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NEW OPERATIONAL MATRIX OF RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE OF ORTHONORMAL BERNOULLI POLYNOMIALS FOR THE NUMERICAL SOLUTION OF SOME DISTRIBUTED-ORDER TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract In this article, the orthonormal Bernoulli polynomials (OBPs) and their properties are applied for concluding a general technique for forming a new operational matrix of the distributed-order (DO) fractional derivative. Then, we apply tau approach and obtained operational matrix to solve some DO time-fractional partial differential equations including distributed-order Rayleigh-Stokes problem (DRSP) for a generalized second-grade fluid and DO anomalous sub-diffusion equation. Our methodology reduces the solution of these problems to a set of algebraic equations. By analysis the error of approximation by the obtained matrix and comparing between the numerical solutions and exact result, we can conclude that this operational matrix is valid to solve the mentioned equations. Also, to confirm the accuracy and the validity of our technique three examples are provided. Finally, we compare obtained results from this approach with the achieved results from relevant studies.

Keywords Distributed-order, Riemann-Liouville derivative, orthonormal Bernoulli polynomials, fractional differential equations, Rayleigh-Stokes problem.

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1. Introduction

Recently, the study of distributed-order fractional differential (DOFD) equations has generated a great deal of interest among researchers. DOFD equations provide a more precise tool to build and illstrate some adequate models for certain dynamical systems [29]. When the differential orders are integrated within the range of values, so the DOFD operators appeared [8]. Most analytical methods are complicated because the exact solution in these techniques obtained with the special functions

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and infinite series. So for numerical solutions, they are inconvenient. To date, there are several papers published that have been widely studied on how to solve DOFD equations. Generally, the construction of the new scheme becomes a two-stage process. In the first step, a quadrature formulae was utilized to estimate the DOFD equation to a multi-term fractional differential equation (FDE). In the second step, we solve this multi-term FDE by a suitable numerical technique. Nevertheless, analytical methods for solving DOFD equations are complicated forms, so in recent decades, there seems to be a growing interest in the expansion of numerical methods to solve DOFD equations. For example, Mashayekhi and Razzaghi [35] applied hybrid functions for solving DOFD equations. Heydari et al. [27] introduced the orthonormal piecewise Jacobi functions to solve the DO time-fractional Schrödinger equation. Ye et al. [53] used a compact difference method for a DO time-fractional diffusion-wave equation. Gao et al. [17] suggested two different methods of both one-dimensional and two-dimensional DOFD equation. Heydari et al. presented a numerical method for distributed-order time fractional 2D Sobolev equation in [25]. We also refer the interested reader to [1, 4, 7, 13–16, 23, 24, 26, 37, 39]. For approximated the integral term, in definition of (DO) fractional derivative, Diethelm and Ford applied the trapezoidal rule [12] and the midpoint rule was used in [37,54]. Also, the Simpson's rule and the composite trapezoidal rule were used in [17, 18].

In 2019, for the first time, we constructed operational matrix based on Legendre polynomials for DOFD equations [41]. Also, we constructed the generalize of this matrix by use of Müntz-Legendre polynomials in 2022 [42]. It is noted that we work on these papers on the concept of Caputo fractional derivative. As far as we know, to date there isn't any operational matrix has been constructed to solve DOFD equations by Riemann-Liouville (R-L) fractional derivative. In this study for the first time, we construct operational matrix by orthonormal Bernoulli polynomials for DOFD equations in the R-L type. The spectral tau approach and calculated operational matrix are used to solve some DO time-fractional partial differential equations. Our methodology converts the solution of this problem to a system of algebraic equations. For this main we study a class of DO time-fractional partial differential equations as follows:

$$\frac{\partial q(z,\mathfrak{T})}{\partial \mathfrak{T}} = \left(\lambda + D_{\mathfrak{T}}^{\eta(\vartheta)}\right) \frac{\partial^2 q(z,\mathfrak{T})}{\partial z^2} + \mathfrak{h}(z,\mathfrak{T}), \qquad z \in [0,\mathsf{L}],\ \mathfrak{T} \in [0,\varpi], \tag{1.1}$$

with initial condition:

$$q(z,0) = \mathfrak{f}(z), \tag{1.2}$$

and the boundary conditions:

$$q(0,\mathfrak{T}) = g_1(\mathfrak{T}), \qquad q(\mathsf{L},\mathfrak{T}) = g_2(\mathfrak{T}), \tag{1.3}$$

where $D_{\mathfrak{T}}^{\eta(\vartheta)}$ denoted DO fractional derivative and $\eta(\vartheta)$ is a non-negative smooth weight function.

For $\lambda = 1$ in Eqs. (1.1)-(1.3), we obtain the DRSP for a heated generalized second-grade fluid with DO time-fractional derivative. Hafez et al. in [20] applied Jacobi spectral Galerkin method to solve this problem. As pointed by [20], in order to study the behavior of the solution to DRSP, great attention was paid to determine a closed-form solution for the particular case $\eta(\vartheta) = \delta(\vartheta + \gamma - 1)$, where $\gamma \in (0, 1)$ and $\delta(.)$ display Dirac delta function. Recently, much attention has been paid to the fractional Rayleigh-Stokes problem [9,11,52]. Certain classical problems can be considered as a special case of this model. The authors of [47] obtained the exact solution of the velocity and temperature fields for this problem by using the Fourier sine transform and the fractional Laplace transform. To solve this model, effective numerical methods studies by the researcher, for example, the finite element approach to the two-dimensional fractional Rayleigh-Stokes model was proposed by in [11]. Mohebbi et al. [36] compared radial basis functions meshless technique and compact difference method of two-dimensional fractional Rayleigh-Stokes problem. Although, the spectral meshless radial point interpolation approach [48], compact finite difference approximation [9], implicit numerical approximation scheme [50], reproducing kernel method [33] and other methods [3, 6, 51, 55] have been utilized for solving fractional Rayleigh-Stokes problem.

For $\lambda = 0$ in Eqs. (1.1)-(1.3), we obtain the modified DO anomalous sub-diffusion equation [2, 32]. This model can be seen as an extension of the anomalous sub-diffusion equation. The authors of [32] used backward difference method in time and Galerkin finite element method in space, for solving this equation. Also, the authors of [2] proposed meshless Galerkin method based upon the shape functions of reproducing kernel particle method to solve this problem.

A plan of this article organized is as follows: Some basic properties and definitions of OBPs and fractional derivative operators are prepared in Section 2. In Section 3 by using OBPs the new operational matrix for FDEs and DOFD equations are constructed. Section 4 is devoted to the numerical method for solving the problem given in Eqs. (1.1)-(1.3). In Section 5 we obtain the upper error bound for the operational matrix. Section 6, include our numerical findings and demonstrate the good performance of the developed approach. Finally, in Section 7 we conclude with a few concluding remarks.

2. Bernoulli polynomials

Bernoulli polynomials have received considerable attention in numerical analysis. The classical Bernoulli polynomial of $\mathfrak{u}$th degree is defined on the interval $[0, 1]$ as [19]

$$B_{\mathfrak{u}}(\mathfrak{T}) = \sum_{\varsigma=0}^{\mathfrak{u}} \frac{\mathfrak{u}!}{(\mathfrak{u}-\varsigma)!\varsigma!} \beta_{\varsigma} \mathfrak{T}^{\mathfrak{u}-\varsigma},$$

in which $\beta_{\varsigma}, \varsigma = 0, ..., \mathfrak{u}$ are Bernoulli numbers and determined by

$$\frac{\mathfrak{T}}{\exp(\mathfrak{T}) - 1} = \sum_{\varsigma=0}^{\infty} \beta_{\varsigma} \frac{\mathfrak{T}^{\varsigma}}{\varsigma!}.$$

Bernoulli polynomials have a lot of useful properties, but they have no orthogonal properties. In some numerical methods, the orthogonality properties are particularly. So using these polynomials are less appropriate than orthogonal polynomials such as Chebyshev and Legendre polynomials. To overcome this problem the Gram-Schmidt orthonormalization process on sets of Bernoulli polynomials of different degrees are used. For a particular case, on the interval $[0, 1]$ we have [46]

$$\begin{aligned}
&\Theta_0(\mathfrak{T}) = 1, \\
&\Theta_1(\mathfrak{T}) = \sqrt{3}(2\mathfrak{T} - 1), \\
&\Theta_2(\mathfrak{T}) = \sqrt{5}(6\mathfrak{T}^2 - 6\mathfrak{T} + 1),
\end{aligned}$$

$$\Theta_3(\mathfrak{T}) = \sqrt{7}(20\mathfrak{T}^3 - 30\mathfrak{T}^2 + 12\mathfrak{T} - 1),$$
$$\Theta_4(\mathfrak{T}) = 3(70\mathfrak{T}^4 - 140\mathfrak{T}^3 + 90\mathfrak{T}^2 - 20\mathfrak{T} + 1),$$
$$\Theta_5(\mathfrak{T}) = \sqrt{11}(252\mathfrak{T}^5 - 630\mathfrak{T}^4 + 560\mathfrak{T}^3 - 210\mathfrak{T}^2 + 30\mathfrak{T} - 1).$$

We can find a pattern in the coefficients of this polynomial by analyzing these coefficients and then introducing the shifted OBPs. So we have the following definition.

