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# ON BEST PROXIMITY POINT APPROACH TO SOLVABILITY OF A SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

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**Abstract** In this article, a class of cyclic (noncyclic) condensing operators is defined on a Banach space using the notion of measure of noncompactness and $C$-class functions. For these newly defined condensing operators, best proximity point (pair) results are manifested. Then the obtained main results are applied to demonstrate the existence of optimum solutions of a system of fractional differential equations involving $\psi$-Hilfer fractional derivatives.

**Keywords** Best proximity point (pair), measure of noncompactness, Hilfer fractional differential equation, $\psi$-Hilfer fractional derivative, cyclic mapping, noncyclic mapping.

**MSC(2010)** 47H10, 34A08, 47H08, 47H09.

## 1. Introduction and basic concepts

Various physical phenomena are represented mathematically using a mathematical model with the help of different forms of equations. Those equations may be differential, integral, integro differential or that of fractional type and functional equations. The existence of fixed point is equivalent to the existence of solution of an equation. This concept is very well utilised in demonstrating actuality of solutions of mathematical models. This fact made fixed point theory an indispensable tool in mathematics. In the absence of fixed point for a mapping, we search for the points which are most closed with the image of the point under a mapping. Such points are called best proximity points. In the last three decades there is significant development in the field of best proximity point (pair) results. The concept of best proximity point is applicable in establishing existence of optimum solutions for a system of equations. Let us recall the concept of best proximity points (pairs) in brief. Let $\mathfrak{X}$ be a normed linear space (NLS) and $\mathcal{G}$ and $\mathcal{H}$ be its nonempty subsets. It is understood that a pair $(\mathcal{G},\mathcal{H})$ holds a property, if both $\mathcal{G}$ and $\mathcal{H}$ individually hold that property. Let

$$\mathfrak{D}(\mathcal{G},\mathcal{H}) = \inf\{\|g-h\| : g \in \mathcal{G}, h \in \mathcal{H}\}$$

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defines distance between sets $\mathcal{G}$ and $\mathcal{H}$. We define

$$\mathcal{G}_0 = \{g \in \mathcal{G} :\ \exists\, h^\dagger \in \mathcal{H} \mid \|g - h^\dagger\| = \mathfrak{D}(\mathcal{G},\mathcal{H})\},$$
$$\mathcal{H}_0 = \{y \in \mathcal{H} :\ \exists\, x^\dagger \in \mathcal{G} \mid \|x^\dagger - y\| = \mathfrak{D}(\mathcal{G},\mathcal{H})\}.$$

It is proved in [10] that if $(\mathcal{G},\mathcal{H})$ is a pair of nonempty, convex and weakly compact subsets of a Banach space $\mathfrak{X}$ then the pair $(\mathcal{G}_0,\mathcal{H}_0)$ retains these three properties. If $\mathcal{G}_0$ is same as $\mathcal{G}$ and $\mathcal{H}_0$ is same as $\mathcal{H}$ then the pair $(\mathcal{G},\mathcal{H})$ is called proximinal pair in $\mathfrak{X}$. We consider a mapping $f : \mathcal{G}\cup\mathcal{H} \to \mathcal{G}\cup\mathcal{H}$. The mapping $f$ is said to be cyclic mapping if it maps $\mathcal{G}$ into $\mathcal{H}$ and $\mathcal{H}$ into $\mathcal{G}$ (i.e. $f(\mathcal{G}) \subseteq \mathcal{H}$ and $f(\mathcal{H}) \subseteq \mathcal{G}$). Whereas $f$ is said to be noncyclic if image of $\mathcal{G}$ under $f$ lies in $\mathcal{G}$ while image of $\mathcal{H}$ under $f$ lies in $\mathcal{H}$. The mapping $f$ is said to be relatively nonexpansive if it satisfies $\|f(a)-f(b)\| \le \|a-b\|$ where $a \in \mathcal{G}$ and $b \in \mathcal{H}$. A relatively nonexpansive mappings becomes nonexpansive mapping if $\mathcal{G} = \mathcal{H}$. Note that $f$ is said to be compact mapping if $(\overline{f(\mathcal{G})}, \overline{f(\mathcal{H})})$ is compact. A cyclic mapping $f$ can possess a best proximity point and is mathematically defined as a point $w^* \in \mathcal{G}\cup\mathcal{H}$ satisfying $\|w^* - Tw^*\| = \mathfrak{D}(\mathcal{G},\mathcal{H})$. In case of noncyclic mappings we consider existence of best proximity pair mathematically represented as pair $(g,h) \in (\mathcal{G},\mathcal{H})$ such that $g$ is fixed point of $f$ in $\mathcal{G}$, $h$ is fixed point of $f$ in $\mathcal{H}$ and $\|h-g\| = \mathfrak{D}(\mathcal{G},\mathcal{H})$.

Eldred et al. in [10] introduced the cyclic and noncyclic versions of relatively nonexpansive mappings and proved the existence of at least one best proximity point and pair in the setting of Banach spaces using a concept of proximal normal structure (in short, PNS). Gabeleh [12] proved the existence of best proximity points (pairs) in Banach space without the concept of PNS and using some intrinsic properties of space in the following results.

**Theorem 1.1** ( [12]). *A relatively nonexpansive cyclic mapping $f : \mathcal{G}\cup\mathcal{H} \to \mathcal{G}\cup\mathcal{H}$ possesses a best proximity point if $f$ is compact and $\mathcal{G}_0$ is nonempty, where $(\mathcal{G},\mathcal{H})$ is a nonempty, bounded, closed and convex (in short, $\mathcal{NBCC}$) pair in a Banach space $\mathfrak{X}$.*

Following result establishes existence of best proximity pairs for noncyclic mappings on a Banach space which is strictly convex. A Banach space $\mathfrak{X}$ is strictly convex if for $g,h,\alpha \in \mathfrak{X}$ and $\Delta > 0$, following holds

$$\big[\|g-\alpha\| \le \Delta,\ \|h-\alpha\| \le \Delta,\ g \ne h\big] \Rightarrow \Big\|\,\frac{g+h}{2} - \alpha\Big\| < \Delta.$$

**Theorem 1.2** ( [12]). *Let $(\mathcal{G},\mathcal{H})$ be an $\mathcal{NBCC}$ pair in a strictly convex Banach space $\mathfrak{X}$. A relatively nonexpansive noncyclic mapping $f : \mathcal{G}\cup\mathcal{H} \to \mathcal{G}\cup\mathcal{H}$ possesses a best proximity pair if $f$ is compact and $\mathcal{G}_0$ is nonempty.*

Theorems 1.1 and 1.2 are extensions of Schauder fixed point theorem in case of best proximity point and pairs. It is very well known that the compactness condition on mapping $f$ is very strong one. There is a technique called measure of noncompactness (MNC) which allows us to select classes of mappings that are more general than that of compact mappings. Realizing this fact, Darbo [9] and Sadovskii [26] genaralized the Schauder's fixed point theorem using the concept of MNC which is defined axiomatically as follows.

