Journal of Applied Analysis and Computation
Volume 13, Number 6, December 2023, 3374–3402
Website:http://www.jaac-online.com
DOI:10.11948/20230051

A FRACTIONAL LANDWEBER ITERATION METHOD FOR SIMULTANEOUS INVERSION IN A TIME-FRACTIONAL DIFFUSION EQUATION*

Jin Wen[1,†], Chong-Wang Yue[1], Zhuan-Xia Liu[1] and Donal O'Regan[2]

Abstract In the present paper, we study the problem to identify the space-dependent source term and initial value simultaneously for a time-fractional diffusion equation. This inverse problem is ill-posed, and we use the idea of decoupling to turn it into two operator equations based on the Fourier method. To solve the inverse problem, a fractional Landweber regularization method is proposed. Furthermore, we present convergence estimates between the exact solution and the regularized solution by using the a-priori and the a-posteriori parameter choice rules. In order to verify the accuracy and efficiency of the proposed method, several numerical examples are constructed.

Keywords Time-fractional diffusion equation, simultaneous inversion, fractional Landweber iteration, a-priori and a-posteriori regularization parameters.

MSC(2010) 41A28, 65N21, 35R11.

1. Introduction

In recent years, the study of time-fractional diffusion has attracted some attention due to its successful application in the fields of anomalous diffusion and mechanics and we note that related mathematical theories and numerical methods of the anomalous diffusion equation have often been used; we refer the reader to [18, 19, 22, 26, 29] and the references therein. Due to the non-local nature of fractional differential operators, we note that time-fractional diffusion models have better properties than integer-order diffusion models in simulating real super-diffusion and sub-diffusion processes [3, 22, 30], and many researchers have studied the direct and inverse problems of the time-fractional diffusion (wave) equation (see for example [1, 4, 6, 13, 29, 32, 46]). However, the inverse problem of simultaneous multi-

†The corresponding author.
1Department of Mathematics, Northwest Normal University, Gansu 730070, China
2School of Mathematical and Statistical Sciences, University of Galway, Galway, Ireland
*The authors were supported by the NNSF of China (12261082, 11326234), NSF of Gansu Province (145RJZA099), Scientific research project of Higher School in Gansu Province (2014A-012), and Project of NWNU-LKQN2020-08.
Email: wenj@nwnu.edu.cn/wenjin0421@163.com(J. Wen),
yuechongwangsd@163.com(C. Yue), liu11200810@163.com(Z. Liu),
donal.oregan@nuigalway.ie(D. O'Regan)

parameter inversion in time-fractional diffusion equations have not been extensively studied and we list here some references on this topic [27, 28, 31, 45].

On the physical background, we can refer to [11], Hatano and Hatano used the continuous-time random walk to better simulations for the anomalous diffusion in an underground environmental problem. Ginoa, Cerbelli and Roman proposed a fractional diffusion equation that describe the relaxation of complex viscoelastic materials in [9]. Mainardi states that the fractional wave equation governs the propagation of mechanical diffusion waves in viscoelastic media in [21]. These new fractional order models are more adequate than previous integer-order models because fractional derivatives and integrals can describe the memory and genetic properties of different substances. This is the most significant advantage of fractional order models over integer-order models, and integer-order models ignore these effects. In physics, fractional space derivatives are used to model anomalous diffusion or dispersion, in which particles disperse at rates inconsistent with classical models of Brownian motion [5].

In this paper, we investigate the following time-fractional diffusion equation with the homogeneous Dirichlet boundary condition:

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} + Lu(x,t) = f(x), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = \phi(x), & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is the Caputo fractional derivative of order α, which is given by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\eta)^{-\alpha} \frac{\partial u}{\partial \eta} d\eta, \quad 0 < \alpha < 1,$$

here, $\Gamma(\cdot)$ is a Gamma function.

Let $\Omega \in \mathbb{R}^d (1 \le d \le 3)$ be an open bounded domain with its sufficiently smooth boundary $\partial\Omega$. Here, L is a symmetric strongly elliptic operator of the order $\alpha(\alpha \in (0,1))$ that is defined by

$$L(u) = -\sum_{i=1}^d \frac{\partial}{\partial x_i} \left(\sum_{j=1}^d \theta_{i,j} \frac{\partial}{\partial x_j} u(x) \right) + c(x)u(x).$$

We assume that $\theta_{i,j} = \theta_{j,i}, c(x) \ge 0, \forall x \in \overline{\Omega}$, and we suppose that L is uniformly elliptic on $\overline{\Omega}$ and its coefficients are smooth: where $v > 0$ is a constant, such that

$$v \sum_{i=1}^d \xi_i^2 \le \sum_{i,j=1}^d \theta_{i,j}(x) \xi_i \xi_j, \quad \forall x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^d.$$

If given the source term $f(x)$ and initial data $\phi(x)$, the problem (1.1) is called the direct problem. The inverse problem for (1.1) is not well known. Inverse problems appear when there is no given data (initial data, source term, boundary value or diffusion coefficient). By adding some additional data, we can get an inverse problem.

In this paper, we reconstruct the initial condition $u(x,0)=\phi(x)$ and the source term $f(x)$ under the noisy measurement, and we assume that $T_2 > T_1 > 0$,

$$g_1(\cdot)=u(\cdot,T_1), \quad g_2(\cdot)=u(\cdot,T_2).$$

Due to the noisy measurement being unavoidable, we denote the noisy measurement of g_1 and g_2 as $g_1^\delta(\cdot)$ and $g_2^\delta(\cdot)$, which satisfy

$$\|u(\cdot,T_1)-g_1^\delta(\cdot)\| \le \delta, \quad \|u(\cdot,T_2)-g_2^\delta(\cdot)\| \le \delta,$$

where $\|\cdot\|$ is the L^2 norm. When $\alpha=1$, the above inverse problem is called the simultaneous inversion of a standard parabolic equation [15].

About the physical background of the above model, especially in the real-world applications, one may not know the initial value and source term of the pollution simultaneously. Hence, the determination of initial value and source term is very important in underground environmental problem, nuclear pollution crisis and so on.

Recently, several authors used iterative methods to solve ill-posed problems, and the reader is referred to [25,41,43]. In 1951, Landweber proposed a fixed point iteration method to solve the first kind of Fredholm integral equation in [17] and for more recent applications we refer the reader to [7,14,23].

However, the classical Landweber iterative method has its own drawback in that the approximate solution is too smooth to reconstruct the exact solution. Therefore, researchers use some effective improved regularization methods to obtain stable numerical algorithms for ill-posed problems. Later, it was found that fractional regularization methods can overcome the shortcomings of over-smoothing to some extent. The fractional Landweber method was first proposed by Klann and Ramlau [16] when considering general regularization techniques for solving linear inverse problems. The fractional Landweber method was studied for the ill-posed operator equation $Kx=y$, where the forward operator K is a compact operator with a known singular system and Han et al. applied the fractional Landweber regularization method to solve the backward time-fractional diffusion problem in [10]. In [12], Le et al. considered the fractional Landweber method to solve the initial inverse problem of time-fractional wave equations, in [43], Yang et al. used the fractional Landweber method to solve an inverse problem for identifying the source term of nonhomogeneous time-fractional diffusion equation with the fractional Laplacian in a nonlocal boundary, in [2], Babaei et al. applied a softening regularization method to solve the unknown nonlinear boundary condition problem of the time-fractional diffusion equation, in [40], Yang et al. studied the inverse problem using the fractional Landweber method to determine the unknown source term in the time-fractional diffusion equation with variable coefficients in a general bound domain, and in [39], Xiong et al. studied an ill-posed problem and investigated a modified Landweber iterative method through the gradient flow equation induced by the weighted least squares functional. For the advantages of fractional Landweber iteration method, we can find that the modified method in [39] not only overcomes the over-smoothness problem of the approximate solutions, but also can reduce the total number of iterations compared to the traditional Landweber method. In comparing with the classical Landweber method, the fractional Landweber method not only reduces the total number of iterations but also can overcome the problems of oversmoothness of the approximate solutions. According to [36], we can see that the fractional Landweber regularization algorithm requires fewer steps.

To the best of the authors' knowledge, there is only a few papers on fractional diffusion equations where the source term and the initial value are determined simultaneously. In [27], the authors investigated the standard Tikhonov regularization method for solving the above inverse problem and gave the conditional stability. In [36], Wen et al. constructed the solution of the corresponding conjugate operator equation problem using the Landweber iteration method. Besides, we can only refer to [35, 37, 38]. Inspired by this, our paper applies the fractional Landweber iterative method to identify the source term and initial value of the time-fractional diffusion equation simultaneously. We will use the fractional Landweber iterative method of operator equations to solve the inverse problem on the basis of the Fourier method.

The rest of the paper is structured as follows: In Section 2, we present preliminaries needed for the upcoming discussion. The conditional stability and ill-posed analysis of the simultaneously inverse problem are given in Section 3. In Section 4, we use the fractional Landweber iteration method to solve the problem and give the convergence rates under both the a-priori and the a-posteriori parameter selection rules. In Section 5, several numerical examples are constructed to verify accuracy and efficiency of the proposed method. Finally, we give a brief conclusion in Section 6.

2. Preliminaries

In order to facilitate the forthcoming proofs and theoretical derivations, we give the following definition and properties.

Definition 2.1 ([24]). For arbitrary constants $\alpha > 0$ and $\beta \in \mathbb{R}$, we consider the Mittag-Leffler function defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}.$$

Lemma 2.1 ([24]). *(a) For $0 < \alpha < 1$ and $\eta > 0$,*

$$0 \leq E_{\alpha,1}(-\eta) < 1, \quad \frac{d^\alpha}{d\eta^\alpha} E_{\alpha,1}(-\lambda\eta^\alpha) = -\lambda E_{\alpha,1}(-\lambda\eta^\alpha).$$

In addition, $E_{\alpha,1}(-\eta)$ is fully monotonic. That is to say $(-1)^n d^n E_{\alpha,1}(-\eta)/d\eta^n \geq 0$, when $\eta \to +\infty$, and $E_{\alpha,1}(-\eta)$ satisfies the following approximation relation:

$$E_{\alpha,1}(-\eta) = \frac{1}{\eta\Gamma(1-\alpha)} + \mathcal{O}(| \eta |^{-2}).$$

(b) For $\lambda > 0$, $\alpha > 0$ and positive integer $m \in \mathbb{N}$,

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \quad t > 0.$$

Lemma 2.2 ([34, 42]). *For any λ_k satisfying $\lambda_k > \lambda_1 > 0$, there exist positive constants $\underline{C}$, $\overline{C}$ and C_1 depending on α, T, λ_1 such that*

$$\frac{\underline{C}}{\lambda_k} \le E_{\alpha,1}(-\lambda_k T^\alpha) \le \frac{\overline{C}}{\lambda_k},$$
$$\frac{C_1}{\lambda_k T^\alpha} \le E_{\alpha,1+\alpha}(-\lambda_k T^\alpha) \le \frac{1}{\lambda_k T^\alpha},$$

where $C_1(\alpha, T, \lambda_1) = 1 - E_{\alpha,1}(-\lambda_1 T^\alpha)$.

