Journal of Applied Analysis and Computation
Volume 13, Number 6, December 2023, 3403–3421
Website:http://www.jaac-online.com
DOI:10.11948/20230070

EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR DOUBLE PHASE PROBLEM WITH INDEFINITE SINGULAR TERMS*

Yu Cheng[1], Baoyuan Shan[1] and Zhanbing Bai[1,†]

Abstract In this article, the solvability of a class of periodic boundary value problems with double phase operators and mixed singular terms is considered. By applying the continuation theorem of Manásevich-Mawhin and techniques of a prior estimates, some existence results of positive solutions are obtained. Several numerical examples are given to illustrate the main results.

Keywords Double phase operator, continuation theorem, periodic solution, singular term.

MSC(2010) 34B16, 34C25.

1. Introduction

For the past few years, lots of scholars have been studying the issues of differential equations with double phase operators, the existence and multiplicity of solutions have been widely studied. The double phase problem originated from nonlinear materials science. Zhikov found that the hardening properties of materials varied dramatically with the point known as the Lavrentiev's phenomenon. In order to provide the materials with strong anisotropy for the model, using the homogenization theory, they studied the following functional in [24-27]:

$$x \mapsto \int_{\Omega} \left(|\nabla x|^p + a(t)|\nabla x|^q\right) dt,$$

where, in general, $1 < p < q < N$ and $a(\cdot) \geq 0$. The modulation coefficient $a(t)$ determines the geometry of the composite material consisting of two different materials with hardening exponents p and q, respectively. Since then, double phase problems have been studied extensively using variational methods and regularity theory (see, e.g. [1, 2, 4, 15, 16, 21]).

In [21], Perera and Squassina proved the existence of solutions to the following

†The corresponding author.
[1]College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China
*The authors were supported by Natural Science Foundation of China (No. 12371173), Natural Science Foundation of Shandong Province (No. ZR2021MA064), and the Taishan Scholar Project.
Email: chengyu9660@163.com(Y. Cheng), xiaoxvyuan@outlook.com(B. Shan), zhanbingbai@163.com(Z. Bai)

double phase problem:

$$\begin{cases} -\operatorname{div}\left(|\nabla x|^{p-2}\nabla x + a(t)|\nabla x|^{q-2}\nabla x\right) = f(t,x), \text{ in } \Omega, \\ x = 0, \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

They used a cohomological local split to solve an estimate of the critical group at zero. In [2], Colasuonno and Squassina studied an eigenvalue problem in the framework of double phase variational integration, introduced nonlinear eigenvalue sequences through the minimax method, and established the continuity results for nonlinear eigenvalues to phase changes. Using variational methods, Liu and Dai [15-16] discussed the existence and multiplicity of solutions to the equation with double phase operators, and considered the case of sign-changing solutions.

In fact, in the past few years, due to its strong application background in physics and science (see [5, 22] and the reference therein), periodic problems with singular terms have also been received extensive attention from scholars (see [3, 6-14, 17-18, 20]). The first study to periodic solutions of the second-order differential equation with singular terms was published by Nagumo in 1943 (see [20]). After that, in 1987, Lazer and Solimini considered existence results to second order differential equations in [12], which is a landmark work in the field of periodic problems with singular terms. They discussed two equations as follows,

$$x''(t) + \frac{1}{x^{\zeta}(t)} = h(t),$$

$$x''(t) - \frac{1}{x^{\zeta}(t)} = h(t),$$

where $\zeta \in (0,\infty)$ is a constant, $h \in \mathbb{C}(\mathbb{R},\mathbb{R})$ with T-periodic. Using the upper and lower solution method, topological degree theory, and truncation function technique, they discussed the singularity of repulsive type and attractive type respectively, and considered the case under $\zeta \in (0,1)$ (which is called the strong force singularity) and $\zeta \geq 1$ (which is called the weak force singularity).

Lately, Hakl and Torres in [7] considered the existence results for second-order differential equations with both attractive and repulsive singular terms as follows:

$$\begin{cases} x''(t) = \dfrac{h(t)}{x^{\mu}(t)} - \dfrac{g(t)}{x^{\lambda}(t)} + f(t), \text{ a.e. } t \in [0,T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

where $h, g \in L(\mathbb{R},\mathbb{R}^+)$, $f \in L(\mathbb{R},\mathbb{R})$ are T-periodic functions, constants $\mu, \lambda > 0$. By using upper and lower solution method, they obtained the existence results of differential equations with mixed singular terms.

In [11], Jebelean and Mawhin considered the existence of periodic solutions with the p-Laplacian and singular term as follows:

$$\begin{cases} (\phi_p(x'))' + f(x)x' + g(x) = h(t), \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

with $\phi_p(s) = |s|^{p-2}s, \quad p > 1$, g singular at 0. They used upper and lower solution method and continuation theorem, extended results of Lazer and Solimini [12].