Definition 2.1. The OBPs on the interval $[0, \varpi]$ are defind as [22, 46]

$$\Theta_{\mathfrak{u},\varpi}(\mathfrak{T}) = \sqrt{\frac{1+2\mathfrak{u}}{\varpi}} \sum_{\varsigma=0}^{\mathfrak{u}} (-1)^{\varsigma} \begin{pmatrix} \mathfrak{u} \\ \varsigma \end{pmatrix} \begin{pmatrix} 2\mathfrak{u}-\varsigma \\ \mathfrak{u}-\varsigma \end{pmatrix} \frac{\mathfrak{T}^{\mathfrak{u}-\varsigma}}{\varpi^{\mathfrak{u}-\varsigma}}, \quad \mathfrak{u} \geq 0. \tag{2.1}$$

Thus, these polynomials satisfy the orthogonality properly as follows [46]

$$\int_0^{\varpi} \Theta_{r,\varpi}(\mathfrak{T})\Theta_{s,\varpi}(\mathfrak{T}) d\mathfrak{T} = \delta_{rs}, \quad r, s = 0, 1, 2, \ldots,$$

where δ_{rs} displays the Kronecker delta function.

Remark 2.1. Some of the advantages of Bernoulli basis functions are: (i) as said in [5] Bernoulli basis functions, in comparison with some basis functions provide more accurate approximations of the problem solution with a fewer number of basis functions. (ii) The OBPs are simple basis functions, so the implementation of the Bernoulli operational matrices method is easy. (iii) As said in [34] the Bernoulli polynomials have fewer terms than shifted Legendre polynomials (SLP). Also, the coefficient of individual terms in Bernoulli polynomials is smaller than the coefficient of individual terms in the SLP.

Definition 2.2. The left R-L fractional derivative of order ϑ of $q(\mathfrak{T})$ is given by the following formulae [40]

$${}^{RL}D_{\mathfrak{T}}^{\vartheta} q(\mathfrak{T}) = \frac{1}{\Gamma(\mathsf{n}-\vartheta)} \frac{d^{\mathsf{n}}}{d\mathfrak{T}^{\mathsf{n}}} \int_0^{\mathfrak{T}} (\mathfrak{T}-s)^{\mathsf{n}-\vartheta-1} q(s) ds, \quad \mathsf{n}-1 \leq \vartheta < \mathsf{n}, \quad \mathsf{n} \in \mathbb{N}.$$

Here, $\Gamma(.)$ is the the Gamma function.

It is noted that the R-L fractional derivative of the power function satisfies [31, 40]

$${}^{RL}D_{\mathfrak{T}}^{\vartheta} \mathfrak{T}^{\delta} = \frac{\Gamma(\delta+1)}{\Gamma(\delta+1-\vartheta)} \mathfrak{T}^{\delta-\vartheta}, \quad \mathsf{n}-1 \leq \vartheta < \mathsf{n}, \quad \delta > -1, \quad \delta \in \mathbb{R}. \tag{2.2}$$

Also, the R-L fractional derivative is a linear operator.

Definition 2.3. The Caputo fractional derivative of order ϑ of $q(\mathfrak{T})$ is given by [21, 40, 45]

$${}^{C}D_{\mathfrak{T}}^{\vartheta} q(\mathfrak{T}) = \begin{cases} \dfrac{1}{\Gamma(\mathsf{n}-\vartheta)} \int_0^{\mathfrak{T}} (\mathfrak{T}-s)^{\mathsf{n}-\vartheta-1} q^{(\mathsf{n})}(s) ds, & \mathsf{n}-1 < \vartheta < \mathsf{n}, \quad \mathsf{n} \in \mathbb{N}, \\ \frac{d^{\mathsf{n}}}{d\mathfrak{T}^{\mathsf{n}}} q(\mathfrak{T}), & \vartheta = \mathsf{n} \in \mathbb{N}. \end{cases}$$

Lemma 2.1. *The R-L fractional derivative and the Caputo fractional derivative satisfy the following relation [31]*

$$ {}^{RL}D_{\mathfrak{T}}^{\vartheta}q(\mathfrak{T}) = {}^{C}D_{\mathfrak{T}}^{\vartheta}q(\mathfrak{T}) + \sum_{\varsigma=0}^{\mathsf{n}-1} \frac{\mathfrak{T}^{\varsigma-\vartheta}q^{(\varsigma)}(0)}{\Gamma(\varsigma+1-\vartheta)}, \tag{2.3} $$

so these two fractional derivatives are equivalent if and only if $q^{(\varsigma)}(0) = 0, \quad \varsigma = 0, \dots, \mathsf{n}-1$.

Definition 2.4. For $\vartheta \in (0,1), \eta(\vartheta) \geq 0,\ \eta(\vartheta) \not\equiv 0$ and $0 < \int_0^1 \eta(\vartheta)d\vartheta < \infty$, the distributed-order fractional derivatives in the R-L is defined as [29]

$$ D_{\mathfrak{T}}^{\eta(\vartheta)}q(\mathfrak{T}) = \int_0^1 \eta(\vartheta)\,{}^{RL}D_{\mathfrak{T}}^{1-\vartheta}q(\mathfrak{T})d\vartheta, \tag{2.4} $$

similar to the R-L fractional derivative, the DO fractional derivative is a linear operator.

For the approximation of the integral in Eq. (2.4) we use the Gauss-Legendre quadrature formula on the interval $(0,1)$ as

$$ \int_0^1 \eta(\vartheta)\,{}^{RL}D_{\mathfrak{T}}^{1-\vartheta}q(\mathfrak{T}) \cong \sum_{r=0}^{Y} w_r \eta(\epsilon_r)\,{}^{RL}D_{\mathfrak{T}}^{1-\epsilon_r}q(\mathfrak{T}), \tag{2.5} $$

where $\{w_r\}_{r=0}^{Y}$ are the corresponding quadrature weights, and $\{\epsilon_r\}_{r=0}^{Y}$ are Gauss-Legendre quadrature nodes on the interval $(0,1)$ [28].

Let $G = L^2[0,\varpi]$ and $q(\mathfrak{T})$ be a square integrable function defined over G, then $q(\mathfrak{T})$ may be expressed in an OBP series as [22]

$$ q(\mathfrak{T}) = \sum_{\mathfrak{u}=0}^{\infty} \alpha_{\mathfrak{u}} \Theta_{\mathfrak{u},\varpi}(\mathfrak{T}), $$

where $\alpha_{\mathfrak{u}} = \int_0^{\varpi} q(\mathfrak{T})\Theta_{\mathfrak{u},\varpi}(\mathfrak{T})d\mathfrak{T}, \quad \mathfrak{u} = 0,1,\dots$. We can consider the following truncated series for $q(\mathfrak{T})$ as

$$ q(\mathfrak{T}) \simeq q_R(\mathfrak{T}) = \sum_{\mathfrak{u}=0}^{R} \alpha_{\mathfrak{u}} \Theta_{\mathfrak{u},\varpi}(\mathfrak{T}) = \mathbf{\Lambda}^T \mathbf{\Pi}_{R,\varpi}(\mathfrak{T}), \tag{2.6} $$

where

$$ \mathbf{\Lambda} = [\alpha_0, \alpha_1, \dots, \alpha_R]^T, \quad \mathbf{\Pi}_{R,\varpi}(\mathfrak{T}) = [\Theta_{0,\varpi}(\mathfrak{T}), \Theta_{1,\varpi}(\mathfrak{T}), \dots, \Theta_{R,\varpi}(\mathfrak{T})]^T. \tag{2.7} $$