**Definition 1.1.** [3, 5, 6] Let $\mathbb{B}(\mathfrak{X})$ be a collection of bounded subsets in metric space $\mathfrak{X}$. An MNC is a mapping $\varkappa : \mathbb{B}(\mathfrak{X}) \to [0,\infty)$ that satisfies the following axioms:

1. $\varkappa(\mathcal{G}) = 0$ if and only if $\mathcal{G}$ is relatively compact,
2. $\varkappa(\mathcal{G}) = \varkappa(\overline{\mathcal{G}})$, $\mathcal{G} \in \mathbb{B}(\mathfrak{X})$,
3. $\varkappa(\mathcal{G} \cup \mathcal{H}) = \max\{\varkappa(\mathcal{G}), \varkappa(\mathcal{H})\}$, where $\mathcal{G}, \mathcal{H} \in \mathbb{B}(\mathfrak{X})$.

An MNC $\varkappa$ on $\mathbb{B}(\mathfrak{X})$ satisfies following properties:

(a) $\mathcal{G} \subset \mathcal{H}$ implies $\varkappa(\mathcal{G}) \leq \varkappa(\mathcal{H})$.

(b) $\varkappa(\mathcal{G}) = 0$ if $\mathcal{G}$ is a finite set.

(c) $\varkappa(\mathcal{G} \cap \mathcal{H}) = \min\{\varkappa(\mathcal{G}), \varkappa(\mathcal{H})\}$, for all $\mathcal{G}, \mathcal{H} \in \mathbb{B}(\mathfrak{X})$.

(d) If $\lim\limits_{n\to\infty} \varkappa(\mathcal{G}_n) = 0$ for a nonincreasing sequence $\{\mathcal{G}_n\}$ of nonempty, bounded and closed subsets of $\mathfrak{X}$, then $\mathcal{G}_\infty = \cap_{n\geq 1}\mathcal{G}_n$ is nonempty and compact.

$\varkappa$ satisfies the following properties on a Banach space $\mathfrak{X}$:

(i) $\varkappa(\overline{con}(\mathcal{H})) = \varkappa(\mathcal{H})$, for all $\mathcal{H} \in \mathbb{B}(\mathfrak{X})$;

(ii) $\varkappa(\lambda\mathcal{H}) = |\lambda|\varkappa(\mathcal{H})$ for any number $\lambda$ and $\mathcal{H} \in \mathbb{B}(\mathfrak{X})$;

(iii) $\varkappa(\mathcal{G} + \mathcal{H}) \leq \varkappa(\mathcal{G}) + \varkappa(\mathcal{H})$.

In particular, if $\mathcal{B}(\xi, \rho)$ denotes closed ball of radius $\rho$ with center $\xi$ and $diam(\mathcal{G})$ denotes diameter of the set $\mathcal{G}$, then the numbers

$$\alpha(\mathcal{G}) = \inf\{\gamma > 0 : \mathcal{G} \subset \cup_{i=1}^{N} G_i,\ diam(G_i) \leq \gamma, i = 1, 2, \cdots N\}$$

and

$$\beta(\mathcal{C}) = \inf\{\gamma > 0 : \mathcal{G} \subset \cup_{i=1}^{N} B(\xi_i, r), \xi_i \in \mathfrak{X}, i = 1, ..., N\},$$

assigned with a bounded subset $\mathcal{G}$ of a metric space $\mathfrak{X}$ are called Kuratowski MNC and Hausdorff MNC, respectively.

MNC is used to generalize the Schauder fixed point theorem by Darbo [9] and Sadovskii [26]. The combined statement of theorems of both the mathematicians is as follows:

**Theorem 1.3.** *Let $\mathcal{G}$ be a $\mathcal{NBCC}$ subset of a Banach space $\mathfrak{X}$ and $\varkappa$ be an MNC on $\mathfrak{X}$. A mapping $f$ on $\mathcal{G}$ has at least one fixed point in $\mathcal{G}$ if it is continuous and for every $\mathcal{M} \subset \mathcal{G}$ it satisfies any one of the following:*

*(D)* $\exists\ 0 \leq L < 1$ *such that* $\varkappa(f(\mathcal{M})) \leq L\varkappa(\mathcal{M})$,

*(S)* $\varkappa(\mathcal{M}) > 0$, $\varkappa(f(\mathcal{M})) < \varkappa(\mathcal{M})$.

A mapping satisfying condition $(D)$ is called $L$-set contraction (Darbo [9]) whereas satisfying $(S)$ is called as $\varkappa$-condensing (Sadovskii [26]).

Motivated by the Theorem 1.3, Gabeleh and Markin in [14] used the concept of MNC and generalized the Theorems 1.1 and 1.2 by relaxing the condition of compactness on mapping $f$. They also applied the obtained results to prove the existence of the optimum solutions of system of differential equations. Recently, the results of [14] have been generalized further in different directions in [13, 15–17, 20–24]) in which best proximity point and pairs results are obtained using MNC.

In this article, we prove best proximity point and pair theorems for a new class of cyclic and non-cyclic operators facilitated by MNC and $\mathcal{C}$-class functions. We apply the obtained results to prove existence of optimal solutions of system of fractional differential equation with initial value involving $\psi$-Hilfer fractional derivative. This is achieved by means of defining an operator from integral equations equivalent to the system of differential equations and proving that this operator has at least one best proximity point.

## 2. Main results

In this section, we prove our main results for best proximity points and pairs. We consider the following abstract function known as $\mathcal{C}$-class function.

**Definition 2.1.** [1,25] A function $\mathfrak{C} : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is called $\mathcal{C}$-class function if it satisfies the following conditions:

$(C_1)$ $\mathfrak{C}$ is continuous;

$(C_2)$ $\mathfrak{C}(\phi,\chi) \le \phi$;

$(C_3)$ $\mathfrak{C}(\phi,\chi) = \phi$ implies that either $\phi = 0$ or $\chi = 0$ for every $\phi,\chi \in \mathbb{R}$.

**Example 2.1.** Let us consider

(i) $\mathfrak{C}_1(\phi,\chi) = \phi - \chi$;

(ii) $\mathfrak{C}_2(\phi,\chi) = \lambda\phi$, $0 \le \lambda < 1$.

Then $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are $\mathcal{C}$-class functions.

Following theorem is our first main result. Some part of the proof and the concept of $T$-invariant pair is adopted from [16].

**Theorem 2.1.** *Let $(\mathcal{G},\mathcal{H})$ be a nonempty and convex pair in a Banach space $\mathfrak{X}$ with $\mathcal{G}_0$ being nonempty and $\varkappa$ being an MNC on $\mathfrak{X}$. A relatively nonexpansive cyclic mapping $f : \mathcal{G}\cup\mathcal{H} \to \mathcal{G}\cup\mathcal{H}$ has at least one best proximity point if for every $\mathcal{NBCC}$, proximinal and $f$ invariant pair $(\mathfrak{M}_1,\mathfrak{M}_2) \subseteq (\mathcal{G},\mathcal{H})$ with $\mathfrak{D}(\mathfrak{M}_1,\mathfrak{M}_2) = \mathfrak{D}(\mathcal{G},\mathcal{H})$, $f$ satisfies*

$$\begin{aligned}&\kappa\Big(\int_0^{\varkappa(f(\mathfrak{M}_1)\cup f(\mathfrak{M}_2))}\varphi(\xi)d\xi\Big)\\ \le&\mathfrak{C}\Big[\kappa\Big(\int_0^{\varkappa(\mathfrak{M}_1\cup\mathfrak{M}_2)}\varphi(\xi)d\xi\Big),\psi\Big(\kappa\big(\int_0^{\varkappa(\mathfrak{M}_1\cup\mathfrak{M}_2)}\varphi(\xi)d\xi\big)\Big)\Big],\end{aligned}\tag{2.1}$$

