frac{a^{2p}d_2^2 - 2a^p(p+1+ka^p)d_1 d_2 + (1-p+ka^p)^2 d_1^2}{(1+ka^p)^2}. \end{aligned}$$

Define the quadratic function

$$h(z) = a^{2p}z^2 - 2a^p(p+1+ka^p)z + (1-p+ka^p)^2.$$

The discriminant of $h(z)$ is

$$\widetilde{\Delta} = [2a^p(p+1+ka^p)]^2 - 4a^{2p}(1-p+ka^p)^2 = 16p(1+ka^p)a^{2p} > 0.$$

Hence, the equation $h(z) = 0$ has two different real positive roots

$$z_1 = \frac{1 + p + ka^p - 2\sqrt{p(1 + ka^p)}}{a^p}, \quad z_2 = \frac{1 + p + ka^p + 2\sqrt{p(1 + ka^p)}}{a^p}. \tag{2.4}$$

If $z_1 < z < z_2$, then $h(z) < 0$ and we have $D_n(k) > 0$ for all $n \in \mathbb{N}_0$. Note that $z_1 < d_2/d_1 < k_0 + 1 - k < z_2$ and E^* is stable when $d_2/d_1 \geq k_0 + 1 - k$. So we know that the equilibrium E^* is stable when $d_2/d_1 > z_1$.

Theorem 2.2. *Assume that $p > 1$.*

(i) *If $k \geq k_0 + 1$, the equilibrium E^* is asymptotically stable for system* (1.2)*;*

(ii) *If $\max\{0, k_0\} < k < k_0 + 1$, the equilibrium E^* is asymptotically stable for system* (1.2) *when $d_2/d_1 > z_1$, where z_1 is given by* (2.4).

Remark 2.1. The expression of k_0 tells that if $0 < a < \sqrt[p]{p-1}$, then $k_0 > 0$ and if $a \geq \sqrt[p]{p-1}$, then $k_0 \leq 0$, that is

$$\max\{0, k_0\} = \begin{cases} k_0, & \text{if } \ 0 < a < \sqrt[p]{p-1}, \\ 0, & \text{if } \ a \geq \sqrt[p]{p-1}. \end{cases}$$

Now we discuss the stability of the equilibrium E^* when $0 < d_2/d_1 < z_1$. In this case, $\Delta > 0$ and $D_n(k) = 0$ has two real positive roots

$$\lambda_+(d_1, d_2) = \frac{R + \sqrt{R^2 - 4d_1 d_2 S}}{2(1 + ka^p)d_1 d_2}, \qquad \lambda_-(d_1, d_2) = \frac{R - \sqrt{R^2 - 4d_1 d_2 S}}{2(1 + ka^p)d_1 d_2}, \tag{2.5}$$

where

$$R = d_2 A + d_1 N = (p - 1 - ka^p)d_1 - a^p d_2, \qquad S = a^p(1 + ka^p) > 0.$$

Define the function

$$K(d_1) = \frac{d_2}{d_1}A + N + \sqrt{\left(\frac{d_2}{d_1}A + N\right)^2 - \frac{4d_2 S}{d_1}}.$$

Then

$$K'(d_1) = -\left[\frac{d_2}{d_1^2}A + \frac{\left(\frac{d_2}{d_1}A + N\right)\frac{d_2}{d_1^2}A - \frac{2d_2 S}{d_1^2}}{\sqrt{\left(\frac{d_2}{d_1}A + N\right)^2 - \frac{4d_2 S}{d_1}}}\right].$$

Recall that $A < 0$ and $0 < d_2/d_1 < k_0+1-k$. Then we have $R = d_2 A + d_1 N > 0$ and $K'(d_1) > 0$. Therefore, λ_+ is strictly monotonically increasing with respect to d_1. On the other hand, we have $\lambda_+(d_1, d_2)\lambda_-(d_1, d_2) = \dfrac{a^p}{(1 + ka^p)d_1 d_2}$. Differentiating with respect to d_1, we get

$$\lambda_+'(d_1, d_2)\lambda_-(d_1, d_2) + \lambda_+(d_1, d_2)\lambda_-'(d_1, d_2) = -\frac{a^p}{(1 + ka^p)d_1^2 d_2} < 0.$$

Since $\lambda_+'(d_1, d_2) > 0, \lambda_+(d_1, d_2) > 0$ and $\lambda_-(d_1, d_2) > 0$, we obtain $\lambda_-'(d_1, d_2) < 0$. Therefore, λ_- is strictly monotonically decreasing with respect to d_1.

Denote

$$\Phi_1 = \{\lambda | \lambda > 0, \lambda_-(d_1, d_2) < \lambda < \lambda_+(d_1, d_2)\}, \qquad \Phi_2 = \{\lambda_0, \lambda_1, \lambda_2, \cdots\}.$$

If there exists $n_0 \in \mathbb{N}$ such that $\lambda_-(d_1, d_2) < \lambda_{n_0} < \lambda_+(d_1, d_2)$, then $\Phi_1 \cap \Phi_2 \neq \emptyset$ and the equilibrium E^* is unstable. So we focus on the case of large d_1 or small d_2, the inequality $0 < d_2/d_1 < z_1$ hold true.

Fix d_2 and let $d_1 \to \infty$ to get

$$\lim_{d_1 \to \infty} \lambda_-(d_1, d_2) = 0, \qquad \lim_{d_1 \to \infty} \lambda_+(d_1, d_2) = \frac{p - 1 - ka^p}{d_2(1 + ka^p)} = \lambda^*. \tag{2.6}$$

If $\lambda_1 \geq \lambda^*$, then $\Phi_1 \cap \Phi_2 = \emptyset$ and we have $D_n(k) > 0$ for all $n \in \mathbb{N}_0$. On the other hand, if $\lambda_1 < \lambda^*$, then $\Phi_1 \cap \Phi_2 \neq \emptyset$ and the equilibrium E^* is unstable.

Theorem 2.3. *Assume that $p > 1$ and $\max\{0, k_0\} < k < k_0 + 1$. Then there exists large $D_1 > 0$ such that if $d_1 > D_1$, the equilibrium E^* is asymptotically stable for system* (1.2) *when $\lambda_1 \geq \lambda^*$ and is Turing unstable when $\lambda_1 < \lambda^*$.*

If fix d_1 and let $d_2 \to 0$, then

$$\lim_{d_2 \to 0} \lambda_-(d_1, d_2) = \frac{a^p}{d_1(p - 1 - ka^p)}, \qquad \lim_{d_2 \to 0} \lambda_+(d_1, d_2) = \infty.$$

Therefore, there exists small $\delta > 0$ such that $\Phi_1 \cap \Phi_2 \neq \emptyset$ for $0 < d_2 < \delta$, which shows that the equilibrium E^* is unstable.

Theorem 2.4. *Assume that $p > 1$ and $\max\{0, k_0\} < k < k_0 + 1$. Then there exists small $\delta > 0$ such that the equilibrium E^* is Turing unstable for system* (1.2) *if $0 < d_2 < \delta$.*

The positive equilibrium E^* is stable when k is large. The ratio of diffusion coefficient d_2/d_1 affect the stability of the equilibrium E^* when k is in a certain range. The equilibrium E^* is still stable when d_2/d_1 is properly large. The equilibrium E^* may be stable or Turing instability when d_2/d_1 is properly small.

3. A priori estimates and some characters

In this section, we shall use the maximum principle, Poincaré inequality and Hölder inequality to obtain a priori estimates and some properties of positive solutions of (1.3). Start with two useful lemmas.

Lemma 3.1. (see [22]) *Suppose that $g \in C\left(\overline{\Omega} \times \mathbb{R}^1\right)$.*

(i) *Assume that $w(x) \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ and satisfies*

$$\Delta w(x) + g(x, w(x)) \geq 0, \quad x \in \Omega, \quad \partial_\nu w \leq 0, \quad x \in \partial\Omega.$$

If $w(x_0) = \max_{\overline{\Omega}} w(x)$, then $g(x_0, w(x_0)) \geq 0$;

(ii) *Assume that $w(x) \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ and satisfies*

$$\Delta w(x) + g(x, w(x)) \leq 0, \quad x \in \Omega, \quad \partial_\nu w \leq 0, \quad x \in \partial\Omega.$$

If $w(x_0) = \min_{\overline{\Omega}} w(x)$*, then* $g(x_0, w(x_0)) \leq 0$.

Lemma 3.2. (see [20]) *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^N$ *and* Λ *be a positive constant. Suppose that* $w \in W^{1,2}(\Omega)$ *is a nonnegative weak solution of the inequalities*

$$0 \leq -\Delta w(x) + \Lambda w(x), \quad x \in \Omega, \quad \partial_\nu w \leq 0, \quad x \in \partial\Omega.$$

For any $q \in [1, n/(n-2))$*, there is a constant* $C_0 = C_0(q, \Omega, \Lambda)$*, such that*

$$||w||_q \leq C_0 \inf_\Omega w.$$

Now we give a priori estimates for positive solutions of system (1.3).

Theorem 3.1. *Let* (u, v) *be any positive solution of (1.3). Then there exists a positive constant* $C_* = C_*(d_2, \Omega)$ *such that* (u, v) *satisfies*

$$ak \leq u \leq a(k + \frac{C_*^p}{a^p|\Omega|^p}), \qquad \frac{a|\Omega|}{C_*} \leq v \leq a(1 + \frac{C_*^p}{ka^p|\Omega|^p}).$$

Proof. Let $u(x_0) = \min_{\overline{\Omega}} u(x)$. From (ii) of Lemma 3.1, it follows that

$$a - \frac{u(x_0)v^p(x_0)}{1 + kv^p(x_0)} \leq 0,$$

i.e.,

$$u(x_0)v^p(x_0) \geq a(1 + kv^p(x_0)) \geq akv^p(x_0).$$

Thus we get

$$u(x) \geq u(x_0) \geq ak. \tag{3.1}$$

Integrate the first and second equations in (1.3) over Ω, respectively, to get

$$\int_\Omega \frac{uv^p}{1 + kv^p} dx = \int_\Omega v dx = a|\Omega|, \tag{3.2}$$

where $|\Omega|$ is the volume of Ω. Since v satisfies

$$-d_2\Delta v + v \geq 0, \quad x \in \Omega, \quad \partial_\nu v \leq 0, \quad x \in \partial\Omega,$$

by Lemma 3.2, there exists a positive constant $C_* = C_*(d_2, \Omega)$ such that

$$a|\Omega| = \int_\Omega v dx \leq C_* \inf_\Omega v,$$

which leads to

$$v(x) \geq \inf_\Omega v \geq \frac{a|\Omega|}{C_*}. \tag{3.3}$$

From (3.2), we have $|\Omega|/C_* \leq 1$.

Let $u(x_1) = \max_{\overline{\Omega}} u(x)$. From (i) of Lemma 3.1, it follows that

$$a - \frac{u(x_1)v^p(x_1)}{1 + kv^p(x_1)} \geq 0.$$

Thus, by the lower bound for v in (3.3), we have

$$u(x) \leq u(x_1) \leq a(k + \frac{1}{v^p(x_1)}) \leq a(k + \frac{C_*^p}{a^p|\Omega|^p}). \tag{3.4}$$

Let $v(x_2) = \max_{\overline{\Omega}} v(x)$. Form (i) of Lemma 3.1 and the upper bound for u in (3.4), we have

$$v(x) \leq v(x_2) \leq \frac{u(x_2)v^p(x_2)}{1 + kv^p(x_2)} \leq \frac{u(x_1)}{k} \leq a(1 + \frac{C_*^p}{ka^p|\Omega|^p}). \tag{3.5}$$

The proof is completed with (3.1), (3.3)-(3.5). □

Theorem 3.2. *Assume* $0 < p < 1$. *Then any positive solution of system* (1.3) *satisfies*

$$ak \leq u \leq a(k + \frac{1}{C_{**}^p}), \qquad C_{**} \leq v \leq a(1+\frac{1}{kC_{**}^p}),$$

where C_{**} *is the unique positive root of the equation* $1 + ks^p - aks^{p-1} = 0$.

Proof. Let $v(y_1) = \min_{\overline{\Omega}} v(x)$. From (ii) of Lemma 3.1, it follows that

$$\frac{u(y_1)v^p(y_1)}{1 + kv^p(y_1)} - v(y_1) \leq 0.$$

From Theorem 3.1, we know $u \geq ak$ and thus

$$1 + kv^p(y_1) \geq u(y_1)v^{p-1}(y_1) \geq akv^{p-1}(y_1). \tag{3.6}$$

The function $f(s) = 1 + ks^p - aks^{p-1}$ is increasing in $(0, +\infty)$ when $0 < p < 1$ and $k > 0$. Therefore, $f(s)$ has a unique zero point C_{**} satisfying $0 < C_{**} < a$.

By (3.6), we have $v(y_1) \geq C_{**}$. Similar to the proof in Theorem 3.1, it is easy to verify

$$u \leq a(k + \frac{1}{C_{**}^p}), \qquad v \leq a(1+\frac{1}{kC_{**}^p}).$$

The proof is completed. □

Remark 3.1. The unique zero point C_{**} mentioned in Theorem 3.2 satisfies $C_{**} \to a$ as $k \to \infty$.

Theorem 3.3. *Assume that* $p = 1$ *and* $ak > 1$. *Then any positive solution of system* (1.3) *satisfies*

$$ak \leq u \leq \frac{a^2k^2}{ak - 1}, \qquad \frac{ak - 1}{k} \leq v \leq \frac{a^2k}{ak - 1}.$$

Proof. Choose $f(s) = 1 + ks - ak$ in Theorem 3.2 or directly use Lemma 3.1 yields the estimates. □

Next we give more information on the characterization of positive solutions. Denote their averages over Ω by

$$\bar{u} = \frac{1}{|\Omega|}\int_{\Omega} u\,(x)\,dx, \quad \bar{v} = \frac{1}{|\Omega|}\int_{\Omega} v\,(x)\,dx.$$

It follows from (3.2) that $\bar{v} = a$. Let $\phi = u - \bar{u},\ \psi = v - \bar{v}$. Then

$$\int_\Omega \phi dx = \int_\Omega \psi dx = 0.$$

If (u, v) is a non-constant solution, then ϕ and ψ are non-trivial and change signs in Ω. However, the following result shows $\phi\psi$ and $\nabla\phi \cdot \nabla\psi$ have a negative average over Ω, respectively.