Similarly, a function $q(z,\mathfrak{T}) \in L^2([0,\mathsf{L}] \times [0,\varpi])$ can be approximated by OBPs as

$$ q(z,\mathfrak{T}) \simeq \sum_{\mathfrak{u}=0}^{R} \sum_{j=0}^{R} \alpha_{\mathfrak{u}j} \Theta_{\mathfrak{u},\varpi}(\mathfrak{T}) \Theta_{j,\mathsf{L}}(z) = \mathbf{\Pi}_{R,\varpi}^T(\mathfrak{T}) \mathbf{Q} \mathbf{\Pi}_{R,\mathsf{L}}(z), \tag{2.8} $$

where the OBP vectors $\mathbf{\Pi}_{R,\varpi}(\mathfrak{T})$ and $\mathbf{\Pi}_{R,\mathsf{L}}(z)$ introduce similar to Eq. (2.7). The shifted coefficient matrix $\mathbf{Q}$ is also given by

$$ \mathbf{Q} = \begin{bmatrix} \alpha_{00} & \cdots & \alpha_{0R} \\ \vdots & \ddots & \vdots \\ \alpha_{R0} & \cdots & \alpha_{RR} \end{bmatrix}, \tag{2.9} $$

where

$$\alpha_{\mathfrak{u}j} = \int_0^{\mathsf{L}} \int_0^{\varpi} q(z,\mathfrak{T})\Theta_{\mathfrak{u},\varpi}(\mathfrak{T})\Theta_{j,\mathsf{L}}(z)d\mathfrak{T}dz, \quad j,\mathfrak{u} = 0,1,\ldots,R.$$

Lemma 2.2. *The operational matrices of integer derivative of the vector $\mathbf{\Pi}_{R,\varpi}(\mathfrak{T})$ are defined as:*

$$\frac{d^{\varsigma}\mathbf{\Pi}_{R,\varpi}(\mathfrak{T})}{d\mathfrak{T}^{\varsigma}} = \mathbf{D}^{(\varsigma)}\mathbf{\Pi}_{R,\varpi}(\mathfrak{T}) \quad \textit{where} \quad \mathbf{D}^{(\varsigma)} = (\mathbf{D}^{(1)})^{\varsigma}. \tag{2.10}$$

Here $\mathbf{D}^{(1)}$ is an operational matrix of integer derivative [22].

Also, the integration of $\mathbf{\Pi}_{R,\varpi}(\mathfrak{T})$ from 0 to $\mathfrak{T}$ can be displayed as

$$\int_0^{\mathfrak{T}} \mathbf{\Pi}_{R,\varpi}(\mathfrak{T})d\mathfrak{T} \simeq \mathbf{P}\mathbf{\Pi}_{R,\varpi}(\mathfrak{T}), \tag{2.11}$$

where, $\mathbf{P}$ is an $(R+1)\times(R+1)$ operational matrix of integration and, similar to [49], given by

$$\mathbf{P} = \frac{\varpi}{2}\begin{pmatrix} 1 & \frac{1}{\sqrt{1.3}} & 0 & \cdots & 0 \\ \frac{-1}{\sqrt{1.3}} & 0 & \frac{1}{\sqrt{3.5}} & \cdots & 0 \\ 0 & \frac{-1}{\sqrt{3.5}} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \frac{1}{\sqrt{(2R+1).(2R+3)}} \\ 0 & 0 & \cdots & \frac{-1}{\sqrt{(2R-1).(2R+1)}} & 0 \end{pmatrix}. \tag{2.12}$$

3. Constracting new operational matrices by OBPs

The main contribution of this section is to construct operational matrices for R-L fractional derivative and DO fractional derivative by using of OBPs. At first we obtained operational matrix for FDE as follow:

Theorem 3.1. *Let $\mathbf{\Pi}_{R,\varpi}(\mathfrak{T})$ be orthonormal Bernoulli polynomials vector defined in Eq. (2.7) and $0<\vartheta<1$, then*

$${}^{RL}D_{\mathfrak{T}}^{1-\vartheta}\mathbf{\Pi}_{R,\varpi}(\mathfrak{T}) \simeq \mathfrak{D}^{(1-\vartheta)}\mathbf{\Pi}_{R,\varpi}(\mathfrak{T}), \tag{3.1}$$

where $\mathfrak{D}^{(1-\vartheta)}$ is the orthonormal Bernoulli operational matrix of R-L fractional derivative of order $1-\vartheta$ that defined as

$$\mathfrak{D}^{(1-\vartheta)} \simeq \begin{pmatrix} \chi(0,0) & \chi(0,1) & \cdots & \chi(0,R) \\ \vdots & \vdots & \ddots & \vdots \\ \chi(\mathfrak{u},0) & \chi(\mathfrak{u},1) & \cdots & \chi(\mathfrak{u},R) \\ \vdots & \vdots & \vdots & \vdots \\ \chi(R,0) & \chi(R,1) & \cdots & \chi(R,R) \end{pmatrix}. \tag{3.2}$$

Where $\chi(\mathfrak{u},h)$ is given by

$$\chi(\mathfrak{u},h) = \frac{\sqrt{2\mathfrak{u}+1}\sqrt{2h+1}}{\varpi} \times \sum_{\varsigma=0}^{\mathfrak{u}}\sum_{r=0}^{h} \frac{(-1)^{\varsigma+r}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}\Gamma(1-\varsigma+\mathfrak{u})\begin{pmatrix}h\\ r\end{pmatrix}\begin{pmatrix}2h-r\\ h-r\end{pmatrix}\varpi^{\vartheta}}{(\mathfrak{u}+h-\varsigma-r+\vartheta)\Gamma(\mathfrak{u}+\vartheta-\varsigma)}. \quad (3.3)$$

Proof. Using Eqs. (2.1) and (2.2) we have

$$ {}^{RL}D_{\mathfrak{T}}^{1-\vartheta}\Theta_{\mathfrak{u},\varpi}(\mathfrak{T}) = \sqrt{\frac{1+2\mathfrak{u}}{\varpi}}\sum_{\varsigma=0}^{\mathfrak{u}} \frac{(-1)^{\varsigma}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}}{\varpi^{\mathfrak{u}-\varsigma}}\ {}^{RL}D_{\mathfrak{T}}^{1-\vartheta}\mathfrak{T}^{\mathfrak{u}-\varsigma} \quad (3.4)$$

$$= \sqrt{\frac{1+2\mathfrak{u}}{\varpi}}\sum_{\varsigma=0}^{\mathfrak{u}} \frac{(-1)^{\varsigma}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}\Gamma(1-\varsigma+\mathfrak{u})}{\varpi^{\mathfrak{u}-\varsigma}\Gamma(\mathfrak{u}+\vartheta-\varsigma)}\mathfrak{T}^{\mathfrak{u}+\vartheta-\varsigma-1},$$

$$0 \le \mathfrak{u} \le R.$$

Now, we approximate $\mathfrak{T}^{\mathfrak{u}-\varsigma-1+\vartheta}$ based on OBP series as

$$\mathfrak{T}^{\mathfrak{u}-\varsigma-1+\vartheta} \simeq \sum_{h=0}^{R} d_h \Theta_{h,\varpi}(\mathfrak{T}), \quad (3.5)$$

where

$$d_h = \sqrt{\frac{1+2h}{\varpi}}\sum_{r=0}^{h} \frac{(-1)^{r}\begin{pmatrix}h\\ r\end{pmatrix}\begin{pmatrix}2h-r\\ h-r\end{pmatrix}}{\varpi^{h-r}}\int_0^{\varpi}\mathfrak{T}^{\mathfrak{u}-\varsigma-1+\vartheta}\mathfrak{T}^{h-r}d\mathfrak{T}$$

$$= \sqrt{\frac{1+2h}{\varpi}}\sum_{r=0}^{h} \frac{(-1)^{r}\begin{pmatrix}h\\ r\end{pmatrix}\begin{pmatrix}2h-r\\ h-r\end{pmatrix}\varpi^{\mathfrak{u}+h-\varsigma-r+\vartheta}}{\varpi^{h-r}(\mathfrak{u}+h-\varsigma-r+\vartheta)}, \quad (3.6)$$

so

$$ {}^{RL}D_{\mathfrak{T}}^{1-\vartheta}\Theta_{\mathfrak{u},\varpi}(\mathfrak{T}) = \sqrt{\frac{2\mathfrak{u}+1}{\varpi}}\sum_{\varsigma=0}^{\mathfrak{u}} \frac{(-1)^{\varsigma}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}\Gamma(1-\varsigma+\mathfrak{u})}{\varpi^{\mathfrak{u}-\varsigma}\Gamma(\mathfrak{u}-\varsigma+\vartheta)}$$

$$\times \sum_{h=0}^{R} \sqrt{\frac{1+2h}{\varpi}} \sum_{r=0}^{h} \frac{(-1)^r \begin{pmatrix} h \\ r \end{pmatrix} \begin{pmatrix} 2h-r \\ h-r \end{pmatrix} \varpi^{\mathfrak{u}-\varsigma+h-r+\vartheta}}{\varpi^{h-r}(\mathfrak{u}-\varsigma+h-r+\vartheta)} \Theta_{h,\varpi}(\mathfrak{T})$$

$$= \sum_{h=0}^{R} \chi(\mathfrak{u},h)\Theta_{h,\varpi}(\mathfrak{T}), \tag{3.7}$$

where $\chi(\mathfrak{u},h)$ is given in Eq. (3.3). □

By applying the linear property of DO operator and by utilize of Gauss-Legendre quadrature formulae we present the next lemma:

Lemma 3.1. *Let $\mathfrak{u} \in \mathbb{N}, \varsigma \in \mathbb{Z}$ and $\varsigma < \mathfrak{u}$, then $D^{\eta(\vartheta)}\mathfrak{T}^{\mathfrak{u}-\varsigma}$ can be approximated as*

$$D_{\mathfrak{T}}^{\eta(\vartheta)}\mathfrak{T}^{\mathfrak{u}-\varsigma} \simeq \sum_{r=1}^{Y} d_{\varsigma r}\mathfrak{T}^{\mathfrak{u}-\varsigma-1+\epsilon_r}, \quad d_{\varsigma r} = \frac{w_r \eta(\epsilon_r)\Gamma(\mathfrak{u}-\varsigma+1)}{\Gamma(\mathfrak{u}-\varsigma+\epsilon_r)}, \tag{3.8}$$

where $\{w_r\}_{r=1}^{Y}$ and $\{\epsilon_r\}_{r=1}^{Y}$ are the weights and the points of the Gauss-Legendre quadrature rule respectively.

Proof. According to Eqs. (2.2) and (2.5) this proof is complete. □

Theorem 3.2. *Suppose $Y \in \mathbb{N}, \vartheta > 0$, by applying the Gauss-Legendre quadrature rule we can obtain a new operational matrix for DOFD equation by OBPs as following:*

$$D_{\mathfrak{T}}^{\eta(\vartheta)}\mathbf{\Pi}_{R,\varpi}(\mathfrak{T}) \simeq \widehat{\mathfrak{D}}^{(\eta(\vartheta))}\mathbf{\Pi}_{R,\varpi}(\mathfrak{T}),$$

where

$$\widehat{\mathfrak{D}}^{(\eta(\vartheta))} \simeq \begin{pmatrix} \zeta(0,0) & \zeta(0,1) & \cdots & \zeta(0,R) \\ \zeta(1,0) & \zeta(1,1) & \cdots & \zeta(1,R) \\ \vdots & \vdots & \ddots & \vdots \\ \zeta(\mathfrak{u},0) & \zeta(\mathfrak{u},1) & \cdots & \zeta(\mathfrak{u},R) \\ \vdots & \vdots & \vdots & \vdots \\ \zeta(R,0) & \zeta(R,1) & \cdots & \zeta(R,R) \end{pmatrix}.$$

Here $\widehat{\mathfrak{D}}^{(\eta(\vartheta))}$ is the $(R+1)\times(R+1)$ orthonormal Bernoulli operational matrix for DOFD equation and for $\mathfrak{u} = 0,\dots,R$ and $j = 0,\dots,R$ we have

$$\zeta(\mathfrak{u},j) = \sqrt{\frac{2\mathfrak{u}+1}{\varpi}} \sum_{\varsigma=0}^{\mathfrak{u}} \sum_{r=1}^{Y} \left(\frac{(-1)^\varsigma}{\varpi^{\mathfrak{u}-\varsigma}} \begin{pmatrix} \mathfrak{u} \\ \varsigma \end{pmatrix} \begin{pmatrix} 2\mathfrak{u}-\varsigma \\ \mathfrak{u}-\varsigma \end{pmatrix} \right) \left(\frac{w_r\eta(\epsilon_r)\Gamma(1-\varsigma+\mathfrak{u})}{\Gamma(\mathfrak{u}-\varsigma+\epsilon_r)} \right)$$

$$\times \sqrt{\frac{2j+1}{\varpi}} \sum_{\ell=0}^{j} \frac{(-1)^\ell \begin{pmatrix} j \\ \ell \end{pmatrix} \begin{pmatrix} 2j-\ell \\ j-\ell \end{pmatrix} \varpi^{\mathfrak{u}+\epsilon_r-\varsigma}}{\mathfrak{u}+j-\varsigma-\ell+\epsilon_r}. \tag{3.9}$$

Proof. By considering Lemma 3.1 and linear property of DO operator and by using Eq. (2.1) we have

$$\begin{aligned} & D_{\mathfrak{T}}^{\eta(\vartheta)}\Theta_{\mathfrak{u},\varpi}(\mathfrak{T}) \\ &= \sqrt{\frac{2\mathfrak{u}+1}{\varpi}}\sum_{\varsigma=0}^{\mathfrak{u}}\frac{(-1)^{\varsigma}}{\varpi^{\mathfrak{u}-\varsigma}}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}D_{\mathfrak{T}}^{\eta(\vartheta)}\mathfrak{T}^{\mathfrak{u}-\varsigma} \\ &\simeq \sqrt{\frac{2\mathfrak{u}+1}{\varpi}}\sum_{\varsigma=0}^{\mathfrak{u}}\sum_{r=1}^{Y}\frac{(-1)^{\varsigma}}{\varpi^{\mathfrak{u}-\varsigma}}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}d_{\varsigma r}\mathfrak{T}^{\mathfrak{u}+\epsilon_r-\varsigma-1}, \quad 0\le\mathfrak{u}\le R. \end{aligned} \tag{3.10}$$

Naturally, we can approximate $\mathfrak{T}^{\mathfrak{u}-\varsigma-1+\epsilon_r}$ by truncating series OBPs as

$$\mathfrak{T}^{\mathfrak{u}-\varsigma-1+\epsilon_r}\simeq\sum_{j=0}^{R}b_{\varsigma rj}\Theta_{j,\varpi}(\mathfrak{T}), \tag{3.11}$$

in which

$$\begin{aligned} b_{\varsigma rj} &= \int_0^{\varpi}\mathfrak{T}^{\mathfrak{u}-\varsigma-1+\epsilon_r}\Theta_{j,\varpi}(\mathfrak{T})d\mathfrak{T} \\ &= \sqrt{\frac{2j+1}{\varpi}}\sum_{\ell=0}^{j}\frac{(-1)^{\ell}}{\varpi^{j-\ell}}\begin{pmatrix}j\\ \ell\end{pmatrix}\begin{pmatrix}2j-\ell\\ j-\ell\end{pmatrix}\int_0^{\varpi}\mathfrak{T}^{\mathfrak{u}-\varsigma-1+\epsilon_r}\mathfrak{T}^{j-\ell}d\mathfrak{T} \\ &= \sqrt{\frac{2j+1}{\varpi}}\sum_{\ell=0}^{j}\frac{(-1)^{\ell}\begin{pmatrix}j\\ \ell\end{pmatrix}\begin{pmatrix}2j-\ell\\ j-\ell\end{pmatrix}\varpi^{\mathfrak{u}-\varsigma+\epsilon_r}}{\mathfrak{u}-\varsigma+j-\ell+\epsilon_r}. \end{aligned} \tag{3.12}$$

Combination of Eqs. (3.10)-(3.12) conclude that

$$\begin{aligned} D_{\mathfrak{T}}^{\eta(\vartheta)}\Theta_{\mathfrak{u},\varpi}(\mathfrak{T}) &\simeq \sqrt{\frac{2\mathfrak{u}+1}{\varpi}}\sum_{\varsigma=0}^{\mathfrak{u}}\sum_{r=1}^{Y}\frac{(-1)^{\varsigma}}{\varpi^{\mathfrak{u}-\varsigma}}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}d_{\varsigma r}\sum_{j=0}^{R}b_{\varsigma rj}\Theta_{j,\varpi}(\mathfrak{T}) \\ &= \sum_{j=0}^{R}\left(\sqrt{\frac{2\mathfrak{u}+1}{\varpi}}\sum_{\varsigma=0}^{\mathfrak{u}}\sum_{r=1}^{Y}\frac{(-1)^{\varsigma}}{\varpi^{\mathfrak{u}-\varsigma}}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}d_{\varsigma r}b_{\varsigma rj}\right)\Theta_{j,\varpi}(\mathfrak{T}) \\ &= \sum_{j=0}^{R}\zeta(\mathfrak{u},j)\Theta_{j,\varpi}(\mathfrak{T}), \end{aligned} \tag{3.13}$$

where $\zeta(\mathfrak{u},j)$ is given in Eq. (3.9). So the following vector form is held for Eq. (3.13)

$$D_{\mathfrak{T}}^{\eta(\vartheta)}\Theta_{\mathfrak{u},\varpi}(\mathfrak{T})\simeq[\zeta(\mathfrak{u},0),\zeta(\mathfrak{u},1),\ldots,\zeta(\mathfrak{u},R)]\,\mathbf{\Pi}_{R,\varpi}(\mathfrak{T}), \quad \mathfrak{u}=0,\ldots,R, \tag{3.14}$$

which completes the proof. □

Remark 3.1. Let $\eta(\vartheta) = \delta(\vartheta - s)$, witch $s \in (0,1)$ and $\delta(.)$ is a Dirac delta function. If we consider $Y = r = 1, \epsilon_r = s$ and $w_r = 1$ it is uncomplicated to see that $\widehat{\mathfrak{D}}^{(\eta(\vartheta))} = \mathfrak{D}^{(1-\vartheta)}$. In the other words the operational matrix of DO fractional derivative calculated in Theorem 3.2 is generalized the operational matrix of R-L fractional derivative which is given in Theorem 3.1.