*where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous mapping, $\psi : [0,\infty) \to [0,1)$ is a continuous function and $\kappa : [0,\infty) \to [0,\infty)$ is a nondecreasing and continuous function with $\kappa(t) = 0$ if and only if $t = 0$.*

**Proof.** Since $\mathcal{G}_0$ is nonempty yields us $(\mathcal{G}_0,\mathcal{H}_0)$ is nonempty. It is evident that $(\mathcal{G}_0,\mathcal{H}_0)$ is convex, closed, $f$-invariant and proximinal pair considering conditions on $f$ (for more details see [16]). For $\phi \in \mathcal{G}_0$, there is a $\chi \in \mathcal{H}_0$ such that $\|\phi-\chi\| = \mathfrak{D}(\mathcal{G},\mathcal{H})$. Since $f$ is relatively non-expansive cyclic mapping

$$\|f\phi - f\chi\| \le \|\phi-\chi\| = \mathfrak{D}(\mathcal{G},\mathcal{H}),$$

which gives $f\phi \in \mathcal{G}_0$, that is, $f(\mathcal{G}_0) \subseteq \mathcal{H}_0$. Similarly, $f(\mathcal{H}_0) \subseteq \mathcal{G}_0$ and so $f$ is cyclic on $\mathcal{G}_0 \cup \mathcal{H}_0$.

We define a pair $(\mathfrak{G}_n,\mathfrak{H}_n)$ as follows $\mathfrak{G}_n = \overline{con}(f(\mathfrak{G}_{n-1}))$ and $\mathfrak{H}_n = \overline{con}(f(\mathfrak{H}_{n-1}))$, $n \ge 1$ with $\mathfrak{G}_0 = \mathcal{G}_0$ and $\mathfrak{H}_0 = \mathcal{H}_0$. We claim that $\mathfrak{G}_{n+1} \subseteq \mathfrak{H}_n$ and $\mathfrak{H}_n \subseteq \mathfrak{G}_{n-1}$ for all $n \in \mathbb{N}$. We have $\mathfrak{H}_1 = \overline{con}(f(\mathfrak{H}_0)) = \overline{con}(f(\mathcal{H}_0))) = \overline{con}(\mathcal{G}_0) \subseteq \mathcal{G}_0 = \mathfrak{G}_0$. Therefore, $f(\mathfrak{H}_1) \subseteq f(\mathfrak{G}_0)$. So $\mathfrak{H}_2 = \overline{con}(f(\mathfrak{H}_1)) \subseteq \overline{con}(f(\mathfrak{G}_0)) = \mathfrak{G}_1$. In general, we get $\mathfrak{H}_n \subseteq \mathfrak{G}_{n-1}$ by using induction. Similarly, we have $\mathfrak{G}_{n+1} \subseteq \mathfrak{H}_n$ for all $n \in \mathbb{N}$. Thus $\mathfrak{G}_{n+2} \subseteq \mathfrak{H}_{n+1} \subseteq \mathfrak{G}_n \subseteq \mathfrak{G}_{n-1}$ for all $n \in \mathbb{N}$. Hence, we get a decreasing sequence $\{(\mathfrak{G}_{2n}, \mathfrak{H}_{2n})\}$ of nonempty, closed and convex pairs in $\mathcal{G}_0 \times \mathcal{H}_0$. Moreover, $f(\mathfrak{H}_{2n}) \subseteq f(\mathfrak{G}_{2n-1}) \subseteq \overline{con}(f(\mathfrak{G}_{2n-1})) = \mathfrak{G}_{2n}$ and $f(\mathfrak{G}_{2n}) \subseteq f(\mathfrak{H}_{2n-1}) \subseteq \overline{con}(f(\mathfrak{H}_{2n-1})) = \mathfrak{H}_{2n}$. Therefore for all $n \in \mathbb{N}$, the pair $(\mathfrak{G}_{2n}, \mathfrak{H}_{2n})$ is $f$-invariant.

Now, if $(\mu, \nu) \in \mathcal{G}_0 \times \mathcal{H}_0$ is a proximinal pair then

$$\mathfrak{D}(\mathfrak{G}_{2n}, \mathfrak{H}_{2n}) \leq \|f^{2n}\mu - f^{2n}\nu\| \leq \|\mu - \nu\| = \mathfrak{D}(\mathcal{G}, \mathcal{H}).$$

Next, to show that the pair $(\mathfrak{G}_n, \mathfrak{H}_n)$ is proximinal. We use mathematical induction for this purpose. Obviously for $n = 0$, the pair $(\mathfrak{G}_0, \mathfrak{H}_0)$ is proximinal. Suppose that $(\mathfrak{G}_k, \mathfrak{H}_k)$ is proximinal. We show that $(\mathfrak{G}_{k+1}, \mathfrak{H}_{k+1})$ is also proximinal. Let $u$ be an arbitrary member in $\mathfrak{G}_{k+1} = \overline{con}(f(\mathfrak{G}_k))$. Then it is represented as $u = \sum_{l=1}^{m} \lambda_l f(u_l)$ with $u_l \in \mathfrak{G}_k$, $m \in \mathbb{N}$, $\lambda_l \geq 0$ and $\sum_{l=1}^{m} \lambda_l = 1$. Due to proximinality of the pair $(\mathfrak{G}_k, \mathfrak{H}_k)$, there exists $v_l \in \mathfrak{H}_k$ for $1 \leq l \leq m$ such that $\|u_l - v_l\| = \mathfrak{D}(\mathfrak{G}_k, \mathfrak{H}_k) = \mathfrak{D}(\mathcal{G}, \mathcal{H})$. Take $v = \sum_{l=1}^{m} \lambda_l f(v_l)$. Then $v \in \overline{con}(f(\mathfrak{H}_k)) = \mathfrak{H}_{k+1}$ and

$$\|u - v\| = \|\sum_{l=1}^{m} \lambda_l f(u_l) - \sum_{l=1}^{m} \lambda_l f(v_l)\| \leq \sum_{l=1}^{m} \lambda_l \|u_l - v_l\| = \mathfrak{D}(\mathcal{G}, \mathcal{H}).$$

This means that the pair $(\mathfrak{G}_{k+1}, \mathfrak{H}_{k+1})$ is proximinal and induction does the rest to prove $(\mathfrak{G}_n, \mathfrak{H}_n)$ is proximinal for all $n \in \mathbb{N}$.