Theorem 3.4. *Let* (u, v) *be a non-constant positive solution of* (1.3). *Then*

$$\int_\Omega \phi\psi dx < 0 \quad \textit{and} \quad \int_\Omega \nabla\phi \cdot \nabla\psi dx < 0.$$

Proof. Let $\omega = d_1 u + d_2 v$. Then ω satisfies

$$-\Delta\omega = a - v = \bar{v} - v = -\psi. \tag{3.7}$$

Multiplying (3.7) by $\omega = d_1 u + d_2 v$ and then integrating Ω by parts, we have

$$\int_\Omega |\nabla\omega|^2 dx = -\int_\Omega \omega\psi dx = -d_1 \int_\Omega \phi\psi dx - d_2 \int_\Omega \psi^2 dx. \tag{3.8}$$

This implies that

$$\int_\Omega \phi\psi dx = -\frac{1}{d_1}\left(\int_\Omega |\nabla\omega|^2 dx + d_2 \int_\Omega \psi^2 dx\right) < 0. \tag{3.9}$$

Multiplying (3.7) by ψ and then integrating by parts, we have

$$\begin{aligned}
-\int_\Omega \psi^2 dx &= -\int_\Omega \psi\Delta\omega dx \\
&= \int_\Omega \nabla\omega \cdot \nabla\psi dx \\
&= \int_\Omega (d_1\nabla u + d_2\nabla v) \cdot \nabla\psi dx \\
&= d_1 \int_\Omega \nabla\phi \cdot \nabla\psi dx + d_2 \int_\Omega |\nabla\psi|^2 dx.
\end{aligned}$$

Hence, we obtain

$$\int_\Omega \nabla\phi \cdot \nabla\psi dx = -\frac{1}{d_1}\left(d_2 \int_\Omega |\nabla\psi|^2 dx + \int_\Omega \psi^2 dx\right) < 0. \tag{3.10}$$

The proof is completed. □

According to Theorem 3.1, there are positive constants C_1 and C_2, depending on a, k, p, d_1, d_2 and Ω, such that

$$|a - \frac{uv^p}{1 + kv^p}| \le C_1, \qquad |\frac{uv^p}{1 + kv^p} - v| \le C_2.$$

Theorem 3.5. *Let* (u, v) *be a non-constant positive solution of (1.3). Then there exists a positive constant* C_1 *such that*

$$\int_\Omega (|\nabla\phi|^2 + \phi^2) dx \le \frac{C_1^2 |\Omega| (1 + \lambda_1)}{d_1^2 \lambda_1^2}.$$

Proof. Multiply the first equation in (1.3) by ϕ and then integrate over Ω by parts to yield

$$d_1 \int_\Omega |\nabla\phi|^2 dx = \int_\Omega (a - \frac{uv^p}{1+kv^p})\phi dx \le C_1 \int_\Omega |\phi| dx. \tag{3.11}$$

Using the Hölder inequality

$$\int_\Omega |\phi| dx \le |\Omega|^{\frac{1}{2}} (\int_\Omega |\phi|^2 dx)^{\frac{1}{2}},$$

and the Poincaré inequality

$$\int_\Omega |\phi|^2 dx \le \lambda_1^{-1} \int_\Omega |\nabla\phi|^2 dx,$$

where λ_1 is the first positive eigenvalue of $-\Delta$ subject to the Neumman boundary condition, we get

$$d_1 \int_\Omega |\nabla\phi|^2 dx \le \frac{C_1|\Omega|^{\frac{1}{2}}}{\sqrt{\lambda_1}} \left(\int_\Omega |\nabla\phi|^2 dx\right)^{\frac{1}{2}}.$$

This shows that

$$\int_\Omega |\nabla\phi|^2 dx \le \frac{C_1^2|\Omega|}{d_1^2\lambda_1}, \qquad \int_\Omega \phi^2 dx \le \frac{C_1^2|\Omega|}{d_1^2\lambda_1^2}.$$

Therefore,

$$\int_\Omega (|\nabla\phi|^2 + \phi^2) dx \le \frac{C_1^2|\Omega|(1+\lambda_1)}{d_1^2\lambda_1^2}.$$

The proof is completed. □

Similarly, we have the follwing result.

Theorem 3.6. *Let (u, v) be a non-constant positive solution of (1.3). Then there exists a positive constant C_2 such that*

$$\int_\Omega (|\nabla\psi|^2 + \psi^2) dx \le \frac{C_2^2|\Omega|(1+\lambda_1)}{d_2^2\lambda_1^2}.$$

Theorem 3.7. *Let (u, v) be a non-constant positive solution of (1.3). Then*

$$d_2^2 \int_\Omega |\nabla v|^2 dx < d_1^2 \int_\Omega |\nabla u|^2 dx \le \frac{4(d_2^2\lambda_1^2 + d_2\lambda_1 + 1)}{3\lambda_1^2} \int_\Omega |\nabla v|^2 dx. \tag{3.12}$$

Proof. From (3.10), it follows that

$$\begin{aligned}
\int_\Omega |\nabla\omega|^2 dx &= d_1^2 \int_\Omega |\nabla\phi|^2 dx + 2d_1d_2 \int_\Omega \nabla\phi \cdot \nabla\psi dx + d_2^2 \int_\Omega |\nabla\psi|^2 dx \\
&= d_1^2 \int_\Omega |\nabla\phi|^2 dx - 2d_2(\int_\Omega \psi^2 dx + d_2 \int_\Omega |\nabla\psi|^2 dx) + d_2^2 \int_\Omega |\nabla\psi|^2 dx \\
&= d_1^2 \int_\Omega |\nabla\phi|^2 dx - d_2^2 \int_\Omega |\nabla\psi|^2 dx - 2d_2 \int_\Omega \psi^2 dx > 0,
\end{aligned}$$

which leads to

$$d_2^2 \int_\Omega |\nabla\psi|^2 dx < d_1^2 \int_\Omega |\nabla\phi|^2 dx. \tag{3.13}$$

The right inequality in (3.12) holds true. On the other hand, we have

$$d_1^2 \int_\Omega |\nabla\phi|^2 dx = \int_\Omega |\nabla\omega|^2 dx + d_2^2 \int_\Omega |\nabla\psi|^2 dx + 2d_2 \int_\Omega \psi^2 dx.$$

Combining (3.8) with the Poincaré inequality and Cauchy inequality, we obtain

$$\begin{aligned} d_1^2 \int_\Omega |\nabla\phi|^2 dx &= d_2^2 \int_\Omega |\nabla\psi|^2 dx + d_2 \int_\Omega \psi^2 dx - d_1 \int_\Omega \phi\psi dx \\ &\leq d_2^2 \int_\Omega |\nabla\psi|^2 dx + \frac{d_2}{\lambda_1} \int_\Omega |\nabla\psi|^2 dx + \frac{1}{\lambda_1} \int_\Omega \psi^2 dx + \frac{d_1^2\lambda_1}{4} \int_\Omega \phi^2 dx \\ &\leq \left(d_2^2 + \frac{d_2}{\lambda_1} + \frac{1}{\lambda_1^2}\right) \int_\Omega |\nabla\psi|^2 dx + \frac{d_1^2}{4} \int_\Omega |\nabla\phi|^2 dx. \end{aligned}$$

Hence, we get

$$\frac{3d_1^2}{4} \int_\Omega |\nabla u|^2 dx \leq \frac{d_2^2\lambda_1^2 + d_2\lambda_1 + 1}{\lambda_1^2} \int_\Omega |\nabla v|^2 dx. \tag{3.14}$$

In view of (3.13) and (3.14), the proof is completed. □

Theorem 3.8. *Let (u, v) be a non-constant positive solution of (1.3). Then*

$$\frac{3d_2^2\lambda_1^3}{4(d_2^2\lambda_1^2 + d_2\lambda_1 + 1)(\lambda_1 + 1)} < \frac{d_2^2 \int_\Omega (|\nabla\psi|^2 + \psi^2) dx}{d_1^2 \int_\Omega (|\nabla\phi|^2 + \phi^2) dx} < \frac{\lambda_1 + 1}{\lambda_1}.$$

Proof. By the Poincaré inequality, we have

$$\int_\Omega (|\nabla\phi|^2 + \phi^2) dx \leq \frac{\lambda_1 + 1}{\lambda_1} \int_\Omega |\nabla\phi|^2 dx.$$

From the right inequality in (3.12), it follows

$$\frac{d_2^2 \int_\Omega (|\nabla\psi|^2 + \psi^2) dx}{d_1^2 \int_\Omega (|\nabla\phi|^2 + \phi^2) dx} > \frac{\lambda_1}{\lambda_1 + 1} \cdot \frac{d_2^2 \int_\Omega |\nabla\psi|^2 dx}{d_1^2 \int_\Omega |\nabla\phi|^2 dx} \geq \frac{3d_2^2\lambda_1^3}{4(d_2^2\lambda_1^2 + d_2\lambda_1 + 1)(\lambda_1 + 1)}.$$

By the left inequality in (3.12), we obtain

$$\frac{d_2^2 \int_\Omega (|\nabla\psi|^2 + \psi^2) dx}{d_1^2 \int_\Omega (|\nabla\phi|^2 + \phi^2) dx} < \frac{\lambda_1 + 1}{\lambda_1} \cdot \frac{d_2^2 \int_\Omega |\nabla\psi|^2 dx}{d_1^2 \int_\Omega |\nabla\phi|^2 dx} < \frac{\lambda_1 + 1}{\lambda_1}.$$

The proof is completed. □

4. Non-constant positive solutions

In this section, we study the non-existence and existence of non-constant positive solutions for system (1.3).

4.1. Non-existence of non-constant positive solutions

In this subsection, we obtain sufficient conditions for the non-existence of non-constant positive solutions of (1.3) as the parameters d_1, d_2 and k are varied. Our analysis deals with the following three situations: small d_1, large d_2 and large k. Here we note that the non-existence results play a key role in the following arguments.

Considering Theorem 3.1, for simplicity and convenience, we denote

$$M_1 := M_1(a,k,p,d_2,\Omega) \text{ and } M_2 := M_2(a,k,p,d_2,\Omega)$$

by the upper bounds of u and v, respectively.

Theorem 4.1. *There exists a positive constant $D_1^* := D_1^*(\lambda_1, M_1, M_2)$ such that (1.3) does not admit a non-constant positive solution when $d_1 \le D_1^*$.*

Proof. Suppose on the contrary that (1.3) has a non-constant positive solution (u,v). Multiplying the first equation of (1.3) by ϕ and integrating over Ω by parts, we have

$$\begin{aligned}
d_1\int_\Omega |\nabla\phi|^2dx &= -\int_\Omega \frac{uv^p}{1+kv^p}\phi dx\\
&= -\int_\Omega \left[\frac{v^p(u-\bar u)}{1+kv^p} + \left(\frac{v^p}{1+kv^p} - \frac{\bar v^p}{1+k\bar v^p}\right)\bar u\right]\phi dx\\
&= -\int_\Omega \frac{v^p(u-\bar u)}{1+kv^p}\phi dx - \int_\Omega \frac{\bar u(v^p-\bar v^p)}{(1+kv^p)(1+k\bar v^p)}\phi dx\\
&= -\int_\Omega \frac{v^p}{1+kv^p}\phi^2 dx - \int_\Omega \frac{\bar u p\gamma^{p-1}}{(1+kv^p)(1+k\bar v^p)}\phi\psi dx,
\end{aligned}$$

where γ lies between v and $\bar v$. From Theorem 3.1, Cauchy inequality and Poincaré inequality, it follows that

$$\begin{aligned}
d_1\int_\Omega |\nabla\phi|^2dx &\le pM_1M_\gamma^{p-1}\int_\Omega|\phi||\psi|dx - M_\alpha\int_\Omega\phi^2dx\\
&\le \frac{p^2M_1^2M_\gamma^{2(p-1)}}{4M_\alpha}\int_\Omega\psi^2dx + M_\alpha\int_\Omega\phi^2dx - M_\alpha\int_\Omega\phi^2dx\\
&\le \frac{p^2M_1^2M_\gamma^{2(p-1)}}{4\lambda_1M_\alpha}\int_\Omega|\nabla\psi|^2dx,
\end{aligned}$$

where

$$M_\gamma = \begin{cases} M_2, & \text{if } p>1,\\ 1, & \text{if } p=1,\\ \alpha, & \text{if } 0<p<1,\end{cases} \qquad M_\alpha = \frac{\alpha^p}{1+k\alpha^p}, \qquad \alpha = \frac{aC_*}{|\Omega|}.$$

By Theorem 3.7, we get

$$\int_\Omega|\nabla\phi|^2dx < \frac{d_1}{D_1^*}\int_\Omega|\nabla\phi|^2dx, \quad \text{where} \quad D_1^* = \frac{4\lambda_1M_\alpha d_2^2}{p^2M_1^2M_\gamma^{2(p-1)}}. \tag{4.1}$$

Therefore, if $d_1 \le D_1^*$, the inequality (4.1) becomes no sense and thus the assumption is not ture. The proof is completed. $\square$

Theorem 4.2. *There exists a positive constant $D_2^* := D_2^*(\lambda_1, M_1, M_2)$ such that (1.3) does not admit a non-constant positive solution when $d_2 \ge D_2^*$.*

Proof. Suppose on the contrary that (1.3) has a non-constant positive solution (u, v). Multiplying the first equation of (1.3) by ψ and integrating over Ω by parts, we have

$$d_2 \int_\Omega |\nabla \psi|^2 dx = \int_\Omega \left[\frac{u(v^p - \bar{v}^p)}{(1+kv^p)(1+k\bar{v}^p)} + \frac{\bar{v}^p(u-\bar{u})}{1+k\bar{v}^p} - (v - \bar{v}) \right] \psi dx$$

$$= \int_\Omega \left[\frac{up\gamma^{p-1}\psi^2}{(1+kv^p)(1+k\bar{v}^p)} + \frac{\bar{v}^p \phi\psi}{1+k\bar{v}^p} - \psi^2 \right] dx.$$

Recall $\int_\Omega \phi\psi dx < 0$ in Theorem 3.4. By Theorem 3.1 and Poincaré inequality, we obtain

$$d_2 \int_\Omega |\nabla \psi|^2 dx < \int_\Omega \frac{p\gamma^{p-1}u\psi^2}{(1+kv^p)(1+k\bar{v}^p)} dx < \frac{pM_1 M_\gamma^{p-1}}{\lambda_1(1+ka^p)} \int_\Omega |\nabla \psi|^2 dx. \tag{4.2}$$

Let $D_2^* = \dfrac{pM_1 M_2^{p-1}}{\lambda_1(1+ka^p)}$. If $d_2 \ge D_2^*$, the inequality (4.2) becomes no sense, which shows there is no non-constant positive solution of (1.3). The proof is completed. $\square$

Next we show that when k is large enough, system (1.3) doesn't have a non-constant positive solution. It is a new phenomenon that the saturation law determines the formation of spatial patterns of system (1.3).

For convenience, we make a variable change $w = u/k$ and system (1.3) becomes

$$\begin{cases} -d_1 k \Delta w = a - \dfrac{kwv^p}{1+kv^p}, \ x \in \Omega, \\ -d_2 \Delta v = \dfrac{kwv^p}{1+kv^p} - v, \quad x \in \Omega, \\ \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega. \end{cases} \tag{4.3}$$

Clearly, system (4.3) has a unique positive constant solution $(w^*, v^*) = (a + a^{1-p}/k, a)$.

Lemma 4.1. *Assume that (w, v) is any positive solution of system (4.3). Then we have a priori estimates for (w, v).*

(i) *If $0 < p < 1$, then*

$$a \le w \le a(1 + \frac{1}{kC_{**}^p}), \qquad C_{**} \le v \le a(1+\frac{1}{kC_{**}^p}),$$

where $C_{**} > 0$ is a constant given in Theorem 3.2;

(ii) *If $p = 1$ and $ak > 1$, then*

$$a \le w \le \frac{a^2 k}{ak-1}, \qquad \frac{ak-1}{k} \le v \le \frac{a^2 k}{ak-1};$$

(iii) *If either* $p > 1$ *or* $p = 1$ *and* $0 < ak \leq 1$, *then*

$$a \leq w \leq a(1 + \frac{C_*^p}{ka^p|\Omega|^p}), \qquad \frac{a|\Omega|}{C_*} \leq v \leq a(1 + \frac{C_*^p}{ka^p|\Omega|^p}),$$

where $C_* > 0$ *is a constant given in Theorem* 3.1.