Remark 2.1. In this paper, we need to use two moments T_1, T_2, according to Lemma 2.2, there exist positive constants $\underline{C}_1$, $\overline{C}_1$ and $\underline{C}_2$, $\overline{C}_2$ such that:

$$\frac{\underline{C}_1}{\lambda_k} \le E_{\alpha,1}(-\lambda_k T_1^\alpha) \le \frac{\overline{C}_1}{\lambda_k},$$

$$\frac{\underline{C}_2}{\lambda_k} \le E_{\alpha,1}(-\lambda_k T_2^\alpha) \le \frac{\overline{C}_2}{\lambda_k}.$$

Lemma 2.3 ([20, 33]). *For $0 < \lambda < 1$, $p > 0$, $m \in \mathbb{N}$, let $r_m(\lambda) := (1-\lambda)^m$, the following inequality holds:*

$$r_m(\lambda)\lambda^p \le \theta_p (m+1)^{-p},$$

where,

$$\theta_p = \begin{cases} 1, & 0 \le p \le 1, \\ p^p, & p > 1. \end{cases}$$

We now prove the following Lemmas:

Lemma 2.4. *For $m \ge 1$, $\sigma_k > 0$, $0 < a\sigma_k^2 < 1$, we have*

$$\sup_{\sigma_k > 0} (1 - a\sigma_k^2)^m \sigma_k^{\frac{p}{2}} \le (\frac{p}{4a})^{\frac{p}{4}} m^{-\frac{p}{4}}.$$

Proof. We introduce a new variable $x := \sigma_k^2 < \frac{1}{a}$ and define a function $f(x) = (1-ax)^m x^{\frac{p}{4}}$. It is easy to verify that there exists a unique $x_0 = \frac{p}{a(4m+p)}$ such that $f'(x_0) = 0$. Then we calculate the second derivative of the function and bring x_0 into the equation which is smaller than 0, hence $f(x_0)$ is the maximum value point. Thus we have

$$\begin{aligned} f(x) \le f(x_0) &= (1 - \frac{p}{4m+p})^m \left(\frac{p}{a(4m+p)}\right)^{\frac{p}{4}} \\ &\le (\frac{p}{a})^{\frac{p}{4}} (\frac{1}{4m+p})^{\frac{p}{4}} \\ &\le (\frac{p}{4a})^{\frac{p}{4}} m^{-\frac{p}{4}}. \end{aligned}$$

□

By linear superposition, the solution $u(x,t)$ which satisfies the problem (1.1) can be grouped into the components $u_1(x,t)$ and $u_2(x,t)$, and they are the solutions of two sub-problems respectively,

$$\begin{cases} \frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} + Lu_1(x,t) = f(x), & (x,t) \in \Omega \times (0,T), \\ u_1(x,t) \mid_{\partial\Omega} = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u_1(x,0) \mid_{t=0} = 0, & x \in \overline{\Omega}. \end{cases} \tag{2.1}$$

$$\begin{cases} \frac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} + Lu_2(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ u_2(x,t) \mid_{\partial\Omega} = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u_2(x,0) \mid_{t=0} = \phi(x), & x \in \overline{\Omega}. \end{cases} \tag{2.2}$$

Hence

$$u(x,t) = u_1(x,t) + u_2(x,t).$$

Consider that L is a symmetric strongly elliptic operator. Assume that L has eigenvalues $\lambda_k \in \mathbb{R}$ and corresponding orthogonal eigenfunctions $\varphi_k(x) \in H^2(\Omega) \bigcap H_0^1(\Omega)$, and we set

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

Using the variable separation method and the results of Lemma 2.1, the formal solutions of direct problem (2.1) and (2.2) can be constructed as follows, respectively:

$$u_1(x,t) = \sum_{k=1}^{\infty} f_k \frac{1 - E_{\alpha,1}(-\lambda_k t^\alpha)}{\lambda_k} \varphi_k(x), \tag{2.3}$$

$$u_2(x,t) = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k t^\alpha) \phi_k \varphi_k(x), \tag{2.4}$$

where $f_k = \langle f(x), \varphi_k(x) \rangle$ and $\phi_k = \langle \phi(x), \varphi_k(x) \rangle$ are Fourier coefficients.

For any given initial function $\phi(x)$ and source term $f(x)$, we can periodically define a pair of linear operators K_1 and K_2 to solve problem (1.1):

$$K_1 : (f,\phi) \mapsto u(x,T_1),$$
$$K_2 : (f,\phi) \mapsto u(x,T_2).$$

Similarly, for problems (2.1) and (2.2), we can define four linear operators respectively.

$$K_{1,i} : f \mapsto u_1(x,T_i), \quad i = 1,2,$$
$$K_{2,i} : \phi \mapsto u_2(x,T_i), \quad i = 1,2.$$

By the solution expressions (2.3) and (2.4), we can obtain operator equations:

$$\left(K_{1,i}(f)\right)(x) = \sum_{k=1}^{\infty} f_k \frac{1 - E_{\alpha,1}(-\lambda_k T_i^\alpha)}{\lambda_k} \varphi_k(x), \quad i = 1,2, \tag{2.5}$$

$$\left(K_{2,i}(\phi)\right)(x) = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k T_i^\alpha) \phi_k \varphi_k(x), \quad i = 1,2. \tag{2.6}$$

Using the property of linear superposition, then we obtain the expressions of the following two operator equations

$$u(x,T_1) = (K_1(f,\phi))(x) = (K_{1,1}(f))(x) + (K_{2,1}(\phi))(x) = g_1(x), \tag{2.7}$$

$$u(x,T_2) = (K_2(f,\phi))(x) = (K_{1,2}(f))(x) + (K_{2,2}(\phi))(x) = g_2(x). \tag{2.8}$$

Now, we hope to solve the inverse problem, to find the pair of functions (f,ϕ) in the problem (2.1) and (2.2). From equations (2.7) and (2.8), we can seek a solution (f,ϕ) to the system:

$$\begin{cases} K_{2,1}\phi + K_{1,1}f = g_1, \\ K_{2,2}\phi + K_{1,2}f = g_2. \end{cases} \tag{2.9}$$

Lemma 2.5. *The two operators are commutative i.e.* $K_{2,2}K_{2,1} = K_{2,1}K_{2,2}$.

Proof. Using equation (2.6)

$$(K_{2,1}(\phi))(x) = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k T_1^{\alpha})\phi_k \varphi_k(x),$$

then

$$K_{2,2}(K_{2,1}(\phi))(x) = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k T_1^{\alpha})\phi_k E_{\alpha,1}(-\lambda_k T_2^{\alpha})\phi_k(x).$$

Similarly,

$$K_{2,1}(K_{2,2}(\phi))(x) = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k T_2^{\alpha})\phi_k E_{\alpha,1}(-\lambda_k T_1^{\alpha})\phi_k(x).$$

Hence , $K_{2,2}K_{2,1} = K_{2,1}K_{2,2}$. □

According to the above Lemma 2.5, applying the operator $K_{2,2}$ to the first equation in the system (2.9) and the operator $K_{2,1}$ to the second one, we obtain:

$$K_{2,2}K_{2,1}\phi + K_{2,2}K_{1,1}f = K_{2,2}g_1, \tag{2.10}$$

$$K_{2,1}K_{2,2}\phi + K_{2,1}K_{1,2}f = K_{2,1}g_2. \tag{2.11}$$

By subtracting equation (2.10) from equation (2.11) and using semi-groups properties, we have

$$(K_{2,1}K_{1,2} - K_{2,2}K_{1,1})f = K_{2,1}g_2 - K_{2,2}g_1.$$

Similarly, we apply the operator $K_{1,2}$ to the first equation in the system (2.9) and $K_{1,1}$ to the second one, we get

$$K_{1,2}K_{2,1}\phi + K_{1,2}K_{1,1}f = K_{1,2}g_1, \tag{2.12}$$

$$K_{1,1}K_{2,2}\phi + K_{1,1}K_{1,2}f = K_{1,1}g_2. \tag{2.13}$$

We subtract the equation (2.12) from (2.13) as follows

$$(K_{1,2}K_{2,1} - K_{1,1}K_{2,2})\phi = K_{1,2}g_1 - K_{1,1}g_2.$$

Thus, (2.9) is equivalent to the system

$$\begin{cases} Kf = \eta_1, \\ K\phi = \eta_2. \end{cases} \tag{2.14}$$

Here

$$K = K_{1,2}K_{2,1} - K_{1,1}K_{2,2},\ \eta_1 = K_{2,1}g_2 - K_{2,2}g_1,\ \eta_2 = K_{1,2}g_1 - K_{1,1}g_2.$$

Through the properties of singular values, we obtain the singular values of the operators $K_{1,1}, K_{1,2}, K_{2,1}, K_{2,2}$, respectively

$$\begin{aligned} \sigma_{1k} &= \frac{1 - E_{\alpha,1}(-\lambda_k T_1^\alpha)}{\lambda_k}, \\ \sigma_{2k} &= \frac{1 - E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k}, \\ \sigma_{3k} &= E_{\alpha,1}(-\lambda_k T_1^\alpha), \\ \sigma_{4k} &= E_{\alpha,1}(-\lambda_k T_2^\alpha). \end{aligned}$$

It is not difficult to get the singular values of the linear, compact and self-adjoint operator K:

$$\sigma_k = \frac{E_{\alpha,1}(-\lambda_k T_1^\alpha) - E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k}, \quad k = 1, 2 \cdots . \tag{2.15}$$

Remark 2.2. In this paper, the study of problems (2.7)-(2.8) is reduced to the study of the system (2.14), that is, the study of the first class of operator equations in $L^2(\Omega)$ of the form

$$Kb = \eta.$$

From the injectivity of K, we have

$$b = K^{-1}\eta = \sum_{k=1}^{\infty} \frac{1}{\sigma_k}(\eta, \varphi_k)\varphi_k.$$

Furthermore, since the measured data g_1 and g_2 are never known accurately in practice, our goal is to construct stable approximate solutions of f and ϕ in the system

$$\begin{cases} Kf = \eta_1^\delta, \\ K\phi = \eta_2^\delta, \end{cases} \tag{2.16}$$

where $\eta_1^\delta = K_{2,1}g_2^\delta - K_{2,2}g_1^\delta$, $\eta_2^\delta = K_{1,2}g_1^\delta - K_{1,1}g_2^\delta$, g_1^δ and g_2^δ are the perturbed data functions which satisfying

$$\|g_1(\cdot) - g_1^\delta(\cdot)\| + \|g_2(\cdot) - g_2^\delta(\cdot)\| \leq \delta + \delta = 2\delta. \tag{2.17}$$

3. Ill-posedness and conditional stability for the simultaneous inversion problem

Definition 3.1. For arbitrary $\chi \in L^2(\Omega)$, we have the Hilbert space

$$D\left((L)^p\right) = \left\{ \chi \in L^2(\Omega) : \left(\sum_{k=1}^{\infty} \lambda_k^{2p} \mid \langle \chi, \varphi_k \rangle \mid^2 \right)^{\frac{1}{2}} < \infty \right\}, \tag{3.1}$$

where

$$\|\chi\|_{D((L)^p)} = (\lambda_k^{2p} \mid \langle \chi, \varphi_k \rangle \mid^2)^{\frac{1}{2}}. \tag{3.2}$$

Theorem 3.1. *If $u \in C([0,T]; L^2(\Omega)) \bigcap C((0,T]; H^2(\Omega) \bigcap H_0^1(\Omega))$ is the solution that satisfies problem (1.1), $\phi(x) \in L^2(\Omega)$, $f(x) \in L^2(\Omega)$, and $u(x,T_1) = u(x,T_2) \equiv 0$, then:*

$$f = \phi = 0.$$

Proof. The solution to (1.1) is

$$u(x,t) = \sum_{k=1}^{\infty} \frac{1 - E_{\alpha,1}(-\lambda_k t^\alpha)}{\lambda_k} f_k \varphi_k + \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k t^\alpha) \phi_k \varphi_k,$$

then

$$g_i(x) = u(x,T_i) = \sum_{k=1}^{\infty} \frac{1 - E_{\alpha,1}(-\lambda_k T_i^\alpha)}{\lambda_k} f_k \varphi_k + \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k T_i^\alpha) \phi_k \varphi_k, \; i = 1,2.$$

Integrate φ_k on both sides of the above equation at the same time,

$$g_{ik} = \frac{1 - E_{\alpha,1}(-\lambda_k T_i^\alpha)}{\lambda_k} f_k + E_{\alpha,1}(-\lambda_k T_i^\alpha) \phi_k, \; i = 1,2,$$

where

$$g_{ik} = (g_i, \varphi_k) = \int_\Omega g_i(x) \varphi_k(x) dx, \; i = 1,2,$$

so, the coefficients for $f(x)$ and $\phi(x)$, respectively

$$\begin{aligned} f_k &= \frac{E_{\alpha,1}(-\lambda_k T_2^\alpha) g_{1k} - E_{\alpha,1}(-\lambda_k T_1^\alpha) g_{2k}}{E_{\alpha,1}(-\lambda_k T_2^\alpha) - E_{\alpha,1}(-\lambda_k T_1^\alpha)} \lambda_k, \\ \phi_k &= \frac{\left(1 - E_{\alpha,1}(-\lambda_k T_1^\alpha)\right) g_{2k} - \left(1 - E_{\alpha,1}(-\lambda_k T_2^\alpha)\right) g_{1k}}{E_{\alpha,1}(-\lambda_k T_1^\alpha) - E_{\alpha,1}(-\lambda_k T_2^\alpha)}. \end{aligned}$$

Because $0 < T_1 < T_2$, so $E_{\alpha,1}(-\lambda_k T_1^\alpha) \neq E_{\alpha,1}(-\lambda_k T_2^\alpha)$. According to Lemma 2.1, $1 - E_{\alpha,1}(-\lambda_k T_1^\alpha) \neq 0$ and $E_{\alpha,1}(-\lambda_k T_2^\alpha) - 1 \neq 0$. Hence, if $u(x,T_1) = u(x,T_2) = 0$, then $g_{1k} = g_{2k} = 0$. Combine the above results, we have $f_k = \phi_k = 0$, ŒŽ $f = \phi = 0$. □

Theorem 3.2. *If $f(x)$ and $\phi(x) \in D\left((L)^p\right) \subset H^p$ satisfy the a-priori bound condition*

$$\max\left\{ \|f(x)\|_{D((L)^p)}, \|\phi(x)\|_{D((L)^p)} \right\} \leq E, \tag{3.3}$$

where p is a nonnegative constant, then we obtain $u(\cdot,t) \in D\left((L)^{p+1}\right)$, and $\|u(\cdot,t)\|_{D((L)^{p+1})} \leq \left(1 + \frac{C}{t^\alpha}\right) E$ for any $t > 0$.