4. Solving problem (1.1)-(1.3)

In the current section, we applay the new operational matrix $\widehat{\mathfrak{D}}^{(\eta(\vartheta))}$ together with tau method to solve DO time-fractional partial differential equations given in Eqs. (1.1)-(1.3). According to Eq. (2.8), we approximate $q(z,\mathfrak{T})$ by OBPs as $q(z,\mathfrak{T}) \simeq \mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})\mathbf{Q}\mathbf{\Pi}_{R,\mathsf{L}}(z)$. So, by using Eq. (2.10), we can obtained

$$\frac{\partial^2 q(z,\mathfrak{T})}{\partial z^2} \simeq \mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})\mathbf{Q}\mathbf{D}^{(2)}\mathbf{\Pi}_{R,\mathsf{L}}(z). \tag{4.1}$$

Also, we approximate $\mathfrak{h}(z,\mathfrak{T})$ as

$$\mathfrak{h}(z,\mathfrak{T}) \simeq \mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})\mathbf{H}\mathbf{\Pi}_{R,\mathsf{L}}(z). \tag{4.2}$$

Here $\mathbf{H}$ is the known matrix and $\mathbf{Q}$ is an $(R+1)\times(R+1)$ unknown matrix. Now, by integration Eq. (1.1) from 0 to $\mathfrak{T}$ and using Eq. (1.2) we have

$$q(z,\mathfrak{T}) - \mathfrak{f}(z) = \lambda \int_0^{\mathfrak{T}} \frac{\partial^2 q(z,\mathfrak{T})}{\partial z^2} d\mathfrak{T} + \int_0^{\mathfrak{T}} D^{\eta(\vartheta)} \left(\frac{\partial^2 q(z,\mathfrak{T})}{\partial z^2} \right) d\mathfrak{T} + \int_0^{\mathfrak{T}} \mathfrak{h}(z,\mathfrak{T}) d\mathfrak{T}. \tag{4.3}$$

Expanding $\mathfrak{f}(z)$ by OBPs we obtain

$$\mathfrak{f}(z;\mathfrak{T}) \simeq \sum_{j=0}^{R} \mathfrak{f}_j \Theta_{j,\mathsf{L}}(z) = \mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})\mathbf{F}\mathbf{\Pi}_{R,\mathsf{L}}(z), \tag{4.4}$$

here $\mathbf{F}$ is a known $(R+1)\times(R+1)$ matrix and can be displayed as

$$\mathbf{F} = \begin{pmatrix} \mathfrak{f}_0 & \mathfrak{f}_1 & \cdots & \mathfrak{f}_R \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Now, by using Eqs. (2.11) and (4.1) , we have

$$\begin{aligned} \int_0^{\mathfrak{T}} \frac{\partial^2 q(z,\mathfrak{T})}{\partial z^2} d\mathfrak{T} &\simeq \int_0^{\mathfrak{T}} \mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})\mathbf{Q}\mathbf{D}^{(2)}\mathbf{\Pi}_{R,\mathsf{L}}(z) d\mathfrak{T} \\ &= \left(\int_0^{\mathfrak{T}} \mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T}) d\mathfrak{T} \right) \mathbf{Q}\mathbf{D}^{(2)}\mathbf{\Pi}_{R,\mathsf{L}}(z) \\ &\simeq \mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})\mathbf{P}^T\mathbf{Q}\mathbf{D}^{(2)}\mathbf{\Pi}_{R,\mathsf{L}}(z). \end{aligned} \tag{4.5}$$

Similarly by using Eqs. (2.11) and (4.2) we obtain

$$\int_0^{\mathfrak{T}} \mathfrak{h}(z,\mathfrak{T})d\mathfrak{T} \simeq \int_0^{\mathfrak{T}} \mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})\mathbf{H}\mathbf{\Pi}_{R,\mathsf{L}}(z)d\mathfrak{T} \simeq \mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})\mathbf{P}^T\mathbf{H}\mathbf{\Pi}_{R,\mathsf{L}}(z). \tag{4.6}$$

Employing Eq. (2.11) and Theorem 3.2, we get

$$\begin{aligned}\int_0^{\mathfrak{T}} D_{\mathfrak{T}}^{\eta(\vartheta)}\left(\frac{\partial^2 q(z,\mathfrak{T})}{\partial z^2}\right)d\mathfrak{T} &\simeq \left(\int_0^{\mathfrak{T}} D_{\mathfrak{T}}^{\eta(\vartheta)}\mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})d\mathfrak{T}\right)\mathbf{Q}\mathbf{D}^{(2)}\mathbf{\Pi}_{R,\mathsf{L}}(z)\\ &\simeq \left(\int_0^{\mathfrak{T}} \mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})d\mathfrak{T}\right)(\widehat{\mathfrak{D}}^{(\eta(\vartheta))})^T\mathbf{Q}\mathbf{D}^{(2)}\mathbf{\Pi}_{R,\mathsf{L}}(z)\\ &\simeq \mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})\mathbf{P}^T(\widehat{\mathfrak{D}}^{(\eta(\vartheta))})^T\mathbf{Q}\mathbf{D}^{(2)}\mathbf{\Pi}_{R,\mathsf{L}}(z). \end{aligned} \tag{4.7}$$

By combination of Eqs. (4.3)-(4.7) we can write residual $U_R(z,\mathfrak{T})$ for Eq. (1.1) as

$$\begin{aligned}U_R(z,\mathfrak{T}) &= \mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})\left[\mathbf{Q}-\mathbf{F}-\lambda\mathbf{P}^T\mathbf{Q}\mathbf{D}^{(2)}-\mathbf{P}^T(\widehat{\mathfrak{D}}^{(\eta(\vartheta))})^T\mathbf{Q}\mathbf{D}^{(2)}-\mathbf{P}^T\mathbf{H}\right]\mathbf{\Pi}_{R,\mathsf{L}}(z)\\ &= \mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})\mathbf{E}\mathbf{\Pi}_{R,\mathsf{L}}(z),\end{aligned}$$

where

$$\mathbf{E} = \mathbf{Q}-\mathbf{F}-\lambda\mathbf{P}^T\mathbf{Q}\mathbf{D}^{(2)}-\mathbf{P}^T(\widehat{\mathfrak{D}}^{(\eta(\vartheta))})^T\mathbf{Q}\mathbf{D}^{(2)}-\mathbf{P}^T\mathbf{H}.$$

By considering the tau method we create the following linear algebraic equations

$$\mathbf{E}_{\mathfrak{u}j} = 0, \quad 0 \le \mathfrak{u} \le R, \quad 0 \le j \le R-2. \tag{4.8}$$

On the other hand by substituting Eq. (2.8) to boundary conditions given in Eq. (1.3) we get

$$\mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})\mathbf{Q}\mathbf{\Pi}_{R,\mathsf{L}}(0) = g_1(\mathfrak{T}), \tag{4.9}$$

$$\mathbf{\Pi}^T_{R,\varpi}(\mathfrak{T})\mathbf{Q}\mathbf{\Pi}_{R,\mathsf{L}}(\mathsf{L}) = g_2(\mathfrak{T}). \tag{4.10}$$

Now we collocate Eqs. (4.9) and (4.10) at the shifted Legendre roots $\mathfrak{T}_{\mathfrak{u}}$, where $1 \le \mathfrak{u} \le R+1$. Thus by combination Eqs. (4.8)-(4.10) we obtain $(R+1)\times(R+1)$ algebraic equations. The number of the unknown coefficients $\alpha_{\mathfrak{u}j}$ is equal to $(R+1)\times(R+1)$ and by using any standard numerical algorithm to solve the calculated system, we can obtain $\alpha_{\mathfrak{u}j}$. In this work we apply the *fsolve* command in Maple to solve this algebraic system. Consequently $q(z,\mathfrak{T})$ can be computed via Eq. (2.8).

5. Upper error bound for operational matrix $\widehat{\mathfrak{D}}^{(\eta(\vartheta))}$

In this section, we obtain an upper error bound for operational matrix $\widehat{\mathfrak{D}}^{(\eta(\vartheta))}$. Here, we assume $\varpi = 1$ and $0 < \vartheta < 1$.