Proof. The proof can be obtained in the same way as in Theorems 3.1-3.3. $\square$

Lemma 4.2. *Fix* $d_1, d_2 > 0$ *and assume that* (w_k, v_k) *is a positive solution of system* (4.3)*. Then* $(w_k, v_k) \to (a, a)$ *in* $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ *as* $k \to \infty$.

Proof. From the embedding theory and standard elliptic theorems using the equations in (4.3) and Lemma 4.1, we know that there exists a sequence k_i with $k_i \to \infty$ as $i \to \infty$, and $(\tilde{w}, \tilde{v}) \in C^2(\overline{\Omega})$, such that $(\tilde{w}_{k_i}, \tilde{v}_{k_i}) \to (\tilde{w}, \tilde{v})$ in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ as $i \to \infty$, where $(\tilde{w}_{k_i}, \tilde{v}_{k_i})$ are the corresponding positive solutions of system (4.3), and $\tilde{w}$ is a constant and $\tilde{v}$ satisfies

$$-d_2 \Delta \tilde{v} = \tilde{w} - \tilde{v}, \quad x \in \Omega, \quad \partial_\nu \tilde{v} = 0, \quad x \in \partial\Omega.$$

Thus, $\tilde{v}$ is also a positive constant, which combined with (3.2) or $\bar{v} = a$ implies $\tilde{w} = \tilde{v} = a$. The proof is completed. $\square$

Theorem 4.3. *Fix* $d_1, d_2 > 0$*. There exists* $K > 0$*, which depends on* p, d_1, d_2 *and* Ω*, such that* (4.3) *does not admit a non-constant positive solution when* $k \geq K$.

Proof. We first write $w = \xi + h$ with $\bar{h} = 0$ and $\xi = \mathbb{R}^+$. Denote

$$L_0^2(\Omega) = \{g \in L^2(\Omega) |\ \bar{g} = 0\}, \qquad W_\nu^{2,2}(\Omega) = \{g \in W^{2,2}(\Omega) |\ \partial_\nu g = 0,\ x \in \partial\Omega\}.$$

It is easy to find that discussing the solution of system (4.3) is equivalent to finding the solution of the following system

$$\begin{cases} \Delta h + \dfrac{r}{d_1} P\big[a - \dfrac{(\xi + h)v^p}{r + v^p}\big] = 0,\ x \in \Omega, \\ \displaystyle\int_\Omega \big[a - \frac{(\xi + h)v^p}{r + v^p}\big] dx = 0, \\ \Delta v + \dfrac{1}{d_2}\big[\dfrac{(\xi + h)v^p}{r + v^p} - v\big] = 0, \quad x \in \Omega, \\ \dfrac{\partial h}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega, \\ \xi > 0,\ v(x) > 0, \qquad x \in \Omega, \end{cases} \tag{4.4}$$

where $r = k^{-1}$ and $Pz = z - \bar{z}$. P is the projective operator from $L^2(\Omega)$ to $L_0^2(\Omega)$. Obviously, $(0, a + a^{1-p}/k, a)$ is a solution of (4.4). Note that when $r = 0$, system (4.4) has a unique solution $(0, a, a)$. It suffices to prove that if $r > 0$ is small enough, then $(0, a, a)$ is the unique solution of (4.4). For this purpose, we further define

$$\begin{aligned} F(r, h, \xi, v) = (f_1, f_2, f_3)(r, h, \xi, v) &: \mathbb{R}^+ \times (L_0^2(\Omega) \cap W_\nu^{2,2}(\Omega)) \times \mathbb{R}^+ \times W_\nu^{2,2}(\Omega) \\ &\to L_0^2(\Omega) \times \mathbb{R}^+ \times L^2(\Omega), \end{aligned}$$

where

$$f_1(r,h,\xi,v) = \Delta h + \frac{r}{d_1} P\big[a - \frac{(\xi+h)v^p}{r+v^p}\big],$$
$$f_2(r,h,\xi,v) = \int_\Omega \big[a - \frac{(\xi+h)v^p}{r+v^p}\big]dx,$$
$$f_3(r,h,\xi,v) = \Delta v + \frac{1}{d_2}\big[\frac{(\xi+h)v^p}{r+v^p} - v\big].$$

It is easy to see that finding solution of (4.3) is equivalent to solving $F(r,h,\xi,v) = 0$. Note that system (4.4) has a unique solution $(0,a,a)$ when $r = 0$. Simple computations give

$$D_{(h,\xi,v)}F(0,0,a,a) : (L_0^2(\Omega) \cap W_\nu^{2,2}(\Omega)) \times \mathbb{R} \times W_\nu^{2,2}(\Omega) \to L_0^2(\Omega) \times \mathbb{R} \times L^2(\Omega),$$

and

$$D_{(h,\xi,v)}F(0,0,a,a)(y,\tau,z) = \begin{pmatrix} \Delta y \\ -\displaystyle\int_\Omega (y+\tau)dx \\ \Delta z - \dfrac{1}{d_2}z + \dfrac{1}{d_2}(y+\tau) \end{pmatrix}.$$

Since $\Delta : L_0^2(\Omega) \cap W_\nu^{2,2}(\Omega) \to L_0^2(\Omega)$ is invertible, $D_{(h,\xi,v)}F(0,0,a,a)$ is also invertible. Moreover, $D_{(h,\xi,v)}F(0,0,a,a)$ can be verified to be surjective by simple calculations.

By the implicit function theorem and Lemma 4.2, there exist positive constants r_0 and δ_0 such that for each $r \in [0,r_0]$, $(0,a,a)$ is the unique solution $F(r,h,\xi,v) = 0$ in $B_{\delta_0}(0,a,a)$, where $B_{\delta_0}(0,a,a)$ is the ball centered at $(0,a,a)$ with radius δ_0. The proof is completed. $\square$

4.2. The existence of non-constant positive solutions

In this subsection, we shall discuss the existence of non-constant positive solutions of (1.3) using the theory of Leray-Schauder topological degree.

For later arguments, we define the function space $\boldsymbol{X} = \{(u,v) \in C^2(\Omega) \cap C^1(\overline{\Omega}) : \partial_\nu u = \partial_\nu v = 0 \text{ on } \partial\Omega\}$, $\boldsymbol{X}_* = \{\boldsymbol{u} = (u,v) \in \boldsymbol{X} : \underline{C} < u, v < \overline{C} \text{ on } \overline{\Omega}\}$ with $\underline{C}, \overline{C} > 0$ which can be obtained in Section 3. Let $E(\lambda_i)$ be the corresponding eigenspace of λ_i and $\{\phi_{ij} : j = 1,2,\cdots,\mathrm{dimE}(\lambda_i)\}$ be an orthonormal basis for $E(\lambda_i)$, and $\boldsymbol{X}_{ij} = \{c\phi_{ij} : c \in \mathbb{R}^2\}$. We decompose $\boldsymbol{X}$ as

$$\boldsymbol{X} = \bigoplus_{i=0}^{\infty} \boldsymbol{X}_i \quad \text{and} \quad \boldsymbol{X}_i = \bigoplus_{j=0}^{\dim E(\lambda_i)} \boldsymbol{X}_{ij}.$$

Define

$$G(\boldsymbol{u}) = \begin{pmatrix} d_1^{-1}(a - \dfrac{uv^p}{1+kv^p}) \\ d_2^{-1}(\dfrac{uv^p}{1+kv^p} - v) \end{pmatrix}.$$

Then (1.3) can be written as

$$-\Delta\boldsymbol{u} = G(\boldsymbol{u}), \quad x \in \Omega, \quad \partial_\nu \boldsymbol{u} = 0, \quad x \in \partial\Omega, \tag{4.5}$$

and $\boldsymbol{u}$ is a positive solution of (4.5) if and only if

$$\mathcal{L}(\boldsymbol{u}) \equiv \boldsymbol{u} - (-\Delta + \mathcal{I})^{-1}\{G(\boldsymbol{u}) + \boldsymbol{u}\} = 0$$

has a positive solution, where $\mathcal{I}$ is the identity operator. Note that $\mathcal{L}(\cdot)$ is a compact perturbation of the identity operator, and so the Leray-Schauder degree $\deg(\mathcal{L}(\cdot), 0, X_*)$ is well defined because of $\mathcal{L}(\cdot) \neq 0$ on ∂X_*. Furthermore, we observe that

$$D_{\boldsymbol{u}}\mathcal{L}(\boldsymbol{u}^*) = \mathcal{I} - (-\Delta + \mathcal{I})^{-1}(\mathcal{A} + \mathcal{I}),$$

where

$$\mathcal{A} := D_{\boldsymbol{u}}G(\boldsymbol{u}^*) = \begin{pmatrix} -\dfrac{a^p}{d_1(1+ka^p)} & -\dfrac{p}{d_1(1+ka^p)} \\ \dfrac{a^p}{d_2(1+ka^p)} & \dfrac{p-1-ka^p}{d_2(1+ka^p)} \end{pmatrix}, \quad \boldsymbol{u}^* = (u^*, v^*).$$

If $D_{\boldsymbol{u}}\mathcal{L}(\boldsymbol{u}^*)$ is invertible, then it follows from [3] that the index of $\mathcal{L}$ at $\boldsymbol{u}^*$ is defined as

$$\text{index}(\mathcal{L}(\cdot), \boldsymbol{u}^*) = (-1)^{\zeta}, \tag{4.6}$$

where ζ is the sum of algebraic multiplicities of the negative eigenvalues of $D_{\boldsymbol{u}}\mathcal{L}(\boldsymbol{u}^*)$. A direct calculation shows that, for each integer $i \geq 0$, X_i is invariant under $D_{\boldsymbol{u}}\mathcal{L}(\boldsymbol{u}^*)$, and ξ is an eigenvalue of $D_{\boldsymbol{u}}\mathcal{L}(\boldsymbol{u}^*)$ on X_i if and only if it is an eigenvalue of the matrix $\lambda_i\mathcal{I} - \mathcal{A}$.

Denote

$$\mathcal{H}(\lambda) = \det(\lambda\mathcal{I} - \mathcal{A}) = \lambda^2 + \frac{d_2a^p + d_1(1 + ka^p - p)}{d_1d_2(1 + ka^p)}\lambda + \frac{a^p}{d_1d_2(1 + ka^p)}. \tag{4.7}$$

From the discussions in Section 2, we know that if $p > 1$, $\max\{0, k_0\} < k < k_0 + 1$ and $0 < d_2/d_1 < z_1$, then $\mathcal{H}(\lambda) = 0$ has two positive roots $\lambda_{\pm}(d_1, d_2)$, where z_1 and $\lambda_{\pm}(d_1, d_2)$ are given by (2.4) and (2.5), respectively. Moreover, by (2.6), we have $\lambda_-(d_1, d_2) \to 0$, $\lambda_+(d_1, d_2) \to \lambda^*$ when $d_1 \to \infty$.

Theorem 4.4. *Assume that* $p > 1$, $\max\{0, k_0\} < k < k_0 + 1$ *and* $0 < d_2/d_1 < z_1$, *where* z_1 *is given by* (2.4). *If* $\lambda^* = \dfrac{p-1-ka^p}{d_2(1+ka^p)} \in (\lambda_m, \lambda_{m+1})$ *for some integer* $m \geq 1$ *and* $\sum_{i=1}^{m} \dim E(\lambda_i)$ *is odd, then there exists* $\overline{D}_1 > 0$ *such that (1.3) has at least one non-constant positive solution provided that* $d_1 \geq \overline{D}_1$.

Proof. From the above arguments and $\lambda^* \in (\lambda_m, \lambda_{m+1})$ for some $m \geq 1$, it follows that there exists a constant $\overline{D}_1 > 0$ such that

$$\lambda_+(d_1, d_2) \in (\lambda_m, \lambda_{m+1}), \quad \lambda_-(d_1, d_2) \in (\lambda_0, \lambda_1) \text{ for all } d_1 \geq \overline{D}_1.$$

We aim to show that (1.3) has at least one non-constant positive solution if $d_1 \geq \overline{D}_1$. Prove the conclusion by contradiction and assume that the assertion is not true for some $d_1 = \widehat{d}_1 \geq \overline{D}_1$. By Theorem 4.2, there exists $D_2^* > 0$ such that (1.3) has no non-constant positive solution for all $d_2 \geq D_2^*$. Moreover, we can choose $d_2 = \widehat{d}_2 \geq D_2^*$ sufficiently large such that $\lambda^* < \lambda_1$. Then we have

$$0 < \lambda_-(d_1, d_2) < \lambda_+(d_1, \widehat{d}_2) < \lambda_1, \quad \overline{D}_1 \leq d_1 \leq \widehat{d}_1. \tag{4.8}$$

For $t \in [0, 1]$, we define

$$G(\boldsymbol{u}; t) = \begin{pmatrix} [td_1 + (1-t)\widehat{d}_1]^{-1}(a - \frac{uv^p}{1+kv^p}) \\ [td_2 + (1-t)\widehat{d}_2]^{-1}(\frac{uv^p}{1+kv^p} - v) \end{pmatrix},$$

and let

$$\begin{aligned} \mathcal{A}(t) :=& D_{\boldsymbol{u}} G(\boldsymbol{u}^*; t) \\ =& \begin{pmatrix} -\frac{a^p}{(1+ka^p)[td_1 + (1-t)\widehat{d}_1]} & -\frac{p}{(1+ka^p)[td_1 + (1-t)\widehat{d}_1]} \\ \frac{a^p}{(1+ka^p)[td_2 + (1-t)\widehat{d}_2]} & \frac{p-1-ka^p}{(1+ka^p)[td_2 + (1-t)\widehat{d}_2]} \end{pmatrix}. \end{aligned}$$

Think about the following problem

$$-\Delta \boldsymbol{u} = G(\boldsymbol{u}; t), \quad x \in \Omega, \quad \partial_\nu \boldsymbol{u} = 0, \quad x \in \partial\Omega. \tag{4.9}$$

Obviously, system (4.9) has a unique positive constant solution $\boldsymbol{u}_*$. Note that $\boldsymbol{u}$ is a positive solution of system (1.3) if and only if it is a positive solution of (4.9) for $t = 1$. Since the operator $(\mathcal{I} - \Delta)^{-1} : C(\overline{\Omega}) \to C(\overline{\Omega})$ exists and is compact, we know that $\boldsymbol{u}$ is a positive solution of (4.9) if and only if $\boldsymbol{u}$ satisfies

$$\mathcal{L}(\boldsymbol{u}; t) \equiv \boldsymbol{u} - (-\Delta + \mathcal{I})^{-1}\{G(\boldsymbol{u}; t) + \boldsymbol{u}\} = 0 \text{ on } X. \tag{4.10}$$