The proof can be found in [27].

From Theorem 3.2, it can be seen that $u(x,T_1)$, $u(x,T_2) \in D\left((L)^1\right)(\Omega)$ if f, $\phi \in L^2(\Omega)$. The operator $K : \left(L^2(\Omega)\right)^2 \to \left(L^2(\Omega)\right)^2$ is compact because $D\left((L)^1\right)$ is compactly imbedded into $L^2(\Omega)$. Accordingly, the problem is ill-posed.

Next, we will give the conditional stability of the ill-posed problem in the following theorem.

Theorem 3.3. *Let $g_i = u(\cdot, T_i)$ for $i = 1, 2$. Suppose f, $\phi \in D\left((L)^p\right)$ satisfy the a-priori bound condition (3.3), in which p is a positive constant, then*

$$\|f\| \le CE^{\frac{2}{p+2}} \left(\|g_1\| + (\frac{T_2}{T_1})^{\alpha} \|g_2\| \right)^{\frac{p}{p+2}}$$

and

$$\|\phi\| \le CE^{\frac{2}{p+2}} \left(\frac{1}{1 - E_{\alpha,1}(-\lambda_1 T_1^{\alpha})} \|g_1\| + \|g_2\| \right)^{\frac{p}{p+2}}.$$

The proof can be found in [27].

4. Fractional Landweber iteration and convergence analysis

In this section, we will use the fractional Landweber iteration regularisation method to solve the ill-posed problem, and present the convergence analysis under two regularization parameter choice rules. The standard theory of the fractional Landweber iteration method can be found in [16].

Now, we rewrite the system (2.14) as

$$\begin{pmatrix} f \\ \phi \end{pmatrix} = (I - aK^*K) \begin{pmatrix} f \\ \phi \end{pmatrix} + aK^* \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \tag{4.1}$$

for some $\alpha > 0$. We set the initial value $\begin{pmatrix} f_0 \\ \phi_0 \end{pmatrix} = \begin{pmatrix} f_{initial} \\ \phi_{initial} \end{pmatrix}$, then iterate this equation

$$\begin{pmatrix} f_m \\ \phi_m \end{pmatrix} = (I - aK^*K) \begin{pmatrix} f_{m-1} \\ \phi_{m-1} \end{pmatrix} + aK^* \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \quad m = 1, 2, 3 \cdots. \tag{4.2}$$

Here m is the number of iterations, which plays the role of regularization parameter, and a is the relaxation factor satisfying

$$0 < a < min \left\{ \frac{1}{\|K_{1,1}\|^2}, \frac{1}{\|K_{1,2}\|^2}, \frac{1}{\|K_{2,1}\|^2}, \frac{1}{\|K_{2,2}\|^2} \right\}.$$

In this case, K is a self-adjoint operator, $K_{i,j} (i, j = 1, 2, 3, 4)$ are linear compact operators.

We define the operator $R_m : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$ as follows:

$$\begin{pmatrix} f_m^\delta \\ \phi_m^\delta \end{pmatrix} = R_m \begin{pmatrix} \eta_1^\delta \\ \eta_2^\delta \end{pmatrix}.$$

Here

$$R_m := a \sum_{k=0}^{m-1} (I - aK^*K)^k K^*, \quad for\ m = 1, 2, \cdots.$$

Then we have

$$\begin{cases} f_m^\delta = R_m \eta_1^\delta = \sum\limits_{k=1}^{\infty} \frac{1}{\sigma_k} \left[1 - (1 - a\sigma_k^2)^m\right]^\gamma \left(\eta_1^\delta, \varphi_k\right) \varphi_k, \\ \phi_m^\delta = R_m \eta_2^\delta = \sum\limits_{k=1}^{\infty} \frac{1}{\sigma_k} \left[1 - (1 - a\sigma_k^2)^m\right]^\gamma \left(\eta_2^\delta, \varphi_k\right) \varphi_k, \end{cases} \tag{4.3}$$

where $\gamma \in (\frac{1}{2}, 1]$, we denote $\eta_{1,k} = (\eta_1, \varphi_k)$, $\eta_{2,k} = (\eta_2, \varphi_k)$, and we define

$$\left(K_1(f, \phi)\right)(x) = \left(K_{1,1}(f)\right)(x) + \left(K_{2,1}(\phi)\right)(x) = h_1(x) + h_3(x), \tag{4.4}$$
$$\left(K_2(f, \phi)\right)(x) = \left(K_{1,2}(f)\right)(x) + \left(K_{2,2}(\phi)\right)(x) = h_2(x) + h_4(x). \tag{4.5}$$

Now $K_{i,j}(i, j = 1, 2, 3, 4)$ are linear compact operators, so we define the operators $R_{1,m}$ and $R_{2,m}$: $L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$, which are given by

$$R_{1,m} g_1 = \sum_{k=1}^{\infty} \frac{1}{\sigma_{1k}} \left[1 - (1 - a\sigma_{1k}^2)^m\right]^\gamma h_{1,k} \varphi_k + \sum_{k=1}^{\infty} \frac{1}{\sigma_{3k}} \left[1 - (1 - a\sigma_{3k}^2)^m\right]^\gamma h_{3,k} \varphi_k, \tag{4.6}$$

$$R_{2,m} g_2 = \sum_{k=1}^{\infty} \frac{1}{\sigma_{2k}} \left[1 - (1 - a\sigma_{2k}^2)^m\right]^\gamma h_{2,k} \varphi_k + \sum_{k=1}^{\infty} \frac{1}{\sigma_{4k}} \left[1 - (1 - a\sigma_{4k}^2)^m\right]^\gamma h_{4,k} \varphi_k, \tag{4.7}$$

where $h_{i,k} = (h_i, \varphi_k)$. $(i = 1, 2, 3, 4.)$

We can refer to the specific iterative process in [44]. When $\gamma = 1$, this is the classical Landweber iterative method. The convergence results are given in the following theorems.

4.1. A-priori regularization parameter choice rule

Theorem 4.1. *Let $g_1 \in L^2(\Omega)$, $g_2 \in L^2(\Omega)$ and u, f and ϕ are the unique exact solution for the inverse problem (1.1). Assume that a satisfies $0 < a < \min\left\{\frac{1}{\|K_{1,1}\|^2}, \frac{1}{\|K_{1,2}\|^2}, \frac{1}{\|K_{2,1}\|^2}, \frac{1}{\|K_{2,2}\|^2}\right\}$, and u_m, f_m, and ϕ_m are the m-th fractional Landweber iterative regularization approximation solutions in the above iterative procedure. Then we have*

$$\lim_{m\to\infty} \|f(\cdot) - f_m(\cdot)\|_{L^2(\Omega)} = 0 \ and \ \lim_{m\to\infty} \|\phi(\cdot) - \phi_m(\cdot)\|_{L^2(\Omega)} = 0$$

for every initial function $f_0 \in L^2(\Omega)$ and $\phi_0 \in L^2(\Omega)$.

We know that the proposed iterative procedure is a regularization method, hence it is related to inexact data. We assume that the exact solutions f and ϕ are obtained, i.e., there exist functions $f \in L^2(\Omega)$ and $\phi \in L^2(\Omega)$ satisfying

$$u(\cdot, T_1; f, \phi) = g_1(\cdot), \quad u(\cdot, T_2; f, \phi) = g_2(\cdot),$$

and the noise level δ as an upper bound takes the following forms

$$\|g_1^\delta(\cdot) - g_1(\cdot)\| \le \delta, \quad \|g_2^\delta(\cdot) - g_2(\cdot)\| \le \delta,$$

where the observation is known a-priori.

Theorem 4.2. *Suppose the a-priori condition (3.3) and noise assumption* (2.17) *hold. If we choose regularization parameter $m = [b]$, where:*

$$b = \left(\frac{E}{\delta}\right)^{\frac{4}{p+2}}, \tag{4.8}$$

then we have the following estimates:

$$\left\|f_m^\delta(\cdot) - f(\cdot)\right\| \le C_5 E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}},$$

$$\left\|\phi_m^\delta(\cdot) - \phi(\cdot)\right\| \le C_6 E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}},$$

where $[b]$ denotes the largest integer not exceeding b and C_5, C_6 are positive constants.

Proof. Using the triangle inequality, we have

$$\|f_m^\delta(\cdot) - f(\cdot)\| \le \|f_m^\delta(\cdot) - f_m(\cdot)\| + \|f_m(\cdot) - f(\cdot)\|. \tag{4.9}$$

From (2.17) and Remark 2.1, we have

$$\begin{aligned}
&\left\|f_m^\delta(\cdot) - f_m(\cdot)\right\| \\
&= \left\|\sum_{k=1}^{\infty} \frac{1}{\sigma_k}[1-(1-a\sigma_k^2)^m]^\gamma \eta_{1,k}^\delta \varphi_k - \sum_{k=1}^{\infty} \frac{1}{\sigma_k}[1-(1-a\sigma_k^2)^m]^\gamma \eta_{1,k}\varphi_k\right\| \\
&= \left\|\sum_{k=1}^{\infty} \frac{1}{\sigma_k}[1-(1-a\sigma_k^2)^m]^\gamma \left(\eta_{1,k}^\delta - \eta_{1,k}\right)\varphi_k\right\| \\
&= \left\|\sum_{k=1}^{\infty} \frac{1}{\sigma_k}[1-(1-a\sigma_k^2)^m]^\gamma (K_{2,1}g_{2,k}^\delta - K_{2,2}g_{1,k}^\delta - K_{2,1}g_{2,k} + K_{2,2}g_{1,k})\varphi_k\right\| \\
&\le \left(\left\|K_{2,1}(g_{2,k}^\delta - g_{2,k})\right\| + \left\|K_{2,2}(g_{1,k}^\delta - g_{1,k})\right\|\right)\left(\sup_{\sigma_k>0} \frac{1}{\sigma_k}[1-(1-a\sigma_k^2)^m]^\gamma\right) \\
&\le \sup_{\lambda_k>0}\left(E_{\alpha,1}(-\lambda_k T_1^\alpha) + E_{\alpha,1}(-\lambda_k T_2^\alpha)\right)\left(\sup_{\sigma_k>0} G(n)\right)\delta,
\end{aligned}$$

where,

$$E_{\alpha,1}(-\lambda_k T_1^\alpha) + E_{\alpha,1}(-\lambda_k T_2^\alpha) \le \frac{\overline{C}_1}{\lambda_k} + \frac{\overline{C}_2}{\lambda_k} = \frac{C_3}{\lambda_k} \le \frac{C_3}{\lambda_1}.$$

Now $\overline{C}_1$ depends on α, T_1, $\overline{C}_2$ depends on α, T_2, and $C_3 = \overline{C}_1 + \overline{C}_2$.