Theorem 5.1. *Suppose that Γ be a Hilbert space and S be a closed subspace with finite dimensions of Γ and $\{\hat{s_1}, \hat{s_2}, \cdots, \hat{s_R}\}$ is any basis of S. Let x is an arbitrary element in Γ, then it has a unique best approximation out of S such as z^* and we have [10, 30, 43]*

$$\|x - z^*\|_2^2 = \frac{G(x, \hat{s_1}, \hat{s_2}, \ldots, \hat{s_R})}{G(\hat{s_1}, \hat{s_2}, \ldots, \hat{s_R})},$$

where

$$G(x,\hat{s_1},\hat{s_2},\cdots,\hat{s_R})=\begin{bmatrix} <x,x> & <x,\hat{s_1}> & \cdots & <x,\hat{s_R}> \\ <\hat{s_1},x> & <\hat{s_1},\hat{s_1}> & \cdots & <\hat{s_1},\hat{s_R}> \\ \vdots & \vdots & \cdots & \vdots \\ <\hat{s_R},x> & <\hat{s_R},\hat{s_1}> & \cdots & <\hat{s_R},\hat{s_R}> \end{bmatrix}.$$

Theorem 5.2 ([44]). *Assume $q(\mathfrak{T})\in L^2[0,1]$ and $q_R(\mathfrak{T})$ is the best approximation of $q(\mathfrak{T})$ out of $\{\Theta_{0,1}(\mathfrak{T}),\Theta_{1,1}(\mathfrak{T}),\cdots,\Theta_{R,1}(\mathfrak{T})\}$ as*

$$q_R(\mathfrak{T})\simeq\sum_{\mathfrak{u}=0}^{R}a_{\mathfrak{u}}\Theta_{\mathfrak{u},1}(\mathfrak{T})=\mathbf{\Lambda}^T\mathbf{\Pi}_{R,1}(\mathfrak{T}),$$

where $\mathbf{\Lambda}$ and $\mathbf{\Pi}_{R,1}(\mathfrak{T})$ are defined in Eq. (2.7). So we have

$$\lim_{R\to\infty}\|q(\mathfrak{T})-q_R(\mathfrak{T})\|_2=0.$$

Also, the error vector $E^{\vartheta}_{\widehat{\mathfrak{D}}^{(\eta(\vartheta))}}$ is an approximation of the DO fractional derivative of $\mathbf{\Pi}_{R,1}(\mathfrak{T})$ by employing $\widehat{\mathfrak{D}}^{(\eta(\vartheta))}$ is given by

$$E^{\vartheta}_{\widehat{\mathfrak{D}}^{(\eta(\vartheta))}}=D^{\eta(\vartheta)}\mathbf{\Pi}_{R,1}(\mathfrak{T})-\widehat{\mathfrak{D}}^{(\eta(\vartheta))}\mathbf{\Pi}_{R,1}(\mathfrak{T})=\begin{bmatrix} e^{\vartheta}_{d_0} \\ e^{\vartheta}_{d_1} \\ \vdots \\ e^{\vartheta}_{d_R} \end{bmatrix}.$$

According to Eq. (3.11) when we approximated $\mathfrak{T}^{\mathfrak{u}-\varsigma-1+\epsilon_r}$ we have

$$\mathfrak{T}^{\mathfrak{u}-\varsigma-1+\epsilon_r}\simeq\sum_{j=0}^{R}b_{\varsigma rj}\Theta_{j,1}(\mathfrak{T}),$$

where $b_{\varsigma rj}$ calculated by the best approximation. By consider Theorem 5.1 we get

$$\left\|\mathfrak{T}^{\mathfrak{u}-\varsigma+\epsilon_r-1}-\sum_{j=0}^{R}b_{\varsigma rj}\Theta_{j,1}(\mathfrak{T})\right\|_2 =\left(\frac{G(\mathfrak{T}^{\mathfrak{u}-\varsigma-1+\epsilon_r},\Theta_{0,1}(\mathfrak{T}),\Theta_{1,1}(\mathfrak{T}),\ldots,\Theta_{R,1}(\mathfrak{T}))}{G(\Theta_{0,1}(\mathfrak{T}),\Theta_{1,1}(\mathfrak{T}),\ldots,\Theta_{R,1}(\mathfrak{T}))}\right)^{\frac{1}{2}}. \tag{5.1}$$

Furthermore, by considering the integration error, Eq. (3.10) can be written as

$$D^{\eta(\vartheta)}\Theta_{\mathfrak{u},1}(\mathfrak{T}) =\sqrt{1+2\mathfrak{u}}\sum_{\varsigma=0}^{\mathfrak{u}}(-1)^{\varsigma}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}D^{\eta(\vartheta)}\mathfrak{T}^{\mathfrak{u}-\varsigma}$$

$$= \sqrt{1+2\mathfrak{u}}\sum_{\varsigma=0}^{\mathfrak{u}}(-1)^{\varsigma}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}\left[\sum_{r=1}^{Y} d_{\varsigma r}\mathfrak{T}^{\mathfrak{u}+\epsilon_r-\varsigma-1}+M_{\varsigma}(\mathfrak{T})\right] \tag{5.2}$$

$$= \sum_{\varsigma=0}^{\mathfrak{u}}\sum_{r=1}^{Y} B_{\mathfrak{u}\varsigma r}\mathfrak{T}^{\mathfrak{u}-\varsigma+\epsilon_r-1}+\sqrt{1+2\mathfrak{u}}\sum_{\varsigma=0}^{\mathfrak{u}}(-1)^{\varsigma}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}\times M_{\varsigma}(\mathfrak{T}),$$

where

$$B_{\mathfrak{u}\varsigma r} = \sqrt{1+2\mathfrak{u}}(-1)^{\varsigma}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}\times d_{\varsigma r}.$$

It is noted that $M_{\varsigma}(\mathfrak{T})$ is the error of estimate integral by Y-point Gauss-Legendre quadrature formula. M_{ς} may be estimated as [38, 41].

$$M_{\varsigma}(\mathfrak{T}) \simeq \frac{\pi}{4^{Y}}\frac{d^{2Y}}{d\vartheta^{2Y}}\phi_{\varsigma}(\mathfrak{T},\xi), \quad \xi\in[0,1], \tag{5.3}$$

where $\phi_{\varsigma}(\mathfrak{T},\xi) = \eta(\vartheta)\ {}^{RL}D^{1-\vartheta}\mathfrak{T}^{\mathfrak{u}-\varsigma}$. For fixed $\mathfrak{T}\in[0,1]$ and by this assumption that $\phi_{\varsigma}(\mathfrak{T},\xi)\in C^{2Y}([0,1])$ we have

$$\|M_{\varsigma}(\mathfrak{T})\|_2^2 = \int_0^1 |M_{\varsigma}(\mathfrak{T})|^2 d\mathfrak{T} \simeq \int_0^1 \frac{\pi^2}{4^{2Y}}\left|\frac{d^{2Y}\phi_{\varsigma}(\mathfrak{T},\xi)}{d\vartheta^{2Y}}\right|^2 d\mathfrak{T} \le \frac{\pi^2}{4^{2Y}}\beta_{\varsigma}^2, \tag{5.4}$$

where

$$\beta_{\varsigma} = \max\left\{\left|\frac{d^{2Y}\phi_{\varsigma}(\mathfrak{T},\xi)}{d\vartheta^{2Y}}\right|, 0<\vartheta,\mathfrak{T}<1\right\}, \quad \varsigma=0,\dots,\mathfrak{u}.$$