Further calculations to get

$$D_{\boldsymbol{u}}\mathcal{L}(\boldsymbol{u}^*; 0) = \mathcal{I} - (-\Delta + \mathcal{I})^{-1}(\mathcal{A}(0) + \mathcal{I}),$$
$$D_{\boldsymbol{u}}\mathcal{L}(\boldsymbol{u}^*; 1) = \mathcal{I} - (-\Delta + \mathcal{I})^{-1}(\mathcal{A}(1) + \mathcal{I}),$$

where

$$\mathcal{A}(0) = \begin{pmatrix} -\frac{a^p}{\widehat{d}_1(1+ka^p)} & -\frac{p}{\widehat{d}_1(1+ka^p)} \\ \frac{a^p}{\widehat{d}_2(1+ka^p)} & \frac{p-1-ka^p}{\widehat{d}_2(1+ka^p)} \end{pmatrix},$$
$$\mathcal{A}(1) = \begin{pmatrix} -\frac{a^p}{d_1(1+ka^p)} & -\frac{p}{d_1(1+ka^p)} \\ \frac{a^p}{d_2(1+ka^p)} & \frac{p-1-ka^p}{d_2(1+ka^p)} \end{pmatrix}.$$

Next, we calculate the number of negative eigenvalues of $D_{\boldsymbol{u}}\mathcal{L}(\boldsymbol{u}^*; 1)$ on X. Since X is composed by $\overset{\infty}{\underset{i=0}{\oplus}} X_i$, we have

$$\sum_{i \geq 0,\, \mathcal{H}(\lambda_i) < 0} \dim E(\lambda_i) = \sum_{i=1}^{m} \dim E(\lambda_i).$$

According to (4.7), we can obtain

$$\mathcal{H}(\lambda_0, \widehat{d}_1) = \frac{a^p}{\widehat{d}_1 D_2^*(1+ka^p)} > 0,$$

$$\mathcal{H}(\lambda_i, \widehat{d}_1) > 0 \text{ for all } i \geq m+1, \qquad \mathcal{H}(\lambda_j, \widehat{d}_1) < 0 \text{ for all } 0 < j \leq m.$$

Thus, 0 is not an eigenvalue of the matrix $\lambda_i \mathcal{I} - \mathcal{A}(1)$ for any $i \geq 0$, and $\sum_{i=1}^{m} \dim E(\lambda_i)$ is odd by the hypothesis. For $t = 0$, Theorem 4.2 implies that $\mathcal{L}(\boldsymbol{u}; 0)$ has the only positive solution $\boldsymbol{u}^*$ in X, and from (4.8), it is easy to see that $\mathcal{H}(\lambda_i) > 0$ for all $i \geq 0$ and $\sum_{i=1}^{m} \dim E(\lambda_i) = 0$.

By Theorem 3.1, there exist positive constants $\overline{C}$ and $\underline{C}$ depending on p, k, d_2 and $|\Omega|$, such that any positive solution (u, v) of (1.3) satisfies

$$\overline{C} < u(x), v(x) < \underline{C} \text{ on } \overline{\Omega}.$$

Let $\Omega_1 = \{\boldsymbol{u} \in X | \underline{C} < \boldsymbol{u} < \overline{C} \text{ on } \overline{\Omega}\}$, then $\mathcal{L}(\boldsymbol{u}; t) \neq 0$ for $\boldsymbol{u} \in \partial\overline{\Omega}_1$. By the homotopy invariance of topological degree, we have

$$\deg(\mathcal{L}(\cdot; 0), 0, \Omega_1) = \deg(\mathcal{L}(\cdot; 1), 0, \Omega_1). \tag{4.11}$$

According to our hypothesis, both equations $\mathcal{L}(\boldsymbol{u}; 0) = 0$ and $\mathcal{L}(\boldsymbol{u}; 1) = 0$ have only the non-negative solution $\boldsymbol{u}^*$ in Ω_1, and from the formula (4.6), we have

$$\deg(\mathcal{L}(\cdot; 0), 0, \Omega_1) = \text{index}(\mathcal{L}(\cdot; 0); \boldsymbol{u}^*) = (-1)^{\sum_{\mathrm{i}=1}^{\mathrm{m}} \dim \mathrm{E}(\lambda_\mathrm{i})} = 1,$$
$$\deg(\mathcal{L}(\cdot; 1), 0, \Omega_1) = \text{index}(\mathcal{L}(\cdot; 1); \boldsymbol{u}^*) = (-1)^{\sum_{i=1}^{m} \dim E(\lambda_i)} = -1,$$

which is contradictory with (4.11) and thus the proof is completed. □

5. Steady-state bifurcation

In this section, we focus on a detailed qualitative analysis on the steady-state bifurcation for system (1.3) in one-dimensional space. Related works can be found in [9, 10, 13, 19, 38]. Assume $\Omega = (0, \pi)$ and consider the steady-state problem

$$\begin{cases} -d_1 \Delta u = a - \dfrac{uv^p}{1 + kv^p}, \; x \in (0, \pi), \\ -d_2 \Delta v = \dfrac{uv^p}{1 + kv^p} - v, \; x \in (0, \pi), \\ u' = v' = 0, \qquad x = 0, \pi. \end{cases} \tag{5.1}$$

Furthermore, we shall give the structure and direction of bifurcation solutions. For convenience, we translate (u^*, v^*) to the origin by the translation $(\tilde{u}, \tilde{v}) = (u - u^*, v - v^*)$ and still denote $\tilde{u}, \tilde{v}$ by u, v, respectively. Then (5.1) turns to the following system

$$\begin{cases} -d_1 u'' = a - \dfrac{(u + u^*)(v + v^*)^p}{1 + k(v + v^*)^p}, & x \in (0, \pi), \\ -d_2 v'' = \dfrac{(u + u^*)(v + v^*)^p}{1 + k(v + v^*)^p} - (v + v^*), & x \in (0, \pi), \\ u' = v' = 0, & x = 0, \pi. \end{cases} \tag{5.2}$$

5.1. Local steady-state bifurcation

In this subsection, taking d_1 as a bifurcation parameter, we shall prove the existence of positive solutions bifurcating from E^*. The Crandall-Rabinowitz bifurcation theorem is used to derive bifurcations from simple eigenvalues. For the case of double eigenvalues, we resort to some space decomposition techniques and the implicit function theorem.

Let $X = \{(u,v) \in W^{2,2}(0,\pi) \times W^{2,2}(0,\pi) : u' = v' = 0, \ x = 0,\pi\}$ and $Y = L^2(0,\pi) \times L^2(0,\pi)$. Define the map $F : \mathbb{R}^+ \times X \to Y$ by

$$F(d_1, U) = \begin{pmatrix} d_1 u'' + a - \dfrac{(u+u^*)(v+v^*)^p}{1+k(v+v^*)^p} \\ d_2 v'' + \dfrac{(u+u^*)(v+v^*)^p}{1+k(v+v^*)^p} - (v+v^*) \end{pmatrix}, \quad U = (u,v).$$

Then the solutions of (5.2) are exactly zeros of this map. Clearly, $F(d_1,(0,0)) = 0$. By calculations, the Fréchet derivative of F with respect to U at $(0,0)$ can be given by

$$L(d_1) = F_U\left(d_1,(0,0)\right) = \begin{pmatrix} d_1\Delta - \dfrac{a^p}{1+ka^p} & -\dfrac{p}{1+ka^p} \\ \dfrac{a^p}{1+ka^p} & d_2\Delta + \dfrac{p}{1+ka^p} - 1 \end{pmatrix}, \quad \Delta = \frac{d^2}{dx^2},$$

whose characteristic equation is given by (2.3) in Section 2.

Define

$$\text{(H1): } p > 1 + ka^p \text{ and } d_2 < \frac{p-1-ka^p}{1+ka^p}.$$

In this section, we always assume that (H1) is true. Then there exists a largest integer $i^* \geq 1$ such that $d_2\lambda_i < \dfrac{p-1-ka^p}{1+ka^p}$ for $1 \leq i \leq i^*$. Let $\mu = 0$ in (2.3). Then we have

$$d_1{=}d_{1,i} := \frac{a^p(1+d_2\lambda_i)}{\lambda_i[p-(1+d_2\lambda_i)(1+ka^p)]}, \quad \lambda_i = i^2, \quad 1 \leq i \leq i^*. \tag{5.3}$$

If we set

$$d_1^* = d_1^*(p, d_1, d_2) = \min_{1\leq i\leq i^*} d_{1,i}, \tag{5.4}$$

the local stability of E^* is presented in the following.

Theorem 5.1. *Assume $p > 1$ and $k > \max\{0, k_0\}$ so that E^* is locally asymptotically stable for (2.1). Then the equilibrium E^* is locally asymptotically stable for (1.2) if $d_2 \geq (p-1-ka^p)/(1+ka^p)$, or $d_2 < (p-1-ka^p)/(1+ka^p)$ and $0 < d_1 < d_1^*$; the equilibrium E^* is unstable for (1.2) if $d_2 < (p-1-ka^p)/(1+ka^p)$ and $d_1 > d_1^*$.*

Proof. If $d_2 \geq (p-1-ka^p)/(1+ka^p)$, we have

$$D_i(k) = d_1\lambda_i[\frac{1}{1+ka^p}p - (1+d_2\lambda_i)] - \frac{a^p(1+d_2\lambda_i)}{1+ka^p} > 0,$$

for $i \geq 1$. This implies that $Re\mu < 0$ for all eigenvalues μ of $L(d_1)$ and E^* is locally asymptotically stable for (1.2).

If $d_2 < (p-1-ka^p)/(1+ka^p)$ and $0 < d_1 < d_1^*$, we see $d_2\lambda_i < (p-1-ka^p)/(1+ka^p)$ and $d_1 < d_{1,i}$ for $1 \le i \le i^*$, which leads to $D_i(k) > 0$ for $1 \le i \le i^*$. For $i > i^*$, we see $d_2\lambda_i \ge (p-1-ka^p)/(1+ka^p)$ and then still obtain $D_i(k) > 0$. Hence, we have $D_i(k) > 0$ for all $i \ge 1$, which shows the asymptotical stability of E^* for (1.2).

If $d_2\lambda_1 < (p-1-ka^p)/(1+ka^p)$ and $d_1 > d_1^*$, let the minimum in (5.3) be attained at $j \in [1, i^*]$ and thus we have $d_1 > d_{1,j}$, which implies $D_j(k) = d_1\lambda_j[\frac{1}{1+ka^p}p - (1+d_2\lambda_j)] - \frac{a^p(1+d_2\lambda_j)}{1+ka^p} < 0$. Hence, E^* is unstable for (1.2). $\square$

Note that $d_{1,i}$ may be equal or not equal to $d_{1,j}$ when $i \neq j$. To obtain the bifurcation from the point $(d_{1,i}, (0,0))\,(1 \le i \le i_0)$, our arguments will be divided into two different cases, corresponding to that from the simple and double eigenvalues, respectively.

Theorem 5.2. *Assume that (H1) is satisfied. Then the following statements are true:*

(i) *If $i \neq j$ implies $d_{1,i} \neq d_{1,j}$ for arbitrary integers $i, j \in [1, i_*]$, then $(d_{1,i}, (0,0))$ is a bifurcation point of $F(d_1, U) = 0$. Moreover, there is a curve of non-constant solutions $(d_1(s), (u(s), v(s)))$ of $F(d_1, U) = 0$ for $|s|$ sufficiently small, satisfying $d_1(0) = d_{1,i}$, $(u(0), v(0)) = (0,0)$, $u(s) = s\phi_i + o(s), v(s) = se_i\phi_i + o(s)$, where $\phi_i = \sqrt{\frac{2}{\pi}}\cos ix$, $e_i = \frac{a^p}{(1+ka^p)(1+d_2\lambda_i)-p}$, and $d_1(s), u(s), v(s)$ are continuously differential functions with respect to s;*

(ii) *Suppose that there exist positive integers $i, j \in [1, i^*]$ and $i \neq j$ such that $d_{1,i} = d_{1,j} = \widetilde{d}_1$. Let*

$$\Phi_i = \begin{pmatrix} 1 \\ e_i \end{pmatrix} \phi_i, \qquad \Phi_i^* = \begin{pmatrix} 1 \\ e_i^* \end{pmatrix} \phi_i, \tag{5.5}$$

$$X_2 = \{(y,z) \in X : \int_0^\pi (y + e_i z)\phi_i dx = \int_0^\pi (y + e_j z)\phi_j dx = 0\}, \tag{5.6}$$

$$A_1 = -\frac{1}{2}c_2 {e_i}^2 - c_1 e_i, \quad A_2 = -c_2 e_i e_j - c_1 e_i - c_1 e_j, \quad A_3 = -\frac{1}{2}c_2 {e_j}^2 - c_1 e_j, \tag{5.7}$$

where

$$e_i = \frac{a^p}{(1+ka^p)(1+d_2\lambda_i)-p}, \qquad {e_i}^* = \frac{p}{p-(1+ka^p)(1+d_2\lambda_i)}, \tag{5.8}$$

and

$$c_1 = \frac{pa^p}{a(1+ka^p)^2}, \qquad c_2 = \frac{p(p-1)-kp(p+1)a^p}{a(1+ka^p)^2}.$$

If $1 + e_i e_i^* \neq 0$, $1 + e_j e_j^* \neq 0$ and $j = 2i$ (resp. $i = 2j$), then $(\widetilde{d}_1, (0,0))$ is a bifurcation point of $F(d_1, U) = 0$. Moreover, there is a curve of non-constant solutions $(d_1(\omega), s(\omega)(\cos\omega\Phi_i + \sin\omega\Phi_j + W(\omega)))$ of $F(d_1, U) = 0$ for $|\omega - \omega_0|$ sufficiently small, satisfying $d_1(\omega_0) = \widetilde{d}_1, s(\omega_0) = 0, W(\omega_0) = 0$, where ω_0 is any constant satisfying

$$\tan^2\omega_0 \neq \frac{A_1(e_j^*-1)i^2}{A_2(e_i^*-1)j^2} \quad (\textit{resp.}\ \tan^2\omega_0 \neq \frac{A_2(e_j^*-1)i^2}{A_3(e_i^*-1)j^2}), \tag{5.9}$$

and $d_1(\omega), s(\omega), W(\omega)$ are continuously differentiable functions with respect to ω.

Proof. It is obvious that the linear operators F_{d_1}, F_U and F_{d_1U} are continuous. Recall that the operator

$$L(d_{1,i}) = F_U(d_{1,i},(0,0)) = \begin{pmatrix} d_{1,i}\Delta - \dfrac{a^p}{1+ka^p} & -\dfrac{p}{1+ka^p} \\ \dfrac{a^p}{1+ka^p} & d_2\Delta + \dfrac{p}{1+ka^p} - 1 \end{pmatrix}.$$

By simple calculations, we have

$$\ker L(d_{1,i}) = \operatorname{span}\{\Phi_i\}, \quad \Phi_i = \begin{pmatrix} 1 \\ e_i \end{pmatrix}\phi_i,$$

where $e_i = \dfrac{a^p}{(1+ka^p)(1+d_2\lambda_i)-p}$. The adjoint operator is defined by

$$L^*(d_{1,i}) = \begin{pmatrix} d_{1,i}\Delta - \dfrac{a^p}{1+ka^p} & \dfrac{a^p}{1+ka^p} \\ -\dfrac{p}{1+ka^p} & d_2\Delta + \dfrac{p}{1+ka^p} - 1 \end{pmatrix}.$$

Similarly, we can obtain

$$\ker L^*(d_{1,i}) = \operatorname{span}\{\Phi_i^*\}, \quad \Phi_i^* = \begin{pmatrix} 1 \\ e_i^* \end{pmatrix}\phi_i,$$

where ${e_i}^* = \dfrac{p}{p-(1+ka^p)(1+d_2\lambda_i)}$. Since $R(L(d_{1,i})) = (\ker L^*(d_{1,i}))^\perp$, we have

$$\operatorname{codim}\ R(L(d_{1,i})) = \dim\ker L(d_{1,i}) = 1.$$

Finally, we see

$$F_{d_1\boldsymbol{u}}(d_{1,i},(0,0))\Phi_i = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}\Phi_i = \begin{pmatrix} -\lambda_i\phi_i \\ 0 \end{pmatrix},$$

and

$$\langle F_{d_1U}(d_{1,i},(0,0))\Phi_i, \Phi_i^*\rangle = -\lambda_i\int_0^\pi \phi_i^2 dx = -\lambda_i \neq 0, \tag{5.10}$$

which implies $F_{d_1U}(d_{1,i},(0,0))\Phi_i \notin R(L(d_{1,i}))$. The proof of (i) is completed.