We assume that $\varkappa(\mathfrak{G}_{2n} \cup \mathfrak{H}_{2n}) > 0$, else there is nothing left to prove. We prove that the sequence $\left\{ \int_0^{\varkappa(\mathfrak{G}_{2n} \cup \mathfrak{H}_{2n})} \varphi(\xi) d\xi \right\}$ is non-negative and non-increasing. Considering the construction of $\mathfrak{G}_{2n}$ and $\mathfrak{H}_{2n}$, we have

$$\begin{aligned}
\varkappa(\mathfrak{G}_{2n+1} \cup \mathfrak{H}_{2n+1}) &= \max\{\varkappa(\mathfrak{G}_{2n+1}), \varkappa(\mathfrak{H}_{2n+1})\} \\
&= \max\{\varkappa(\overline{con}(f(\mathfrak{H}_{2n}))), \varkappa(\overline{con}(f(\mathfrak{G}_{2n})))\} \\
&= \max\{\varkappa(f(\mathfrak{H}_{2n})), \varkappa(f(\mathfrak{G}_{2n}))\} = \varkappa(f(\mathfrak{G}_{2n}) \cup f(\mathfrak{H}_{2n})).
\end{aligned}$$

Keeping this fact in view and using (2.1), we have

$$\begin{aligned}
&\kappa\left( \int_0^{\varkappa(\mathfrak{G}_{2n+1} \cup \mathfrak{H}_{2n+1})} \varphi(\xi) d\xi \right) \\
&= \kappa\left( \int_0^{\varkappa(f(\mathfrak{G}_{2n}) \cup f(\mathfrak{H}_{2n}))} \varphi(\xi) d\xi \right) \\
&\leq \mathfrak{C}\left[ \kappa\left( \int_0^{\varkappa(\mathfrak{G}_{2n} \cup \mathfrak{H}_{2n})} \varphi(\xi) d\xi \right), \psi\left( \kappa\Big( \int_0^{\varkappa(\mathfrak{G}_{2n} \cup \mathfrak{H}_{2n})} \varphi(\xi) d\xi \Big) \right) \right].
\end{aligned} \tag{2.2}$$

Using Property $C_2$ of $\mathcal{C}$-class function, we get

$$\kappa\left( \int_0^{\varkappa(\mathfrak{G}_{2n+1} \cup \mathfrak{H}_{2n+1})} \varphi(\xi) d\xi \right) \leq \kappa\left( \int_0^{\varkappa(\mathfrak{G}_{2n} \cup \mathfrak{H}_{2n})} \varphi(\xi) d\xi \right).$$

As $\kappa$ is nondecreasing function, we get

$$\int_0^{\varkappa(\mathfrak{G}_{2n+1} \cup \mathfrak{H}_{2n+1})} \varphi(\xi) d\xi \leq \int_0^{\varkappa(\mathfrak{G}_{2n} \cup \mathfrak{H}_{2n})} \varphi(\xi) d\xi.$$

Thus, $\left\{ \int_0^{\varkappa(\mathfrak{G}_{2n} \cup \mathfrak{H}_{2n})} \varphi(\xi) d\xi \right\}$ is non-negative and non-increasing sequence bounded below. Therefore it converges to $r \geq 0$.

Passing to the limit $n \to \infty$, we get

$$\kappa(r) \le \mathfrak{C}(\kappa(r), \psi(\kappa(r))) \le \kappa(r),$$

which means

$$\mathfrak{C}(\kappa(r), \psi(\kappa(r))) = \kappa(r).$$

Property $(C_3)$ of $\mathfrak{C}$ yields $\kappa(r) = 0$. Hence by assumption of $\kappa$, $r = 0$. This gives us

$$\lim_{n\to\infty} \varkappa(\mathfrak{G}_{2n} \cup \mathfrak{H}_{2n}) = 0.$$

That is, $\lim_{n\to\infty} \varkappa(\mathfrak{G}_{2n} \cup \mathfrak{H}_{2n}) = \max\{\lim_{n\to\infty} \varkappa(\mathfrak{G}_{2n}), \lim_{n\to\infty} \varkappa(\mathfrak{H}_{2n})\} = 0$. Now let $\mathfrak{G}_\infty = \cap_{n=0}^{\infty}\mathfrak{G}_{2n}$ and $\mathfrak{H}_\infty = \cap_{n=0}^{\infty}\mathfrak{H}_{2n}$. By property $(d)$ of MNC, the pair $(\mathfrak{G}_\infty, \mathfrak{H}_\infty)$ is nonempty, convex, compact and $f$-invariant with $\mathfrak{D}(\mathfrak{G}_\infty, \mathfrak{H}_\infty) = \mathfrak{D}(\mathcal{G}, \mathcal{H})$. All this is sufficient to ensure that $f$ admits a best proximity point. □

Next result is an analogous of the above theorem for relatively nonexpansive noncyclic mapping which constitute second main result of the section.

**Theorem 2.2.** *Let $(\mathcal{G}, \mathcal{H})$ be a nonempty and convex pair in a strictly convex Banach space $\mathfrak{X}$ with $\mathcal{G}_0$ being nonempty and $\varkappa$ being an MNC on $\mathfrak{X}$. A relatively nonexpansive noncyclic mapping $f : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ has at least one best proximity pair if for every $\mathcal{NBCC}$, proximinal and $f$ invariant pair $(\mathfrak{M}_1, \mathfrak{M}_2) \subseteq (\mathcal{G}, \mathcal{H})$ with $\mathfrak{D}(\mathfrak{M}_1, \mathfrak{M}_2) = \mathfrak{D}(\mathcal{G}, \mathcal{H})$ equation (2.1) is satisfied.*

**Proof.** It is evident that $(\mathcal{G}_0, \mathcal{H}_0)$ is $\mathcal{NBCC}$ pair which is proximinal and $f$ invariant (see [16] for more details on proof). Let $(u, v) \in \mathcal{G}_0 \times \mathcal{H}_0$ be such that $\|u - v\| = \mathfrak{D}(\mathcal{G}, \mathcal{H})$. Since $f$ is relatively nonexpansive noncyclic mapping

$$\|fu - fv\| \le \|u - v\| = \mathfrak{D}(\mathcal{G}, \mathcal{H}),$$

which gives $fu \in \mathcal{G}_0$, that is, $f(\mathcal{G}_0) \subseteq \mathcal{G}_0$. Similarly, $f(\mathcal{H}_0) \subseteq \mathcal{H}_0$ and so $f$ is noncyclic on $\mathcal{G}_0 \cup \mathcal{H}_0$.

Let us define a pair $(\mathfrak{G}_n, \mathfrak{H}_n)$ as $\mathfrak{G}_n = \overline{con}(f(\mathfrak{G}_{n-1}))$ and $\mathfrak{H}_n = \overline{con}(f(\mathfrak{H}_{n-1}))$, $n \ge 1$ with $\mathfrak{G}_0 = \mathcal{G}_0$ and $\mathfrak{H}_0 = \mathcal{H}_0$. We have $\mathfrak{H}_1 = \overline{con}(f(\mathfrak{H}_0)) = \overline{con}(f(\mathcal{H}_0)) \subseteq \mathcal{H}_0 = \mathfrak{H}_0$. Therefore, $f(\mathfrak{H}_1) \subseteq f(\mathfrak{H}_0)$. Thus $\mathfrak{H}_2 = \overline{con}(f(\mathfrak{H}_1)) \subseteq \overline{con}(f(\mathfrak{H}_0)) = \mathfrak{H}_1$. Continuing this pattern, we get $\mathfrak{H}_n \subseteq \mathfrak{H}_{n-1}$ by using induction. Similarly, $\mathfrak{G}_{n+1} \subseteq \mathfrak{G}_n$ for all $n \in \mathbb{N}$. Hence we get a decreasing sequence $\{(\mathfrak{G}_n, \mathfrak{H}_n)\}$ of nonempty, closed and convex pairs in $\mathcal{G}_0 \times \mathcal{H}_0$. Also, $f(\mathfrak{H}_n) \subseteq f(\mathfrak{H}_{n-1}) \subseteq \overline{con}(f(\mathfrak{H}_{n-1})) = \mathfrak{H}_n$ and $f(\mathfrak{G}_n) \subseteq f(\mathfrak{G}_{n-1}) \subseteq \overline{con}(f(\mathfrak{G}_{n-1})) = \mathfrak{G}_n$. Therefore, for all $n \in \mathbb{N}$, the pair $(\mathfrak{G}_n, \mathfrak{H}_n)$ is $f$-invariant. From the proof of Theorem 2.1, we have that $(\mathfrak{G}_n, \mathfrak{H}_n)$ is a proximinal pair such that $\mathfrak{D}(\mathfrak{G}_n, \mathfrak{H}_n) = \mathfrak{D}(\mathcal{G}, \mathcal{H})$ for all $n \in \mathbb{N} \cup \{0\}$.

