CONTINUITY OF SOLUTIONS IN $H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ FOR STOCHASTIC REACTION-DIFFUSION EQUATIONS AND ITS APPLICATIONS TO PULLBACK ATTRACTOR

Wenqiang Zhao^{1,†} and Zhi Li¹

Abstract In this paper, we consider the continuity of solutions for nonautonomous stochastic reaction-diffusion equation driven by additive noise over a Wiener probability space. It is proved that the solutions are strongly continuous in $H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ with respect to the L^2 -initial data and the samples in the double limit sense. As applications of the results on the continuity we obtain that the pullback random attractor for this equation is measurable, compact and attracting in the topology of the space $H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ under a weak assumption on the forcing term and the noise coefficient. More precisely, the continuity of solutions in the initial data implies the asymptotic compactness of system and therefore the attraction of attractor, and the continuity in the samples indicates its measurability. The main technique employed here is the difference estimate method, by which an appropriate multiplier is carefully selected.

Keywords Stochastic reaction-diffusion equations, continuity, pullback random attractor, additive noises, measurability.

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1. Introduction

The continuous dependence of solutions on initial data is definitely an important topic for the problems arising from physical applications. As stated by Evans [11, p7], we would prefer that our solutions change only a little when the conditions specifying the problem change a little. In the framework of the dynamical system, the system is always required to be continuous with respect to the initial data in the initial space [6, 18, 21]. The continuity in the initial space, a sufficient condition on the invariance of the attractor, is relatively easy to obtain for most concrete problems. Nevertheless, the continuity of most of the evolution equations in the related regular spaces is unknown and its meaning on the attractor is not sufficiently analysed in the literature. As we will see, to prove the continuity of solutions in the regular spaces is not an easy thing and often requires many intricate calculations.

In this paper, we consider the continuity of solutions in $H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ for the following non-autonomous stochastic reaction-diffusion equation driven by additive

[†]The corresponding author.

¹School of Mathematics and Statistics, Chongqing Technology and Business

University, Chongqing 400067, China

Email: zhaowq@ctbu.edu.cn(W. Zhao), 13419343162@163.com(Z. Li)

white noise on the whole space $\mathbb{R}^N, N \geq 1$,

$$\frac{du}{dt} + \lambda u - \Delta u = F(t, x, u) + g(t, x) + \phi(x)\dot{W}(t), \quad t > \tau,$$
(1.1)

with the initial condition

$$u(\tau, x) = u_{\tau}, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $\lambda > 0, \tau \in \mathbb{R}, g \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^N)), W(t)$ is a two-sided real-valued Brownian motion defined on a probability space specified later, and ϕ is a given function on \mathbb{R}^N . The nonlinear function f is continuous from $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ to \mathbb{R} satisfying the following conditions: for all $t, u_1, u_2 \in \mathbb{R}$ and $x \in \mathbb{R}^N$, F(t, x, 0) = 0 and

$$(F(t, x, u_1) - F(t, x, u_2))(u_1 - u_2) \le -\alpha |u_1 - u_2|^p + \psi_1(t, x)|u_1 - u_2|^2, \quad (1.3)$$

$$|F(t, x, u_1) - F(t, x, u_2)| \le \psi_2(t, x)|u_1 - u_2|(1 + |u_1|^{p-2} + |u_2|^{p-2}),$$
(1.4)

where $\alpha > 0, p \ge 2$ and $\psi_1, \psi_2 \in L^{\infty}(\mathbb{R}^{N+1})$.

It is well-known, see e.g. [23], that for every initial value $u_{\tau} \in L^2(\mathbb{R}^N)$ the initial problem (1.1)-(1.2) has a unique weak solution u satisfying

$$u(.,\tau,\omega,u_{\tau}) \in C([\tau,+\infty); L^{2}(\mathbb{R}^{N})) \cap L^{p}_{loc}(\tau,+\infty; L^{p}(\mathbb{R}^{N}))$$
$$\cap L^{2}_{loc}(\tau,+\infty; H^{1}(\mathbb{R}^{N})).$$
(1.5)

Furthermore, u is continuously dependent of the initial data u_{τ} in the initial space $L^2(\mathbb{R}^N)$.

We recall some results in the bibliography about the continuity on the initial data for the reaction-diffusion equation. In the deterministic case, if the state space $\mathcal{O} \subset \mathbb{R}^N$ is bounded, Robinson [18, p227-p231] proved that the strong solution $u: H_0^1(\mathcal{O}) \cap L^p(\mathcal{O}) \mapsto H_0^1(\mathcal{O})$ is continuous for $N \leq 2$, where $p \geq 2$ is the order of nonlinearity with polynomial grow. However, if N = 3, the proof in [18] required $p \leq 4$. Up to 2008, Trujillo and Wang [22] proved that the strong solutions $u: H_0^1(\mathcal{O}) \cap L^p(\mathcal{O}) \mapsto H_0^1(\mathcal{O})$ is continuous for any $N \geq 1$ and $p \geq 2$, which largely extended the result in [18]. The key point in that paper is to derive the estimate that $t\frac{du}{dt} \in L^{\infty}(0,T; L^2(\mathcal{O}))$ by differentiating with respect to t on both sides of the equations. However, since the Browian motion is not necessary differentiable, then the method used in the deterministic case is not applicable to the stochastic differential equations like problem (1.1).

In 2015, in the random case, by using the Sobolev critical embedding that $H_0^1(\mathcal{O}) \hookrightarrow L^{\frac{2N}{N-2}}(\mathcal{O})$ and a boot strap technique, Cao et al [2] proved the continuity of solutions from $H_0^1(\mathcal{O}) \cap L^p(\mathcal{O})$ to $H_0^1(\mathcal{O})$ with $N \ge 3$ and $p \ge 2$. This method, successfully surmounting the obstacle of non-differentiability of the white noises, was also used to obtain the higher-order integrability of attractor for stochastic *p*-Laplacian with multiplicative noise on unbounded domain; see [26]. Then, Zhu and Zhou [31] generalized this technique to derive the continuity of solutions from $L^2(\mathbb{R}^N)$ to $H^1(\mathbb{R}^N)$ for deterministic reaction-diffusion equations on unbounded domain. However, since the proof in [2, 26, 31] deeply depends on the Sobolev critical embedding inequalities, the dimension $N \ge 3$ for reaction-diffusion equations (rep. N > p for *p*-Lapacian equations) is required, and thus the technique can not applied directly to the general case $N \ge 1$, especially on unbounded domains. It is

noticed that the continuity of solutions for some other equations is still an interesting topic. For example, for the Hölder continuity of gradient of a unique weak solution for unsteady generalized Navier-Stokes equations was studied in [20], and the boundedness and Hölder continuity of quasilinear elliptic problems involving variable exponents were showed in [13].

Most recently, Zhao [24, 25, 27] dramatically modified the induction method of Cao et al. [2] by means of an iteration of a nonlinear term, which can be used to a variety of stochastic differential equations on \mathbb{R}^N , such as *p*-Laplacian equation [25], FitzHugh-Nagumo system [24] and semi-linear degenerate parabolic equation [27] for any dimension $N \geq 1$. Moreover, the continuity of solutions in $L^p(\mathbb{R}^N)$ was also derived without employing the critical embedding inequality. However, there we only considered the continuous dependence of solutions on the initial data, whereas the continuity on the sample ω was not involved, which is definitely of significance to analyse the measurability of the solution and the attractor in the regular spaces [10].

In this paper, we first prove that if $g \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^N))$, the solutions may map $L^2(\mathbb{R}^N)$ into $H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. We then present a new method to prove the strong continuity of solutions u for problem (1.1)-(1.2) in $H^1(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ respectively, with respect to the L^2 -initial data and the sample ω . More specifically, we show that the mapping $u: \Omega_k \times L^2(\mathbb{R}^N) \mapsto H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ is continuous in the double limit sense (jointed continuous), where $\Omega_k, k = 1, 2, ...$ is a countable splitting of the sample space Ω consisting of continuous functions. It is noted that we do not use the Sobolev critical embedding like in [2, 26, 31] and thus there are no restrictions on space dimension $N \geq 1$. The crucial idea is to estimate the difference of solutions by an appropriate multiplier. This is greatly different from the way used in the mentioned papers [2, 24–27, 31], where the continuity in the initial data is considered as a single limit sense, and therefore the structure of the corresponding difference equations is not changed much.

However, in the case of random (stochastic) differential equations, we need additional measurability properties to assure that the solutions are well-posed. This leads to an additional and great difference in comparison with the non-random case as argued in [3]. It is often feasible to derive the measurability property by proving the continuity of solutions with respect to the sample ω in the targeted space like the work in [10], in which a special nonlinearity is considered. Thus, the continuity for our problem is in the sense of double limit, which is more complicated than the single limit. But it seems impossible to obtain directly the continuity of solutions with respect to the sample over the given metric dynamical system $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$. We overcome this problem by considering the continuity in a Polish space $\Omega_k, k = 1, 2, ...,$ which is a countable decomposition of Ω . The space Ω_k has good properties, for example, the Ornstein-Uhlenbeck process over Ω_k is continuous and uniformly bounded and etc. Finally, as applications of our continuity results, it offers that the attractor established in $L^2(\mathbb{R}^N)$ is measurable, compact and attracting in the regular space $H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$.

The framework of the paper is arranged as follows. The next section is concerned with the notion of bi-spatial pullback random attractors and bi-spatial continuous random dynamical systems. In section 3, we make an Ornstein-Uhlenbeck transformation of the original equations so that we can carry out a series of estimates of solutions. In section 4, through a series of careful calculations, we obtain the estimates of difference of the solutions in $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. As a result, we establish the bi-spatial $(L^2(\mathbb{R}^N), L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N))$ -continuity of solutions. In section 5, the results on continuity are applied to establish the existence of a measurable pullback attractor in $L^p(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$.

2. Preliminaries

In this section, we list some basic concepts about the pullback random attractor and the existence theorem including the measurability in the regular spaces. For a comprehensive information of this topic, the reader is referred to the monographs [1,7] and the original work [8,9,19]. For the recent studies on random attractors the following several papers deserve to be mentioned. For example, Gess et al. [12] obtained the random attractors for locally monotone stochastic partial differential equations perturbed by additive Lévy noise. Lin [17] generalized the Morse decompositions on uniform attractors to the random case for non-autonomous random dynamical systems. The backward compact random attractor for lattice FitzHugh-Nagumo system was studied in [14].