(ii) Suppose that there are positive integers $i, j \in [1, i_*]$ and $i \neq j$ such that $d_{1,i} = d_{1,j} = \widetilde{d}_1$. Then we have $\ker L(\widetilde{d}_1) = \operatorname{span}\{\Phi_i, \Phi_j\}$, $\ker L^*(\widetilde{d}_1) = \operatorname{span}\{{\Phi_i}^*, {\Phi_j}^*\}$ and

$$R(L(\widetilde{d}_1)) = \{(y,z)^T \in Y : \int_0^\pi (y + e_i^* z)\phi_i dx = \int_0^\pi (y + e_j^* z)\phi_j dx = 0\}.$$

Obviously, we have $\dim\ker L(\widetilde{d}_1)=\mathrm{codim}R(L(\widetilde{d}_1))=2$. The Crandall-Rabinowitz bifurcation theorem does not hold in this case. Next, we shall use the techniques of space decomposition and the implicit function theorem to deal with the case of double eigenvalues.

Rewrite the map $F:\mathbb{R}^+\times X\to Y$ as

$$\begin{aligned}F(d_1,(u,v))&=\begin{pmatrix} d_1u''+a-\dfrac{(u+u^*)(v+v^*)^p}{1+k(v+v^*)^p}\\ d_2v''+\dfrac{(u+u^*)(v+v^*)^p}{1+k(v+v^*)^p}-(v+v^*)\end{pmatrix}\\ &=L(d_1)\begin{pmatrix}u\\ v\end{pmatrix}+\begin{pmatrix}F^1(u,v)\\ F^2(u,v)\end{pmatrix},\end{aligned}$$

where $F^2(u,v)=-F^1(u,v)$ and

$$F^1(u,v)=-c_1uv-\frac{1}{2}c_2v^2-\frac{1}{2}c_3uv^2-\frac{1}{6}c_4v^3+O(|u||v|^3,|v|^4), \tag{5.11}$$

$$\begin{aligned}c_1&=\frac{pa^p}{a(1+ka^p)^2},\\ c_2&=\frac{p(p-1)-kp(p+1)a^p}{a(1+ka^p)^2},\\ c_3&=\frac{p(p-1)a^p-kp(p+1)a^{2p}}{a^2(1+ka^p)^3},\\ c_4&=\frac{p(p-1)(p-2)-4kp(p-1)(p+1)a^p+k^2p(p+1)(p+2)a^{2p}}{a^2(1+ka^p)^3}.\end{aligned}$$

We make the decomposition $X=X_1\oplus X_2$ and look for solutions of $F=0$ in the form

$$(u,v)^T=s(\cos\omega\Phi_i+\sin\omega\Phi_j+W),\quad W=(w_1,w_2)^T,$$

where $X_1=\mathrm{span}\{\Phi_i,\Phi_j\}$, X_2 is defined by (5.6) and $s,\omega\in\mathbb{R}$ are parameters. Define a projection P on Y by

$$P\begin{pmatrix}u\\ v\end{pmatrix}=\frac{1}{1+e_ie_i^*}\left[\int_0^\pi(u+e_i^*v)\phi_idx\right]\Phi_i+\frac{1}{1+e_je_j^*}\left[\int_0^\pi(u+e_j^*v)\phi_jdx\right]\Phi_j.$$

Based on the assumption in conclusion (ii), we have $1+e_ie_i{}^*\neq0$ and $1+e_je_j{}^*\neq0$. By simple computations, we have $R(P)=\mathrm{span}\{\Phi_i,\Phi_j\}=X_1\subset Y$, $P^2=P$. Hence, P is the projection from Y to $X_1\subset Y$, and then decompose Y as $Y=Y_1\oplus Y_2$ with $Y_1=R(P)$ and $Y_2=\ker P=R(L(\widetilde{d}_1))$.

Next, we use the implicit function theorem to prove the existence of non-constant pairs (u,v). Fix $\omega_0\in\mathbb{R}$ for the time being and define a nonlinear mapping $\mathcal{K}(d_1,s,W;\omega):\mathbb{R}\times\mathbb{R}\times X_2\times(\omega_0-\delta,\omega_0+\delta)\to Y$ by

$$\begin{aligned}\mathcal{K}(d_1,s,W;\omega)&=s^{-1}\widetilde{F}(d_1,s(\cos\omega\Phi_i+\sin\omega\Phi_j+W))\\ &=L(d_1)(\cos\omega\Phi_i+\sin\omega\Phi_j+W)+s(\widetilde{F}_1,\widetilde{F}_2)^T,\end{aligned}$$

where $\widetilde{F}_2 = -\widetilde{F}_1$ and

$$\begin{aligned}\widetilde{F}_1 = &- c_1(\cos\omega\phi_i + \sin\omega\phi_j + w_1)(e_i\cos\omega\phi_i + e_j\sin\omega\phi_j + w_2)\\ &- \frac{1}{2}c_3(\cos\omega\phi_i + \sin\omega\phi_j + w_1)(e_i\cos\omega\phi_i + e_j\sin\omega\phi_j + w_2)^2\\ &- \frac{1}{2}c_2(e_i\cos\omega\phi_i + e_j\sin\omega\phi_j + w_2)^2\\ &- \frac{1}{6}c_4(e_i\cos\omega\phi_i + e_j\sin\omega\phi_j + w_2)^3 + o(|s|^2).\end{aligned}$$

Obviously, $\mathcal{K}(\widetilde{d}_1, 0, 0; \omega_0) = 0$. The Fréchet derivative of $\mathcal{K}(d_1, s, W; \omega)$ with respect to (d_1, s, W) at $(\widetilde{d}_1, 0, 0; \omega_0)$ is the linear mapping

$$\begin{aligned}&\mathcal{K}_{(d_1,s,W)}(\widetilde{d}_1, 0, 0; \omega_0)(d_1, s, W)\\ &= L(\widetilde{d}_1)W - d_1\lambda_i\cos\omega_0\begin{pmatrix}\phi_i\\ 0\end{pmatrix} - d_1\lambda_j\sin\omega_0\begin{pmatrix}\phi_j\\ 0\end{pmatrix}\\ &\quad + sA_1\cos^2\omega_0\begin{pmatrix}\phi_i^2\\ -\phi_i^2\end{pmatrix} + sA_2\cos\omega_0\sin\omega_0\begin{pmatrix}\phi_i\phi_j\\ -\phi_i\phi_j\end{pmatrix} + sA_3\sin^2\omega_0\begin{pmatrix}\phi_j^2\\ -\phi_j^2\end{pmatrix},\end{aligned}$$

where A_1, A_2 and A_3 are given in (5.7).

We further prove that $\mathcal{K}_{(d_1,s,W)}(\widetilde{d}_1, 0, 0; \omega_0) : \mathbb{R}\times\mathbb{R}\times X_2\times(\omega_0-\delta, \omega_0+\delta) \to Y$ is an isomorphism. We can rewrite

$$\mathcal{K}_{(d_1,s,W)}(\widetilde{d}_1, 0, 0; \omega_0)(d_1, s, W) = \mathscr{Y}_1 + \mathscr{Y}_2, \quad \mathscr{Y}_1 \in Y_1 \text{ and } \mathscr{Y}_2 \in Y_2,$$

and decompose

$$\begin{pmatrix}\phi_i\\ 0\end{pmatrix} = h_1\Phi_i + \begin{pmatrix}u_1\\ v_1\end{pmatrix}, \quad \begin{pmatrix}\phi_j\\ 0\end{pmatrix} = h_2\Phi_j + \begin{pmatrix}u_2\\ v_2\end{pmatrix},$$

where

$$h_1 = \frac{1-e_i}{1+e_ie_i^*} \neq 0, \quad \begin{pmatrix}u_1\\ v_1\end{pmatrix} = \begin{pmatrix}1-h_1\\ -h_1e_i\end{pmatrix}\phi_i,$$

$$h_2 = \frac{1-e_j}{1+e_je_j^*} \neq 0, \quad \begin{pmatrix}u_2\\ v_2\end{pmatrix} = \begin{pmatrix}1-h_2\\ -h_2e_j\end{pmatrix}\phi_j,$$

and it is clear that $\begin{pmatrix}u_1\\ v_1\end{pmatrix}, \begin{pmatrix}u_2\\ v_2\end{pmatrix} \in Y_2$.

Now, we divide our discussion into two cases $j = 2i$ and $i = 2j$.

Case I. $j = 2i$.

In this case, we easily get

$$\int_0^\pi \phi_i^2 \phi_j dx = \sqrt{\frac{1}{2\pi}}, \quad \int_0^\pi \phi_i \phi_j^2 dx = 0 \quad \text{and} \quad \int_0^\pi \phi_i^3 dx = \int_0^\pi \phi_j^3 dx = 0.$$

Then we can obtain $\begin{pmatrix} \phi_j^2 \\ -\phi_j^2 \end{pmatrix} \in Y_2$ and we further need to decompose

$$\begin{pmatrix} \phi_i^2 \\ -\phi_i^2 \end{pmatrix} = h_3 \Phi_j + \begin{pmatrix} u_3 \\ v_3 \end{pmatrix}, \qquad \begin{pmatrix} \phi_i \phi_j \\ -\phi_i \phi_j \end{pmatrix} = h_4 \Phi_i + \begin{pmatrix} u_4 \\ v_4 \end{pmatrix},$$

where

$$\begin{aligned}
h_3 &= \frac{e_j^* - 1}{e_j e_j^* + 1} \int_0^\pi \phi_i^2 \phi_j dx = \sqrt{\frac{1}{2\pi}} \frac{e_j^* - 1}{e_j e_j^* + 1} \neq 0, \\
\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} &= \begin{pmatrix} \phi_i^2 - h_3 \phi_j \\ -\phi_i^2 - h_3 e_j \phi_j \end{pmatrix} \in Y_2, \\
h_4 &= \frac{e_i^* - 1}{e_i e_i^* + 1} \int_0^\pi \phi_i^2 \phi_j dx = \sqrt{\frac{1}{2\pi}} \frac{e_i^* - 1}{e_i e_i^* + 1} \neq 0, \\
\begin{pmatrix} u_4 \\ v_4 \end{pmatrix} &= \begin{pmatrix} \phi_i \phi_j - h_4 \phi_i \\ -\phi_i \phi_j - h_4 e_i \phi_i \end{pmatrix} \in Y_2.
\end{aligned}$$

Some arrangements give

$$\mathcal{K}_{(d_1,s,W)}(\widetilde{d}_1, 0, 0; \omega_0)(d_1, s, W) = \mathscr{Y}_1 + \mathscr{Y}_2,$$

where

$$\begin{aligned}
\mathscr{Y}_1 =& (-d_1 h_1 \lambda_i \cos\omega_0 + s h_4 A_2 \cos\omega_0 \sin\omega_0) \Phi_i \\
&+ (-d_1 h_2 \lambda_j \sin\omega_0 + s h_3 A_1 \cos^2\omega_0) \Phi_j \in Y_1, \\
\mathscr{Y}_2 =& L(\widetilde{d}_1) W - d_1 \lambda_i \cos\omega_0 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - d_1 \lambda_j \sin\omega_0 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \\
&+ s A_1 \cos^2\omega_0 \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} + s A_2 \cos\omega_0 \sin\omega_0 \begin{pmatrix} u_4 \\ v_4 \end{pmatrix} + s A_3 \sin^2\omega_0 \begin{pmatrix} \phi_j^2 \\ -\phi_j^2 \end{pmatrix} \in Y_2.
\end{aligned}$$

Let

$$\mathcal{K}_{(d_1,s,W)}(\widetilde{d}_1, 0, 0; \omega_0)(d_1, s, W) = 0. \tag{5.12}$$

Note that $L(\widetilde{d}_1)$ is an isomorphism from X_2 to Y_2. Then (5.12) is equivalent to $\mathscr{Y}_1 = 0$ and $\mathscr{Y}_2 = 0$. Based on the condition in (5.9), we get $d_1 = 0, s = 0$ from $\mathscr{Y}_1 = 0$. Substituting them into $\mathscr{Y}_2 = 0$, we have $W = 0$, which implies $\mathcal{K}_{(d_1,s,W)}(\widetilde{d}_1, 0, 0; \omega_0)$ is injective.

We further prove $\mathcal{K}_{(d_1,s,W)}(\widetilde{d}_1, 0, 0; \omega_0)$ is surjective. For any $\begin{pmatrix} u \\ v \end{pmatrix} \in Y$, we find $(d_1, s, W) \in \mathbb{R} \times \mathbb{R} \times X_2$ such that

$$\mathcal{K}_{(d_1,s,W)}(\widetilde{d}_1, 0, 0; \omega_0)(d_1, s, W) = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{5.13}$$

According to the decomposition of Y, there exist $\alpha, \beta \in \mathbb{R}$ and $(u_0, v_0) \in Y_2$ such that

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \alpha\Phi_i + \beta\Phi_j.$$

Substitute it into (5.13) to get

$$\begin{cases} -d_1 h_1 \lambda_i \cos\omega_0 + s h_4 A_2 \cos\omega_0 \sin\omega_0 = \alpha, \\ -d_1 h_2 \lambda_j \sin\omega_0 + s h_3 A_1 \cos^2\omega_0 = \beta, \\ L(\widetilde{d}_1)W - d_1\lambda_i \cos\omega_0 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - d_1 \lambda_j \sin\omega_0 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} + sA_1\cos^2\omega_0 \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} \\ +sA_2 \cos\omega_0 \sin\omega_0 \begin{pmatrix} u_4 \\ v_4 \end{pmatrix} + sA_3 \sin^2\omega_0 \begin{pmatrix} \phi_j^2 \\ -\phi_j^2 \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \end{cases} \tag{5.14}$$

Due to ω_0 satisfying (5.9) when $j = 2i$, we have

$$d_1 = \bar{d}_1 := \frac{\alpha h_3 A_1 \cos\omega_0 - \beta h_4 A_2 \sin\omega_0}{h_2 h_4 \lambda_j A_2 \sin^2\omega_0 - h_1 h_3 \lambda_i A_1 \cos^2\omega_0},$$

$$s = \bar{s} =: \frac{\alpha h_2 \lambda_j \tan\omega_0 - \beta h_1 \lambda_i}{h_2 h_4 \lambda_j A_2 \sin^2\omega_0 - h_1 h_3 \lambda_i A_1 \cos^2\omega_0}.$$

Substituting $\bar{d}_1$ and $\bar{s}$ into the third equation of (5.14), we obtain $W = L^{-1}\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \in Y_2$, where

$$\begin{aligned} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} &= \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \bar{d}_1 \lambda_i \cos\omega_0 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \bar{d}_1 \lambda_j \sin\omega_0 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} - \bar{s} A_1 \cos^2\omega_0 \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} \\ &\quad - \bar{s} A_2 \cos\omega_0 \sin\omega_0 \begin{pmatrix} u_4 \\ v_4 \end{pmatrix} - \bar{s} A_3 \sin^2\omega_0 \begin{pmatrix} \phi_j^2 \\ -\phi_j^2 \end{pmatrix} \\ &:= \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \in Y_2. \end{aligned}$$

Then we find

$$(d_1, s, W) = (\bar{d_1}, \bar{s}, L^{-1}\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix}),$$

satisfying (5.12). This shows that $\mathcal{K}_{(d_1,s,W)}(\widetilde{d_1}, 0, 0; \omega_0)$ is surjective.