Now, by using Eqs. (3.13) and (5.2) we get

$$\begin{aligned}
&\left\|e_{d_{\mathfrak{u}}}^{\vartheta}\right\|_2 \\
&= \left\|D^{\eta(\vartheta)}\Theta_{\mathfrak{u},1}(\mathfrak{T})-\sum_{j=0}^{R}\zeta(\mathfrak{u},j)\Theta_{j,1}(\mathfrak{T})\right\|_2 \\
&= \Bigg\|\sum_{\varsigma=0}^{\mathfrak{u}}\sum_{r=1}^{Y} B_{\mathfrak{u}\varsigma r}\mathfrak{T}^{\mathfrak{u}-\varsigma+\epsilon_r-1}+\sqrt{1+2\mathfrak{u}}\sum_{\varsigma=0}^{\mathfrak{u}}(-1)^{\varsigma}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}M_{\varsigma}(\mathfrak{T}) \\
&\quad -\sum_{j=0}^{R}\zeta(\mathfrak{u},j)\Theta_{j,1}(\mathfrak{T})\Bigg\|_2 \\
&\le \left\|\sum_{\varsigma=0}^{\mathfrak{u}}\sum_{r=1}^{Y} B_{\mathfrak{u}\varsigma r}\mathfrak{T}^{\mathfrak{u}-\varsigma+\epsilon_r-1}-\sum_{j=0}^{R}\zeta(\mathfrak{u},j)\Theta_{j,1}(\mathfrak{T})\right\|_2 \\
&\quad +\left\|\sqrt{1+2\mathfrak{u}}\sum_{\varsigma=0}^{\mathfrak{u}}(-1)^{\varsigma}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}M_{\varsigma}(\mathfrak{T})\right\|_2 \\
&\le \left\|\sum_{\varsigma=0}^{\mathfrak{u}}\sum_{r=1}^{Y} B_{\mathfrak{u}\varsigma r}\mathfrak{T}^{\mathfrak{u}-\varsigma+\epsilon_r-1}-\sum_{j=0}^{R}\sum_{\varsigma=0}^{\mathfrak{u}}\sum_{r=1}^{Y} B_{\mathfrak{u}\varsigma r}b_{\varsigma rj}\Theta_{j,1}(\mathfrak{T})\right\|_2
\end{aligned}$$

$$+\left\|\sqrt{1+2\mathfrak{u}}\sum_{\varsigma=0}^{\mathfrak{u}}(-1)^{\varsigma}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}M_{\varsigma}(\mathfrak{T})\right\|_2$$

$$\leq\sum_{\varsigma=0}^{\mathfrak{u}}\sum_{r=1}^{Y}B_{\mathfrak{u}\varsigma r}\left\|\mathfrak{T}^{\mathfrak{u}-\varsigma-1+\epsilon_r}-\sum_{j=0}^{R}b_{\varsigma rj}\Theta_{j,1}(\mathfrak{T})\right\|_2$$

$$+\left\|\sqrt{1+2\mathfrak{u}}\sum_{\varsigma=0}^{\mathfrak{u}}(-1)^{\varsigma}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}M_{\varsigma}(\mathfrak{T})\right\|_2. \tag{5.5}$$

Also, by employing Eq. (5.4) we have

$$\left\|\sqrt{1+2\mathfrak{u}}\sum_{\varsigma=0}^{\mathfrak{u}}(-1)^{\varsigma}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}M_{\varsigma}(\mathfrak{T})\right\|_2$$

$$\leq\sqrt{1+2\mathfrak{u}}\sum_{\varsigma=0}^{\mathfrak{u}}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}\frac{\pi}{4^{Y}}\Delta, \tag{5.6}$$

where $\Delta=\max\{\beta_{\varsigma},\varsigma=0,\ldots,\mathfrak{u}\}$. Finally, using Eqs. (5.1), (5.5) and (5.6) we get a switable bound for $\|e^{\vartheta}_{d_{\mathfrak{u}}}\|_2$ as following

$$\|e^{\vartheta}_{d_{\mathfrak{u}}}\|_2\leq\sum_{\varsigma=0}^{\mathfrak{u}}\sum_{r=1}^{Y}B_{\mathfrak{u}\varsigma r}\left(\frac{G(\mathfrak{T}^{\mathfrak{u}-\varsigma-1+\epsilon_r},\Theta_{0,1}(\mathfrak{T}),\Theta_{1,1}(\mathfrak{T}),\ldots,\Theta_{R,1}(\mathfrak{T}))}{G(\Theta_{0,1}(\mathfrak{T}),\Theta_{1,1}(\mathfrak{T}),\ldots,\Theta_{R,1}(\mathfrak{T}))}\right)^{\frac{1}{2}}$$

$$+\sqrt{2\mathfrak{u}+1}\sum_{\varsigma=0}^{\mathfrak{u}}\begin{pmatrix}\mathfrak{u}\\ \varsigma\end{pmatrix}\begin{pmatrix}2\mathfrak{u}-\varsigma\\ \mathfrak{u}-\varsigma\end{pmatrix}\frac{\Delta\pi}{4^{Y}},\quad \mathfrak{u}=0,\cdots,R. \tag{5.7}$$

By considering the above discussion and Theorem 5.1, it can be concluded that by increasing the number of OBP bases and the number of quadrature points, the vector $E^{\vartheta}_{\widehat{\mathfrak{D}}(\eta(\vartheta))}$ tends to zero.

6. Numerical results

In this section, to illustrate the efficiency of our numerical method, we give three examples. All the symbolic and numerical computations were performed by using Maple 17 in a personal computer with 2.20 GHz, Core i7, and 8 GB of memory.

Example 6.1. Firstly, we apply the proposed method on the next modified DO anomalous sub-diffusion equation on a region $\Omega=(0,1)\times(0,0.5)$ as follow [32]

$$\frac{\partial q(z,\mathfrak{T})}{\partial\mathfrak{T}}=\int_0^1\Gamma(\vartheta+2)\ {}^{RL}D^{1-\vartheta}_{\mathfrak{T}}\left(\frac{\partial^2 q(z,\mathfrak{T})}{\partial z^2}\right)d\vartheta+\mathfrak{h}(z,\mathfrak{T}),$$

with boundary conditions

$$q(1,\mathfrak{T})=q(0,\mathfrak{T})=0,$$

and with initial condition

$$q(z,0)=0,$$

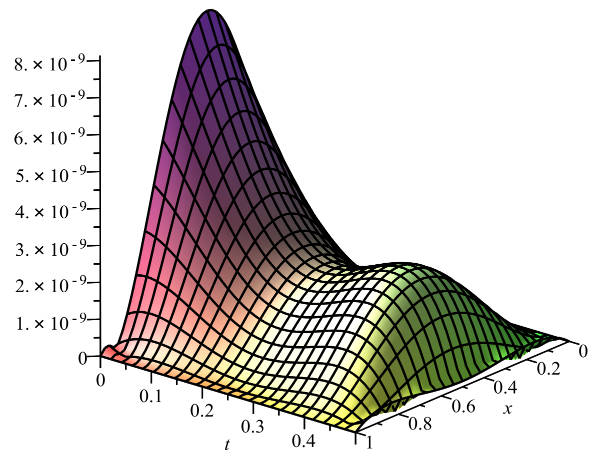

Figure 1. The absolute error function for $R = 4, Y = 6$, of Example 6.1.

where $\mathfrak{h}(z, \mathfrak{T}) = 2\mathfrak{T}z^2(1-z)^2 - 4\frac{\mathfrak{T}^2-\mathfrak{T}}{\ln \mathfrak{T}}(6z^2 - 6z + 1)$. The exact solution of this example is $q(z, \mathfrak{T}) = z^2\mathfrak{T}^2(1-z)^2$.

Table 1. Comparison of the L^2-error for the present method with the method in [32] for Example 6.1

	present method ($R = 4$)		Method in [32]
Y	L^2-error	τ	L^2-error
2	2.90×10^{-6}	1/16	7.92×10^{-5}
3	3.43×10^{-8}	1/32	1.97×10^{-5}
4	1.26×10^{-9}	1/64	4.78×10^{-6}
5	1.60×10^{-9}	1/128	1.08×10^{-6}

The graph for the absolute error function $|q(z, \mathfrak{T}) - q_R(z, \mathfrak{T})|$ with $R = 4$ and $Y = 6$ is plotted in Figure 1. Also, the comparison of L^2-error for our method and the result obtained by using the Galerkin finite element method given in [32] reported in Table 1. From this table, we see that our method has better results if compared with the method in [32] which uses mid-point quadrature rule with 40 points. For discretized the appeared integral we use Y-point Gauss-Legendre quadrature formula. Note that this formula is more accurate than mid-point quadrature rule.

Example 6.2. To demonstrate the ability and reliability of the presented method for the DRSP, we consider the following test [20].

$$\frac{\partial q(z, \mathfrak{T})}{\partial \mathfrak{T}} = \int_0^1 \eta(\vartheta)\ ^{RL}D_{\mathfrak{T}}^{1-\vartheta}(\frac{\partial^2 q(z, \mathfrak{T})}{\partial z^2})d\vartheta + \frac{\partial^2 q(z, \mathfrak{T})}{\partial z^2} + \mathfrak{h}(z, \mathfrak{T}),$$
$$(z, \mathfrak{T}) \in [0, 1] \times [0, 1],$$

In this example, by considering the exact non-smooth solution in time direction

$q(z, \mathfrak{T}) = e^z \mathfrak{T}^{k+2}$, we can obtain the boundary conditions, initial condition, and the function $\mathfrak{h}(z, \mathfrak{T})$. We consider three cases for weight function as follow:

- Case 1: $\eta(\vartheta) = \delta(\vartheta + \alpha - 1),\ 0 < \alpha < 1,\ k = 1$,
- Case 2: $\eta(\vartheta) = \delta(\vartheta + \alpha - 1),\ 0 < \alpha = k < 1$,
- Case 3: $\eta(\vartheta) = \Gamma(\vartheta + k + 2),\ k = 0.5, 1.5, 2$.