Now since $\{\varkappa(\mathfrak{G}_n \cup \mathfrak{H}_n)\}$ is a positive nonincreasing sequence and following the proof of Theorem 2.1, we can prove that $\{\varkappa(\mathfrak{G}_n \cup \mathfrak{H}_n)\}$ converges to 0.

Therefore, $\varkappa(\mathfrak{G}_n \cup \mathfrak{H}_n) \to 0$ as $n \to \infty$. That is,

$$\lim_{n\to\infty} \varkappa(\mathfrak{G}_n \cup \mathfrak{H}_n) = \max\{\lim_{n\to\infty} \varkappa(\mathfrak{G}_n), \lim_{n\to\infty} \varkappa(\mathfrak{H}_n)\} = 0.$$

Now, let $\mathfrak{G}_\infty = \cap_{n=0}^{\infty}\mathfrak{G}_n$ and $\mathfrak{H}_\infty = \cap_{n=0}^{\infty}\mathfrak{H}_n$. By property $(d)$ of MNC, $(\mathfrak{G}_\infty, \mathfrak{H}_\infty)$ is nonempty, convex, compact and $f$-invariant pair with $dist(\mathfrak{G}_\infty, \mathfrak{H}_\infty) = dist(\mathcal{G}, \mathcal{H})$. All this is sufficient to ensures that $f$ admits a best proximity pair. □

Now, we give some consequences of above theorems as corollaries.

**Corollary 2.1.** *Let $(\mathcal{G},\mathcal{H})$ be a nonempty and convex pair in a (strictly convex) Banach space $\mathfrak{X}$ with $\mathcal{G}_0$ being nonempty and $\varkappa$ being an MNC on $\mathfrak{X}$. A relatively nonexpansive cyclic (noncyclic) mapping $f:\mathcal{G}\cup\mathcal{H}\to\mathcal{G}\cup\mathcal{H}$ has at least one best proximity point (pair) if for every $\mathcal{NBCC}$, proximinal and $f$ invariant pair $(\mathfrak{M}_1,\mathfrak{M}_2)\subseteq(\mathcal{G},\mathcal{H})$ with $\mathfrak{D}(\mathfrak{M}_1,\mathfrak{M}_2)=\mathfrak{D}(\mathcal{G},\mathcal{H})$, $f$ satisfies*

$$\kappa\big(\varkappa(f(\mathfrak{M}_1)\cup f(\mathfrak{M}_2))\big)\leq\mathfrak{C}\big[\kappa\big(\varkappa(\mathfrak{M}_1\cup\mathfrak{M}_2)\big),\psi\big(\kappa\big(\varkappa(\mathfrak{M}_1\cup\mathfrak{M}_2)\big)\big)\big],\tag{2.3}$$

*where $\psi:[0,\infty)\to[0,1)$ is a continuous function and $\kappa:[0,\infty)\to[0,\infty)$ is a nondecreasing and continuous function with $\kappa(t)=0$ if and only if $t=0$.*

**Proof.** Taking $\varphi(\xi)=1$ in Theorem 2.1 (Theorem 2.2), we obtain the desired result. □

**Corollary 2.2.** *Let $(\mathcal{G},\mathcal{H})$ be a nonempty and convex pair in a (strictly convex) Banach space $\mathfrak{X}$ with $\mathcal{G}_0$ being nonempty and $\varkappa$ being an MNC on $\mathfrak{X}$. A relatively nonexpansive cyclic (noncyclic) mapping $f:\mathcal{G}\cup\mathcal{G}\to\mathcal{G}\cup\mathcal{H}$ has at least one best proximity point (pair) if for every $\mathcal{NBCC}$, proximinal and $f$ invariant pair $(\mathfrak{M}_1,\mathfrak{M}_2)\subseteq(\mathcal{G},\mathcal{H})$ with $\mathfrak{D}(\mathfrak{M}_1,\mathfrak{M}_2)=\mathfrak{D}(\mathcal{G},\mathcal{H})$, $f$ satisfies*

$$\kappa\big(\varkappa(f(\mathfrak{M}_1)\cup f(\mathfrak{M}_2))\big)\leq\psi\big(\kappa\big(\varkappa(\mathfrak{M}_1\cup\mathfrak{M}_2)\big)\big)\kappa\big(\varkappa(\mathfrak{M}_1\cup\mathfrak{M}_2)\big),\tag{2.4}$$

*where $\psi:[0,\infty)\to[0,1)$ is a continuous function and $\kappa:[0,\infty)\to[0,\infty)$ is a nondecreasing and continuous function with $\kappa(t)=0$ if and only if $t=0$.*

**Proof.** If we take $\mathfrak{C}(u,v)=vu$ in Corollary 2.1, we obtain the desired result. □

**Corollary 2.3.** *Let $(\mathcal{G},\mathcal{H})$ be a nonempty and convex pair in a (strictly convex) Banach space $\mathfrak{X}$ with $\mathcal{G}_0$ being nonempty and $\varkappa$ being an MNC on $\mathfrak{X}$. A relatively nonexpansive cyclic (noncyclic) mapping $f:\mathcal{G}\cup\mathcal{H}\to\mathcal{G}\cup\mathcal{H}$ has at least one best proximity point (pair) if for every $\mathcal{NBCC}$, proximinal and $f$ invariant pair $(\mathfrak{M}_1,\mathfrak{M}_2)\subseteq(\mathcal{G},\mathcal{H})$ with $\mathfrak{D}(\mathfrak{M}_1,\mathfrak{M}_2)=\mathfrak{D}(\mathcal{G},\mathcal{H})$, $f$ satisfies*

$$\kappa\big(\varkappa(f(\mathfrak{M}_1)\cup f(\mathfrak{M}_2))\big)\leq\kappa\big(\varkappa(\mathfrak{M}_1\cup\mathfrak{M}_2)\big)-\psi\big(\kappa\big(\varkappa(\mathfrak{M}_1\cup\mathfrak{M}_2)\big)\big),\tag{2.5}$$

*where $\psi:[0,\infty)\to[0,1)$ is a continuous function and $\kappa:[0,\infty)\to[0,\infty)$ is a nondecreasing and continuous function with $\kappa(t)=0$ if and only if $t=0$.*

**Proof.** If we take $\mathfrak{C}(u,v)=u-v$ in Corollary 2.1, we obtain the desired result. □

The following result is the main result of [14].