In the following

$$G(n) = \frac{1}{\sigma_k}[1-(1-a\sigma_k^2)^m]^\gamma, \quad \sigma_k = \frac{E_{\alpha,1}(-\lambda_k T_1^\alpha) - E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k}.$$

Let $v = a^{\frac{1}{2}}\sigma_k$, and $\psi(v) = v^{-2}[1-(1-v^2)^m]^{2\gamma}$. Because $0 < a < \frac{1}{\|K\|^2}$, we have $0 < a\sigma_k < 1$. Hence, the function is continuous when $v \in (0,1)$. For $\gamma \in (\frac{1}{2}, 1)$ and $v \in (0,1)$, using Lemma 3.3 in [16]:

$$\psi(v) \le m.$$

Thus,

$$\sup_{\sigma_k > 0} G(n) \le \sqrt{am}. \tag{4.10}$$

Then

$$\left\| f_m^\delta(\cdot) - f_m(\cdot) \right\| \le \frac{C_3}{\lambda_1}\sqrt{am}\delta. \tag{4.11}$$

For the second term on the right side of (4.9), using a-priori bound condition (3.3), we obtain:

$$\begin{aligned}
\|f(\cdot) - f_m(\cdot)\| &= \left\| \sum_{k=1}^{\infty} \frac{1}{\sigma_k}\eta_{1,k}\varphi_k - \sum_{k=1}^{\infty} \frac{1}{\sigma_k}\left[1-\left(1-a\sigma_k^2\right)^m\right]^\gamma \eta_{1,k}\varphi_k \right\| \\
&= \left\| \sum_{k=1}^{\infty} \left[1-\left(1-(1-a\sigma_k^2)^m\right)^\gamma\right] \frac{1}{\sigma_k}\eta_{1,k}\varphi_k \right\| \\
&\le \left\| \sum_{k=1}^{\infty} \left(1-a\sigma_k^2\right)^m \lambda_k^{-p}\lambda_k^p f_k \right\| \\
&\le E \sup_{\lambda_k>0, \sigma_k>0} \left(1-a\sigma_k^2\right)^m \lambda_k^{-p}.
\end{aligned}$$

According to Remark 2.1 , we have

$$\frac{\underline{C}_1}{\lambda_k} - \frac{\overline{C}_2}{\lambda_k} \le E_{\alpha,1}(-\lambda_k T_1^\alpha) - E_{\alpha,1}(-\lambda_k T_2^\alpha) \le \frac{\overline{C}_1}{\lambda_k} - \frac{\underline{C}_2}{\lambda_k}.$$

We define $\frac{\underline{C}_1}{\lambda_k} - \frac{\overline{C}_2}{\lambda_k} = \frac{C_4}{\lambda_K}$, hence $\frac{C_4}{\lambda_k^2} \le \frac{E_{\alpha,1}(-\lambda_k T_1^\alpha) - E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k} = \sigma_n$.

Hence, using Lemma 2.4, we get:

$$\|f_m(\cdot) - f(\cdot)\| \le E \sup_{\sigma_k>0} (1-a\sigma_k^2)^m C_4^{-\frac{p}{2}} \sigma_k^{\frac{p}{2}} \le C_4^{-\frac{p}{2}} (\frac{p}{4a})^{\frac{p}{4}} m^{-\frac{p}{4}} E. \tag{4.12}$$

Combining (4.11) and (4.12), we select $m = [b]$ and obtain:

$$\|f_m^\delta(\cdot) - f(\cdot)\| \le C_5 E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}},$$

where $C_5 = \frac{C_3}{\lambda_k}\sqrt{a} + C_4^{-\frac{p}{2}}(\frac{p}{4a})^{\frac{p}{4}}$.

By the same calculation used to obtain (4.12) we have

$$\|\phi(\cdot) - \phi_m(\cdot)\| \le C_4^{-\frac{p}{2}} (\frac{p}{4a})^{\frac{p}{4}} m^{-\frac{p}{4}} E.$$

On the other hand,

$$
\begin{aligned}
&\left\|\phi_m^\delta(\cdot)-\phi_m(\cdot)\right\| \\
&=\left\|\sum_{k=1}^{\infty}\frac{1}{\sigma_k}\left[1-(1-a\sigma_k^2)^m\right]^\gamma\eta_{2,k}^\delta\varphi_k-\sum_{k=1}^{\infty}\frac{1}{\sigma_k}\left[1-(1-a\sigma_k^2)^m\right]^\gamma\eta_{2,k}\varphi_k\right\| \\
&=\left\|\sum_{k=1}^{\infty}\frac{1}{\sigma_k}\left[1-(1-a\sigma_k^2)^m\right]^\gamma\left(\eta_{2,k}^\delta-\eta_{2,k}\right)\varphi_k\right\| \\
&=\left\|\sum_{k=1}^{\infty}\frac{1}{\sigma_k}\left[1-(1-a\sigma_k^2)^m\right]^\gamma\left(K_{1,2}g_{1,k}^\delta-K_{1,1}g_{2,k}^\delta-K_{1,2}g_{1,k}+K_{1,1}g_{2,k}\right)\varphi_k\right\| \\
&\le\left(\|K_{1,2}(g_{1,k}^\delta-g_{1,k})\|+\|K_{1,1}(g_{2,k}-g_{2,k}^\delta)\|\right)\sup_{\sigma_k>0}\left(\frac{1}{\sigma_k}[1-(1-a\sigma_k^2)^m]^\gamma\right)\delta \\
&\le\left(\frac{1-E_{\alpha,1}(-\lambda_k T_1^\alpha)}{\lambda_k}+\frac{1-E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k}\right)\sqrt{am}\delta,
\end{aligned}
$$

using Lemma 2.2, we have:

$$
\|\phi_m^\delta(\cdot)-\phi_m(\cdot)\|\le\frac{2}{\lambda_k}\sqrt{am}\delta<\frac{2}{\lambda_1}\sqrt{am}\delta. \tag{4.13}
$$

If we select $m=[b]$, then one has

$$
\|\phi_m^\delta(\cdot)-\phi(\cdot)\|\le C_6E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}},
$$

where $C_6=\frac{2\sqrt{a}}{\lambda_1}+C_4^{-\frac{p}{2}}(\frac{p}{4a})^{\frac{p}{4}}$. □

4.2. A-posteriori regularization choice rule

We know that iterative-type regularization methods of ill-posed problems have semi-convergence character. For this reason, we need a reliable stopping rule to detect critical changes from convergence to divergence.

Now we use the Morozov discrepancy principle [8] to determine the regularization parameter m by using the a-posteriori choice rule, and we give the convergent error estimate for the regularized solution.

The general a-posteriori rule can be summarized as follows:

$$
\tau_1\delta\le\|u_m^\delta(\cdot,T_1)-g_1^\delta(\cdot)\|+\|u_m^\delta(\cdot,T_2)-g_2^\delta(\cdot)\|\le\tau_2\delta. \tag{4.14}
$$

Here $\tau_1>2$, $\tau_2>2$ are constants independent of δ, f_m^δ, ϕ_m^δ are the $m-th$ fractional Landweber approximation solutions defined in (4.3).

Lemma 4.1. *Let* $\rho(m)=\|u_m^\delta(\cdot,T_1)-g_1^\delta(\cdot)\|+\|u_m^\delta(\cdot,T_2)-g_2^\delta(\cdot)\|$. *Then we have the following results:*
(1) $\rho(m)$ *is a continuous function;*
(2) $\lim_{m\to0}\rho(m)=\|g_1^\delta(\cdot)\|+\|g_2^\delta(\cdot)\|$*;*
(3) $\lim_{m\to\infty}\rho(m)=0$*;*
(4) $\rho(m)$*is a strictly decreasing function over* $(0,\infty)$.

Proof. From (4.3), (4.4)and (4.5), we obtain:

$$\begin{aligned}\rho(m)=&\left(\sum_{k=1}^{\infty}\left(\left[1-\left(1-(1-a\sigma_{1k}^2)^m\right)^{\gamma}\right]h_{1,k}^{\delta}+\left[1-\left(1-(1-a\sigma_{3k}^2)^m\right)^{\gamma}\right]h_{3,k}^{\delta}\right)^2\right)^{\frac{1}{2}}\\&+\left(\sum_{k=1}^{\infty}\left(\left[1-\left(1-(1-a\sigma_{2k}^2)^m\right)^{\gamma}\right]h_{2,k}^{\delta}\right.\right.\\&\left.\left.+\left[1-\left(1-(1-a\sigma_{4k}^2)^m\right)^{\gamma}\right]h_{4,k}^{\delta}\right)^2\right)^{\frac{1}{2}}.\end{aligned}$$

Hence,

$$\lim_{m\to 0}\rho(m)=\left(\sum_{k=1}^{\infty}(h_{1,k}^{\delta}+h_{3,k}^{\delta})^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}(h_{2,k}^{\delta}+h_{4,k}^{\delta})^2\right)^{\frac{1}{2}}=\left\|g_1^{\delta}(\cdot)\right\|+\left\|g_2^{\delta}(\cdot)\right\|.$$

The above results (3)-(4) are easily obtained. □

Lemma 4.2. *Assume m makes (4.14) hold at the first time. The following inequality holds:*

$$(ma)^{\frac{1}{2}}\le\left(\frac{C_7}{\tau_2-2}\right)^{\frac{1}{p+1}}\left(\frac{E}{\delta}\right)^{\frac{1}{p+1}},$$

where $C_7=\left(2+(\overline{C}_1)^{-p}+(\overline{C}_2)^{-p}\right)\theta_{\frac{p+1}{2}}$.

Proof. From the definition of m, and using the triangle inequality we obtain:

$$\begin{aligned}\tau_2\delta&\le\left\|u_{m-1}^{\delta}(\cdot,T_1)-g_1^{\delta}(\cdot)\right\|+\left\|u_{m-1}^{\delta}(\cdot,T_2)-g_2^{\delta}(\cdot)\right\|\\&=\left\|\sum_{k=1}^{\infty}\left[1-\left[1-(1-a\sigma_{1k}^2)^{m-1}\right]^{\gamma}\right]h_{1,k}^{\delta}\varphi_k\right.\\&\quad\left.+\left[1-\left[1-(1-a\sigma_{3k}^2)^{m-1}\right]^{\gamma}\right]h_{3,k}^{\delta}\varphi_k\right\|\\&\quad+\left\|\sum_{k=1}^{\infty}\left[1-\left[1-(1-a\sigma_{2k}^2)^{m-1}\right]^{\gamma}\right]h_{2,k}^{\delta}\varphi_k\right.\\&\quad\left.+\left[1-\left[1-(1-a\sigma_{4k}^2)^{m-1}\right]^{\gamma}\right]h_{4,k}^{\delta}\varphi_k\right\|\\&\le\left\|\sum_{k=1}^{\infty}(1-a\sigma_{1k}^2)^{m-1}\left(h_{1,k}^{\delta}-h_{1,k}\right)\varphi_k+\sum_{k=1}^{\infty}(1-a\sigma_{1k}^2)^{m-1}h_{1,k}\varphi_k\right.\\&\quad\left.+\sum_{k=1}^{\infty}(1-a\sigma_{3k}^2)^{m-1}\left(h_{3,k}^{\delta}-h_{3,k}\right)\varphi_k+\sum_{k=1}^{\infty}(1-a\sigma_{3k}^2)^{m-1}h_{3,k}\varphi_k\right\|\\&\quad+\left\|\sum_{k=1}^{\infty}(1-a\sigma_{2k}^2)^{m-1}\left(h_{2,k}^{\delta}-h_{2,k}\right)\varphi_k+\sum_{k=1}^{\infty}(1-a\sigma_{2k}^2)^{m-1}h_{2,k}\varphi_k\right.\\&\quad\left.+\sum_{k=1}^{\infty}(1-a\sigma_{4k}^2)^{m-1}\left(h_{4,k}^{\delta}-h_{4,k}\right)\varphi_k+\sum_{k=1}^{\infty}(1-a\sigma_{4k}^2)^{m-1}h_{4,k}\varphi_k\right\|.\end{aligned}$$