Let $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ be two completely separable metric spaces, where X serves as the initial space, and Y as the target space or regular space, whose intersection is nontrivial. We assume that each of them is continuously imbedded in a Hausdorff topological vector space \mathcal{X} . This implies that for every sequence $x_n \in X \cap Y, n = 1, 2, ...,$ such that $x_n \to x$ in X and $x_n \to y$ in Y respectively, then we have $x = y \in X \cap Y$. Let $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t \in \mathbb{R}})$ be a metric (or measurable) dynamical system (briefly, MDS ϑ), $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$, and 2^X be the collection of all subsets of X.

Definition 2.1. A family of single-valued mappings $\varphi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \mapsto X, (t, \tau, \omega, x) \mapsto \varphi(t, \tau, \omega, x)$ is called a (non-autonomous) random dynamical system (briefly a cocycle) on X over an MDS ϑ if for all $s, t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$, the following statements are satisfied:

- $\varphi(., \tau, ., .) : \mathbb{R}^+ \times \Omega \times X \mapsto X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$ -measurable;
- $\varphi(0, \tau, \omega, .)$ is the identity on X;
- $\varphi(t+s,\tau,\omega,.) = \varphi(t,\tau+s,\vartheta_s\omega,\varphi(s,\tau,\omega,.)).$

A random dynamical system φ on X is said to take its values into the target space Y if $\varphi(t, \tau, \omega, .)$ maps X into Y for every $t > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$.

Definition 2.2 (Definition 2.2, [28]). Let φ be a random dynamical system on X over an MDS ϑ , taking its values into Y. φ is said to be (X, X)-continuous (briefly, continuous in X) if the mapping $\varphi(t, \tau, \omega, .) : X \mapsto X$ is continuous for each $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$. A random cocycle φ is said to be (X, Y)-continuous if φ is (X, X)-continuous, and in addition the mapping $\varphi(t, \tau, \omega, .) : X \mapsto Y$ is continuous for each $t > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$.

Definition 2.3. Let $D : \mathbb{R} \times \Omega \to 2^X \setminus \emptyset$; $D : (\tau, \omega) \mapsto D(\tau, \omega) \in 2^X$ be a set-valued mapping with closed image. We say $D : (\tau, \omega) \mapsto D(\tau, \omega)$ is measurable with respect to \mathcal{F} (briefly, measurable) in X if for every fixed $x \in X$ and $\tau \in \mathbb{R}$, the mapping

$$\omega \to d_X(x, D(\tau, \omega)) = \inf_{z \in D(\tau, \omega)} d_X(x, z)$$

is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable. If D is measurable, then the family of its images $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is also called a random set. If further for every fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the image $D(\tau, \omega)$ is compact in X, then the family $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called a compact random set in X.

Definition 2.4 (Definition 2.5, [28]). Let \mathfrak{D} be a collection of some families of nonempty closed subsets of the initial space X. Let φ be a random dynamical system on X over an MDS ϑ .

(i) A family of sets $\mathcal{A} = {\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega} \in \mathfrak{D}$ is called a (X, X)-pullback random attractor for φ if the next three statements hold:

• \mathcal{A} is a random set in X and $\mathcal{A}(\tau, \omega)$ is compact in X for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$;

• \mathcal{A} is invariant, that is, $\varphi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \vartheta_t \omega)$ for every $t \ge 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$;

• \mathcal{A} is attracting in X, namely, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and $D \in \mathfrak{D}$,

 $\lim_{t \to \infty} \operatorname{dist}_X(\varphi(t, \tau - t, \vartheta_{-t}\omega, D(\tau - t, \vartheta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0.$

(*ii*) Suppose further that φ takes its values into Y. Then a family of sets $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ is called a (X, Y)-pullback random attractor for φ if \mathcal{A} is a (X, X)-pullback random attractor, and in addition there hold

- \mathcal{A} is a random set in Y, $\mathcal{A}(\tau, \omega)$ is compact in Y for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$;
- \mathcal{A} is attracting in Y, namely, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and $D \in \mathfrak{D}$,

$$\lim_{t \to \infty} \operatorname{dist}_Y(\varphi(t, \tau - t, \vartheta_{-t}\omega, D(\tau - t, \vartheta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0,$$

where $dist_Y$ is the Hausdorff semi-metric in 2^Y with

$$\operatorname{dist}_Y(A,B) = \sup_{x \in A} \inf_{y \in B} d_Y(x,y).$$

The existence theorem of bi-spatial random attractor seems to be established in [15] for the autonomous case and [30] for the non-autonomous case. The interesting thing of this result is that the absorption in the target space Y used ever in the literature [16, 29] is omitted. However, the measurability of random attractor is assumed to hold on the initial space X, not considering the measurability in the target space Y. In fact, the measurability of random attractor \mathcal{A} in the target space Y is closely related to the measurability of the cocycle φ in ω and the continuity of the cocycle φ in Y with respect to the initial values belonging to X. This consequence was proved in [28].

Theorem 2.1 (Theorem 2.10, [28]). Let φ be a random dynamical system on X over an MDS ϑ which is (X, Y)-continuous, and \mathfrak{D} be inclusion closed universe in the initial space X. Suppose that

(i) φ has a \mathfrak{D} -pullback random absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ in X, namely, K is a random set and for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathfrak{D}$, there exists an absorbing time $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,

$$\varphi(t,\tau-t,\vartheta_{-t}\omega,D(\tau-t,\vartheta_{-t}\omega))\subseteq K(\tau,\omega);$$

(ii) φ is (X, X)-pullback asymptotically compact in X, namely, for every $\tau \in \mathbb{R}, \omega \in \Omega, D \in \mathfrak{D}$ and whenever $t_n \to \infty, x_n \in D(\tau - t_n, \vartheta_{-t_n}\omega)$, the sequence

$$\{\varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$$
 is precompact in X;

(iii) For every fixed $t > 0, \tau \in \mathbb{R}, x \in X, \varphi(t, \tau, ., x) : \Omega \mapsto Y$ is $(\mathcal{F}, \mathcal{B}(Y))$ -measurable.

Then the random dynamical system φ possesses a unique (X, Y)-pullback attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$, where

$$\mathcal{A}(\tau,\omega) = \bigcap_{s>0} \overline{\bigcup_{t\geq s} \varphi(t,\tau-t,\vartheta_{-t}\omega,K(\tau-t,\vartheta_{-t}\omega))}^X.$$

Furthermore, it can also be structured by the Y-metric, namely,

$$\mathcal{A}(\tau,\omega) = \bigcap_{s>0} \overline{\bigcup_{t\geq s} \varphi(t,\tau-t,\vartheta_{-t}\omega,K(\tau-t,\vartheta_{-t}\omega))}^{r}.$$

3. Non-autonomous stochastic reaction-diffusion equations on \mathbb{R}^N with additive noise

In this section, we present some settings about the non-autonomous stochastic reaction-diffusion equation (1.1). For the noise coefficient, we assume that ϕ satisfies for any $p \geq 2$,

$$\phi \in H^2(\mathbb{R}^N) \cap L^{2p-2}(\mathbb{R}^N). \tag{3.1}$$

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Then by the Sobolev interpolation, we have $\phi \in L^p(\mathbb{R}^N)$. It is noted that we do not require that the noise coefficient $\phi \in L^{\infty}(\mathbb{R}^N)$ as in the literature. We further assume that

$$\lambda > \mu = 2(\|\psi_1\|_{\infty} + \|\psi_2\|_{\infty}), \tag{3.2}$$

where ψ_1 and ψ_2 are given in (1.3) and (1.4). Here $\|.\|_{\infty} = \|.\|_{L^{\infty}(\mathbb{R}^{N+1})}$.

We introduce the probability space (Ω, \mathcal{F}, P) , where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$. \mathcal{F} is the Borel σ -algebra on Ω with respect to the compact-open topology and P is the Wiener measure on (Ω, \mathcal{F}) . The Brownian motion W(t) is identified as $\omega(t)$, i.e., $W(t) = \omega(t)$. The Wiener shift $\{\vartheta_t\}_{t\in\mathbb{R}}$ on the probability space (Ω, \mathcal{F}, P) is defined by $\vartheta_t \omega(.) = \omega(. + t) - \omega(t), t \in \mathbb{R}$. Then the quadruple form $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ forms a metric dynamical system (MDS); see [1].

Since the Brownian motion $\omega(t)$ is not necessarily differentiable in t, the stochastic equation need to be transformed into an equation with random coefficient, usually by an Ornstein-Uhlenbeck process over the MDS $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t \in \mathbb{R}})$, which is a stationary process

$$y(\vartheta_t \omega) = -\lambda \int_{-\infty}^0 e^{\lambda r} \vartheta_t \omega(r) dr, \quad t \in \mathbb{R},$$
(3.3)

satisfying the following stochastic differential equation:

$$dy(\vartheta_t\omega) + \lambda y(\vartheta_t\omega)dt = dW(t).$$

In addition, it follows from [1], that there exits a ϑ_t -invariant set $\tilde{\Omega} \subset \Omega$ of full measure such that $y(\vartheta_t \omega)$ is pathwise continuous in t for every fixed $\omega \in \tilde{\Omega}$ (so that the process has continuous trajectories) and

$$\lim_{|t| \to +\infty} \frac{|W(t)|}{t} = 0$$

for every $\omega \in \tilde{\Omega}$. In the sequel, all arguments are understood to hold on this $\tilde{\Omega}$, but we keep the notation Ω for $\tilde{\Omega}$.

Let $z(\vartheta_t \omega) = \phi(x)y(\vartheta_t \omega)$ and write $v(t) = u(t) - z(\vartheta_t \omega)$, where u(t) satisfies the equations (1.1) and (1.2). Then v(t) satisfies

$$\frac{dv}{dt} + \lambda v - \Delta v = F(t, x, v + z(\vartheta_t \omega)) + g(t, x) + \Delta z(\vartheta_t \omega),$$
(3.4)

with the initial condition

$$v(\tau, x) = v_{\tau} = u_{\tau} - z(\vartheta_{\tau}\omega). \tag{3.5}$$

Throughout the paper, the letter c is a generic positive constant that may change its value from line to line, even in the same line, and $\|.\|_p$ denotes the norm in $L^p(\mathbb{R}^N)$. For p = 2, we write $\|.\|_2 = \|.\|$.