Therefore, $\mathcal{K}_{(d_1,s,W)}(\widetilde{d_1}, 0, 0; \omega_0)$ is an isomorphism from $R^+ \times R \times X_2$ to Y. Apply the implicit theorem for

$$\mathcal{K}(d_1, s, W; \omega) = 0, \tag{5.15}$$

and we know that there is a curve of non-constant solutions $(d_1(\omega), s(\omega), W(\omega))$ of (5.15) (i.e. $F = 0$) in a small neighborhood of ω_0, where $d_1(\omega), s(\omega), W(\omega)$ are continuously differentiable functions with respect to ω satisfying $d_1(\omega_0) = \widetilde{d_1}, s(\omega_0) = 0, W(\omega_0) = 0$ and $W \in X_2$. Therefore, $(d_1(\omega), s(\omega)(\cos\omega\Phi_i + \sin\omega\Phi_j + W(\omega)))$ are non-constant solutions of $F(d_1, (u, v)) = 0$.

Case II. $i = 2j$.

In this case, we can easily get $\int_0^\pi \phi_i^2\phi_j dx = 0$, and $\int_0^\pi \phi_i\phi_j^2 dx = \sqrt{\frac{1}{2\pi}} \neq 0$. Then $\begin{pmatrix}-\phi_i^2\\ \phi_i^2\end{pmatrix} \in Y_2$ and we decompose

$$\begin{pmatrix}\phi_j^2\\ -\phi_j^2\end{pmatrix} = h_5\Phi_i + \begin{pmatrix}u_5\\ v_5\end{pmatrix}, \qquad \begin{pmatrix}\phi_i\phi_j\\ -\phi_i\phi_j\end{pmatrix} = h_6\Phi_j + \begin{pmatrix}u_6\\ v_6\end{pmatrix},$$

where

$$\begin{pmatrix}u_5\\ v_5\end{pmatrix} = \begin{pmatrix}\phi_j^2 - h_5\phi_i\\ -\phi_j^2 - h_5 e_i\phi_i\end{pmatrix}, \quad h_5 = \sqrt{\frac{1}{2\pi}}\frac{e_i^* - 1}{e_i e_i^* + 1} = h_4,$$

$$\begin{pmatrix}u_6\\ v_6\end{pmatrix} = \begin{pmatrix}\phi_i\phi_j - h_6\phi_j\\ -\phi_i\phi_j - h_6 e_j\phi_j\end{pmatrix}, \quad h_6 = \sqrt{\frac{1}{2\pi}}\frac{e_j^* - 1}{e_j e_j^* + 1} = h_3.$$

Therefore, we can assume that

$$\begin{aligned}
&\mathcal{K}_{(d_1,s,W)}(\widetilde{d_1}, 0, 0; \omega_0)(d_1, s, W)\\
=&L(\widetilde{d_1})W + (-d_1h_1\lambda_i\cos\omega_0 + sh_4A_3\sin^2\omega_0)\Phi_i\\
&+ (-d_1h_2\lambda_j\sin\omega_0 + sh_3A_2\cos\omega_0\sin\omega_0)\Phi_j - d_1\lambda_i\cos\omega_0\begin{pmatrix}u_1\\ v_1\end{pmatrix}\\
&- d_1\lambda_j\sin\omega_0\begin{pmatrix}u_2\\ v_2\end{pmatrix} + sA_3\sin^2\omega_0\begin{pmatrix}u_5\\ v_5\end{pmatrix}\\
&+ sA_2\cos\omega_0\sin\omega_0\begin{pmatrix}u_6\\ v_6\end{pmatrix} + sA_1\cos^2\omega_0\begin{pmatrix}\phi_i^2\\ -\phi_i^2\end{pmatrix}.
\end{aligned}$$

By similar arguments as in **Case I**, we can prove that $\mathcal{K}_{(d_1,s,W)}(\widetilde{d}_1, 0, 0; \omega_0)$ is an isomorphism if ω_0 satisfies (5.9). The conclusion will be derived from the implicit function theorem. The whole proof is finished. $\square$

Remark 5.1. It follows from the expression of $d_{1,i}$ that if d_2, i, j and p satisfy

$$d_2^2(1+ka^p)i^2j^2 + d_2(1+ka^p)(i^2+j^2) - p = 0,$$

then for $i \neq j$, we have $d_{1,i} = d_{1,j}$. For example, taking $a = 1, p = 3, k = 1.5$ and $d_2 = 0.039$ leads to $d_{1,1} = d_{1,2}$, which can be seen in Fig.9.

Remark 5.2. When $j \neq 2i$ and $i \neq 2j$ is not established, the existence of non-constant positive solutions of (5.1). In this case, we have $\int_0^\pi \phi_i^2\phi_j dx = \int_0^\pi \phi_j^2\phi_i dx = 0$, which implies

$$\begin{pmatrix} -\phi_i^2 \\ \phi_i^2 \end{pmatrix}, \begin{pmatrix} -\phi_j^2 \\ \phi_j^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\phi_i\phi_j \\ \phi_i\phi_j \end{pmatrix} \in Y_2,$$

and we do not need any more decompositions. However, $\mathcal{K}_{(d_1,s,W)} : (\widetilde{d}_1, 0, 0; \omega_0)R^+ \times R \times X_2 \to Y$ is not an isomorphism at this time. So we can not use the implicit function theorem to obtain the existence result for the case that $j \neq 2i$ and $i \neq 2j$.

5.2. Global bifurcation structure

In this subsection, we extend the local bifurcation obtained in Theorem 5.2 (i) to the global one. Let J be the closure of the non-constant solution set of system (5.1) and Γ_i the connected component of $J \cup \{(d_{1,i}, (u^*, v^*))\}$. We further study the global bifurcation structure and get more information on the bifurcation curve Γ_i. Our method is based on the global bifurcation theory of Rabinowitz and Leray-Schauder degree theory for compact operators.

Theorem 5.3. *Under the same hypothesis of Theorem 5.2 (i), the projection of the bifurcation curve Γ_i can be extended to infinity in $(d_{1,i}, +\infty)$. Furthermore, if $d_1 > d_1^*$ and $d_1 \neq d_{1,j}$ for any integer $j > 0$, system (5.1) has at least one non-constant positive solution, where d_1^* is defined by (5.4).*

Proof. We first rewrite system (5.1) as

$$\begin{cases} -d_1u'' = -\dfrac{a^p}{1+ka^p}u - \dfrac{p}{1+ka^p}v + F^1(u,v), \quad x \in (0,\pi), \\ -d_2v'' = \dfrac{a^p}{1+ka^p}u + \dfrac{p-1-ka^p}{1+ka^p}v + F^2(u,v), \quad x \in (0,\pi), \end{cases} \tag{5.16}$$

where $F^2(u,v) = -F^1(u,v)$ and $F^1(u,v)$ can be found in (5.11).

Denote $G_{d_1} : l \to \theta$ by the operator for the following problem

$$-d_1\theta'' + \frac{a^p}{1+ka^p}\theta = l \quad \text{in} \quad (0,\pi), \quad \theta' = 0 \quad \text{at} \quad x = 0, \pi,$$

and $G_{d_2} : l \to \theta$ for

$$-d_2\theta'' + \frac{p-1-ka^p}{1+ka^p}\theta = l \quad \text{in} \quad (0,\pi), \quad \theta' = 0 \quad \text{at} \quad x = 0, \pi.$$

Let $U = (u, v)^T$, then we have

$$\mathcal{K}(d_1)U = (-\frac{p}{1+ka^p}G_{d_1}(v), \ \ \frac{a^p}{1+ka^p}G_{d_2}(u) + \frac{2(p-1-ka^p)}{1+ka^p}G_{d_2}(v)),$$

and

$$H(U) = (G_{d_1}(F^1(u, v)), G_{d_2}(F^2(u, v))).$$

Then the equations in (5.16) can be transmitted to

$$U = \mathcal{K}(d_1)U + H(U). \tag{5.17}$$

Note that $\mathcal{K}(d_1)$ is a compact linear operator on X for any fixed $d_1 > 0$. $H(U) = o(|U|)$ for U near zero uniformly on closed d_1 sub-intervals of $(0, \infty)$, and it is also a compact operator on X.

To apply the global bifurcation theorem in [31], we first prove that 1 is an eigenvalue of $\mathcal{K}(d_{1,i})$ with algebraic multiplicity one. From Theorem 5.2 (i), it is easy to see that $\ker(K(d_{1,i}) - I) = \ker(L(d_{1,i})) = \mathrm{span}\{\Phi_i\}$. Hence, 1 is an eigenvalue of $\mathcal{K}(d_{1,i})$, and $\dim\ker(\mathcal{K}(d_{1,i}) - I) = 1$. Since the algebraic multiplicity of the eigenvalue 1 is equal to the dimension of the generalized null space $\cup_{n=1}^{\infty} \ker\left(\mathcal{K}(d_{1,i}) - I\right)^n$, we only need to verify that $\ker(\mathcal{K}(d_{1,i}) - I) \cap R(\mathcal{K}(d_{1,i}) - I) = \{0\}$. Let $\mathcal{K}^*(d_{1,i})$ be the adjoint operator of $\mathcal{K}(d_{1,i})$. For any $(\varphi, \chi) \in \ker(\mathcal{K}^*(d_{1,i}) - I)$, we have

$$\frac{a^p}{1+ka^p}G_{d_2}(\chi) = \varphi, \qquad -\frac{p}{1+ka^p}G_{d_{1,i}}(\varphi) + \frac{2(p-1-ka^p)}{1+ka^p}G_{d_2}(\chi) = \chi.$$

By the definitions of G_{d_1} and G_{d_2}, we have

$$\begin{aligned}
-d_2\varphi'' &= -\frac{p-1-ka^p}{1+ka^p}\varphi + \frac{a^p}{1+ka^p}\chi, \\
-d_{1,i}\chi'' &= \frac{p-2-2ka^p}{1+ka^p}\varphi - \frac{2(p-1-ka^p)d_{1,i}}{a^p(1+ka^p)d_2}\varphi + \frac{2(p-1-ka^p)d_{1,i} - a^p d_2}{(1+ka^p)d_2}\chi.
\end{aligned}$$

Simple calculations lead to

$$\ker(\mathcal{K}^*(d_{1,i}) - I) = \widetilde{\Phi}_i, \qquad \widetilde{\Phi}_i = \begin{pmatrix} \dfrac{a^p}{1+ka^p} \\ \dfrac{p-1-ka^p}{1+ka^p} + d_2\lambda_i \end{pmatrix}\phi_i.$$

In addition, we know

$$\int_0^\pi {\Phi_i}^T \widetilde{\Phi}_i dx = \frac{2d_2a^p(1+ka^p)}{(1+ka^p)[(1+d_2\lambda_i)(1+ka^p) - p]} < 0,$$

which means that $\Phi_i \notin (\ker(\mathcal{K}^*(d_{1,i}) - I))^\perp = R(\mathcal{K}(d_{1,i}) - I)$. Hence, we have $\ker(\mathcal{K}(d_{1,i}) - I) \cap R(\mathcal{K}(d_{1,i}) - I) = \{0\}$ and the algebraic multiplicity of the eigenvalue 1 is one.

If $d_1 \neq d_{1,i}$ is in a small neighborhood of $d_{1,i}$, then the linear operator $I - \mathcal{K}(d_1) : X \to X$ is a bijection and $(0, 0)$ is an isolated solution of (5.17). Define

$$i(I - \mathcal{K}(d_1) - H, (d_1, 0)) = deg(I - \mathcal{K}(d_1), B, 0) = (-1)^p,$$

where B is a sufficiently small ball centered at 0, and p is the sum of the algebraic multiplicities of the eigenvalues of $\mathcal{K}(d_1)$ that are greater than one. We next show that the index changes when d_1 crosses $d_{1,i}$, which implies for $\varepsilon > 0$ sufficiently small,

$$i(I - \mathcal{K}(d_{1,i} - \varepsilon) - H, (d_{1,i} - \varepsilon, 0)) \neq i(I - \mathcal{K}(d_{1,i} + \varepsilon) - H, (d_{1,i} + \varepsilon, 0)). \quad (5.18)$$

Suppose that μ is the eigenvalue of $\mathcal{K}(d_{1,i})$ with eigenfunction (φ, χ). Then (φ, χ) satisfies

$$-\mu d_1 \varphi'' = -\frac{a^p}{1+ka^p}\mu\varphi - \frac{p}{1+ka^p}\chi,$$

$$-\mu d_2 \chi'' = \frac{a^p}{1+ka^p}\varphi + \frac{2(p-1-ka^p)}{1+ka^p}\chi - \frac{p-1-ka^p}{1+ka^p}\mu\chi.$$

Using the Fourier cosine series $\varphi = \sum\limits_{0\le j\le\infty} a_j\phi_j$ and $\chi = \sum\limits_{0\le j\le\infty} b_j\phi_j$, we have

$$\sum_{0\le j\le\infty}\begin{pmatrix} -(\frac{a^p}{1+ka^p} + d_1\lambda_j)\mu & -\frac{p}{1+ka^p} \\ \frac{a^p}{1+ka^p} & \frac{2(p-1-ka^p)}{1+ka^p} - (\frac{p-1-ka^p}{1+ka^p} + d_2\lambda_j)\mu \end{pmatrix}$$
$$\times \begin{pmatrix} a_j \\ b_j \end{pmatrix}\phi_j = 0.$$

The characteristic equation is given by

$$\mu^2 - 2(p-1-ka^p)\mu + \frac{pa^p}{a^p + d_1\lambda_j(1+ka^p)} = 0, \quad (5.19)$$

where the integer j is from zero to ∞. For $d_1 = d_{1,i}$, if $\mu = 1$ is a root of (5.19), we find that $d_{1,i} = d_{1,j}$ by the definition of $d_{1,i} = d_{1,j}$ and so $i = j$ by the assumption. Hence, without counting the eigenvalues corresponding to $j \neq i$ in (5.19), $\mathcal{K}(d_1)$ has the same number of eigenvalues greater than 1 for all d_1 close to $d_{1,i}$. Moreover, they have the same algebraic multiplicities. On the other hand, for $j = i$ in (5.19), let $\mu(d_1)$, $\widetilde{\mu}(d_1)$ be the two roots of (5.19). Then we have

$$\mu(d_{1,i}) = 1 \quad \text{and} \quad \widetilde{\mu}(d_{1,i}) = \frac{p-1-ka^p - d_2\lambda_i(1+ka^p)}{p-1-ka^p + d_2\lambda_i(1+ka^p)} < 1.$$

If d_1 is close to $d_{1,i}$, then we have $\widetilde{\mu}(d_1) < 1$. Since $\mu(d_1)$ is an increasing function with respect to d_1, we have

$$\mu(d_{1,i} + \varepsilon) > 1 \quad \text{and} \quad \mu(d_{1,i} - \varepsilon) < 1.$$

Therefore, $\mathcal{K}(d_{1,i} + \varepsilon)$ has exactly one more eigenvalue which is greater than 1 than $\mathcal{K}(d_{1,i} - \varepsilon)$ does. By a similar argument above, the algebraic multiplicity of this eigenvalue is also one. Hence, we verify that (5.18) holds true. Therefore, using Theorem 1.3 in [31], we conclude that Γ_i either meets infinity in $\mathbb{R} \times X$ or meets $(d_{1,j}, (u^*, v^*))$ for some $j \neq i$, $d_{1,j} > 0$. Furthermore, by using of the idea of Nishiura [28] and Takagi [35], the bifurcating curve Γ_i must be extended to infinity in $\mathbb{R} \times X$. The proof is completed. □

5.3. Bifurcation direction

In this subsection, the direction of the steady-state bifurcation from simple eigenvalues obtained in Theorem 5.2 (i) is investigated.