Due to $\left\|1-a\sigma_{i,k}^2\right\|<1$, we get:

$$\begin{aligned}
\tau_2\delta \leq & \left\|\sum_{k=1}^{\infty}\left(h_{1,k}^{\delta}-h_{1,k}+h_{3,k}^{\delta}-h_{3,k}\right)\varphi_k\right\| \\
& +\left\|\sum_{k=1}^{\infty}(1-a\sigma_{1k}^2)^{m-1}h_{1,k}\varphi_k+\sum_{k=1}^{\infty}(1-a\sigma_{3k}^2)^{m-1}h_{3,k}\varphi_k\right\| \\
& +\left\|\sum_{k=1}^{\infty}(1-a\sigma_{2k}^2)^{m-1}h_{2,k}\varphi_k+\sum_{k=1}^{\infty}(1-a\sigma_{4k}^2)^{m-1}h_{4,k}\varphi_k\right\| \\
\leq & \left\|\sum_{k=1}^{\infty}\left(g_{1,k}^{\delta}-g_{1,k}\right)\varphi_k\right\|+\left\|\sum_{k=1}^{\infty}\left(g_{1,k}^{\delta}-g_{1,k}\right)\varphi_k\right\| \\
& +\left\|\sum_{k=1}^{\infty}(1-a\sigma_{1k}^2)^{m-1}\sigma_{1k}f_k\varphi_k+\sum_{k=1}^{\infty}(1-a\sigma_{3k}^2)^{m-1}\sigma_{3k}\phi_k\varphi_k\right\| \\
& +\left\|\sum_{k=1}^{\infty}(1-a\sigma_{2k}^2)^{m-1}\sigma_{2k}f_k\varphi_k+\sum_{k=1}^{\infty}(1-a\sigma_{4k}^2)^{m-1}\sigma_{4k}\phi_k\varphi_k\right\|,
\end{aligned}$$

using Lemma 2.3 and Remark 2.1,

$$\begin{aligned}
\tau_2\delta \leq\ & 2\delta+\left\|\sum_{k=1}^{\infty}\left(1-a\sigma_{1k}^2\right)^{m-1}\sigma_{1k}\lambda_k^{-p}\lambda_k^{p}f_k\varphi_k+\sum_{k=1}^{\infty}\left(1-a\sigma_{3k}^2\right)^{m-1}\sigma_{3k}\lambda_k^{-p}\lambda_k^{p}\phi_k\varphi_k\right\| \\
& +\left\|\sum_{k=1}^{\infty}\left(1-a\sigma_{2k}^2\right)^{m-1}\sigma_{2k}\lambda_k^{-p}\lambda_k^{p}f_k\varphi_k+\sum_{k=1}^{\infty}\left(1-a\sigma_{4k}^2\right)^{m-1}\sigma_{4k}\lambda_k^{-p}\lambda_k^{p}\phi_k\varphi_k\right\| \\
\leq\ & 2\delta+E\left(\sup_{\sigma_{1k}>0}\frac{(1-a\sigma_{1k}^2)^{m-1}\sigma_{1k}}{\lambda_k^p}+\sup_{\sigma_{3k}>0}\frac{(1-a\sigma_{3k}^2)^{m-1}\sigma_{3k}}{\lambda_k^p}\right) \\
& +E\left(\sup_{\sigma_{2k}>0}\frac{(1-a\sigma_{2k}^2)^{m-1}\sigma_{2k}}{\lambda_k^p}+\sup_{\sigma_{4k}>0}\frac{(1-a\sigma_{4k}^2)^{m-1}\sigma_{4k}}{\lambda_k^p}\right) \\
\leq\ & 2\delta+E\left(\sup_{\sigma_{1k}>0}(1-a\sigma_{1k}^2)^{m-1}(a\sigma_{1k}^2)^{\frac{p+1}{2}}a^{-\frac{p+1}{2}}\right. \\
& \left.+\overline{C}_1^{-p}\sup_{\sigma_{3k}>0}(1-a\sigma_{3k}^2)^{m-1}(a\sigma_{3k}^2)^{\frac{p+1}{2}}a^{-\frac{p+1}{2}}\right) \\
& +E\left(\sup_{\sigma_{2k}>0}(1-a\sigma_{2k}^2)^{m-1}(a\sigma_{2k}^2)^{\frac{p+1}{2}}a^{-\frac{p+1}{2}}\right. \\
& \left.+\overline{C}_2^{-p}\sup_{\sigma_{4k}>0}(1-a\sigma_{4k}^2)^{m-1}(a\sigma_{4k}^2)^{\frac{p+1}{2}}a^{-\frac{p+1}{2}}\right) \\
\leq\ & 2\delta+E\left(\theta_{\frac{p+1}{2}}(ma)^{-\frac{p+1}{2}}+(\overline{C}_1)^{-p}\theta_{\frac{p+1}{2}}(ma)^{-\frac{p+1}{2}}\right) \\
& +E\left(\theta_{\frac{p+1}{2}}(ma)^{-\frac{p+1}{2}}+(\overline{C}_2)^{-p}\theta_{\frac{p+1}{2}}(ma)^{-\frac{p+1}{2}}\right) \\
\leq\ & 2\delta+E\left(2\theta_{\frac{p+1}{2}}+(\overline{C}_1)^{-p}\theta_{\frac{p+1}{2}}+(\overline{C}_2)^{-p}\theta_{\frac{p+1}{2}}\right)(ma)^{-\frac{p+1}{2}},
\end{aligned}$$

where, $\theta_{\frac{p+1}{2}}=\begin{cases}1, & 0\leq p\leq 1,\\ \left(\frac{p+1}{2}\right)^{\frac{p+1}{2}}, & p>1.\end{cases}$, and we have used Remark 2.1.

We denote $C_7 = \left(2 + (\overline{C}_1)^{-p} + (\overline{C}_2)^{-p}\right) \theta_{\frac{p+1}{2}}$, then

$$\tau_2 \delta \le 2\delta + C_7 E (ma)^{-\frac{p+1}{2}},$$

and by a simple calculation:

$$(ma)^{\frac{1}{2}} \le \left(\frac{C_7}{\tau_2 - 2}\right)^{\frac{1}{p+1}} \left(\frac{E}{\delta}\right)^{\frac{1}{p+1}}.$$

□

Theorem 4.3. *Assume the a-priori condition (3.3) and the noise assumption (2.17) hold, and the regularization parameter m is given by Lemma 4.2. We obtain the error estimates as follows:*

$$\|f_m^\delta(\cdot) - f(\cdot)\| \le \left(C_8 + (C_9)^{\frac{1}{p+1}}\right) E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}},$$
$$\|\phi_m^\delta(\cdot) - \phi(\cdot)\| \le \left((C_{10})^{\frac{p}{p+1}} + C_{11}\right) E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}},$$

where $C_8 = \frac{C_3}{\lambda_1}\left(\frac{C_7}{\tau_2 - 2}\right)^{\frac{1}{p+1}}$, $C_9 = \left(\frac{1}{\frac{E_{\alpha,1}(-\lambda_1 T_1^\alpha)}{E_{\alpha,1}(-\lambda_1 T_2^\alpha)} - 1}\right)\left[1 + \left(\frac{T_2}{T_1}\right)^\alpha + \tau_2\right]$,
$C_{10} = \left(\frac{1}{\underline{C}_2\left(\frac{E_{\alpha,1}(-\lambda_1 T_1^\alpha)}{E_{\alpha,1}(-\lambda_1 T_2^\alpha)} - 1\right)}\right)\left[1 + \left(\frac{1}{1 - E_{\alpha,1}(-\lambda_1 T_1^\alpha)}\right) + \tau_2\right]$, $C_{11} = \frac{2}{\lambda_1}\left(\frac{C_7}{\tau_2 - 2}\right)^{\frac{1}{p+1}}$.

Proof. By the triangle inequality, we get

$$\|f_m^\delta(\cdot) - f(\cdot)\| \le \|f_m^\delta(\cdot) - f_m(\cdot)\| + \|f_m(\cdot) - f(\cdot)\|. \tag{4.15}$$

Using (4.11) and Lemma 4.2, we obtain:

$$\|f_m^\delta(\cdot) - f_m(\cdot)\| \le \frac{C_3}{\lambda_1}\sqrt{am}\,\delta \le C_8 E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}, \tag{4.16}$$

where $C_8 = \frac{C_3}{\lambda_1}\left(\frac{C_7}{\tau_2 - 2}\right)^{\frac{1}{p+1}}$.

For the second item on the right side of (4.15), we have

$$\begin{aligned}
&\|f(\cdot) - f_m(\cdot)\|^2 \\
=& \left\| \sum_{k=1}^{\infty} \frac{1}{\sigma_k} \eta_{1,k} \varphi_k - \sum_{k=1}^{\infty} \left[1 - (1 - a\sigma_k^2)^m\right]^\gamma \frac{1}{\sigma_k} \eta_{1,k} \varphi_k \right\|^2 \\
=& \left\| \sum_{k=1}^{\infty} \frac{1}{\sigma_k} \left[1 - [1 - (1 - a\sigma_k^2)^m]^\gamma\right] \eta_{1,k} \varphi_k \right\|^2 \\
\le& \left\| \sum_{k=1}^{\infty} \frac{1}{\sigma_k} (1 - a\sigma_k^2)^m \eta_{1,k} \varphi_k \right\|^2 \\
=& \sum_{k=1}^{\infty} \frac{\left[(1 - a\sigma_k^2)^m (g_{1,k} - g_{2,k}\sigma_{3k}/\sigma_{4k})\right]^{\frac{2}{p+1}}}{(\sigma_{1k} - \sigma_{2k}\sigma_{3k}/\sigma_{4k})^2}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\sum_{k=1}^{\infty} \left[(1 - a\sigma_k^2)^m (g_{1,k} - g_{2,k}\sigma_{3k}/\sigma_{4k}) \right]^2 \right)^{\frac{2p}{p+1}} \\
&\le \sum_{k=1}^{\infty} \frac{(g_{1,k} - g_{2,k}\sigma_{3k}/\sigma_{4k})^{\frac{2}{p+1}}}{(\sigma_{1k} - \sigma_{2k}\sigma_{3k}/\sigma_{4k})^2} \left(\sum_{k=1}^{\infty} (g_{1,k} - g_{2,k}\sigma_{3k}/\sigma_{4k})^2 \right)^{\frac{2p}{p+1}} \\
&\le \left(\sum_{k=1}^{\infty} \frac{(g_{1,k} - g_{2,k}\sigma_{3k}/\sigma_{4k})^2}{(\sigma_{1k} - \sigma_{2k}\sigma_{3k}/\sigma_{4k})^{2(p+1)}} \right)^{\frac{1}{p+1}} \left(\sum_{k=1}^{\infty} (g_{1,k} - g_{2,k}\sigma_{3k}/\sigma_{4k})^2 \right)^{\frac{p}{p+1}} \\
&\le \left(\sum_{k=1}^{\infty} \frac{1}{(\sigma_{1k} - \sigma_{2k}\sigma_{3k}/\sigma_{4k})^{2p}} f_k^2 \right)^{\frac{1}{p+1}} \\
&\quad \times \left\{ \left(\sum_{k=1}^{\infty} \left(g_{1,k} - g_{2,k}\sigma_{3k}/\sigma_{4k} - g_{1,k}^{\delta} + g_{2,k}^{\delta}\sigma_{3k}/\sigma_{4k} \right)^2 \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left(\sum_{k=1}^{\infty} (g_{1,k}^{\delta} - g_{2,k}^{\delta}\sigma_{3k}/\sigma_{4k})^2 \right)^{\frac{1}{2}} \right\}^{\frac{2p}{p+1}},
\end{aligned}
$$

where we have used the Hölder inequality, and we also have

$$
\mid \sigma_{1k} - \sigma_{2k}\sigma_{3k}/\sigma_{4k} \mid = \frac{1}{\lambda_k} \left(\frac{E_{\alpha,1}(-\lambda_k T_1^{\alpha})}{E_{\alpha,1}(-\lambda_k T_2^{\alpha})} - 1 \right).
$$