4. Continuity of solutions

4.1. Countable decomposition of the sample space Ω

In order to prove the continuity of solutions in the regular spaces $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ with respect to the sample ω , we need to analyse the topological property and the structure of the sample space Ω and the exponent growth property of the Ornstein-Uhlenbeck process defined by (3.3).

It is well-known that the sample space Ω with metric defined by (4.2) is a Polish space (complete separable metric space); see [1, P544]. For $k \in \mathbb{N}$, consider the set

$$\Omega_k = \{ \omega \in \Omega : |\omega(t)| \le k e^{\zeta |t|}, t \in \mathbb{R} \}, \quad \forall \ k \in \mathbb{N},$$
(4.1)

where $0 < \zeta(2p-2) < \lambda$. It is easy to check that $\Omega = \bigcup_{k \in \mathbb{N}} \Omega_k$ and $\Omega_k \in \mathcal{F}$.

Let \mathcal{F}_{Ω_k} be the trace σ -algebra of \mathcal{F} with respect to Ω_k , and let $B_{\Omega_k}(\omega_0, r), \omega_0 \in \Omega_k, r > 0$ be a ball in Ω_k . This ball can be generated by $B_{\Omega}(\omega_0, r) \cap \Omega_k$, where $B_{\Omega}(\omega_0, r)$ is a ball in Ω , namely, $B_{\Omega_k}(\omega_0, r) = B_{\Omega}(\omega_0, r) \cap \Omega_k$. The same is true for the all open sets in Ω_k . Therefore \mathcal{F}_{Ω_k} is just the σ -algebra of Ω_k . Furthermore, since $\Omega_k \in \mathcal{F}$, then we have $\mathcal{F}_{\Omega_k} \subset \mathcal{F}$; see [5].

The following Lemma 4.1 and Lemma 4.2 are concerning some important properties about $\Omega_k, k \in \mathbb{N}$, see [4,5].

Lemma 4.1. (i) For every $k \in \mathbb{N}$, the space (Ω_k, ρ) is a Polish space, namely, a completely separable metric space, where ρ is the Frechét metric defined by

$$\rho(\omega_1, \omega_2) \triangleq \sum_{n \in \mathbb{N}} \frac{1}{2^n} \frac{\|\omega_1 - \omega_2\|_n}{1 + \|\omega_1 - \omega_2\|_n}, \quad \|\omega_1 - \omega_2\|_n \triangleq \sup_{-n \le t \le n} |\omega_1(t) - \omega_2(t)|.$$
(4.2)

(ii) For every $k \in \mathbb{N}$, the mapping $(\Omega_k, \rho) \ni \omega \to y(\vartheta_t \omega)$ is uniformly continuous from (Ω_k, ρ) to \mathbb{R} on any bounded intervals $[\tau, T]$, where $y(\vartheta_t \omega)$ is the Ornstein-Uhlenbeck process defined by (3.3).

To prove the continuity of solutions in ω , we also need the following lemma about the tempered property of $y(\vartheta_t \omega)$. The proof is similar to [5].

Lemma 4.2. For every $k \in \mathbb{N}$, there exists a constant L > 0 depending only on p and k such that

$$J(\vartheta_t \omega) = |y(\vartheta_t \omega)|^2 + |y(\vartheta_t \omega)|^p + |y(\vartheta_t \omega)|^{2p-2} \le Le^{\lambda|t|}, \quad t \in \mathbb{R},$$
(4.3)

for any $\omega \in \Omega_k$.

4.2. Regularity of solutions in $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$

We first present a lemma in [27] which is useful for the following estimates.

Lemma 4.3 (Lemma 4.2, [27]). Let $\xi(t)$, g and h be three nonnegative and locally integrable functions on \mathbb{R} such that $\frac{d\xi}{dt}$ is also locally integrable and

$$\frac{d\xi(t)}{dt} + \nu\xi(t) + g(t) \le h(t), \quad t \in \mathbb{R},$$

for some constant $\nu > 0$. Then

(i) for arbitrary a > 0 and $\tau \in \mathbb{R}$, $\xi(\tau) \leq \frac{e^{-\nu\tau}}{a} \int_{\tau-a}^{\tau} e^{\nu s} \xi(s) ds + \int_{\tau-a}^{\tau} e^{\nu s} h(s) ds$. (ii) for arbitrary $a, \varepsilon > 0$ and $\sigma \in [\tau - a, \tau]$,

$$\begin{split} \xi(\sigma) + e^{-\nu\tau} \int_{\tau-a}^{\tau} e^{\nu s} g(s) ds &\leq \frac{(e^{\nu a} + 1)e^{-\nu\tau}}{\varepsilon} \int_{\tau-a-\varepsilon}^{\tau} e^{\nu s} \xi(s) ds \\ &+ (e^{\nu a} + 2)e^{-\nu\tau} \int_{\tau-a-\varepsilon}^{\tau} e^{\nu s} h(s) ds. \end{split}$$

In particular, (i) and (ii) hold for $\nu = 0$.

By using the above lemma, we can obtain higher regularity of solutions in $H^1 \cap L^p$ and L^{2p-2} without increasing the conditions on the forcing g and the nonlinearity f.

Lemma 4.4. Let (1.3), (1.4) and (3.1) hold. Then for every $T > \tau \in \mathbb{R}$ and $\omega \in \Omega_k$, there exists a positive constant $C_{T-\tau}$ depending on $||u_{\tau}||^2$ and $\int_{\tau}^{T} ||g(s, .)||^2 ds$ but independent of $\omega \in \Omega_k$, such that the solution of problem (1.1)-(1.2) satisfies for all $t \in (\tau, T]$,

$$(t-\tau)\|u(t,\tau,\omega,u_{\tau})\|_{p}^{p} + (t-\tau)\int_{\frac{t+\tau}{2}}^{t}\|u(s)\|_{2p-2}^{2p-2}ds \le C_{T-\tau},$$
(4.4)

and

$$(t-\tau) \|\nabla u(t,\tau,\omega,u_{\tau}))\|^2 \le C_{T-\tau}.$$
 (4.5)

Proof. Take the inner product of (3.4) in $L^2(\mathbb{R}^N)$ with v, using the assumptions (1.3) and (1.4), we deduce that

$$\frac{d}{dt}\|v\|^2 + \|\nabla v\|^2 + \frac{\alpha}{2^p}\|v\|_p^p \le c\|g(t,.)\|^2 + cJ(\theta_t\omega),$$
(4.6)

where $J(\theta_t \omega)$ is as in (4.3). Integrating (4.6) from τ to t, using Lemma 4.2, we get that for all $t \in (\tau, T]$,

$$\|v(t)\|^{2} + \int_{\tau}^{t} (\|\nabla v(s)\|^{2} + \|v(s)\|_{p}^{p}) ds$$

$$\leq c \int_{\tau}^{t} (\|g(s,.)\|^{2} + J(\theta_{s}\omega))ds + \|v_{\tau}\|^{2}$$

$$\leq c (\int_{\tau}^{t} \|g(s,.)\|^{2}ds + \int_{\tau}^{t} J(\theta_{s}\omega)ds + |y(\theta_{\tau}\omega)|^{2} + \|u_{\tau}\|^{2})$$

$$\leq c (\int_{\tau}^{T} \|g(s,.)\|^{2}ds + L \int_{\tau}^{T} e^{\lambda|s|}ds + \|u_{\tau}\|^{2} + Le^{\lambda|\tau|}).$$
(4.7)

Take the inner product of (3.4) in $L^2(\mathbb{R}^N)$ with $|v|^{p-2}v$, we obtain

$$\frac{1}{p}\frac{d}{dt}\|v\|_{p}^{p} + \lambda\|v\|_{p}^{p} + (-\Delta v, |v|^{p-2}v) + \int_{\mathbb{R}^{N}} F(t, x, u)|v|^{p-2}vdx \\
= \int_{\mathbb{R}^{N}} g(t, x)|v|^{p-2}vdx + (\Delta z(\vartheta_{t}\omega), |v|^{p-2}v).$$
(4.8)

It is clear that $(-\Delta v, |v|^{p-2}v) \ge 0$. By (3.1), $\phi \in H^2(\mathbb{R}^N)$, then we get

$$(\Delta z(\vartheta_t \omega), |v|^{p-2}v) \le \frac{\alpha}{2^{p+3}} \|v\|_{2p-2}^{2p-2} + c|y(\vartheta_t \omega)|^2.$$
(4.9)

By (1.3), (1.4) and (3.1), using Young inequality repeatedly, we infer that

$$\int_{\mathbb{R}^{N}} F(t,x,u) |v|^{p-2} v \geq \frac{\alpha}{2^{p}} ||v||^{2p-2} - 2 \int_{\mathbb{R}^{N}} (|\psi_{1}(t,x)| + |\psi_{2}(t,x)|) |v|^{p} dx \\
- \int_{\mathbb{R}^{N}} (2|\psi_{1}(t,x)| + 3|\psi_{2}(t,x)| |\phi(x)y(\theta_{t}\omega)|^{2} |v|^{p-2} dx \\
- \int_{\mathbb{R}^{N}} (c|\psi_{2}(t,x)|^{p} + \frac{\alpha}{2}) |\phi(x)y(\theta_{t}\omega)|^{p} |v|^{p-2} dx \\
\geq \frac{\alpha}{2^{p+1}} ||v||^{2p-2}_{2p-2} - c ||v||^{p}_{p} - |y(\vartheta_{t}\omega)|^{p} - |y(\vartheta_{t}\omega)|^{2p-2}. \quad (4.10)$$

On the other hand, we have

$$\left|\int_{\mathbb{R}^{N}} g(t,x)|v|^{p-2}vdx\right| \le \frac{\alpha}{2^{p+3}} \|v\|_{2p-2}^{2p-2} + c\|g(t,.)\|^{2}.$$
(4.11)

Hence, by (4.8)-(4.11) we get

$$\frac{d}{dt} \|v\|_p^p + c\|v\|_{2p-2}^{2p-2} \le c(\|v\|_p^p + \|g(t,.)\|^2 + J(\theta_t\omega)).$$
(4.12)