In [18], Lu and Yu studied the following equations with mixed singular terms:

$$\begin{cases} x''(t) + f(x(t))x'(t) + \sigma(t)x^m(t) - \dfrac{\zeta(t)}{x^\mu(t)} + \dfrac{\beta(t)}{x^\xi(t)} = 0, \\ x(0) = x(T),\ x'(0) = x'(T), \end{cases}$$

where $f \in C((0,\infty),\mathbb{R})$, σ, ζ, β are T-periodic functions, m, μ, ξ are constants. Using the continuation theorem of Manásevich and Mawhin and techniques of a priori estimates, the existence of the solution of the problem was given.

Inspired by the above research, this work discusses the periodic solution of the following double phase problem:

$$\begin{cases} (\phi_p(x'(t)))' + \big(a \cdot \phi_q(x'(t))\big)' + g(x(t))x'(t) + \sigma(t)x^v(t) - \dfrac{\xi(t)}{x^\mu(t)} + \dfrac{\zeta(t)}{x^m(t)} = 0, \\ x(0) = x(T),\ x'(0) = x'(T), \end{cases} \tag{1.2}$$

where $f \in C((0,+\infty),\mathbb{R})$, σ, ζ and ξ are T-periodic functions with $\zeta, \xi \in L([0,T],\mathbb{R})$, and $\sigma \in L(\mathbb{R},\mathbb{R}^+)$, $p \geq q > 1,\ a \geq 0,\ \mu > m > 0$ and $v > 0$ are constants.

The interesting points of this paper are as follows. Firstly, as far as we know, few articles discuss the existence of periodic solutions to ordinary differential equations with double phase operators with mixed singular nonlinear terms. Secondly, the nonlinear term $-\frac{\xi(t)}{x^\mu(t)} + \frac{\zeta(t)}{x^m(t)}$ includes attractive singular terms with strong or weak singularities, repulsive singular terms with strong or weak singularities, and mixed singular terms when $\xi(t) \neq 0$ and $\zeta(t) \neq 0$, and when $x = 0$, $g(x)$ may have a singularity. This is an extension and supplement to [11]. Thirdly, in the double phase problem, the prior estimation of the solution in this paper is more difficult than the p-Laplacian operator or the second order operator, especially the boundedness of $|x'(t)|$.

The structure of this work is organized as follows. In section 2, we give several inequalities to prove prior estimates, and some necessary lemmas for subsequent proofs. In section 3, we obtain the existence of solutions to the problem considered, which is the main result of our work. In section 4, some numerical examples are given to illustrate main results.

2. Preliminaries and lemmas

Throughout this article, let:

$$C_T = \{x \in C(\mathbb{R},\mathbb{R}):\ x(t+T) = x(t) \text{ for all } t \in \mathbb{R}\},$$
$$\|x\|_\infty = \max_{t\in[0,T]} |x(t)|,$$

and

$$C_T^1 = \{x \in C^1(\mathbb{R},\mathbb{R}):\ x(t+T) = x(t) \text{ for all } t \in \mathbb{R}\},$$
$$\|x\|_{C_T^1} = \max\{\|x\|_\infty, \|x'\|_\infty\}.$$

For $z \in L([0,T],\mathbb{R})$, define

$$z_+(t) = \max\{z(t), 0\}, \quad z_-(t) = -\min\{z(t), 0\}, \quad \text{and } \overline{z} = \frac{1}{T}\int_0^T z(t)dt.$$

Clearly, $z(t) = z_+(t) - z_-(t)$ for all $t \in \mathbb{R}$.

Lemma 2.1. *Suppose that there are three positive constants K_0, K_1, and K_2, with $0 < K_0 < K_1$, such that*
(H_1) *For all $\lambda \in (0, 1]$, any possible positive T-periodic solution x of*

$$(\phi_p(x'(t)))' + (a \cdot \phi_q(x'(t)))' + \lambda g(x(t))x'(t) + \lambda\sigma(t)x^v(t) - \lambda\frac{\xi(t)}{x^\mu(t)} + \lambda\frac{\zeta(t)}{x^m(t)} = 0,$$

satisfies $K_0 < x(t) < K_1$, and $|x'(t)| < K_2$, for any $t \in [0, T]$.
(H_2) *Any possible solution z of problem* (1.2) *satisfies*

$$\frac{\bar{\xi}}{z^\mu} - \frac{\bar{\zeta}}{z^m} - \overline{\sigma} z^v = 0.$$

(H_3) *The inequality*

$$\left(\frac{\bar{\xi}}{K_0^\mu} - \frac{\bar{\zeta}}{K_0^m} - \overline{\sigma} K_0^v\right)\left(\frac{\bar{\xi}}{K_1^\mu} - \frac{\bar{\zeta}}{K_1^m} - \overline{\sigma} K_1^v\right) < 0$$

holds.