It is not difficult to verify that $\frac{E_{\alpha,1}(-T_1^{\alpha} t)}{E_{\alpha,1}(-T_2^{\alpha} t)}$ is a nondecreasing function greater than 1 for any $t > 0$, hence

$$
\mid \sigma_{1k} - \sigma_{2k}\sigma_{3k}/\sigma_{4k} \mid = \frac{1}{\lambda_k} \left(\frac{E_{\alpha,1}(-\lambda_k T_1^{\alpha})}{E_{\alpha,1}(-\lambda_k T_2^{\alpha})} - 1 \right) \ge \frac{1}{\lambda_k} \left(\frac{E_{\alpha,1}(-\lambda_1 T_1^{\alpha})}{E_{\alpha,1}(-\lambda_1 T_2^{\alpha})} - 1 \right),
$$

and $\sigma_{3k}/\sigma_{4k} = \frac{E_{\alpha,1}(-T_1^{\alpha}\lambda_k)}{E_{\alpha,1}(-T_2^{\alpha}\lambda_k)} \le \lim_{t\to\infty} \frac{E_{\alpha,1}(-T_1^{\alpha} t)}{E_{\alpha,1}(-T_2^{\alpha} t)} = \left(\frac{T_2}{T_1}\right)^{\alpha}$, so we obtain:

$$
\begin{aligned}
&\|f_m(\cdot) - f(\cdot)\|^2 \\
&\le \left(\sum_{k=1}^{\infty} \frac{1}{(\sigma_{1k} - \sigma_{2k}\sigma_{3k}/\sigma_{4k})^{2p}} f_k^2 \right)^{\frac{1}{p+1}} \\
&\quad \times \left\{ \left(\sum_{k=1}^{\infty} \left(g_{1,k} - g_{2,k}\sigma_{3k}/\sigma_{4k} - g_{1,k}^{\delta} + g_{2,k}^{\delta}\sigma_{3k}/\sigma_{4k} \right)^2 \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left(\left(\sum_{k=1}^{\infty} (g_{1,k}^{\delta})^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} (g_{2,k}^{\delta})^2 (\sigma_{3k}/\sigma_{4k})^2 \right)^{\frac{1}{2}} \right) \right\}^{\frac{2p}{p+1}} \\
&\le \left(\frac{\lambda_k^{2p} f_k^2}{\left(\frac{E_{\alpha,1}(-\lambda_1 T_1^{\alpha})}{E_{\alpha,1}(-\lambda_1 T_2^{\alpha})} - 1 \right)^{2p}} \right)^{\frac{1}{P+1}} \left\{ \left(\sum_{k=1}^{\infty} (g_{1,k} - g_{1,k}^{\delta})^2 \right)^{\frac{1}{2}} \right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left(\sum_{k=1}^{\infty}(\sigma_{3k}/\sigma_{4k})^2(g_{2,k}-g_{2,k}^{\delta})^2\right)^{\frac{1}{2}} + \tau_2\delta\Bigg\}^{\frac{2p}{p+1}} \\
&\le \left(\frac{1}{\frac{E_{\alpha,1}(-\lambda_1 T_1^{\alpha})}{E_{\alpha,1}(-\lambda_1 T_2^{\alpha})}-1}\right)^{\frac{2p}{P+1}} E^{\frac{2}{p+1}}\left[1+\left(\frac{T_2}{T_1}\right)^{\alpha}+\tau_2\right]^{\frac{2p}{p+1}}\delta^{\frac{2p}{p+1}} \\
&\le (C_9)^{\frac{2p}{p+1}} E^{\frac{2}{p+1}}\delta^{\frac{2p}{p+1}}.
\end{aligned}
$$

Thus,

$$\|f_m(\cdot)-f(\cdot)\| \le (C_9)^{\frac{p}{p+1}} E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}},$$

where $C_9 = \left(\frac{1}{\frac{E_{\alpha,1}(-\lambda_1 T_1^{\alpha})}{E_{\alpha,1}(-\lambda_1 T_2^{\alpha})}-1}\right)\left[1+\left(\frac{T_2}{T_1}\right)^{\alpha}+\tau_2\right]$.

Therefore, according to (4.16) and the above result we have:

$$\|f_m^{\delta}(\cdot)-f(\cdot)\| \le \left(C_8+(C_9)^{\frac{1}{p+1}}\right) E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}.$$

By the same calculation, we obtain

$$
\begin{aligned}
&\|\phi(\cdot)-\phi_m(\cdot)\|^2 \\
&= \left\|\sum_{k=1}^{\infty}\frac{1}{\sigma_k}\eta_{2,k}\varphi_k - \sum_{k=1}^{\infty}\left[1-(1-a\sigma_k^2)^m\right]^{\gamma}\frac{1}{\sigma_k}\eta_{2,k}\varphi_k\right\|^2 \\
&= \left\|\sum_{k=1}^{\infty}\frac{1}{\sigma_k}\left[1-\left[1-(1-a\sigma_k^2)^m\right]^{\gamma}\right]\eta_{2,k}\varphi_k\right\|^2 \\
&\le \left\|\sum_{k=1}^{\infty}\frac{1}{\sigma_k}(1-a\sigma_k^2)^m\eta_{2,k}\varphi_k\right\|^2 \\
&= \sum_{k=1}^{\infty}\frac{\left[(1-a\sigma_k^2)^m(g_{2,k}-g_{1,k}\sigma_{2k}/\sigma_{1k})\right]^{\frac{2}{p+1}}}{(\sigma_{4k}-\sigma_{2k}\sigma_{3k}/\sigma_{1k})^2} \\
&\quad \times \left(\sum_{k=1}^{\infty}\left[(1-a\sigma_k^2)^m(g_{2,k}-g_{1,k}\sigma_{2k}/\sigma_{1k})\right]^2\right)^{\frac{2p}{p+1}} \\
&\le \left(\sum_{k=1}^{\infty}\frac{(g_{2,k}-g_{1,k}\sigma_{2k}/\sigma_{1k})^2}{(\sigma_{4k}-\sigma_{2k}\sigma_{3k}/\sigma_{1k})^{2(p+1)}}\right)^{\frac{1}{p+1}}\left(\sum_{k=1}^{\infty}(g_{2,k}-g_{1,k}\sigma_{2k}/\sigma_{1k})^2\right)^{\frac{p}{p+1}} \\
&\le \left(\sum_{k=1}^{\infty}\frac{1}{(\sigma_{4k}-\sigma_{2k}\sigma_{3k}/\sigma_{1k})^{2p}}\phi_k^2\right)^{\frac{1}{p+1}} \\
&\quad \times \Bigg\{\left(\sum_{k=1}^{\infty}\left(g_{2,k}-g_{1,k}\sigma_{2k}/\sigma_{1k}-g_{2,k}^{\delta}+g_{1,k}^{\delta}\sigma_{2k}/\sigma_{1k}\right)^2\right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{k=1}^{\infty}(g_{2,k}^{\delta}-g_{1,k}^{\delta}\sigma_{2k}/\sigma_{1k})^2\right)^{\frac{1}{2}}\Bigg\}^{\frac{2p}{p+1}},
\end{aligned}
$$

and we have,

$$\begin{aligned}| \sigma_{4k} - \sigma_{2k}\sigma_{3k}/\sigma_{1k} | &= \frac{E_{\alpha,1}(-\lambda_k T_1^\alpha) - E_{\alpha,1}(-\lambda_k T_2^\alpha)}{1 - E_{\alpha,1}(-\lambda_k T_1^\alpha)} \\ &= \left(\frac{E_{\alpha,1}(-\lambda_k T_1^\alpha)}{E_{\alpha,1}(-\lambda_k T_2^\alpha)} - 1 \right) \frac{E_{\alpha,1}(-\lambda_k T_2^\alpha)}{1 - E_{\alpha,1}(-\lambda_k T_1^\alpha)} \\ &\geq \frac{C_2}{\lambda_k} \left(\frac{E_{\alpha,1}(-\lambda_1 T_1^\alpha)}{E_{\alpha,1}(-\lambda_1 T_2^\alpha)} - 1 \right),\end{aligned}$$

so, we can obtain

$$\|\phi_m(\cdot) - \phi(\cdot)\| \leq (C_{10})^{\frac{p}{p+1}} E^{\frac{1}{p+2}} \delta^{\frac{p}{p+1}}, \tag{4.17}$$

where $C_{10} = \left(\frac{1}{C_2 \left(\frac{E_{\alpha,1}(-\lambda_1 T_1^\alpha)}{E_{\alpha,1}(-\lambda_1 T_2^\alpha)} - 1 \right)} \right) \left[1 + \left(\frac{1}{1 - E_{\alpha,1}(-\lambda_1 T_1^\alpha)} \right) + \tau_2 \right]$.

By (4.13) and Lemma 4.2,

$$\|\phi_m(\cdot) - \phi_m^\delta(\cdot)\| \leq \frac{2}{\lambda_1} \sqrt{am}\delta \leq C_{11} E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}, \tag{4.18}$$

where $C_{11} = \frac{2}{\lambda_1} \left(\frac{C_7}{\tau_2 - 2} \right)^{\frac{1}{p+1}}$.

Thus, from (4.17) and (4.18), we have

$$\|\phi_m^\delta(\cdot) - \phi(\cdot)\| \leq \left((C_{10})^{\frac{p}{p+1}} + C_{11} \right) E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}.$$

$\square$

5. Numerical examples

In this section, we use Matlab software to give several numerical examples to verify the effectiveness of the fractional Landweber iterative regularization method, and the specific algorithm is as follows.

Algorithm source term $f_m^\delta(x)$ and initial data $\phi_m^\delta(x)$

Input: $\alpha, \beta, \gamma, T_1, T_2, k, m, b, \varepsilon$.
Output: $f_m^\delta, \phi_m^\delta$.
1: $E_3 \Leftarrow \frac{E_{\alpha,1}(-\lambda_k T_1^\alpha) - E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k}$.
2: for $i = 1 : length(k), j = 1 : b$, do
3: $K_{11}(i,j) \Leftarrow (\frac{1 - E_{\alpha,1}(-\lambda_k T_1^\alpha)}{j\pi})^2 \cdot \sqrt{2}\sin(j\pi x(i)); K_{12}, K_{21}, K_{22}$.
4: $f_k \Leftarrow \frac{1}{b-1} \cdot f \cdot \sqrt{2}\sin(j\pi x(i)), \phi_k \Leftarrow \frac{1}{b-1} \cdot \phi \cdot \sqrt{2}\sin(j\pi x(i))$.
5: $g_1 \Leftarrow f_k \cdot K_{11} + \phi_k \cdot K_{21}$, $g_2 \Leftarrow \phi_k \cdot K_{12} + \phi_k \cdot K_{22}$.
6: $g_1^\delta \Leftarrow g_1 + \varepsilon \cdot (2 \cdot randn(size(g_1) - 1))$, $g_2^\delta \Leftarrow g_2 + \varepsilon \cdot (2 \cdot randn(size(g_2) - 1))$.
7: $h_1 \Leftarrow trapz(x \cdot g_2^(\delta) \cdot K_{21} - g_1^\delta \cdot K_{22})$, $h_2 \Leftarrow trapz(x \cdot g_2^(\delta) \cdot K_{11} - g_1^\delta \cdot K_{12})$.
8: end for
$f_m^\delta \Leftarrow f + (1 - (1 - a \cdot E_3^2)^m)^\gamma \cdot \frac{h_1}{E_3} \cdot \sqrt{2}\sin(j\pi x(i))$,
$\phi_m^\delta \Leftarrow \phi + (1 - (1 - a \cdot E_3^2)^m)^\gamma \cdot \frac{h_2}{E_3} \cdot \sqrt{2}\sin(j\pi x(i))$.

And we give there one-dimensional and two two-dimensional numerical examples. The main objective of the paper is to analyze the numerical error of the fractional Landweber method for solving the simultaneous inversion problem. In the following two one-dimensional examples, when computing the solution to the regularized solution, we take a truncation number of 15, and truncate 100 and 50 items for x. We then perform computational analysis using the fractional Landweber iterations.

The noisy data are generated by adding a random perturbation, i.e.

$$g_i^\delta = g_i + \varepsilon \cdot g_i \cdot (2randn(size(g_i)) - 1). \quad (i = 1, 2.)$$

In our calculations, for the one-dimensional case, the observation times are selected as $T_1 = 1/2$ and $T_2 = 1$, and m represents the number of iteration steps in all figures. To measure the accuracy of numerical solution, we use the discrete L^2 error as

$$E\,(f, \varepsilon) = \|f_m^\delta(\cdot) - f(\cdot)\|,$$
$$E\,(\phi, \varepsilon) = \|\phi_m^\delta(\cdot) - \phi(\cdot)\|,$$

and the relative error in $L^2(\Omega)$ norm denoted by

$$\varepsilon_f = \|f_m^\delta(\cdot) - f(\cdot)\|/\|f(\cdot)\|,$$
$$\varepsilon_\phi = \|\phi_m^\delta(\cdot) - \phi(\cdot)\|/\|\phi(\cdot)\|,$$

where δ is the noisy level.