Applying Lemma 4.3 (*ii*) to (4.12) over the interval $[\frac{t+\tau}{2}, t]$ with $\nu = 0$ and $a = \varepsilon = \frac{t-\tau}{2}$, we get

$$\|v(t)\|_{p}^{p} + \int_{\frac{t+\tau}{2}}^{t} \|v(s)\|_{2p-2}^{2p-2} ds$$

$$\leq \frac{c}{t-\tau} \int_{\tau}^{t} \|v(s)\|_{p}^{p} ds + c \int_{\tau}^{t} (\|g(s,.)\|^{2} + J(\theta_{s}\omega)) ds, \qquad (4.13)$$

which by $u(t) = v(t) + \phi(x)y(\vartheta_t\omega)$ obviously implies that

$$(t-\tau)\|u(t)\|_p^p + (t-\tau)\int_{\frac{t+\tau}{2}}^t \|u(s)\|_{2p-2}^{2p-2}ds$$

$$\leq c \int_{\tau}^{t} \|v(s)\|_{p}^{p} ds + c(t-\tau) (\int_{\tau}^{t} \|g(s,.)\|^{2} ds + \int_{\tau}^{t} J(\theta_{s}\omega) ds).$$
(4.14)

Thus by (4.14) and (4.7), using Lemma 4.2, we get for all $\omega \in \Omega_k$ and $t \in (\tau, T]$,

$$\begin{aligned} (t-\tau) \|u(t)\|_{p}^{p} + (t-\tau) \int_{\frac{t+\tau}{2}}^{t} \|u(s)\|_{2p-2}^{2p-2} ds \\ \leq & c (\int_{\tau}^{T} \|g(s,.)\|^{2} ds + L \int_{\tau}^{T} e^{\lambda |s|} ds + \|u_{\tau}\|^{2} + L e^{\lambda |\tau|}) \\ & + c (t-\tau) (\int_{\tau}^{t} \|g(s,.)\|^{2} + \int_{\tau}^{t} J(\theta_{s}\omega) ds) \\ \leq & c (T-\tau+1) (\int_{\tau}^{T} \|g(s,.)\|^{2} ds + L \int_{\tau}^{T} e^{\lambda |s|} ds) + c (\|u_{\tau}\|^{2} + L e^{\lambda |\tau|}), \qquad (4.15) \end{aligned}$$

which gives (4.4).

Take the inner product of (3.4) in $L^2(\mathbb{R}^N)$ with $-\Delta v$, by (1.4) and (3.1), we infer that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla v\|^2 + \|\Delta v\|^2 &\leq -\int_{\mathbb{R}^N} F(x,t,u) \Delta v dx - \int_{\mathbb{R}^N} g(t,.) \Delta v dx - \int_{\mathbb{R}^N} \Delta z(\vartheta_t \omega) \Delta v dx \\ &\leq \frac{1}{2} \|\Delta v\|^2 + c(\|u\|^2 + \|u\|_{2p-2}^{2p-2} + \|g(t,.)\|^2 + |y(\vartheta_t \omega)|^2), \end{aligned}$$

which clearly gives that

$$\frac{d}{dt} \|\nabla v\|^2 + \|\Delta v\|^2 \le c(\|v\|^2 + \|u\|_{2p-2}^{2p-2} + \|g(t,.)\|^2 + |y(\vartheta_t \omega)|^2).$$
(4.16)

Applying Lemma 4.3 (*ii*) over the interval $\left[\frac{3t+\tau}{4}, t\right]$ with $\nu = 0, a = \varepsilon = \frac{t-\tau}{4}$ to (4.16), we get that for all $t \in (\tau, T]$,

$$\begin{aligned} &(t-\tau) \|\nabla v(t)\|^2 + (t-\tau) \int_{\frac{3t+\tau}{4}}^t \|\Delta v(s)\|^2 ds \\ &\leq 8 \int_{\tau}^t \|\nabla v(s)\|^2 ds + c(t-\tau) \int_{\frac{t+\tau}{2}}^t \|u(s)\|_{2p-2}^{2p-2} ds \\ &+ c \int_{\tau}^t (\|v(s)\|^2 + \|g(s,.)\|^2 + |y(\theta_s \omega)|^2) ds, \end{aligned}$$

from which and (4.7) and (4.15), we infer that (4.5) holds.

Remark 4.1. It is noted that Lemma 4.4 also holds true for the sample $\omega \in \Omega$, with the constant $C_{T-\tau}$ depending on ω in this case. Thus, Lemma 4.4 addresses that for $g \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^N))$ and any initial data $u_{\tau} \in L^2(\mathbb{R}^N)$, the solution to problem (1.1)-(1.2) satisfies that $u \in L^{\infty}(\tau, T; L^p(\mathbb{R}^N)) \cap L^{\infty}(\tau, T; H^1(\mathbb{R}^N)) \cap L^{2p-2}_{loc}(\tau + \varepsilon, +\infty; L^{2p-2}(\mathbb{R}^N))$ for any $T > \tau$ and $\varepsilon > 0$, which seems new comparing with the regularity of solutions in (1.5).

4.3. Estimate the difference of solutions in $L^2(\mathbb{R}^N)$

Given $\omega_n \in \Omega_k$, for any fixed $t \geq \tau, \tau \in \mathbb{R}$ and $v_{\tau,n} \in L^2(\mathbb{R}^N)$, denote by $v_n(t) = v(t, \tau, \omega_n, v_{\tau,n})$ be the solution of problem (1.1)-(1.2) with the initial datum $v_{\tau,n}$ at the sample ω_n and the initial time $\tau, n \in \mathbb{N}$, respectively. Let $V_n(t) =$

 $v(t, \tau, \omega_n, v_{\tau,n}) - v(t, \tau, \omega, v_{\tau})$. Then $V_n(t)$ is a solution to the following difference equation:

$$\frac{dV_n(t)}{dt} + \lambda V_n(t) - \Delta V_n(t) = F(t, x, v_n(t) + z(\vartheta_t \omega_n)) - F(t, x, v(t) + z(\vartheta_t \omega)) + \Delta (z(\vartheta_t \omega_n) - z(\vartheta_t \omega))$$
(4.17)

with the initial value $V_{\tau,n} = v_{\tau,n} - v_{\tau}$.

For convenience, we write $Y_n(t) = y(\vartheta_t \omega_n) - y(\vartheta_t \omega)$ and $U_n(t) = u(t, \tau, \omega_n, u_{\tau,n}) - u(t, \tau, \omega, u_{\tau})$ where $u(t, \tau, \omega_n, u_{\tau,n}) = u_n(t) = v_n(t) + \phi(x)y(\vartheta_t \omega_n)$. Then U_n is regarded as the difference of solutions to the original equation (1.1)-(1.2). The following is concerned with the estimate of the difference in $L^2(\mathbb{R}^N)$. Note that in the sequel we restrict that $\omega \in \Omega_k, k \in \mathbb{N}$.

Lemma 4.5. Suppose that (1.3)-(1.4) and (3.1)-(3.2) hold. Let $T > \tau \in \mathbb{R}, \omega_n, \omega \in \Omega_k$. Then for all $t \in [\tau, T]$,

$$\begin{aligned} \|u(t,\tau,\omega_n,u_{\tau,n}) - u(t,\tau,\omega,u_{\tau})\|^2 &\leq C_{T-\tau} \sup_{s\in[\tau,T]} (|Y_n(s)| + |Y_n(s)|^2 + |Y_n(s)|^p) \\ &+ c(\|u_{\tau,n} - u_{\tau}\|^2 + |Y_n(\tau)|^2), \end{aligned}$$

where $C_{T-\tau}$ is a positive constant independent of n and ω .

Proof. Let $\{u_{\tau,n}\}_{n\in\mathbb{N}}$ be the initial data sequence in $L^2(\mathbb{R}^N)$ such that $u_{\tau,n} \to u_{\tau}$ as $n \to \infty$. Without loss of generality, we set

$$||u_{\tau,n}||^2 \le ||u_{\tau}||^2 + 1, n \in \mathbb{N}.$$
(4.18)

Using the test function V_n in (4.17), we have

$$\frac{1}{2} \frac{d}{dt} \|V_n(t)\|^2 + \lambda \|V_n(t)\|^2 + \|\nabla V_n(t)\|^2$$

$$= \int_{\mathbb{R}^N} (F(t, x, v_n(t) + z(\vartheta_t \omega_n)) - F(t, x, v(t) + z(\vartheta_t \omega))) V_n(t) dx$$

$$+ (\Delta(z(\vartheta_t \omega_n) - z(\vartheta_t \omega)), V_n(t)).$$
(4.19)

We note that $U_n = u(t, \tau, \omega_n, u_{\tau,n}) - u(t, \tau, \omega, u_{\tau}) = V_n(t) + \phi(x)Y_n(t)$. For the nonlinearity in (4.19), by (1.3) and (1.4), we have

$$\begin{aligned} &(F(t,x,u_{n}(t))-F(t,x,u(t)))V_{n}(t) \\ =&(F(t,x,u_{n}(t))-F(t,x,u(t)))U_{n}-(F(t,x,u_{n}(t))-F(t,x,u(t)))Y_{n}(t)\phi(x) \\ &\leq -\alpha|U_{n}(t)|^{p}+\psi_{1}(t,x)|U_{n}(t)|^{2} \\ &+\psi_{2}(t,x)U_{n}(t)(1+|u_{n}(t)|^{p-2}+|u(t)|^{p-2})|Y_{n}(t)||\phi(x)| \\ &\leq -\alpha|U_{n}(t)|^{p}+(\psi_{1}(t,x)+\psi_{2}(t,x))|U_{n}(t)|^{2}+\psi_{2}(t,x)|Y_{n}|^{2}|\phi(x)|^{2} \\ &+\psi_{2}(t,x))U_{n}(t)(|u_{n}(t)|^{p-2}+|u(t)|^{p-2})|Y_{n}(t)||\phi(x)| \\ &\leq -\frac{\alpha}{2^{p-1}}|V_{n}(t)|^{p}+|\phi(x)|^{p}|Y_{n}|^{p}+(\psi_{1}(t,x)+\psi_{2}(t,x))|U_{n}(t)|^{2} \\ &+\psi_{2}(t,x)|Y_{n}(t)|^{2}|\phi(x)|^{2}+\psi_{2}(t,x)U_{n}(t)(|u_{n}(t)|^{p-2} \\ &+|u(t)|^{p-2})|Y_{n}(t)||\phi(x)|, \end{aligned}$$

$$(4.20)$$

where we employed $|U_n(t)|^p \ge 2^{1-p}|v_n(t)|^p - |\phi(x)|^p|Y_n(t)|^p$ for any $p \ge 2$. By the generalized Young inequality: $|abc| \le \frac{|a|^p}{p} + \frac{|b|^q}{q} + \frac{|c|^r}{r}$ if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ for any p, q, r > 1, we deduce that

$$U_{n}(t)(|u_{n}(t)|^{p-2} + |u(t)|^{p-2})|Y_{n}(t)||\phi(x)|$$

$$\leq (|u_{n}(t)| + |u(t)|)(|u_{n}(t)|^{p-2} + |u(t)|^{p-2})|Y_{n}(t)||\phi(x)|$$

$$= (|u_{n}(t)|^{p-1} + |u(t)|^{p-1} + |u_{n}(t)||u(t)|^{p-2} + |u(t)||u_{n}(t)|^{p-2})\phi(x)|Y_{n}(t)|$$

$$\leq (|u_{n}(t)|^{p} + |u(t)|^{p} + |\phi(x)|^{p})|Y_{n}(t)|.$$
(4.21)