It follows from Theorem 5.2 that

$$\dim \ker F_U(d_{1,i},(0,0)) = \operatorname{codim} R(F_U(d_{1,i},(0,0))) = 1,$$

and $\ker F_U(d_{1,i},(0,0)) = \operatorname{span}\{\Phi_i\}$. So X and Y can be decomposed as

$$X = \ker F_U(d_{1,i},(0,0)) \oplus Z \quad \text{and} \quad Y = R(F_U(d_{1,i},(0,0))) \oplus \bar{Z},$$

where Z and $\bar{Z}$ are the complement of $\ker F_U(d_{1,i},(0,0))$ in X and $R(F_U(d_{1,i},(0,0)))$ in Y, respectively. By (5.10), we get

$$\langle F_{d_1 U}(d_{1,i},(0,0))\Phi_i, \Phi_i^* \rangle = -\lambda_i = -i^2 \neq 0.$$

We first calculate $d_1'(0)$. The expression (4.5) in [32] gives

$$d_1'(0) = -\frac{\langle F_{UU}(d_{1,i},(0,0))\Phi_i^2, \Phi_i^* \rangle}{2\langle F_{d_1 U}(d_{1,i},(0,0))\Phi_i, \Phi_i^* \rangle}.$$

By calculations, we have

$$\langle F_{UU}(d_{1,i},(0,0))\Phi_i^2, \Phi_i^* \rangle = (k_i + l_i e_i^*) \int_0^\pi \phi_i^3 dx = 0,$$

where $k_i = -l_i$ and

$$l_i = 2c_1 e_i + c_2 e_i^2 = \frac{p e_i[2a^p + (p-1)e_i - k(p+1)a^p e_i]}{a(1+ka^p)^2}. \tag{5.20}$$

Thus we get $d_1'(0) = 0$.

Continuing to calculate $d_1''(0)$, which can also be read from [32],

$$d_1''(0) = -\frac{\langle F_{UUU}(d_{1,i},(0,0))\Phi_i^3, \Phi_i^* \rangle + 3\langle F_{UU}(d_{1,i},(0,0))\Phi_i\theta, \Phi_i^* \rangle}{3\langle F_{d_1 U}(d_{1,i},(0,0))\Phi_i, \Phi_i^* \rangle},$$

where θ is the solution of the following problem

$$F_{UU}(d_{1,i},(0,0))\Phi_i^2 + F_U(d_{1,i},(0,0))\theta = 0.$$

Some calculations give

$$\langle F_{UUU}(d_{1,i},(0,0))\Phi_i^3, \Phi_i^* \rangle = \frac{4}{\pi^2}(m_i + n_i e_i^*) \int_0^\pi \cos^4(ix) dx = \frac{3}{2\pi}(m_i + n_i e_i^*),$$

where $m_i = -n_i$ and $n_i = (3c_3 + c_4 e_i)e_i^2$.

Let $\theta = (\theta_1, \theta_2)$. Then it satisfies

$$\begin{cases} d_{1,i}\theta_1'' - \dfrac{a^p}{1+ka^p}\theta_1 - \dfrac{p}{1+ka^p}\theta_2 = -k_i\phi_i^2, \quad x \in (0,\pi), \\ d_2\theta_2'' + \dfrac{a^p}{1+ka^p}\theta_1 + \dfrac{p-1-ka^p}{1+ka^p}\theta_2 = -l_i\phi_i^2, \, x \in (0,\pi), \\ \theta_i'(0) = \theta_i'(\pi) = 0, \quad i = 1,2. \end{cases} \tag{5.21}$$

Integrating (5.21) on $[0, \pi]$, and solving the linear equation group, we have

$$\int_0^\pi \theta_1 dx = -\frac{(1+ka^p)l_i}{a^p}, \qquad \int_0^\pi \theta_2 dx = 0. \tag{5.22}$$

By calculations, we know that

$$\langle F_{UU}(d_{1,i},(0,0))\Phi_i\theta, \Phi_i^*\rangle = C_1^* \int_0^\pi \theta_1\phi_i^2 dx + C_2^* \int_0^\pi \theta_2\phi_i^2 dx,$$

where

$$C_1^* = \frac{pa^p e_i(e_i^*-1)}{a(1+ka^p)^2}, \qquad C_2^* = \frac{pe_i(e_i^*-1)\{a^p + [(p-1)-k(p+1)a^p]\}}{a(1+ka^p)^2}.$$

Multiplying (5.21) by ϕ_i^2, we obtain

$$\begin{cases} d_{1,i}\displaystyle\int_0^\pi \theta_{1''}\phi_i^2 dx - \frac{a^p}{1+ka^p}\int_0^\pi \theta_1\phi_i^2 dx - \frac{p}{1+ka^p}\int_0^\pi \theta_2\phi_i^2 dx = -\frac{3}{2\pi}k_i, \\ d_2\displaystyle\int_0^\pi \theta_{2''}\phi_i^2 dx + \frac{a^p}{1+ka^p}\int_0^\pi \theta_1\phi_i^2 dx + \frac{p-1-ka^p}{1+ka^p}\int_0^\pi \theta_2\phi_i^2 dx = -\frac{3}{2\pi}l_i. \end{cases} \tag{5.23}$$

Using integration by parts, we get

$$\int_0^\pi \theta_j''\phi_i^2 dx = \frac{4i^2}{\pi}\int_0^\pi \theta_j(1-\pi\phi_i^2)dx, \quad j=1,2. \tag{5.24}$$

Substitute (5.22) and (5.24) to (5.23) to get

$$\begin{cases} \quad (\dfrac{a^p}{1+ka^p} + 4i^2 d_{1,i})\displaystyle\int_0^\pi \theta_1\phi_i^2 dx + \frac{p}{1+ka^p}\int_0^\pi \theta_2\phi_i^2 dx \\ = \dfrac{3}{2\pi}k_i - \dfrac{4i^2(1+ka^p)l_i d_{1,i}}{\pi a^p}, \\ \dfrac{a^p}{1+ka^p}\displaystyle\int_0^\pi \theta_1\phi_i^2 dx + (\frac{p-1-ka^p}{1+ka^p} - 4i^2 d_2)\int_0^\pi \theta_2\phi_i^2 dx = -\frac{3}{2\pi}l_i. \end{cases}$$

Solve the system above and then we have

$$\beta_1 \triangleq \int_0^\pi \theta_1\phi_i^2 dx = \frac{l_i(1+ka^p)C_3^*}{2\pi a^p C_4^*}, \qquad \beta_2 \triangleq \int_0^\pi \theta_2\phi_i^2 dx = \frac{2i^2 d_{1,i} l_i(1+ka^p)}{\pi C_4^*},$$

where

$$C_3^* = 4i^2[2d_{1,i}(p-1-ka^p) - 8i^2 d_{1,i}d_2(1+ka^p) - 3d_2a^p] - 3a^p(1+ka^p),$$
$$C_4^* = [a^p - 4i^2 d_{1,i}(p-1-k^p) + 4i^2 d_2 a^p + 16i^4 d_{1,i}d_2(1+ka^p)].$$

In totally, we have

$$d_1''(0) = \frac{m_i(1-e_i^*) + 2\pi(C_1^*\beta_1 + C_2^*\beta_2)}{2\pi\lambda_i}. \tag{5.25}$$

Thus, by the sign of $d_1''(0)$, we can establish the following theorem to determine the local bifurcation direction.

Theorem 5.4. *The bifurcation from $(d_{1,i},(0,0))$ obtained in Theorem 5.2 (i) is subcritical if $d_1''(0) < 0$ and it is supercritical if $d_1''(0) > 0$, where $d_1''(0)$ is given by (5.25).*

6. Numerical simulations

In previous sections, we establish the stability and existence of spatial pattern for system (1.2) by the stability analysis, topological degree theory and bifurcation theory, which indicate the rich dynamics in this system. In this section, we shall give some numerical examples to illustrate the spatiotemporal patterns formation corresponding to the analytical results obtained above. Let $\Omega = (0,\pi)$. Then we have $\lambda_1 = 1$.

Example 1. Take $a = 0.9, p = 2, k = 0.8, d_1 = 0.5, d_2 = 1$ and $(u_0, v_0) = (1.8311 + 0.2\cos x, 0.9 + 0.2\cos x)$. Then $k_0 = 0.2346, z_1 = 0.021$. The conditions $k_0 < k < k_0 + 1$ and $d_2/d_1 > z_1$ hold true. By Theorem 2.2 (ii), the equilibrium E^* is stable for system (1.2). See Fig. 1.

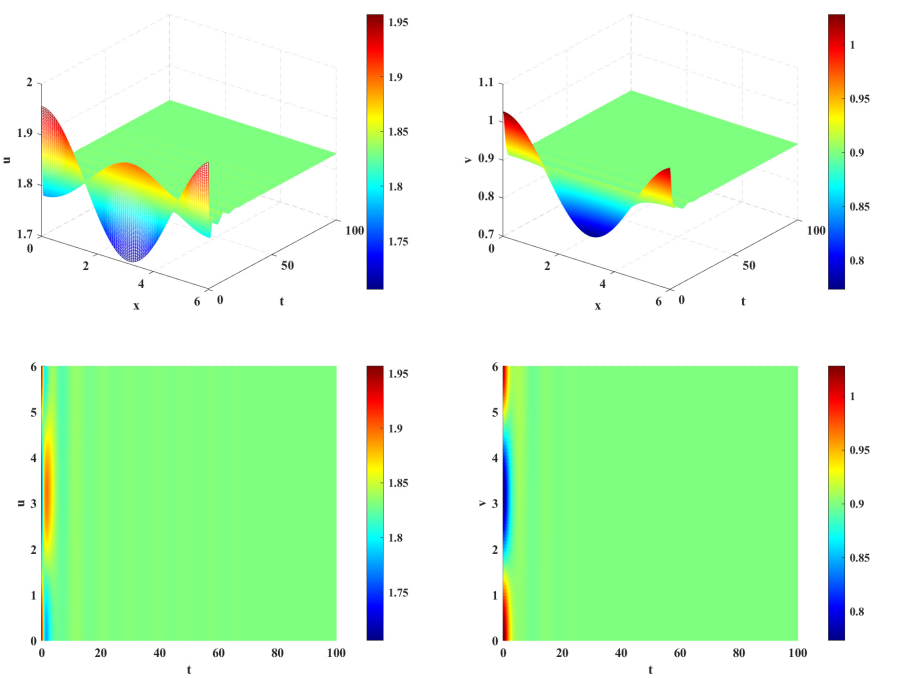

Figure 1. The equilibrium E^* of (1.2) is stable for $a = 0.9, p = 2, k = 0.8, d_1 = 0.5, d_2 = 1$.

Example 2. Take $a = 0.9, p = 2, k = 0.23$ and $(u_0, v_0) = (1.3181 + 0.2\cos x, 0.9 + 0.2\cos x)$. System (1.2) has positive periodic solutions. See Fig. 2 for $d_1 = 0.5$ and $d_2 = 1$.

In addition, we also depict the trajectory graphs of the equilibrium E^* for system (1.2). Obviously, for Example 1, the equilibrium E^* is asymptotically stable. See Fig. 3. However, for Example 2, the equilibrium E^* losses its stability and Hopf bifurcation occurs. See Fig. 4.

Example 3. Take $a = 1, p = 3, k = 1.3, d_1 = 1, d_2 = 0.05$ and $(u_0, v_0) = (2.3 + 0.2\cos x, 1 + 0.2\cos x)$. Then $k_0 = 1$ and from (2.6), $\lambda^* = \dfrac{p - 1 - ka^p}{d_2(1 + ka^p)} = 6.087$ and $\lambda_1 = 1$. Hence, by Theorem 2.3, the equilibrium E^* is Turing unstable for system

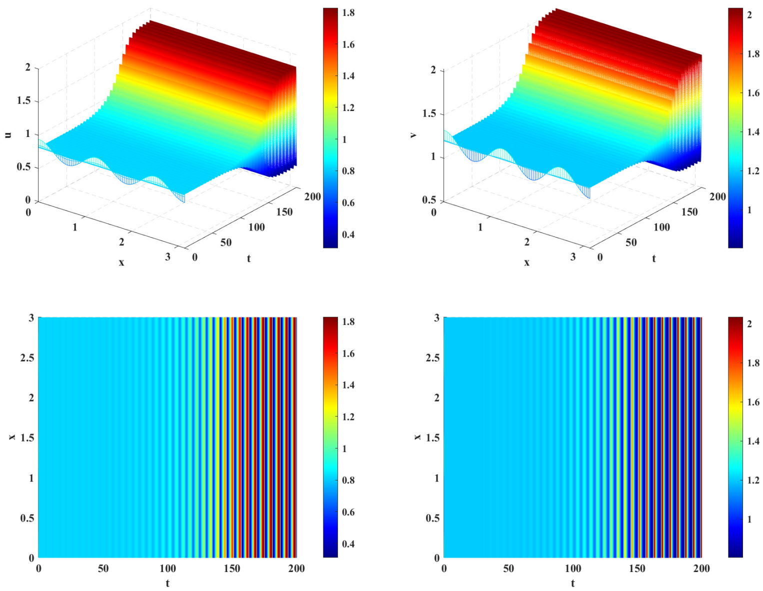

Figure 2. Positive periodic solution of (1.2) for $a = 0.9, p = 2, k = 0.23, d_1 = 0.5, d_2 = 1$.