Example 1. Consider the smooth heat source and initial value:

$$f(x) = 2\sin x,$$

and

$$\phi(x) = \sin x.$$

In this example, the initial value and the source term are sine functions. Then we give an exact solution via $\lambda_k = k^2$ and $\varphi_k = \sqrt{2/\pi}\sin(kx)$,

$$u(x, t) = (2 - E_{\alpha,1}(-t^\alpha))\sin x.$$

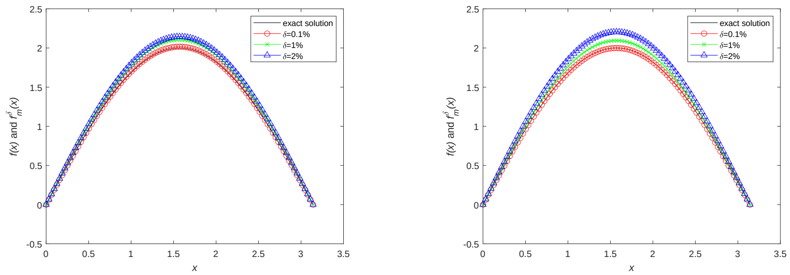
The reconstructed solutions $\phi(x)$ and $f(x)$ with exact input data are shown in Figs. 1-2, where $\alpha = 0.7, \beta = 1$. When $\delta = 0.1\%$, $\delta = 1\%$ and $\delta = 2\%$, we consider input data with different noise levels to test the stability of our algorithm. The reconstructed solutions under the a-priori and the a-posteriori conditions are shown in Fig. 1 and Fig. 2, where a satisfactory estimated solutions was obtained using the noise data. From Fig. 1 and Fig. 2, it can be seen that the data error has little effect on the source term and the initial value.

Example 2. Consider the respective smooth heat source and the non-smooth initial value:

$$f(x) = 2x(1 - x),$$

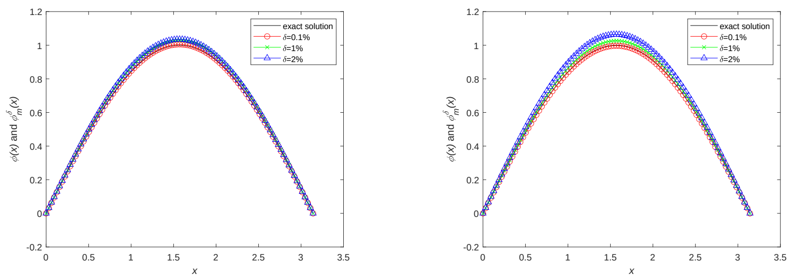
and

$$\phi(x) = \begin{cases} x, & 0 \le x < \frac{1}{2}, \\ (1 - x), & \frac{1}{2} \le x \le 1. \end{cases}$$



(a) a-priori (b) a-posteriori

Figure 1. Numerical results for source term in Example 1



(a) a-priori (b) a-posteriori

Figure 2. Numerical results for initial term in Example 1

In this case, it is easy to see that the source term is a quadratic function, and the initial value is a continuous but not smooth function at $x = 1/2$. Because there is a sharp point at $x = 1/2$, it is usually difficult to reconstruct.

The exact solution and reconstructed solution of $f(x)$ and $\phi(x)$ are obtained from the noisy data $g_1^\delta(x)$ and $g_2^\delta(x)$ under a-priori and a-posteriori conditions, as shown in Fig. 3 and Fig. 4, where $\alpha = 0.9$, $\beta = 1$, and the noise level δ is taken as 0.1%, 0.2% and 1% respectively. We can see that the under the a-priori condition and a-posteriori conditions, the heat source recovers very well, but the shape of the unknown initial temperature value does not recovered well, and the given results are reasonable considering the non-smooth and the ill-posedness of the problem.

Example 3. Consider the respective non-smooth heat source and the smooth initial value:

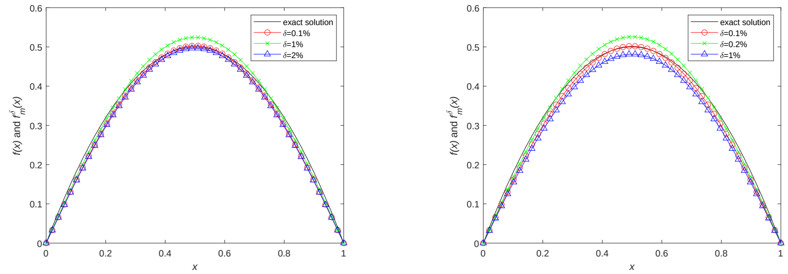
$$f(x) = \begin{cases} x, & 0 \le x < \frac{1}{2}, \\ (1-x), & \frac{1}{2} \le x \le 1, \end{cases}$$

and

$$\phi(x) = 2x(1-x).$$

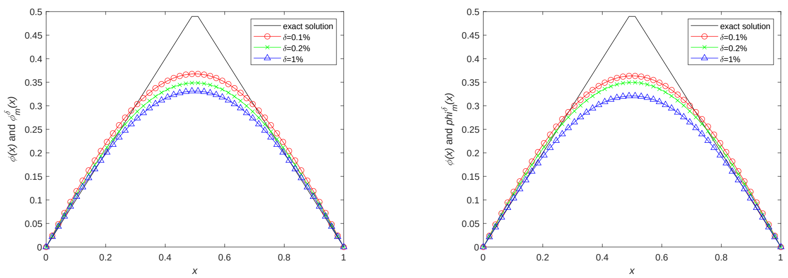
Example 4. Choose the exact source term and initial data

$$f(x,y) = e^{2-x-y}\sin(\pi x)\sin(\pi y),$$



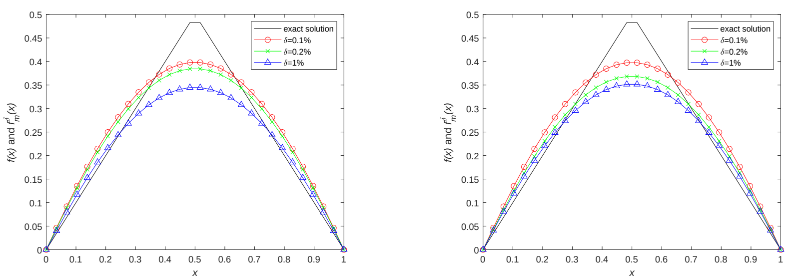
(a) a-priori (b) a-posteriori

Figure 3. Numerical results for source term in Example 2



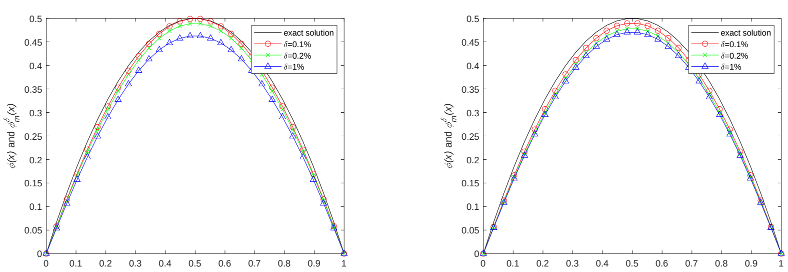
(a) a-priori (b) a-posteriori

Figure 4. Numerical results for initial term in Example 2



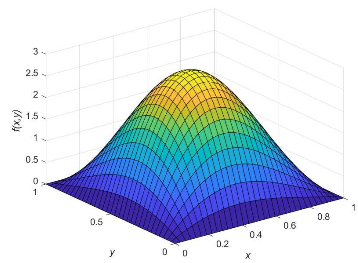
(a) a-priori (b) a-posteriori

Figure 5. Numerical results for source term in Example 3

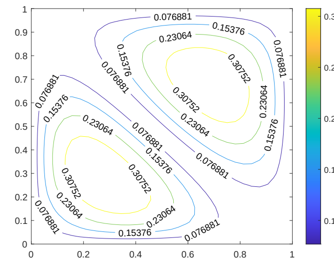


(a) a-priori (b) a-posteriori

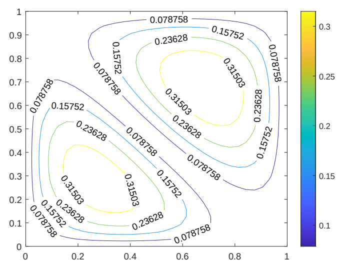
Figure 6. Numerical results for initial term in Example 3



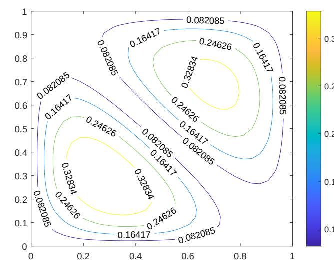
(a) exact solution



(b) error with $\delta = 1\%$



(c) error with $\delta = 2\%$



(d) error with $\delta = 5\%$

Figure 7. Numerical results for source term in Example 4

and

$$\phi(x, y) = xy(1 - x)(1 - y)e^{4-x-y}.$$

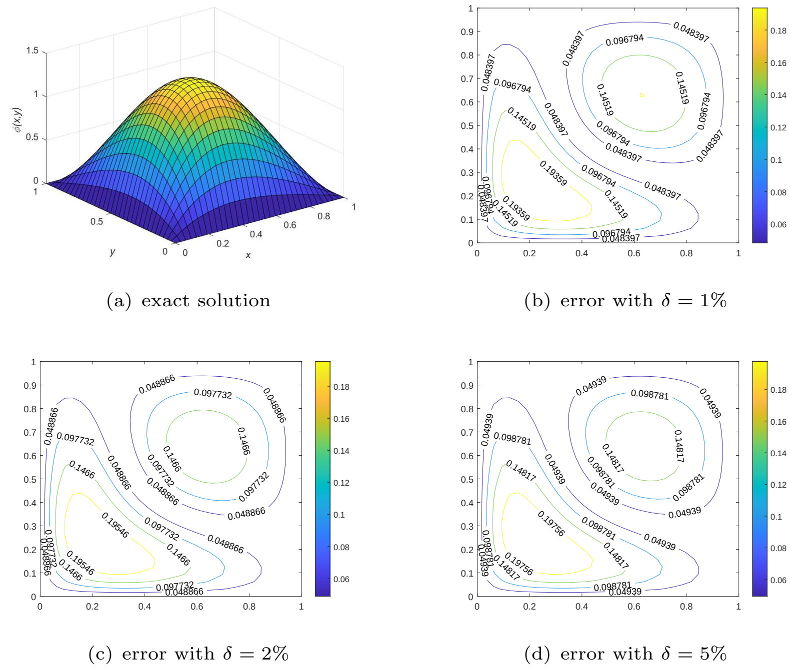
In this example, we truncate 32 terms for x and y respectively, 80 terms for the sum term k. Fig. 7 and Fig. 8 present the exact solution and the regularization solution for various noise levels $\delta = 1\%,\ 2\%,\ 5\%$ in case of $\alpha = 0.8$, $T_1 = 1/2$, $T_2 = 1$.

Example 5. Choose the exact source term and initial data $f(x, y) = 16xy(1 - x)(1 - y)$, $\phi(x, y) = \sin(\pi x)\sin(\pi y)$, respectively. The results are shown in Figs. 9 and 10 for the relative noise levels 1%, 2% and 5%.

Based on the numerical experiments of Example 1−5, we can see that the smaller δ, the better the fitting effect between the exact solution and the regularization solution. Moveover, one-dimensional cases and two-dimensional numerical examples verify the effectiveness and accuracy of the proposed method.