Therefore by (4.20) and (4.21), it follows that

$$(F(t, x, u_n(t)) - F(t, x, u(t)))V_n(t)$$

$$\leq -\frac{\alpha}{2^{p-1}}|V_n(t)|^p + |\phi(x)|^p|Y_n(t)|^p + (\psi_1(t, x) + \psi_2(t, x))|U_n(t)|^2$$

$$+ \psi_2(t, x)|Y_n(t)|^2|\phi(x)|^2 + \psi_2(t, x)(|u_n(t)|^p + |u(t)|^p + |\phi(x)|^p)|Y_n(t)|, \quad (4.22)$$

from which and $\phi \in L^p(\mathbb{R}^N), \psi_i \in L^{\infty}(\mathbb{R}^{N+1}), i = 1, 2$, we derive

$$\int_{\mathbb{R}^{N}} (F(t, x, u_{n}(t)) - F(t, x, u(t))) V_{n}(t) dx$$

$$\leq -\frac{\alpha}{2^{p-1}} \|V_{n}(t)\|_{p}^{p} + \mu \|V_{n}(t)\|^{2} + c|Y_{n}(t)|^{2} + c|Y_{n}(t)|^{p}$$

$$+ c(\|u_{n}(t)\|_{p}^{p} + \|u(t)\|_{p}^{p} + \|\phi\|_{p}^{p})|Y_{n}(t)|, \qquad (4.23)$$

where $\mu = 2(\|\psi_1\|_{\infty} + \|\psi_2\|_{\infty})$. For the second term on the right hand side of (4.19), we have

$$(\Delta(z(\vartheta_t\omega_n) - z(\vartheta_t\omega)), V_n(t)) \le \frac{\lambda - \mu}{2} \|V_n(t)\|^2 + \frac{1}{2} |Y_n(t)|^2.$$
(4.24)

By a combination of (4.19), (4.23) and (4.24), it follows that

$$\frac{d}{dt} \|V_n(t)\|^2 + (\lambda - \mu) \|V_n(t)\|^2 + \|\nabla V_n(t)\|^2 + \frac{\alpha}{2^{p-1}} \|V_n(t)\|_p^p \\
\leq c(|Y_n(t)|^2 + |Y_n(t)|^p) + c(\|u_n(t)\|_p^p + \|u(t)\|_p^p + \|\phi\|_p^p)|Y_n(t)|.$$
(4.25)

Then we rewrite (4.25) as the following:

$$\frac{d}{dt} \|V_n(t)\|^2 \le -\alpha_0 \|V_n(t)\|^2 + c(|Y_n(t)|^2 + |Y_n(t)|^p)
+ c(\|u_n(t)\|_p^p + \|u(t)\|_p^p + \|\phi\|_p^p)|Y_n(t)|,$$
(4.26)

where $\alpha_0 = (\lambda - \mu) > 0$. Multiplying (4.26) by $e^{\alpha_0 t}$ and integrating over the intervals $[\tau, t]$, we get

$$\|V_{n}(t)\|^{2} \leq e^{-\alpha_{0}(t-\tau)} \|v_{\tau,n} - v_{\tau}\|^{2} + ce^{-\alpha_{0}t} \int_{\tau}^{t} e^{\alpha_{0}s} (|Y_{n}(s)|^{2} + |Y_{n}(s)|^{p}) ds$$

+ $ce^{-\alpha_{0}t} \int_{\tau}^{t} e^{\alpha_{0}s} (\|u_{n}(s)\|_{p}^{p} + \|u(s)\|_{p}^{p} + \|\phi\|_{p}^{p}) |Y_{n}(s)| ds$
 $\triangleq I_{1} + I_{2} + I_{3}.$ (4.27)

It is clear to say that for all $t \in [\tau, T]$,

$$I_{1} \triangleq 2e^{-\alpha_{0}(t-\tau)} (\|u_{\tau,n} - u_{\tau}\|^{2} + \|\phi\|^{2}|Y_{n}(\tau)|^{2})$$

$$\leq c(\|u_{\tau,n} - u_{\tau}\|^{2} + |Y_{n}(\tau)|^{2}), \qquad (4.28)$$

and

$$I_{2} \triangleq c e^{-\alpha_{0} t} \int_{\tau}^{t} e^{\alpha_{0} s} (|Y_{n}(s)|^{2} + |Y_{n}(s)|^{p}) ds$$

$$\leq c \sup_{s \in [\tau, T]} (|Y_{n}(s)|^{2} + |Y_{n}(s)|^{p}).$$
(4.29)

In terms of (4.7), we find that, for $\omega_n \in \Omega_k$, $n \in \mathbb{N}$, and $t \in [\tau, T]$,

$$\int_{\tau}^{t} e^{\alpha_{0}s} \|u_{n}(s)\|_{p}^{p} ds \leq e^{\alpha_{0}t} \int_{\tau}^{t} \|u_{n}(s)\|_{p}^{p} ds \\
\leq c e^{\alpha_{0}t} (\int_{\tau}^{T} \|g(s,.)\|^{2} ds + L \int_{\tau}^{T} e^{\lambda|s|} ds + \|u_{\tau}\|^{2} + 1 + L e^{\lambda|\tau|}) \\
\triangleq c e^{\alpha_{0}t} C_{T-\tau},$$
(4.30)

which is independent of $\omega \in \Omega_k$ and $n \in \mathbb{N}$. Then, in view of (4.30) we have

$$I_{3} \triangleq ce^{-\alpha_{0}t} \int_{\tau}^{t} e^{\alpha_{0}s} (\|u_{n}(s)\|_{p}^{p} + \|u(s)\|_{p}^{p} + \|\phi\|_{p}^{p})|Y_{n}(s)|ds$$

$$\leq cC_{T-\tau} \sup_{s \in [\tau,T]} |Y_{n}(s)|.$$
(4.31)

Consequently, it follows from (4.27) to (4.31) that for all $t \in [\tau, T]$,

$$||V_n(t)||^2 \le C_{T-\tau} \sup_{s \in [\tau,T]} (|Y_n(s)| + |Y_n(s)|^2 + |Y_n(s)|^p) + c(||u_{\tau,n} - u_{\tau}||^2 + |Y_n(\tau)|^2).$$
(4.32)

Then we have

$$||U_n(t)||^2 = ||u(t,\tau,\omega_n,u_{\tau,n}) - u(t,\tau,\omega,u_{\tau})||^2 \le 2||V_n(t)||^2 + 2||\phi||^2|Y_n(t)|^2,$$

which and (4.32) together give the desired result.

Lemma 4.6. Suppose that (1.3)-(1.4) and (3.1)-(3.2) hold. Let $V_n(t)$ be the solution to (4.17). Then we have, for all $t \in (\tau, T]$,

$$\int_{\tau}^{t} (\|\nabla V_n(s)\|^2 + \|V_n(s)\|^2) ds \leq C_{T-\tau} \sup_{s \in [\tau,T]} (|Y_n(s)| + |Y_n(s)|^2 + |Y_n(s)|^p) + c(\|u_{\tau,n} - u_{\tau}\|^2 + |Y_n(\tau)|^2), \quad (4.33)$$

and

$$\int_{\tau}^{t} \|V_{n}(s)\|_{p}^{p} ds \leq C_{T-\tau} \sup_{s \in [\tau,T]} (|Y_{n}(s)| + |Y_{n}(s)|^{2} + |Y_{n}(s)|^{p}) + c(\|u_{\tau,n} - u_{\tau}\|^{2} + |Y_{n}(\tau)|^{2}), \quad (4.34)$$

where $C_{T-\tau}$ is a positive constant independent of n and ω .

Proof. Integrate (4.25) from τ to t with $t \in (\tau, T]$ to yield

$$\int_{\tau}^{t} (\alpha_{0} \|V_{n}(s)\|^{2} + \|\nabla V_{n}(s)\|^{2} + \frac{\alpha}{2^{p-1}} \|V_{n}(s)\|_{p}^{p}) ds$$

$$\leq \|V_{n}(\tau)\|^{2} + c \int_{\tau}^{t} (|Y_{n}(s)|^{2} + |Y_{n}(s)|^{p}) ds$$

$$+ c \int_{\tau}^{t} (\|u_{n}(s)\|_{p}^{p} + \|u(s)\|_{p}^{p} + \|\phi\|_{p}^{p}) |Y_{n}(s)| ds.$$

Then by (4.30), the lemma is followed.

4.4. Estimate the difference of solutions in $L^p(\mathbb{R}^N)$

In this subsection, we will obtain some difference estimates of solutions to problem (1.1) and (1.2) in $L^p(\mathbb{R}^N)$ with respect to $u_{\tau} \in L^2(\mathbb{R}^N)$ and $\omega \in \Omega_k$ for every $k \in \mathbb{N}$.