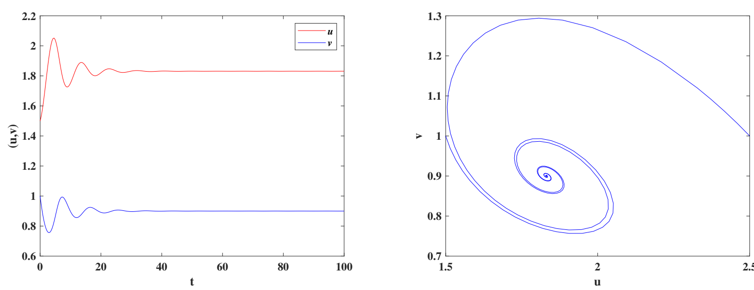

Figure 3. The trajectory graph (left) and phase portrait (right) of (1.2) for $a = 0.9, p = 2, k = 0.8, d_1 = 0.5, d_2 = 1$.

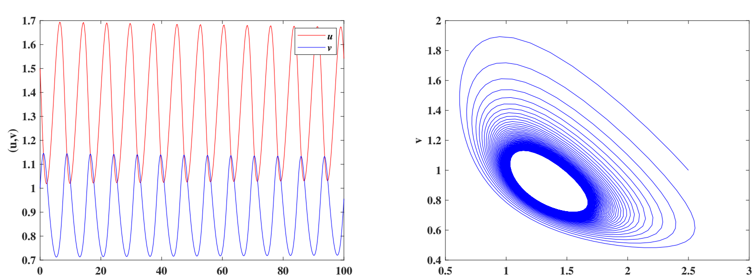

Figure 4. The trajectory graph (left) and phase portrait (right) of (1.2) for $a = 0.9, p = 2, k = 0.23, d_1 = 0.5, d_2 = 1$.

(1.2). See Fig. 5.

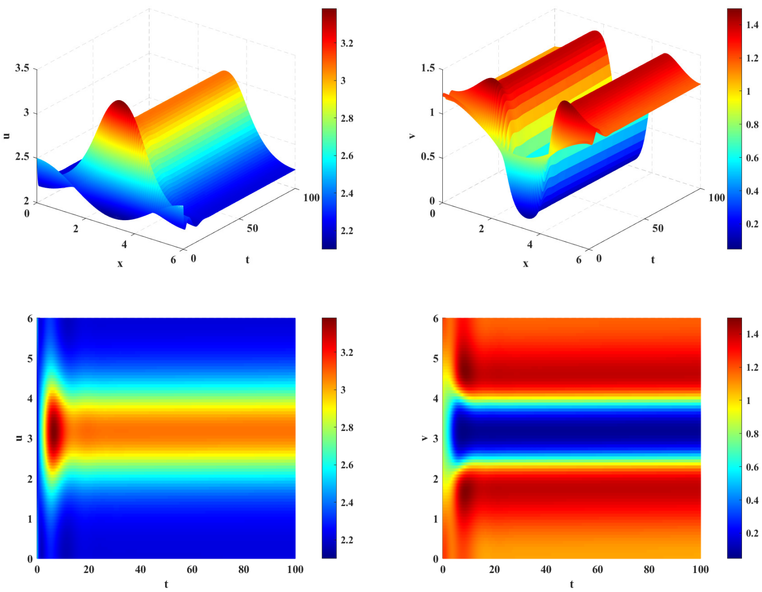

Figure 5. The equilibrium E^* of (1.2) is Turing unstable for $a = 1, p = 4, k = 2, d_1 = 1, d_2 = 0.05$.

Example 4. Take $a = 1, p = 3, k = 1.5, d_2 = 0.045$ and $(u_0, v_0) = (2.5 + 0.2\cos x, 1 + 0.2\cos x)$. It follows from (5.3) that $d_{1,1} = 2.6968$ and $d_{1,2} = 5.9$. By Theorem 5.2 (i), a steady-state bifurcation occurs at $d_{1,i}, i = 1, 2$. And by Theorem 5.4, the direction of bifurcation from $(d_{1,1}, E^*)$ is subcritical and the direction of bifurcation from $(d_{1,2}, E^*)$ is supercritical. See Fig. 6 for $d_1 = 2.7$ and Fig. 7 for $d_1 = 5.9$.

Example 5. Take $a = 1, p = 3, k = 1.5, d_2 = 0.039$ and $(u_0, v_0) = (2.5 + 0.2\cos x, 1+0.2\cos x)$. It follows from (5.3) that $d_{1,1} = d_{1,2} = \widehat{d_1} = 2.5776$. By Theorem 5.2 (ii), a steady-state bifurcation occurs at $d_1 = \widehat{d_1}$, which is shown in Fig. 8, where we choose $d_1 = 2.6$. We must point out that this steady-state bifurcation is from the double eigenvalue.

The neutral curves d_1 with respect to $i \in \mathbb{R}$ are shown in Fig. 9. Clearly, $d_{1,1} \neq d_{1,2}$ in the left of Fig. 9 and $d_{1,1} = d_{1,2}$ in the right of Fig. 9, which implies the steady-state bifurcation in Figs. 6,7 are from the simple eigenvalue and that in Fig. 8 is from the double eigenvalue.

Example 6. Take $a = 1, p = 4, k = 2, d_2 = 0.14$ and $(u_0, v_0) = (3 + 0.2\cos 5x, 1 + 0.2\cos 5x)$. System (1.2) can induce spatially inhomogeneous Hopf bifurcation. See Fig. 10 for $d_1 = 1.9$. Fix $d_2 = 0.14$, we also find that as d_1 increases system (1.2) can still present the steady-state bifurcation. See Fig. 11 for $d_1 = 2.4$.

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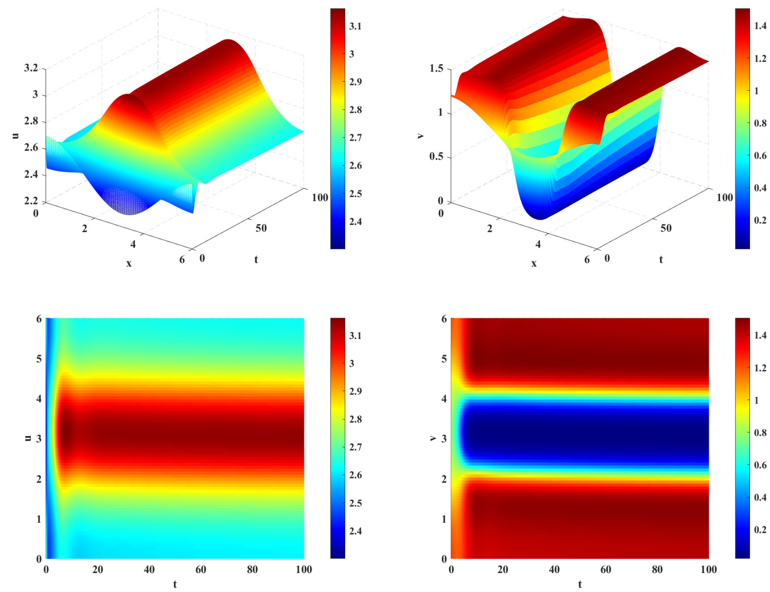
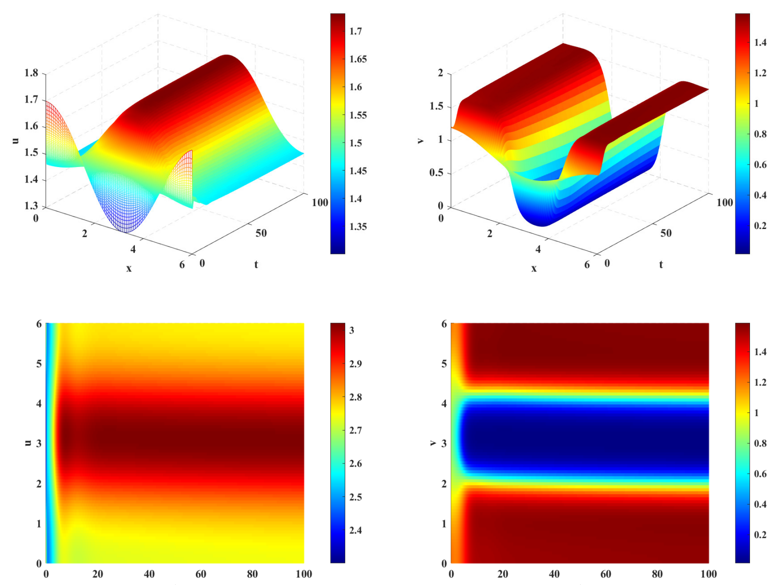


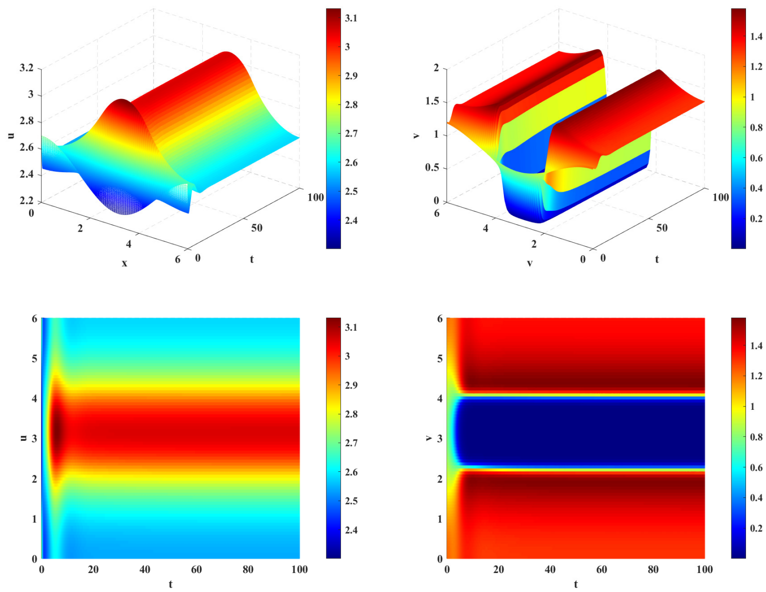
$d_1 = 2.7$ $d_1 = 2.7$

Figure 6. Steady-state bifurcation solution at the simple eigenvalue of (1.3) for $a = 1, p = 3, k = 1.5, d_2 = 0.045$. Here, $d_1 = 2.7$.



$d_1 = 5.9$ $d_1 = 5.9$

Figure 7. Steady-state bifurcation solution at the simple eigenvalue for $a = 1, p = 3, k = 1.5, d_2 = 0.045$. Here $d_1 = 5.9$.



$d_1 = 2.6$ $d_1 = 2.6$

Figure 8. Steady-state bifurcation solution at the double eigenvalue for $a = 1, p = 3, k = 1.5, d_2 = 0.039$. Here $d_1 = 2.6$.

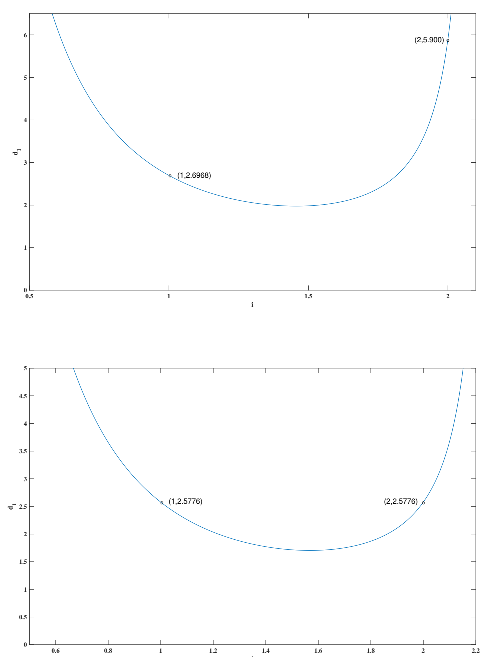
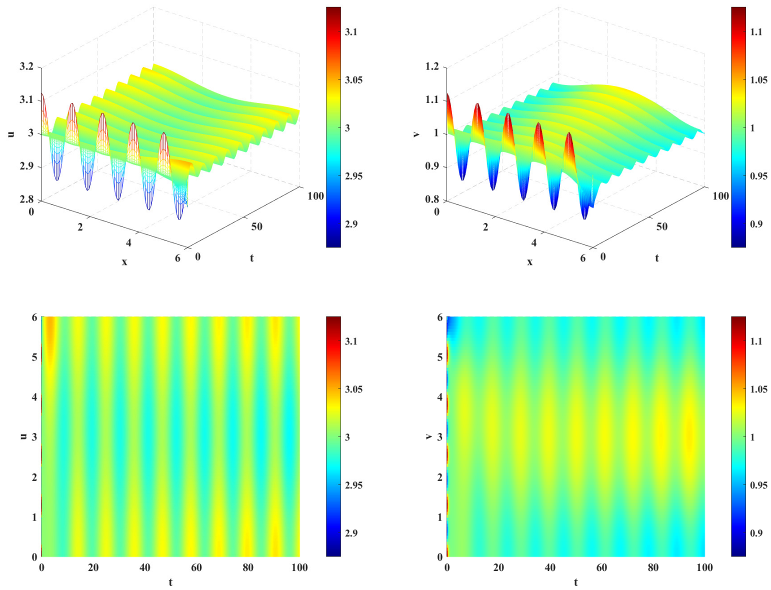
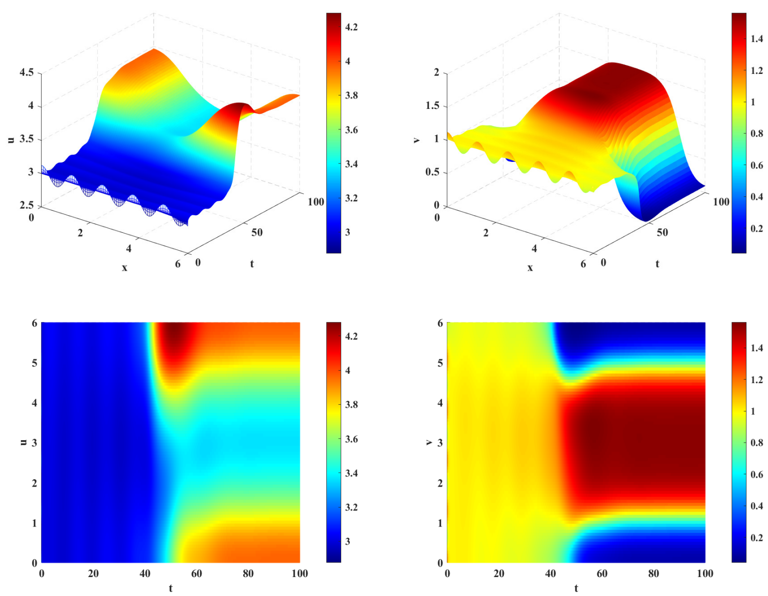


Figure 9. The neutral curves d_1 about $i \in \mathbb{N}$ for $a = 1, p = 3, k = 1.5$. Left: $d_2 = 0.045$; Right: $d_2 = 0.039$.



$d_1 = 1.9$ $d_1 = 1.9$

Figure 10. Positive periodic solution of (1.2) for $a = 1, p = 4, k = 2, d_2 = 0.14$. Here $d_1 = 1.9$.



$d_1 = 2.4$ $d_1 = 2.4$

Figure 11. Positive periodic solution shifts to the steady-state for $a = 1, p = 4, k = 2, d_2 = 0.14$. Here $d_1 = 2.4$.

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