**Corollary 2.4.** *Let $(\mathcal{G},\mathcal{H})$ be a nonempty and convex pair in a (strictly convex) Banach space $\mathfrak{X}$ with $\mathcal{G}_0$ being nonempty and $\varkappa$ being an MNC on $\mathfrak{X}$. A relatively nonexpansive cyclic (noncyclic) mapping $f:\mathcal{G}\cup\mathcal{H}\to\mathcal{G}\cup\mathcal{H}$ has at least one best proximity point (pair) if for every $\mathcal{NBCC}$, proximinal and $f$ invariant pair $(\mathfrak{M}_1,\mathfrak{M}_2)\subseteq(\mathcal{G},\mathcal{H})$ with $\mathfrak{D}(\mathfrak{M}_1,\mathfrak{M}_2)=\mathfrak{D}(\mathcal{G},\mathcal{H})$, $f$ satisfies*

$$\varkappa(f(\mathfrak{M}_1)\cup f(\mathfrak{M}_2))\leq\lambda\varkappa(\mathfrak{M}_1\cup\mathfrak{M}_2);$$

*where $\lambda\in[0,1)$.*

**Proof.** If we take $\psi(t)=\lambda,\ \ 0\leq\lambda<1$ and $\kappa(t)=t$ in corollary 2.2, we obtain the required result.

If we also take $\kappa(t)=e^t$ and $\psi(t)=t-t^\lambda,\ \ 0\leq\lambda<1$ in corollary 2.3, we obtain the required result. □

# 3. Application

In this section, we study the existence of an optimal solution of systems of $\psi$-Hilfer fractional differential equations with initial conditions using the best proximity point results which we have proved in section 2 of this article.

First we recall some concepts and outcomes from fractional calculus. Let $-\infty < a < b < \infty$. We denote the space of all continuous functions on $[a,b]$ by $C[a,b]$. $L^m(a,b)$, $m \geq 1$, denotes the space of Lebesgue integrable functions on $(a,b)$. See [11] for more details. Let $\mu > 0$ and $\psi(x)$ be an increasing and positive monotone function on $[a,b]$, having a continuous derivative $\psi^{'}(t)$ on $(a,b)$.

**Definition 3.1.** [27] Let $g \in L^1(a,b)$. The integral

$$\daleth_{a^+}^{\mu;\psi} g(u) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi^{'}(v)(\psi(u)-\psi(v))^{\mu-1} g(v)dv,\ u > a$$

is called left sided fractional integral of function $g$ w.r.t. function $\psi$ on $[a,b]$ of order $\mu$.

**Definition 3.2.** [27] Let $n-1 < \mu < n$ with $n \in \mathbb{N}$ and $g \in L^1(a,b)$. Let $f, \psi \in \mathbb{C}^n([a,b],\mathbb{R})$ be two functions such that $\psi(x)$ is an increasing and $\psi^{'}(t) \neq 0$ for all $t \in [a,b]$. The left-sided $\psi$-Hilfer fractional derivative of order $p$ and type $0 \leq q \leq 1$ of $g$ is defined as the following expression

$${}^{H}D_{a^+}^{p,q;\psi} g(x) = \daleth_{a^+}^{q(n-p);\psi} \Big(\frac{1}{\psi'(t)}\frac{d}{dx}\Big)^n \daleth_{a^+}^{(1-q)(n-p);\psi} g(x),\ x > a,\ 0 < p < 1,$$

provided the right-hand side exists.

It can be written as:

$${}^{H}D_{a^+}^{p,q;\psi} g(x) = \daleth_{a^+}^{\gamma-p;\psi} D_{a^+}^{\gamma;\psi} g(x)$$

with $\gamma = p + q(n-p)$ and where $D_{a^+}^{\gamma;\psi}$ is the $\psi$-Riemamm-Liouville fractional derivative.

**Remark 3.1.** The $\psi$-Hilfer fractional derivative is considered to be most general and unified defintion of fractional derivative. In fact, by choosing different values of $\psi(x), a$ and taking limit on parameters $p, q$ in definition of $\psi$-Hilfer fractional derivative, we get a wide variety of fractional derivatives in the literature. See [27,28] for more information related to this.

We have following results for the fractional derivatives.

**Lemma 3.1** ( [27]). *For $x > a$, $p > 0$ and $\delta > 0$, we have*

$$\daleth_{a^+}^{p;\psi} (\psi(x)-\psi(a))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(p+\delta)}(\psi(x)-\psi(a))^{p+\delta-1}.$$

**Lemma 3.2** ( [27]). *For $p \geq 0$, $q \geq 0$, and $g \in L^1(a,b)$, we have*

$$\daleth_{a^+}^{p;\psi}\daleth_{a^+}^{q;\psi} g(x) = \daleth_{a^+}^{p+q;\psi} g(x),\ a.e.,\ x \in [a,b].$$

Let $\tau, r$ be positive real numbers, $\mathcal{I} = [a,\tau]$ and $(E, \|\cdot\|)$ be a Banach space. Let $B_1 = B(\alpha_a, r)$ and $B_2 = B(\beta_a, r)$ be closed balls in $E$, where $\alpha_a, \beta_a \in E$.

We consider the following system of right sided $\psi$-Hilfer fractional differential equations of arbitrary order with initial conditions:

$$\begin{cases} {}^H D_{a^+}^{\nu,\mu;\psi} u(t) = f(t, u(t)), t \in (0, \tau] \\ \beth_{a^+}^{(1-\nu)(1-\mu);\psi} u(a) = \alpha_a, \end{cases} \tag{3.1}$$

$$\begin{cases} {}^H D_{a^+}^{\nu,\mu;\psi} v(t) = g(t, v(t)), t \in (0, \tau] \\ \beth_{a^+}^{(1-\nu)(1-\mu);\psi} v(a) = \beta_a, \end{cases} \tag{3.2}$$

where ${}^H D_{a^+}^{\nu,\mu}$ is the left sided Hilfer fractional differential operator of order $0 < \mu < 1$ and type $0 < \nu \leq 1$; $\beth_{a^+}^{(1-\nu)(1-\mu);\psi}$ is the Riemann-Liouville fractional integral of order $(1-\nu)(1-\mu)$; the state $u(\cdot)$ takes the values from Banach space $E$; $f : \mathcal{I} \times B_1 \to E$ and $g : \mathcal{I} \times B_2 \to E$ are given mappings satisfying some assumptions.

The following result establishes the equivalence of (3.1) with the integral equation. Let $\gamma = \mu + \nu - \mu\nu$.