Lemma 4.7. Suppose that (1.3)-(1.4) and (3.1)-(3.2) hold. Let $T > \tau \in \mathbb{R}, \omega_n, \omega \in \Omega_k$ and $V_n(t) = v(t, \tau, \omega_n, v_{\tau,n}) - v(t, \tau, \omega, v_{\tau})$ be a solution to (4.17). Then for all $t \in (\tau, T]$,

$$\frac{t-\tau}{2} \|V_n(t)\|_p^p \leq C_{T-\tau} \sup_{s \in [\tau,T]} (|Y_n(s)| + |Y_n(s)|^2 + |Y_n(s)|^p + |Y_n(s)|^{2p-2} + |Y_n(s)|^{\frac{2p-2}{p}}) + C_{T-\tau} (\|u_{\tau,n} - u_{\tau}\|^2 + |Y_n(\tau)|^2),$$
(4.35)

and

$$\int_{\frac{t+\tau}{2}}^{t} (s - \frac{t+\tau}{2}) \|V_n(s)\|_{2p-2}^{2p-2} ds$$

$$\leq C_{T-\tau} \sup_{s \in [\tau,T]} (|Y_n(s)| + |Y_n(s)|^2 + |Y_n(s)|^p + |Y_n(s)|^{2p-2} + |Y_n(s)|^{\frac{2p-2}{p}}) + C_{T-\tau} (\|u_{\tau,n} - u_{\tau}\|^2 + |Y_n(\tau)|^2), \qquad (4.36)$$

where $C_{T-\tau}$ is a positive constant independent of n and ω .

Proof. Taking the test function $|V_n(t)|^{p-2}V_n(t)$ in (4.17), we have

$$\frac{1}{p}\frac{d}{dt}\|V_{n}(t)\|_{p}^{p} + \lambda\|V_{n}(t)\|_{p}^{p} + (-\Delta V_{n}(t), |V_{n}(t)|^{p-2}V_{n}(t))
= \int_{\mathbb{R}^{N}} (F(t, x, v_{n}(t) + z(\vartheta_{t}\omega_{n})) - F(t, x, v(t) + z(\vartheta_{t}\omega)))|V_{n}(t)|^{p-2}V_{n}(t)dx
+ (\Delta(z(\vartheta_{t}\omega_{n}) - z(\vartheta_{t}\omega)), |V_{n}(t)|^{p-2}V_{n}(t)),$$
(4.37)

where we have $(-\Delta V_n, |V_n|^{p-2}V_n) \ge 0$. We now estimate the terms in (4.37). First, by (4.22), it follows that

$$\begin{aligned} &(F(t,x,u_n(t)) - F(t,x,u(t)))|V_n(t)|^{p-2}V_n(t) \\ &\leq -\frac{\alpha}{2^{p-1}}|V_n(t)|^{2p-2} + |Y_n(t)|^p|\phi|^pV_n(t)|^{p-2} \\ &+ (\psi_1(t,x) + \psi_2(t,x))|U_n(t)|^2)|V_n(t)|^{p-2} + \psi_2(t,x)|Y_n(t)|^2|\phi(x)|^2|V_n(t)|^{p-2} \end{aligned}$$

$$+\psi_2(t,x)(|u_n(t)|^p + |u(t)|^p + |\phi|^p))|V_n(t)|^{p-2}|Y_n(t)|.$$
(4.38)

The terms on the right hand side of (4.38) can be estimated by Young inequality, respectively. Namely, we have

$$\begin{split} &|Y_n(t)|^p |\phi|^p V_n(t)|^{p-2} \leq \frac{\alpha}{2^{p+1}} |V_n(t)|^{2p-2} + c |\phi|^{2p-2} |Y_n(t)|^{2p-2};\\ &\psi_i(t,x)) |U_n(t)|^2) |V_n(t)|^{p-2} \leq \frac{\lambda}{4} |V_n(t)|^p + |\psi_i|^{\frac{p}{2}} |U_n(t)|^p, \ i = 1,2;\\ &\psi_2(t,x) |Y_n(t)|^2 |\phi(x)|^2 |V_n(t)|^{p-2} \leq \frac{\lambda}{4} |V_n(t)|^p + |\psi_2|^{\frac{p}{2}} |\phi|^p |Y_n(t)|^p;\\ &\psi_2(t,x) (|u_n(t)|^p + |u(t)|^p + |\phi|^p)) |V_n(t)|^{p-2} |Y_n(t)|\\ &\leq \frac{\alpha}{2^{p+1}} |V_n|^{2p-2} + c (\psi_2(t,x))^{\frac{2p-2}{p}} (|u_n(t)|^{2p-2} + |u(t)|^{2p-2} + |\phi|^{2p-2}) |Y_n(t)|^{\frac{2p-2}{p}}, \end{split}$$

from which and (4.38), the nonlinearity in (4.37) is estimated as

$$\int_{\mathbb{R}^{N}} (F(t, x, u_{n}(t)) - F(t, x, u(t))) |V_{n}(t)|^{p-2} V_{n}(t) dx$$

$$\leq -\frac{\alpha}{2^{p}} \|V_{n}(t)\|_{2p-2}^{2p-2} + \frac{\lambda}{2} \|V_{n}(t)\|_{p}^{p}$$

$$+ c(\|U_{n}(t)\|_{p}^{p} + |Y_{n}(t)|^{p} + |Y_{n}(t)|^{2p-2} + |Y_{n}(t)|^{\frac{2p-2}{p}})$$

$$+ c(\|u_{n}(t)\|_{2p-2}^{2p-2} + \|u(t)\|_{2p-2}^{2p-2}) |Y_{n}(t)|^{\frac{2p-2}{p}}.$$
(4.39)

On the other hand, for the second term on the right hand of (4.37), we deduce

$$(\Delta(z(\vartheta_t\omega_n) - z(\vartheta_t\omega)), |V_n(t)|^{p-2}V_n(t)) \le \frac{\alpha}{2^{p+1}} \|V_n(t)\|_{2p-2}^{2p-2} + c|Y_n(t)|^2.$$
(4.40)

Therefore by incorporation (4.39)- (4.40) into (4.37), we find that, replacing t by s,

$$\frac{d}{ds} \|V_n(s)\|_p^p + \frac{\alpha}{2^{p+1}} \|V_n(s)\|_{2p-2}^{2p-2}
\leq c \|V_n(s)\|_p^p + c(|Y_n(s)|^2 + |Y_n(s)|^p + |Y_n(s)|^{2p-2} + |Y_n(s)|^{\frac{2p-2}{p}})
+ c(\|u_n(s)\|_{2p-2}^{2p-2} + \|u(s)\|_{2p-2}^{2p-2})|Y_n(s)|^{\frac{2p-2}{p}}.$$
(4.41)

Multiplying (4.41) by $(s - \frac{t+\tau}{2})$ with $s \in [\frac{t+\tau}{2}, t]$, we infer that

$$(s - \frac{t + \tau}{2})\frac{d}{ds} \|V_n(s)\|_p^p + \frac{\alpha}{2^{p+1}} (s - \frac{t + \tau}{2}) \|V_n(s)\|_{2p-2}^{2p-2}$$

$$\leq c(s - \frac{t + \tau}{2}) \|V_n(s)\|_p^p$$

$$+ c(s - \frac{t + \tau}{2}) (|Y_n(s)|^2 + |Y_n(s)|^p + |Y_n(s)|^{2p-2} + |Y_n(s)|^{\frac{2p-2}{p}})$$

$$+ c(s - \frac{t + \tau}{2}) (\|u_n(s)\|_{2p-2}^{2p-2} + \|u(s)\|_{2p-2}^{2p-2}) |Y_n(s)|^{\frac{2p-2}{p}}.$$
(4.42)

Note that for every $s \in [\frac{t+\tau}{2}, t]$,

$$\left(s - \frac{t + \tau}{2}\right) \frac{d}{ds} \|V_n(s)\|_p^p = \frac{d}{ds} \left(\left(s - \frac{t + \tau}{2}\right) \|V_n(s)\|_p^p\right) - \|V_n(s)\|_p^p.$$
(4.43)

Then by (4.42) and (4.43) we get, for all $s \in [\frac{t+\tau}{2}, t]$ with $t \in (\tau, T]$,

$$\frac{d}{ds}\left((s-\frac{t+\tau}{2})\|V_n(s)\|_p^p\right) + \frac{\alpha}{2^{p+1}}\left(s-\frac{t+\tau}{2}\right)\|V_n(s)\|_{2p-2}^{2p-2}$$

$$\leq c(s - \frac{t + \tau}{2}) \|V_n(s)\|_p^p + \|V_n(s)\|_p^p + c(s - \frac{t + \tau}{2})(|Y_n(s)|^2 + |Y_n(s)|^p + |Y_n(s)|^{2p-2} + |Y_n(s)|^{\frac{2p-2}{p}}) + c(s - \frac{t + \tau}{2})(\|u_n(s)\|_{2p-2}^{2p-2} + \|u(s)\|_{2p-2}^{2p-2})|Y_n(s)|^{\frac{2p-2}{p}} \leq C_{T-\tau}(\|V_n(s)\|_p^p + |Y_n(s)|^2 + |Y_n(s)|^p + |Y_n(s)|^{2p-2} + |Y_n(s)|^{\frac{2p-2}{p}}) + c(s - \frac{t + \tau}{2})(\|u_n(s)\|_{2p-2}^{2p-2} + \|u(s)\|_{2p-2}^{2p-2})|Y_n(s)|^{\frac{2p-2}{p}},$$

$$(4.44)$$

where $C_{T-\tau} > 0$ is a positive constant with $T - \tau$ fixed. Integrating (4.44) over the interval $\left[\frac{t+\tau}{2}, t\right]$ for every $t \in (\tau, T]$, by integration by parts for the left term, we get

$$\frac{t-\tau}{2} \|V_n(t)\|_p^p + \frac{\alpha}{2^{p+1}} \int_{\frac{t+\tau}{2}}^t (s - \frac{t+\tau}{2}) \|V_n(s)\|_{2p-2}^{2p-2} ds$$

$$= \int_{\frac{t+\tau}{2}}^t \frac{d}{ds} \Big((s - \frac{t+\tau}{2}) \|V_n(s)\|_p^p \Big) ds + \frac{\alpha}{2^{p+1}} \int_{\frac{t+\tau}{2}}^t (s - \frac{t+\tau}{2}) \|V_n(s)\|_{2p-2}^{2p-2} ds$$

$$\leq C_{T-\tau} \Big(\int_{\tau}^t \|V_n(s)\|_p^p ds + \int_{\tau}^T (|Y_n(s)|^2 + |Y_n(s)|^p + |Y_n(s)|^{2p-2} + |Y_n(s)|^{\frac{2p-2}{p}}) ds \Big)$$

$$+ c(t-\tau) \int_{\frac{t+\tau}{2}}^t (\|u_n(s)\|_{2p-2}^{2p-2} + \|u(s)\|_{2p-2}^{2p-2}) |Y_n(s)|^{\frac{2p-2}{p}} ds.$$
(4.45)