Then, Problem (1.2) *has at least one T-periodic solution x, satisfies $K_0 < x(t) < K_1$, for any $t \in [0, T]$.*

Proof. Define

$$\phi(x) := \phi_p(x) + a \cdot \phi_q(x).$$

For each $x_1, x_2 \in \mathbb{R},\ x_1 \neq x_2$, we have

$$\begin{aligned}
&\langle \phi(x_1) - \phi(x_2), x_1 - x_2 \rangle \\
&= \left(|x_1|^{p-2}x_1 + a|x_1|^{q-2}x_1 - |x_2|^{p-2}x_2 - a|x_2|^{q-2}x_2\right)(x_1 - x_2) \\
&= \left[x_1(|x_1|^{p-2} + a|x_1|^{q-2}) - x_2(|x_2|^{p-2} + a|x_2|^{q-2})\right](x_1 - x_2) \\
&> 0,
\end{aligned}$$

and

$$\begin{aligned}
\langle \phi(x), x \rangle &= \left(|x|^{p-2}x + a|x|^{q-2}x\right) \cdot x \\
&= |x|^p + a|x|^q \\
&= \left(|x|^{p-1} + a|x|^{q-1}\right) \cdot |x|.
\end{aligned}$$

So ϕ is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$, and $|\phi^{-1}(x)| \to +\infty$ as $x \to +\infty$ (see [3]). Therefore, using Theorem 3.1 in [19], Problem (1.2) has at least one T-periodic solution. □

Remark 2.1. [18] If $\bar{\xi},\ \bar{\zeta},\ \sigma > 0$, then we have $\mu > m > 0,\ v > 0$, there are two constants $0 < E_1 < E_2$ such that

$$\frac{\bar{\xi}}{x^\mu} - \frac{\bar{\zeta}}{x^m} - \bar{\sigma} x^v > 0,\ \text{for all } x \in (0, E_1),$$

and

$$\frac{\bar{\xi}}{x^\mu} - \frac{\bar{\zeta}}{x^m} - \bar{\sigma} x^v < 0,\ \text{for all } x \in (E_2, \infty).$$

Lemma 2.2. *Suppose that $x \in AC([0,T],\mathbb{R})$, satisfied $x(0) = x(T)$, and $k > 1$. Then the following inequality hold:*

$$M - m \le \left(\frac{T}{2}\right)^{\frac{k-1}{k}} \left(\int_0^T |x'(t)|^k \, dt\right)^{\frac{1}{k}},$$

where

$$M = \max\{x(t) : t \in [0,T]\}, \qquad m = \min\{x(t) : t \in [0,T]\}.$$

Proof. Let

$$\widetilde{x}(t) := \begin{cases} x(t), & \text{for } t \in [0,T]; \\ x(t-T), & \text{for } t \in (T, 2T]. \end{cases}$$

Then, $\widetilde{x}(t) \in AC([0,2T],\mathbb{R})$. There exist $t_0 \in [0,T]$, $t_1 \in (t_0, t_0+T)$, satisfied

$$\widetilde{x}(t_0) = m, \quad \widetilde{x}(t_1) = M, \quad \widetilde{x}(t_0+T) = m.$$

Therefore,

$$\begin{aligned} M - m &= \int_{t_0}^{t_1} \widetilde{x}'(s) ds, \\ m - M &= \int_{t_1}^{t_0+T} \widetilde{x}'(s) ds. \end{aligned}$$

By Hölder inequality, we get

$$M - m \le |t_1 - t_0|^{\frac{k-1}{k}} \left(\int_{t_0}^{t_1} |\widetilde{x}'(s)|^k \, ds\right)^{\frac{1}{k}}, \tag{2.1}$$

$$M - m \le |t_0 + T - t_1|^{\frac{k-1}{k}} \left(\int_{t_1}^{t_0+T} |\widetilde{x}'(s)|^k \, ds\right)^{\frac{1}{k}}. \tag{2.2}$$

By (2.1) and (2.2), we obtain

$$(M-m)^k \le |t_1 - t_0|^{k-1} \left(\int_{t_0}^{t_1} |\widetilde{x}'(s)|^k \, ds\right), \tag{2.3}$$

$$(M-m)^k \le |t_0 + T - t_1|^{k-1} \left(\int_{t_1}^{t_0+T} |\widetilde{x}'(s)|^k \, ds\right). \tag{2.4}$$