**Lemma 3.3.** *The initial value problem* (3.1) *is equivalent to the following integral equation:*

$$u(t) = \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \alpha_a + \frac{1}{\Gamma(\mu)} \int_a^t \psi'(t)(\psi(t) - \psi(s))^{\mu-1} f(s, u(s))\, ds, \ t \in \mathcal{I}.$$

**Proof.** The proof is similar to the proof of Lemma 3.1 in [28]. So we skip the proof. □

Let $\mathcal{J} \subseteq \mathcal{I}$ and $S = C(\mathcal{J}, E)$ be a Banach space of continuous mappings from $\mathcal{J}$ into $E$ endowed with supremum norm. Let

$$S_1 = \{u \in C(\mathcal{J}, B_1) : \beth^{(1-\nu)(1-\mu)} u(a) = \alpha_a\},$$

$$S_2 = \{u \in C(\mathcal{J}, B_2) : \beth^{(1-\nu)(1-\mu)} u(a) = \beta_a\}.$$

So $(S_1, S_2)$ is a nonempty, bounded, closed and convex pair in $S \times S$. Now for every $u \in S_1$ and $v \in S_2$, we have $\|u - v\| = \sup \|u(s) - v(s)\| \geq \|\alpha_a - \beta_a\|$. Therefore $\text{dist}(S_1, S_2) = \|\alpha_a - \beta_a\|$ which ensures that $(S_1)_0$ is nonempty.

Now, let us define the operator $\Im : S_1 \cup S_2 \to S$ as follows

$$\Im u(t) = \begin{cases} \dfrac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \beta_a + \frac{1}{\Gamma(\mu)} \int_a^t \psi'(t)(\psi(t) - \psi(s))^{\mu-1} f(s, u(s)), & u \in S_1, \\ \dfrac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \alpha_a + \frac{1}{\Gamma(\mu)} \int_a^t \psi'(t)(\psi(t) - \psi(s))^{\mu-1} g(s, u(s)), & u \in S_2. \end{cases} \tag{3.3}$$

**Lemma 3.4.** *The operator $\Im : S_1 \cup S_2 \to S$ defined by* (3.3) *is cyclic if $f$ and $g$ are bounded and continuous such that $f, g \in L^1(a, \tau)$.*

**Proof.** Let $u \in S_1$ and set $\gamma = \mu + \nu - \mu\nu$. We have

$$\Im u(t) = \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \beta_a + \frac{1}{\Gamma(\mu)} \int_a^t \psi'(t)(\psi(t) - \psi(s))^{\mu-1} f(s, u(s))\, ds.$$

Applying $\beth_{a^+}^{1-\gamma;\psi}$ on both sides and applying Lemma 3.1 and 3.2, we get

$$\beth_{a^+}^{1-\gamma;\psi} \Im u(t) = \frac{\beta_a}{\Gamma(\gamma)} \beth_{a^+}^{1-\gamma;\psi} \ (\psi(t) - \psi(a))^{\gamma-1} + \beth_{a^+}^{1-\gamma;\psi} \beth_{a^+}^{\mu;\psi} f(s, u(s))(t)$$

$$= \beta_a + [\beth_{a^+}^{1-\nu(1-\mu);\psi} f(s, u(s))](t)$$
$$= \beta_a + [\beth_{a^+}^{1-\nu(1-\mu);\psi} f(s, u(s))](t).$$

Here $[\beth_{a^+}^{1-\nu(1-\mu);\psi} f(s, u(s))](t) \to 0$ as $t \to a$. Therefore $\beth_{a^+}^{1-\gamma;\psi} \Im u(a) = \beta_a$ which means that $\Im u(t) \in S_2$. Similarly one can show that $\Im u(t) \in S_1$ if $u \in S_2$. Thus $\Im$ is cyclic operator. □

We say that $z \in S_1 \cup S_2$ is an optimal solution for the system (3.1) and (3.2) provided that $\|z - \Im z\| = \text{dist}(S_1, S_2)$, that is $z$ is a best proximity point of the operator $\Im$ defined in (3.3).

**Assumptions.** *We consider the following hypotheses to prove our existence of optimal solutions.*

$(A_1)$ *Let $\varkappa$ be a measure of noncompactness on $E$ such that for any bounded pair $(N_1, N_2) \subseteq (B_1, B_2)$ we have*

$$\varkappa\big(f(\mathcal{J} \times N_1) \cup g(\mathcal{J} \times N_2)\big) < \varkappa(N_1 \cup N_2),$$

$(A_2)$ *For all $(u, v) \in S_1 \times S_2$, we have*

$$\|f(t, u(t)) - g(t, v(t))\| \leq \frac{\Gamma(\mu)}{(\psi(T) - \psi(a))^{\mu}} \big(\|u(t) - v(t)\| - \frac{(\psi(T) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \|\beta_a - \alpha_a\|\big).$$

Following result is the Mean-Value Theorem for fractional differential.

**Theorem 3.1** ( [2, 18]). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $\alpha > 0$. Moreover let $g \in L^1([a, b])$ be a function which does not change its sign on its domain. Then for almost every $t \in [a, b]$, there exists some $\zeta \in (a, x)$ such that*

$$\beth_a^{\alpha}(fg)(t) = f(\zeta) \beth_a^{\alpha} g(t). \tag{3.4}$$

Then we give the following result.

**Theorem 3.2.** *Under notations defined above, the hypotheses of Lemma 3.4 and assumptions $(A_1)$ and $(A_2)$, the system of Hilfer fractional differential equations* (3.1)-(3.2) *has an optimal solution.*

**Proof.** It is clear that, system (3.1)-(3.2) has an optimal solution if the operator $\Im$ defined in (3.3) has a best proximity point.

From Lemma 3.4, $\Im$ is a cyclic operator. It follows trivially that $\Im(S_1)$ is a bounded subset of $S_2$. We prove that $\Im(S_1)$ is also an equicontinuous subset of $S_2$. For $t_1, t_2 \in J$ with $t_1 < t_2$ and $u \in S_1$, we observe that

$$\begin{aligned}
&\|\Im u(t_1) - \Im u(t_2)\| \\
=&\Big\| \frac{(\psi(t_2) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \beta_a + \frac{1}{\Gamma(\mu)} \int_0^{t_2} \psi'(t_2)(\psi(t_2) - \psi(a))^{\mu-1} f(s, u(s))\, ds \\
&- \frac{(\psi(t_1) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \beta_a - \frac{1}{\Gamma(\mu)} \int_a^{t_1} \psi'(t_1)(\psi(t_1) - \psi(a))^{\mu-1} f(s, u(s))\, ds \Big\| \\
=&\Big| \frac{\beta_a}{\Gamma(\gamma)} ((\psi(t_2) - \psi(a))^{\gamma-1} - (\psi(t_1) - \psi(a))^{\gamma-1})
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\mu)} \int_a^{t_1} [\psi^{'}(t_2)(\psi(t_2)-\psi(a))^{\mu-1} - \psi^{'}(t_1)(\psi(t_1)-\psi(a))^{\mu-1}] f(s,u(s))ds \\
&+ \frac{1}{\Gamma(\mu)} \int_{t_1}^{t_2} \psi^{'}(t_2)(\psi(t_2)-\psi(a))^{\mu-1} f(s,u(s))ds| \\
\leq & \frac{\beta_a}{\Gamma(\gamma)} \, |((\psi(t_2)-\psi(a))^{\gamma-1} - (\psi(t_1)-\psi(a))^{\gamma-1})| \\
&+ \frac{M}{\mu} \left| \int_a^{t_1} [\psi^{'}(t_2)(\psi(t_2)-\psi(a))^{\mu-1} - \psi^{'}(t_1)(\psi(t_1)-\psi(a))^{\mu-1}]ds \right| \\
&+ \frac{M}{\mu} \left| \int_{t_1}^{t_2} \psi^{'}(t_2)(\psi(t_2)-\psi(a))^{\mu-1} ds \right|.
\end{aligned}
$$