6. Conclusion

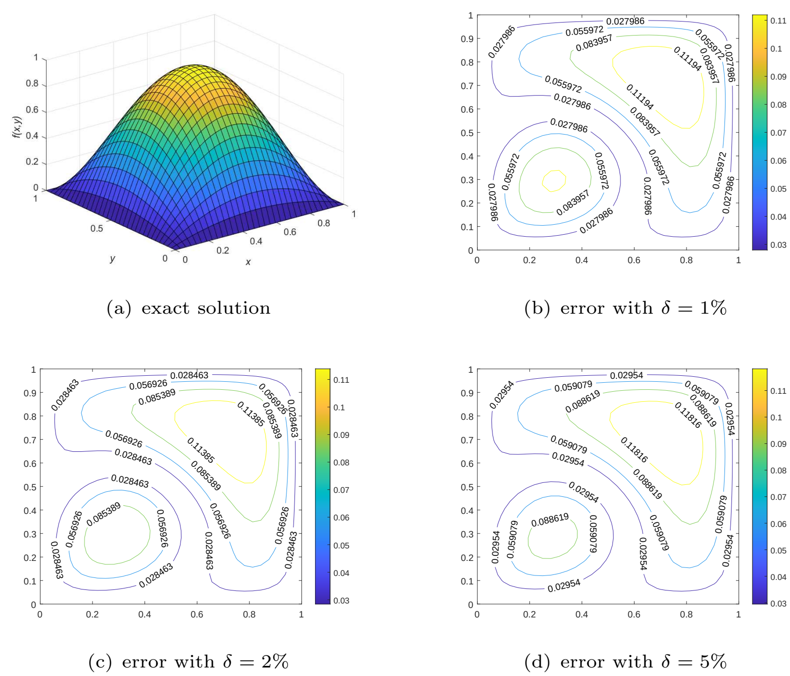
The inverse problem investigated in this paper involves simultaneously identifying the time-independent source function and initial data for a time-fractional diffusion equation. We obtained measurements from the additional temperature of the observation time at both terminals. Based on the Fourier method, the inverse problem was reformulated as a first kind of operator equations. The fractional Landweber iteration regularization method was used to construct the solutions of the proposed inverse problem by decoupling the operator equations. The error estimates between



(a) exact solution (b) error with $\delta = 1\%$

(c) error with $\delta = 2\%$ (d) error with $\delta = 5\%$

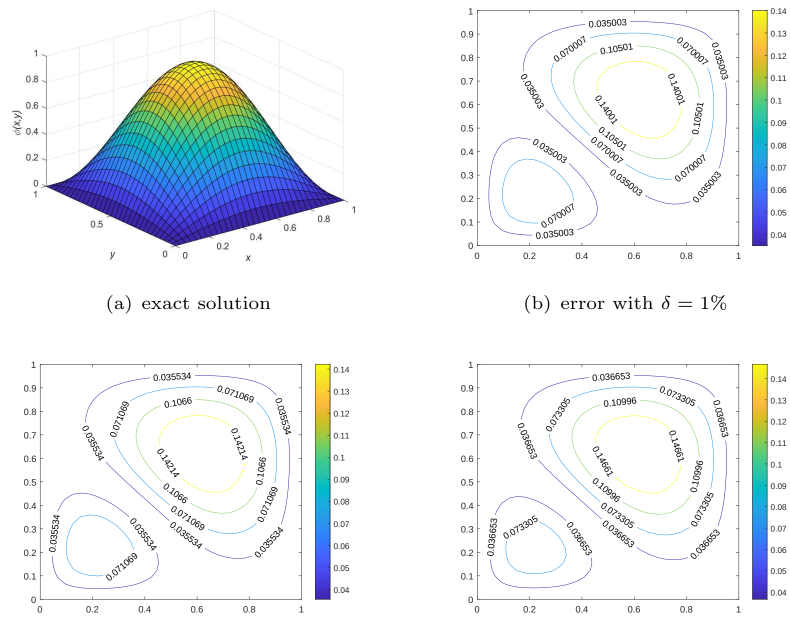
Figure 8. Numerical results for source term in Example 4



(a) exact solution (b) error with $\delta = 1\%$

(c) error with $\delta = 2\%$ (d) error with $\delta = 5\%$

Figure 9. Numerical results for source term in Example 5



(a) exact solution

(b) error with $\delta = 1\%$

(c) error with $\delta = 2\%$

(d) error with $\delta = 5\%$

Figure 10. Numerical results for initial term in Example 5

the exact solution and the regularization solution were given. The numerical experiments for several examples showed the effectiveness and accuracy of our proposed method.

This article only considered the case of $0 < \alpha < 1$, and the case of $1 < \alpha < 2$ can be considered later, which will be our future work.

References

[1] O. P. Agrawal, *Solution for a fractional diffusion-wave equation defined in a bounded domain*, Fractional Order Calculus and its Applications, 2002, 145–155.

[2] A. Babaei, *Solving a time-fractional inverse heat conduction problem with an unknown nonlinear boundary condition*, J. Math. Model., 2019, 7(1), 85–106.

[3] B. Berkowitz, H. Scher and S. E. Silliman, *Anomalous transport in laboratory-scale, heterogeneous porous media*, Water Resources Research, 2000, 36(1), 149–158.

[4] A. Chen and C. Li, *Numerical solution of fractional diffusion-wave equation*, Numer. Funct. Anal. Optim., 2016, 37(1), 19–39.

[5] R. Courant and D. Hilbert, *Methods of mathematical physics - vol.1; vol.2*, New York Interscience Publication, 1953, 305(3–4), 121–132.

[6] R. Du, W. R. Cao and Z. Z. Sun, *A compact difference scheme for the fractional diffusion-wave equation*, Appl. Math. Model., 2010, 34(10), 2998–3007.

[7] H. Egger and A. Neubauer, *Preconditioning Landweber iteration in Hilbert scales*, Numer. Math., 2005, 101(4), 643–662.

[8] H. W. Engl, M. Hanke and A. Neubauer, *Regularization of inverse problems*, 375 of Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 1996.

[9] M. Giona, S. Cerbelli and H. E. Roman, *Fractional Diffusion Equation and Relaxation in Complex Viscoelastic Materials*, Physica A Statistical Mechanics and its Applications, 1992.

[10] Y. Han, X. Xiong and X. Xue, *A fractional Landweber method for solving backward time-fractional diffusion problem*, Comput. Math. Appl., 2019, 78(1), 81–91.

[11] Y. Hatano and N. Hatano, *Dispersive transport of ions in column experiments: An explanation of long-tailed profiles*, Water Resources Research, 1998, 34(5), 1027–1034.

[12] L. N. Huynh, Y. Zhou, D. O'Regan and N. H. Tuan, *Fractional Landweber method for an initial inverse problem for time-fractional wave equations*, Appl. Anal., 2021, 100(4), 860–878.

[13] H. Jiang, F. Liu, I. Turner and K. Burrage, *Analytical solutions for the multi-term time-fractional diffusion-wave/diffusion equations in a finite domain*, Comput. Math. Appl., 2012, 64(10), 3377–3388.

[14] M. Jiang and G. Wang, *Convergence studies on iterative algorithms for image reconstruction*, IEEE Transactions on Medical Imaging, 2003, 22(5), 569–579.

[15] B. T. Johansson and D. Lesnic, *A procedure for determining a spacewise dependent heat source and the initial temperature*, Applicable Analysis, 2008, 87(3), 265–276.

[16] E. Klann and R. Ramlau, *Regularization by fractional filter methods and data smoothing*, Inverse Problems, 2008, 24(2), 025018, 26.

[17] L. Landweber, *An iteration formula for Fredholm integral equations of the first kind*, Amer. J. Math., 1951, 73, 615–624.

[18] X. Li and C. Xu, *A space-time spectral method for the time fractional diffusion equation*, SIAM J. Numer. Anal., 2009, 47(3), 2108–2131.

[19] Y. Lin and C. Xu, *Finite difference/spectral approximations for the time-fractional diffusion equation*, J. Comput. Phys., 2007, 225(2), 1533–1552.

[20] A. K. Louis, *Inverse und Schlecht Gestellte Probleme*, Teubner Studienbücher Mathematik. [Teubner Mathematical Textbooks], B. G. Teubner, Stuttgart, 1989.

[21] F. Mainardi, *Fractional Diffusive Waves in Viscoelastic Solids*, 1995.

[22] R. Metzler and J. Klafter, *Boundary value problems for fractional diffusion equations*, Phys. A, 2000, 278(1–2), 107–125.

[23] A. Neubauer, *On Landweber iteration for nonlinear ill-posed problems in Hilbert scales*, Numer. Math., 2000, 85(2), 309–328.

[24] I. Podlubny, *Fractional Differential Equations*, 198 of Mathematics in Science and Engineering, Academic Press, Inc., San Diego, CA, 1999. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.

[25] C. Ren, X. Xu and S. Lu, *Regularization by projection for a backward problem of the time-fractional diffusion equation*, J. Inverse Ill-Posed Probl., 2014, 22(1), 121–139.

[26] H. E. Roman and P. A. Alemany, *Continuous-time random walks and the fractional diffusion equation*, J. Phys. A, 1994, 27(10), 3407–3410.

[27] Z. Ruan, J. Z. Yang and X. Lu, *Tikhonov regularisation method for simultaneous inversion of the source term and initial data in a time-fractional diffusion equation*, East Asian J. Appl. Math., 2015, 5(3), 273–300.

[28] Z. Ruan and S. Zhang, *Simultaneous inversion of time-dependent source term and fractional order for a time-fractional diffusion equation*, J. Comput. Appl. Math., 2020, 368, 112566, 15.

[29] K. Sakamoto and M. Yamamoto, *Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems*, J. Math. Anal. Appl., 2011, 382(1), 426–447.

[30] I. M. Sokolov and J. Klafter, *From diffusion to anomalous diffusion: a century after Einstein's Brownian motion*, Chaos, 2005, 15(2), 026103, 7.

[31] L. L. Sun, Y. S. Li and Y. Zhang, *Simultaneous inversion of the potential term and the fractional orders in a multi-term time-fractional diffusion equation*, Inverse Problems, 2021, 37(5), Paper No. 055007, 26.

[32] N. H. Tuan, L. N. Huynh, T. B. Ngoc and Y. Zhou, *On a backward problem for nonlinear fractional diffusion equations*, Appl. Math. Lett., 2019, 92, 76–84.

[33] G. M. Vainikko and A. Y. Veretennikov, *Iteration Procedures in Ill-Posed Problems*, Nauka, Moscow, 1986.

[34] J.-G. Wang and T. Wei, *An iterative method for backward time-fractional diffusion problem*, Numer. Methods Partial Differential Equations, 2014, 30(6), 2029–2041.

[35] J. Wen, Z.-X. Liu and S.-S. Wang, *Conjugate gradient method for simultaneous identification of the source term and initial data in a time-fractional diffusion equation*, Applied Mathematics in Science and Engineering, 2022, 30(1), 324–338.

[36] J. Wen, Z.-X. Liu, C.-W. Yue and S.-J. Wang, *Landweber iteration method for simultaneous inversion of the source term and initial data in a time-fractional diffusion equation*, J. Appl. Math. Comput., 2022, 68(5), 3219–3250.

[37] J. Wen, X.-J. Ren and S.-J. Wang, *Simultaneous determination of source term and initial value in the heat conduction problem by modified quasi-reversibility regularization method*, Numerical Heat Transfer, Part B: Fundamentals, 2022, 82(3–4), 112–127.

[38] J. Wen, C.-W. Yue, Z.-X. Liu and S.-J. Wang, *Fractional tikhonov regularization method for simultaneous inversion of the source term and initial data in a time-fractional diffusion equation*, Rocky Mountain Journal of Mathematics, 2023, 53(1), 249–273.

[39] X.-T. Xiong, X.-M. Xue and Z. Qian, *A modified iterative regularization method for ill-posed problems*, Appl. Numer. Math., 2017, 122, 108–128.

[40] F. Yang, J.-L. Fu, P. Fan and X.-X. Li, *Fractional Landweber iterative regularization method for identifying the unknown source of the time-fractional diffusion problem*, Acta Appl. Math., 2021, 175, Paper No. 13, 19.

[41] F. Yang, X. Liu, X.-X. Li and C.-Y. Ma, *Landweber iterative regularization method for identifying the unknown source of the time-fractional diffusion equation*, Adv. Difference Equ., 2017, Paper No. 388, 15.

[42] F. Yang, Y.-P. Ren, X.-X. Li and D.-G. Li, *Landweber iterative method for identifying a space-dependent source for the time-fractional diffusion equation*, Bound. Value Probl., 2017, Paper No. 163, 19.

[43] F. Yang, Y. Zhang and X.-X. Li, *Landweber iterative method for identifying the initial value problem of the time-space fractional diffusion-wave equation*, Numerical Algorithms, 2020, 83(4), 1509–1530.

[44] L. Yang, Z.-C. Deng and Y.-C. Hon, *Simultaneous identification of unknown initial temperature and heat source*, Dynam. Systems Appl., 2016, 25(4), 583–602.

[45] Y. Zhang, T. Wei and Y.-X. Zhang, *Simultaneous inversion of two initial values for a time-fractional diffusion-wave equation*, Numer. Methods Partial Differential Equations, 2021, 37(1), 24–43.

[46] Y. Zhou, J. Wei He, B. Ahmad and N. Huy Tuan, *Existence and regularity results of a backward problem for fractional diffusion equations*, Mathematical Methods in the Applied Sciences, 2019, 42(18), 6775–6790.