Table 2. Case1: Comparison of the L^∞-error of present method in Example 6.2

α	CFDA [9]	SGM [20]		Present method	
	$\tau^2 = h^4 = \frac{1}{625}$	$N = M = 5$	$N = M = 10$	$R = 5$	$R = 10$
0.1	3.05×10^{-5}	2.78×10^{-6}	2.66×10^{-8}	4.5×10^{-6}	3.2×10^{-10}
0.3	5.78×10^{-5}	3.74×10^{-6}	3.77×10^{-8}	4.7×10^{-6}	1.9×10^{-9}
0.5	6.46×10^{-5}	4.28×10^{-6}	3.89×10^{-8}	4.4×10^{-6}	4.7×10^{-9}
0.7	6.43×10^{-5}	3.81×10^{-6}	2.40×10^{-8}	5.2×10^{-6}	6.5×10^{-9}
0.9	6.22×10^{-5}	2.66×10^{-6}	5.96×10^{-9}	5.1×10^{-6}	3.8×10^{-9}

Table 3. Case2: Comparison of the L^∞-error of present method in Example 6.2

$\alpha = k$	INAS [50]	RKM [33]	Present method	
	$\tau^2 = h^4 = \frac{1}{256}$	$\tau^2 = h^4 = \frac{1}{256}$	$R = 8$	$R = 12$
0.5	7.62×10^{-4}	1.63×10^{-4}	1.02×10^{-4}	1.60×10^{-5}
0.6	8.42×10^{-4}	1.78×10^{-4}	7.61×10^{-5}	1.12×10^{-5}
0.7	9.25×10^{-4}	1.91×10^{-4}	5.11×10^{-5}	7.1×10^{-6}
0.8	1.01×10^{-3}	2.03×10^{-4}	2.91×10^{-5}	3.7×10^{-6}
0.9	1.11×10^{-3}	2.17×10^{-4}	1.25×10^{-5}	1.4×10^{-6}

Table 4. Case 3: Comparison of the L^2-error and L^∞-error of present method for Example 6.2 with $Y = 6$

R	$k = 0.5$		$k = 1.5$		$k = 2$	
	L^2-error	L^∞-error	L^2-error	L^∞-error	L^2-error	L^∞-error
5	7.42×10^{-5}	8.21×10^{-4}	3.25×10^{-5}	2.81×10^{-4}	9.52×10^{-7}	5.10×10^{-6}
10	1.60×10^{-6}	3.71×10^{-5}	1.60×10^{-7}	2.90×10^{-6}	1.44×10^{-11}	2.01×10^{-10}
15	1.07×10^{-7}	5.81×10^{-6}	4.29×10^{-9}	2.01×10^{-7}	9.78×10^{-11}	1.80×10^{-9}

The weight functions for cases 1,2 are Dirac delta functions. So according to Remark 3.1 and by considering $r = 1$ we can use operational matrix obtained in Theorem 3.1. In this example, we compare the L^∞-errors of our method for cases 1,2 with the spectral Galerkin method (SGM) [20], method in [9], method in [50] and method in [33] in Tables 2 and 3, respectively. It is obvious in these tables

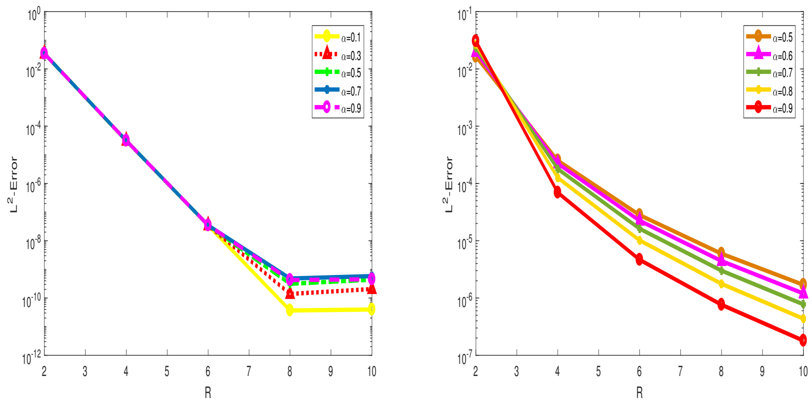

Figure 2. The L^2-error for Case1 (left) and Case2 (right) for Example 6.2.

that even for small choices of R and for special weight functions the results are accurate. Also, Figure 2 shows the graph of the L^2-error for specified values of α and different values of R. Furthermore, Table 4 presents L^2-error and L^∞-error results with applying the proposed method for Case 3. These results are in perfect agreement with the exact solution.

Example 6.3. For the last example we consider another modified DO anomalous sub-diffusion equation as follow:

$$\frac{\partial q(z,\mathfrak{T})}{\partial \mathfrak{T}} = \int_0^1 \Gamma(2+\vartheta)\ {}^{RL}D_{\mathfrak{T}}^{1-\vartheta}(\frac{\partial^2 q(z,\mathfrak{T})}{\partial z^2})d\vartheta + \mathfrak{h}(z,\mathfrak{T}), \quad \Omega \in (0,1)\times(0,0.5), \quad (6.1)$$

with boundary conditions

$$q(1,\mathfrak{T}) = q(0,\mathfrak{T}) = 0,$$

and initial condition

$$q(z,0) = 0.$$

The exact solution of this example is $q(z,\mathfrak{T}) = \mathfrak{T}^2 \sin(2\pi z)$ and

$$\mathfrak{h}(z,\mathfrak{T}) = 2\mathfrak{T}\sin(2\pi z) + \frac{8\pi^2 \sin(2\pi z)\mathfrak{T}(\mathfrak{T}-1)}{\ln(\mathfrak{T})}.$$

Table 5. Comparison of the L^2-error via $Y = 3$ and various values of R in Example 6.3

R	3	5	7	9	11
L^2-error	2.19×10^{-2}	1.15×10^{-3}	3.86×10^{-5}	1.08×10^{-6}	6.63×10^{-7}

Table 5 shows the L^2-error for $Y = 3$ and various values of R. It is found that in Table 5, as R increases, the L^2-error decrease. Also, Figure 3 shows the graph of the absolute error function for $R = 13$ and $Y = 5$.

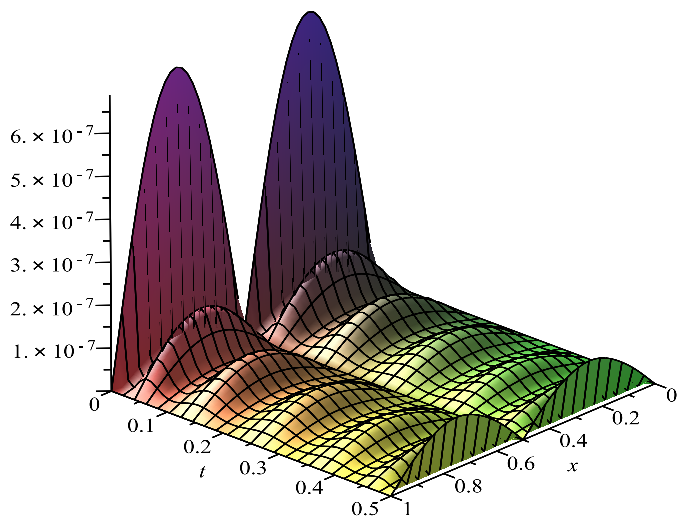

Figure 3. The absolute error function with $R = 13, Y = 5$, for Example 6.3.

7. Conclusion

In this article, by using OBPs, we construct new operational matrices of fractional derivative and DO fractional derivative of the R-L type. By applying these matrices and the spectral tau method, effective and simple numerical method was developed to solve some DO time-fractional partial differential equations. Also, in Section 5, the upper error bound for operational matrix $\widehat{\mathfrak{D}}^{(\eta(\vartheta))}$ was obtained. In this paper, for discretized DO we use of Gauss-Legendre quadrature formula. Some examples were presented in order to confirm the effectiveness of the obtained results. From the results reported for Examples 6.1 and 6.2 we can conclude that the presented method get to the exact solution with fewer quadrature point, while some papers such as [9, 32, 50], use the more number of quadrature points. A direction of future research is to apply an operational matrix obtained in this work for solving other classes of DOFD equations such as DO time-fractional diffusion-wave equation. Also, the technique presented in this paper can be developed to construct an operational matrix of DO fractional derivatives based on other orthogonal functions.

Conflict of interest

All authors declare that they have no conflicts of interest.

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