As $t_2 \to t_1$, right hand side tends to 0. Thus $\|\Im u(t_2) - \Im u(t_1)\| \to 0$ as $t_2 \to t_1$. Thus $\Im(S_1)$ is equicontinuous. With the similar argument we can prove that $\Im(S_2)$ is bounded and equicontinuous subset of $S_1$. Thus with the application of Arzela-Ascoli theorem we can conclude that $(S_1, S_2)$ is relatively compact. Next we show that $\Im$ is relatively nonexpansive. For any $(x, y) \in S_1 \times S_2$, with assumption $(A_2)$ we have

$$
\begin{aligned}
&\|\Im x(t) - \Im y(t)\| \\
=& \Bigg\| \frac{\beta_a}{\Gamma(\gamma)} (\psi(t)-\psi(a))^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^t \psi^{'}(t)(\psi(t)-\psi(a))^{\mu-1} f(s,x(s))\,ds \\
&- \Big( \frac{\alpha_a}{\Gamma(\gamma)} (\psi(t)-\psi(a))^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^t \psi^{'}(t)(\psi(t)-\psi(a))^{\mu-1} g(s,y(s))\,ds \Big) \Bigg\| \\
=& \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} |\beta_a - \alpha_a| \\
&+ \left| \frac{1}{\Gamma(\mu)} \int_a^t \psi^{'}(t)(\psi(t)-\psi(a))^{\mu-1} [f(s,x(s)) - g(s,y(s))] ds \right| \\
\leq & \frac{(\psi(\tau)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} \|\beta_a - \alpha_a\| \\
&+ \frac{1}{\Gamma(\mu)} \int_a^t \frac{\psi^{'}(t)(\psi(t)-\psi(a))^{\mu-1}\Gamma(\mu)}{(\psi(\tau)-\psi(a))^{\mu}} \\
&\times \big[ \|x(s)-y(s)\| - \frac{(\psi(\tau)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} \|\beta_a - \alpha_a\| \big] ds \\
=& \|x-y\|,
\end{aligned}
$$

and thereby, $\|\Im x - \Im y\| \leq \|x-y\|$. Therefore $\Im$ is relatively nonexpansive.

At last, let $(K_1, K_2) \subseteq (S_1, S_2)$ be nonempty, closed, convex and proximinal pair which is $\Im$-invariant and such that $\mathrm{dist}(K_1, K_2) = \mathrm{dist}(S_1, S_2) (= \|\alpha_a - \beta_a\|)$. By using a generalized version of Arzela-Ascoli theorem (see Ambrosetti [4]) and assumption $(A_1)$, we get

$$
\begin{aligned}
&\varkappa(\Im(K_1) \cup \Im(K_2)) \\
=& \max\{\varkappa(\Im(K_1)), \varkappa(\Im(K_2))\} \\
=& \max\big\{ \sup_{t \in \mathcal{J}} \{\varkappa(\{\Im x(t) : x \in K_1\})\}, \sup_{t \in \mathcal{J}} \{\varkappa(\{\Im y(t) : y \in K_2\})\} \big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \max \Big\{ \sup_{t \in \mathcal{J}} \big\{ \varkappa(\{ \frac{\beta_a \, (\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \\
&\quad + \frac{1}{\Gamma(\mu)} \int_a^t \psi^{'}(t)(\psi(t) - \psi(a))^{\mu-1} f(s, x(s)) ds) : x \in K_1 \}) \big\}, \\
&\quad \sup_{t \in \mathcal{J}} \big\{ \varkappa(\{ \frac{\alpha_a \, (\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \\
&\quad + \frac{1}{\Gamma(\mu)} \int_a^t \psi^{'}(t)(\psi(t) - \psi(a))^{\mu-1} g(s, y(s)) ds) : y \in K_2 \}) \big\} \Big\}.
\end{aligned}
$$

So, in view of Theorem (3.1), it follows that

$$
\begin{aligned}
&\varkappa(\Im(K_1) \cup \Im(K_2)) \\
&\leq \max \Big\{ \sup_{t \in \mathcal{J}} \big\{ \varkappa(\{ \frac{\beta_a \ (\psi(\tau) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \\
&\quad + \frac{(\psi(\tau) - \psi(a))^{\mu}}{\Gamma(\mu+1)} \overline{con}(\{ f(\sigma, x(\sigma)) : \sigma \in \mathcal{J} \}) \}) \big\}, \\
&\quad \sup_{t \in \mathcal{J}} \big\{ \varkappa(\{ \frac{\alpha_a \, (\psi(\tau) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \\
&\quad + \frac{(\psi(\tau) - \psi(a))^{\mu}}{\Gamma(\mu+1)} \overline{con}(\{ g(\sigma, x(\sigma)) : \sigma \in \mathcal{J} \}) \}) \big\} \Big\} \\
&= \max \Big\{ \frac{(\psi(\tau) - \psi(a))^{\mu}}{\Gamma(\mu+1)} \varkappa(f(\mathcal{J} \times K_1)), \frac{(\psi(\tau) - \psi(a))^{\mu}}{\Gamma(\mu+1)} \varkappa(g(\mathcal{J} \times K_2)) \Big\} \\
&= \frac{(\psi(\tau) - \psi(a))^{\mu}}{\Gamma(\mu+1)} \varkappa(f(\mathcal{J} \times K_1)) \cup \varkappa(g(\mathcal{J} \times K_2)) \\
&< \frac{(\psi(\tau) - \psi(a))^{\mu}}{\Gamma(\mu+1)} \varkappa(K_1 \cup K_2).
\end{aligned}
$$

Choosing $\lambda = \frac{(\psi(\tau)-\psi(a))^{\mu}}{\Gamma(\mu+1)}$, we get that $\varkappa(\Im(K_1) \cup \Im(K_2)) < \lambda \varkappa(K_1 \cup K_2)$ and $0 \leq \lambda < 1$. Therefore, we conclude that $\Im$ satisfies all the hypotheses of Corollary 2.4 and so the operator $\Im$ has a best proximity point $z \in S_1 \cup S_2$ which is an optimal solution for the system (3.1) and (3.2). □

## 4. Conclusions

The fixed point theory serves as an indipensable tool in nonlinear analysis. When an operator do not possess a fixed point, the case becomes significant from the point of view of existence and leads to the study of best approximation. In the work [7], one can study the characterizations of nearly strongly convex and very convex spaces in terms of best approximation theoretic properties of Banach spaces. The theory further extends to the concept which fairly resembles with the notion of fixed point and called as best proximity point. In this article, we have studied the existence of best proximity points (pairs) by considering a new family of cyclic (noncyclic) condensing operators by using an appropriate measure of noncompactness. To this end we have used $\mathcal{C}$-class functions to introduce such class of mappings. As an application of our main existence result, we have surveyed the existence of an optimal solution of a system of fractional differential equation with initial value involving $\psi$-Hilfer fractional derivative. We further refer to the articles [8, 19] for possible applications of the theory of best proximity points (pairs).

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