On the other hand, in view of (4.4) and (4.18), there exists constant $C_{T-\tau} > 0$ independent of n and $\omega \in \Omega_k$, such that for all $\omega_n \in \Omega_k$ and $t \in (\tau, T]$,

$$(t-\tau)\int_{\frac{t+\tau}{2}}^{t} \|u_n(s)\|_{2p-2}^{2p-2}ds \le C_{T-\tau}, \quad (t-\tau)\int_{\frac{t+\tau}{2}}^{t} \|u(s)\|_{2p-2}^{2p-2}ds \le C_{T-\tau}.$$
 (4.46)

Consequently, by (4.34), (4.45) and (4.46), we deduce that, for all $t \in (\tau, T]$,

$$\frac{t-\tau}{2} \|V_n(t)\|_p^p + \frac{\alpha}{2^{p+1}} \int_{\frac{t+\tau}{2}}^t (s - \frac{t+\tau}{2}) \|V_n(s)\|_{2p-2}^{2p-2} ds$$

$$\leq C_{T-\tau} (\|u_{\tau,n} - u_{\tau}\|^2 + |Y_n(\tau)|^2) + C_{T-\tau} \sup_{s \in [\tau,T]} (|Y_n(s)| + |Y_n(s)|^2 + |Y_n(s)|^{2p-2} + |Y_n(s)|^{\frac{2p-2}{p}}),$$

which concludes the proof.

4.5. Estimate the difference of solutions in $H^1(\mathbb{R}^N)$

The following lemma is concerned with the estimate of the difference of solutions in $H^1(\mathbb{R}^N)$, which is involved in the discussions.

Lemma 4.8. Suppose that (1.3)-(1.4) and (3.1)-(3.2) hold. Let $T > \tau \in \mathbb{R}, \omega_n, \omega \in \Omega_k$ and $V_n = v(t, \tau, \omega_n, v_{\tau,n}) - v(t, \tau, \omega, v_{\tau})$ be a solution to (4.17). Then for all $t \in (\tau, T]$,

$$\left(\frac{t-\tau}{2}\right)^2 \|\nabla V_n(t)\|^2$$

$$\leq C_{T-\tau} \left(\sup_{s \in [\tau,T]} (|Y_n(s)| + |Y_n(s)|^2 + |Y_n(s)|^p) + ||u_{\tau,n} - u_{\tau}||^2 + |Y_n(\tau)|^2 \right) \\ + \left(C_{T-\tau} \right)^{\frac{p-2}{p-1}} \left\{ C_{T-\tau} \left(\sup_{s \in [\tau,T]} (|Y_n(s)| + |Y_n(s)|^2 + |Y_n(s)|^p + |Y_n(s)|^{2p-2} + |Y_n(s)|^{\frac{2p-2}{p}} \right) + ||u_{\tau,n} - u_{\tau}||^2 + |Y_n(\tau)|^2 \right\}^{\frac{1}{p-1}},$$

where $C_{T-\tau}$ is a positive constant independent of n and ω .

Proof. We begin with some estimates. First, by (1.4), we infer that

$$\left| \int_{\mathbb{R}^{N}} (F(t, x, u_{n}(t)) - F(t, x, u(t))) V_{nt} dx \right|$$

$$\leq c \int_{\mathbb{R}^{N}} (1 + |u_{n}(t)|^{p-2} + |u(t)|^{p-2}) |U_{n}(t)| |V_{nt}| dx$$

$$\leq \frac{1}{4} ||V_{nt}||^{2} + c ||U_{n}(t)||^{2} + c \int_{\mathbb{R}^{N}} (|u_{n}(t)|^{2p-4} + |u(t)|^{2p-4}|) U_{n}(t)|^{2} dx$$

$$\leq \frac{1}{4} ||V_{nt}||^{2} + c ||U_{n}(t)||^{2} + c (||u_{n}(t)||^{2p-4}_{2p-2} + ||u(t)||^{2p-4}_{2p-2}) ||U_{n}(t)||^{2}_{2p-2}, \quad (4.47)$$

where V_{nt} is the derivative of $V_n(t)$ with respect to the time t. Using Hölder' inequality and Young' inequality, we get

$$\left| -\Delta(z(\vartheta_t\omega_n) - z(\vartheta_t\omega)), V_{nt}) \right| \le \frac{1}{4} \|V_{nt}\|^2 + c|Y_n(t)|^2, \tag{4.48}$$

where $c = c(\|\Delta\phi\|)$. Taking the inner product of (4.17) in $L^2(\mathbb{R}^N)$ with V_{nt} , from (4.47) and (4.48) we deduce that, with t being replaced by s,

$$\frac{d}{ds} \|\nabla V_n(s)\|^2 \le c \|V_n(s)\|^2 + c|Y_n(s)|^2
+ c(\|u_n(s)\|_{2p-2}^{2p-4} + \|u(s)\|_{2p-2}^{2p-4}) \|U_n(s)\|_{2p-2}^2.$$
(4.49)

Multiply (4.49) by $(s - \frac{t+\tau}{2})^{\frac{p}{p-1}}$ with $s \in [\frac{t+\tau}{2}, t]$ to yield

$$(s - \frac{t + \tau}{2})^{\frac{p}{p-1}} \frac{d}{ds} \|\nabla V_n(s)\|^2$$

$$\leq c(s - \frac{t + \tau}{2})^{\frac{p}{p-1}} \|V_n(s)\|^2 + c(s - \frac{t + \tau}{2})^{\frac{p}{p-1}} |Y_n(s)|^2$$

$$+ c(s - \frac{t + \tau}{2})^{\frac{p}{p-1}} (\|u_n(s)\|_{2p-2}^{2p-4} + \|u(s)\|_{2p-2}^{2p-4}) \|U_n(s)\|_{2p-2}^2$$

$$\leq c(t - \tau)^{\frac{p}{p-1}} \|V_n(s)\|^2 + c(t - \tau)^{\frac{p}{p-1}} |Y_n(s)|^2$$

$$+ c(s - \frac{t + \tau}{2})^{\frac{p}{p-1}} (\|u_n(s)\|_{2p-2}^{2p-4} + \|u(s)\|_{2p-2}^{2p-4}) \|U_n(s)\|_{2p-2}^2.$$
(4.50)

By integrating the last inequality (4.50) over the intervals $\left[\frac{t+\tau}{2}, t\right]$, using the technique of the integration by parts for the left term, we obtain

$$(\frac{t-\tau}{2})^{\frac{p}{p-1}} \|\nabla V_n(t)\|^2 \leq c \int_{\frac{t+\tau}{2}}^t (s - \frac{t+\tau}{2})^{\frac{1}{p-1}} \|\nabla V_n(s)\|^2 ds + c(t-\tau)^{\frac{p}{p-1}} \int_{\frac{t+\tau}{2}}^t (\|V_n(s)\|^2 + |Y_n(s)|^2) ds$$

$$+ c \int_{\frac{t+\tau}{2}}^{t} \left(s - \frac{t+\tau}{2}\right)^{\frac{p}{p-1}} \left(\|u_n(s)\|_{2p-2}^{2p-4} + \|u(s)\|_{2p-2}^{2p-4}\right) \|U_n(s)\|_{2p-2}^{2} ds.$$
(4.51)

Multiply (4.51) by $\left(\frac{t-\tau}{2}\right)^{\frac{p-2}{p-1}}$ to yield $t-\tau$

$$\frac{(t-\tau)^{2}}{2} \|\nabla V_{n}(t)\|^{2} \leq c(t-\tau)^{\frac{p-2}{p-1}} \int_{\frac{t+\tau}{2}}^{t} (s-\frac{t+\tau}{2})^{\frac{p}{p-1}} (\|u_{n}(s)\|_{2p-2}^{2p-4} + \|u(s)\|_{2p-2}^{2p-4}) \|U_{n}(s)\|_{2p-2}^{2} ds + c(t-\tau) \int_{\frac{t+\tau}{2}}^{t} \|\nabla V_{n}(s)\|^{2} ds + c(t-\tau)^{2} \int_{\frac{t+\tau}{2}}^{t} (\|V_{n}(s)\|^{2} + |Y_{n}(s)|^{2}) ds.$$
(4.52)

We now estimate every term on the right hand side of (4.52). For the first term on the right hand side of (4.52), by using Hölder' inequality, we deduce that

$$c(t-\tau)^{\frac{p-2}{p-1}} \int_{\frac{t+\tau}{2}}^{t} \left(s - \frac{t+\tau}{2}\right)^{\frac{p}{p-1}} (\|u_n(s)\|_{2p-2}^{2p-4} + \|u(s)\|_{2p-2}^{2p-4}) \|U_n(s)\|_{2p-2}^{2} ds$$

$$\leq c \left((t-\tau) \int_{\frac{t+\tau}{2}}^{t} (\|u_n(s)\|_{2p-2}^{2p-2} + \|u(s)\|_{2p-2}^{2p-2}) ds \right)^{\frac{p-2}{p-1}} \times \left(\int_{\frac{t+\tau}{2}}^{t} (s - \frac{t+\tau}{2})^p \|U_n(s)\|_{2p-2}^{2p-2} ds \right)^{\frac{1}{p-1}}.$$

Hence it follows from (4.46) and (4.36) that for all $t \in (\tau, T]$,

$$c(t-\tau)^{\frac{p-2}{p-1}} \int_{\frac{t+\tau}{2}}^{t} (s - \frac{t+\tau}{2})^{\frac{p}{p-1}} (\|u_n(s)\|_{2p-2}^{2p-4} + \|u(s)\|_{2p-2}^{2p-4}) \|U_n(s)\|_{2p-2}^{2} ds$$

$$\leq (C_{T-\tau})^{\frac{p-2}{p-1}} \left\{ C_{T-\tau} (\sup_{s \in [\tau,T]} (|Y_n(s)| + |Y_n(s)|^2 + |Y_n(s)|^p + |Y_n(s)|^{2p-2} + |Y_n(s)|^{\frac{2p-2}{p}}) + \|u_{\tau,n} - u_{\tau}\|^2 + |Y_n(\tau)|^2) \right\}^{\frac{1}{p-1}}.$$
(4.53)

By (4.33), we have, for all $t \in (\tau, T]$,

$$c(t-\tau) \int_{\frac{t+\tau}{2}}^{t} \|\nabla V_n(s)\|^2 ds + c(t-\tau)^2 \int_{\frac{t+\tau}{2}}^{t} \|V_n(s)\|^2 ds$$

$$\leq C_{T-\tau} (\sup_{s \in [\tau,T]} (|Y_n(s)| + |Y_n(s)|^2 + |Y_n(s)|^p) + \|u_{\tau,n} - u_{\tau}\|^2 + |Y_n(\tau)|^2). \quad (4.54)$$

Consequently incorporate (4.53) and (4.54) into (4.52) to produce

$$\frac{(t-\tau)^2}{2} \|\nabla V_n(t)\|^2 \leq C_{T-\tau} (\sup_{s\in[\tau,T]} (|Y_n(s)| + |Y_n(s)|^2 + |Y_n(s)|^p) + \|u_{\tau,n} - u_{\tau}\|^2 + |Y_n(\tau)|^2) + (C_{T-\tau})^{\frac{p-2}{p-1}} \Big\{ C_{T-\tau} (\sup_{s\in[\tau,T]} (|Y_n(s)| + |Y_n(s)|^2 + |Y_n(s)|^p + |Y_n(s)|^{2p-2} + |Y_n(s)|^{\frac{2p-2}{p}}) + \|u_{\tau,n} - u_{\tau}\|^2 + |Y_n(\tau)|^2) \Big\}^{\frac{1}{p-1}},$$

for all $t \in (\tau, T]$. This concludes the proof.

3325

4.6. Continuity of solutions in $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ w.r.t. the initial data and samples

Given $k \in \mathbb{N}$, in view of Lemma 4.1, the mapping $y(\vartheta_t \omega) : (\Omega_k, \rho) \ni \omega \to \mathbb{R}$ is uniformly continuous on the arbitrary bounded interval $[\tau, T]$, and consequently $\sup_{s \in [\tau,T]} |Y_n(s)|$ converges to zero as $\omega_n \to \omega$ in (Ω_k, ρ) . Therefore by Lemma 4.5, Lemma 4.7 and Lemma 4.8, we obtain the continuity in $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ of solutions to problem (1.1)-(1.2) with respect to the initial data belonging to $L^2(\mathbb{R}^N)$ and the samples in (Ω_k, ρ) .

Theorem 4.1. Suppose that (1.3)-(1.4) and (3.1)-(3.2) hold. Given $T > \tau \in \mathbb{R}$ and $k \in \mathbb{N}$ fixed, if $u_{\tau,n} \to u_{\tau}$ in $L^2(\mathbb{R}^N)$ and $\omega_n \to \omega$ in (Ω_k, ρ) then for all $t \in (\tau, T]$,

$$u(t,\tau,\omega_n,u_{\tau,n}) \to u(t,\tau,\omega,u_{\tau}) \text{ in } L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$$

where u(.) is the solution to problem (1.1)-(1.2).

Based on Theorem 4.1, we obtain our second main result in this section.

Theorem 4.2. Suppose that (1.3)-(1.4) and (3.1)-(3.2) hold. Given $T > \tau \in \mathbb{R}$ and $\omega \in \Omega$ fixed, if $u_{\tau,n} \to u_{\tau}$ in $L^2(\mathbb{R}^N)$, then for all $t \in (\tau, T]$,

$$u(t,\tau,\omega,u_{\tau,n}) \to u(t,\tau,\omega,u_{\tau}) \text{ in } L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$$

where u(.) is the solution to problem (1.1)-(1.2).

Proof. Given $\omega \in \Omega$, there is a $k \in \mathbb{N}$ such that $\omega \in \Omega_k$. Let $\omega_n = \omega$. Then $\omega_n \to \omega$ in (Ω_k, ρ) and $Y_n \equiv 0$. Therefore Theorem 4.1 is applicable.

5. Applications to the pullback random attractor

In this section, the results on the continuity are applied to analyse the regularity of pullback random attractor. To this purpose, we assume that the non-autonomous term $g \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^N))$ satisfies, for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^{\tau} e^{\alpha_0 s} \|g(s,.)\|^2 ds < \infty,$$
(5.1)

where $\alpha_0 = \lambda - \mu > 0$.

Given $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$ and $u_{\tau} \in L^2(\mathbb{R}^N)$, define a random dynamical system associated with problem (1.1)-(1.2) by

$$\varphi(t,\tau,\omega,u_{\tau}) = u(t+\tau,\tau,\vartheta_{-\tau}\omega,u_{\tau}) = v(t+\tau,\tau,\vartheta_{-\tau}\omega,v_{\tau}) + \phi(x)z(\vartheta_t\omega), \quad (5.2)$$

where $u_{\tau} = v_{\tau} + \phi(x)z(\omega)$. It is well-known that φ is a continuous random dynamical system in $L^2(\mathbb{R}^N)$.

Let $D = \{D(\tau, \omega) \subset L^2(\mathbb{R}^N) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded non-empty closed subsets of $L^2(\mathbb{R}^N)$ such that for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t \to +\infty} e^{-\alpha_0 t} \|D(\tau - t, \vartheta_{-t}\omega)\|^2 = 0,$$
(5.3)

where $\alpha_0 = \lambda - \mu > 0$ and $||D|| = \sup\{||u|| : u \in D\}$. Denote by \mathfrak{D} the collection of all families of bounded nonempty closed subsets of $L^2(\mathbb{R}^N)$ such that (5.3) holds. Then it is obvious that \mathfrak{D} is inclusion closed.

By a similar arguments as in [23, Lemma 4.5, Lemma 4.7], we can prove that the random dynamical system φ defined by (5.2) has a closed \mathfrak{D} -pullback random absorbing set and furthermore φ is \mathfrak{D} -pullback asymptotically compact in $L^2(\mathbb{R}^N)$. We here only present results and omit the detailed proofs.

Lemma 5.1. Suppose that (1.3)-(1.4) and (3.1)-(3.2) hold. Let $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$. Then there exists a closed \mathfrak{D} -pullback random absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ in $L^2(\mathbb{R}^N)$ for the random dynamical system φ defined by (5.2).

Lemma 5.2. Suppose that (1.3)-(1.4) and (3.1)-(3.2) hold. Then the random dynamical system φ defined by (5.2) is \mathfrak{D} -pullback asymptotically compact in $L^2(\mathbb{R}^N)$, namely, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence $\{\varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, u_{0,n})\}_{n=1}^{\infty}$ has a convergent subsequence in $L^2(\mathbb{R}^N)$ whenever $t_n \to \infty$ and $u_{0,n} \in D(\tau - t_n, \vartheta_{-t_n}\omega)$ with $D \in \mathfrak{D}$.

Let Y be a separable Banach space. Then the measurability of a mapping φ from Ω to Y is determined by the measurability as a mapping from the sample space Ω_k to Y, $k \in \mathbb{N}$. Specifically, we have

Lemma 5.3 (Lemma 6.3, [28]). Let φ be a mapping from Ω to Y such that φ : $\Omega_k \to \Lambda$ is $(\mathcal{F}_{\Omega_k}, \mathcal{B}(Y))$ -measurable for every fixed k, then $\varphi : \Omega \to \Lambda$ is $(\mathcal{F}, \mathcal{B}(Y))$ -measurable.

Lemma 5.4. Suppose that (1.3)-(1.4) and (3.1)-(3.2) hold. Then for every fixed $t > 0, \tau \in \mathbb{R}, x \in L^2(\mathbb{R}^N), \varphi(t, \tau, ., x) : \Omega \to L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ is $(\mathcal{F}, \mathcal{B}(L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)))$ -measurable.

Proof. By Theorem 4.1, we know that for every fixed $t > 0, \tau \in \mathbb{R}, x \in L^2(\mathbb{R}^N)$, $\varphi(t,\tau,.,x) : (\Omega_k,\rho) \to L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ is continuous, and thus $\varphi(t,\tau,.,x) : (\Omega_k,\rho) \to \text{is } (\mathcal{F}_{\Omega_k},\mathcal{B}(L^p(\mathbb{R}^N)\cap H^1(\mathbb{R}^N)))$ -measurable. By Lemma 5.3, we get that $\varphi(t,\tau,.,x) : \Omega \to L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ is $(\mathcal{F},\mathcal{B}(L^p(\mathbb{R}^N)\cap H^1(\mathbb{R}^N)))$ -measurable. \Box

By Theorem 4.2, we find that the random dynamical system φ defined by (5.2) is continuous in $L^2(\mathbb{R}^N)$, and furthermore φ is also continuous from $L^2(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. Specifically, we have

Lemma 5.5. Suppose that (1.3)-(1.4) and (3.1)-(3.2) hold. Then the random dynamical system φ defined by (5.2) is bi-spatial $(L^2(\mathbb{R}^N), L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N))$ -continuous with respect to the initial data belonging to $L^2(\mathbb{R}^N)$ for arbitrary $p \geq 2$.

By Lemma 5.1-5.2 and Lemma 5.4-5.5, we know that all the conditions of Theorem 2.1 in the preliminary are satisfied if $X = L^2(\mathbb{R}^N)$ and $Y = L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. Therefore we have the main result in this section.

Theorem 5.1. Suppose that (1.3)-(1.4) and (3.1)-(3.2) hold and $g \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^N))$ satisfying (5.1). Then the random dynamical system φ defined by (5.2) admits a unique $(L^2(\mathbb{R}^N), L^p(\mathbb{R}^N) \cap H^1(\mathbb{R}^N))$ -pullback random attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ for any $p \geq 2$, in the sense of Definition 2.4.

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