Using the inequality $AB \le \frac{1}{4}(A+B)^2$, for any $A, B \in \mathbb{R}$, by (2.3) and (2.4), we obtain

$$\begin{aligned} (M-m)^{2k} &\le (|t_1 - t_0| \cdot |t_0 + T - t_1|)^{p-1} \left(\int_{t_0}^{t_1} |\widetilde{x}'(s)|^k \, ds\right) \left(\int_{t_1}^{t_0+T} |\widetilde{x}'(s)|^k \, ds\right) \\ &\le \frac{1}{4} \left(|t_1 - t_0| + |t_0 + T - t_1|\right)^{2(k-1)} \int_{t_0}^{t_0+T} |\widetilde{x}'(s)|^k \, ds \\ &\le \frac{1}{4} \left(\frac{T}{2}\right)^{2(k-1)} \left(\int_{t_0}^{t_0+T} |x'(t)|^k \, ds\right)^2. \end{aligned}$$

Therefore

$$M - m \le \left(\frac{T}{2}\right)^{\frac{k-1}{k}} \left(\int_0^T |x'(t)|^k \, dt\right)^{\frac{1}{k}}.$$

The proof is complete. □

Now, we embed (1.2) into the following equation family with a parameter $\lambda \in (0,1]$,

$$\begin{aligned} &(\phi_p(x'(t)))' + (a \cdot (\phi_q x'(t)))' + \lambda g(x(t))x'(t) + \lambda\sigma(t)x^v(t) \\ &- \lambda\frac{\xi(t)}{x^\mu(t)} + \lambda\frac{\zeta(t)}{x^m(t)} = 0. \end{aligned} \tag{2.5}$$

Let

$$\begin{aligned} E = \Big\{ &x \in C_T^1 : (\phi_p(x'(t)))' + (a \cdot \phi_q(x'(t)))' + \lambda g(x(t))x'(t) + \lambda\sigma(t)x^v(t) \\ &-\lambda\frac{\xi(t)}{x^\mu(t)} + \lambda\frac{\zeta(t)}{x^m(t)} = 0, \lambda \in (0,1];\ x(t) > 0,\ \forall t \in [0,T] \Big\}, \end{aligned} \tag{2.6}$$

$$G(t) = \int_1^t g(s)ds, \quad H(t) = \int_1^t s^{\mu+m} g(s)ds, \ t \in (0,+\infty), \tag{2.7}$$

$$\xi^* := \max_{t\in[0,T]} \xi(t), \quad \xi_* := \min_{t\in[0,T]} \xi(t),$$
$$\zeta^* := \max_{t\in[0,T]} \zeta(t), \quad \zeta_* := \min_{t\in[0,T]} \zeta(t),$$
$$\sigma^* := \max_{t\in[0,T]} \sigma(t), \quad \sigma_* := \min_{t\in[0,T]} \sigma(t).$$

Lemma 2.3. *Assume that $\mu > m > 0, v > 0$, and suppose the following conditions hold:*
(A_1) $g(t) > 0$ or $g(t) < 0$, for all $t \in \mathbb{R}$.
(A_2) $\sigma \in L([0,T],(0,+\infty))$, and $\xi^* > \zeta_*$.
Then, for any $x \in E$, there exist constants $\eta_0, \eta_1 \in [0,T]$, such that

$$\begin{aligned} x(\eta_0) &\ge \max\left\{1, \left(\frac{\xi_* - \zeta^*}{\sigma^*}\right)^{\frac{1}{m+v}}\right\} := A_0, \\ x(\eta_1) &\le \min\left\{1, \left(\frac{\xi^* - \zeta_*}{\sigma_*}\right)^{\frac{1}{m+v}}\right\} := A_1. \end{aligned} \tag{2.8}$$

Proof. Because of $\int_0^T x'(t)dt = 0$, there exist two points $t_1, t_2 \in [0,T]$, satisfied

$$x'(t_1) \le 0 \quad \text{and} \quad x'(t_2) \ge 0.$$

Thus

$$\begin{aligned} \phi_p(x'(t_1)) \le 0 \quad &\text{and} \quad \phi_p(x'(t_2)) \ge 0, \\ \phi_q(x'(t_1)) \le 0 \quad &\text{and} \quad \phi_q(x'(t_2)) \ge 0. \end{aligned}$$

Let $\bar{t}, \underline{t} \in [0,T]$ be the maximum and minimum points of the (p,q)-Laplacian term $\phi_p(x'(t)) + a \cdot \phi_q(x'(t))$, the above inequalities imply

$$\begin{aligned} \phi_p(x'(\bar{t})) + a \cdot \phi_q(x'(\bar{t})) \ge 0 \quad &\text{and} \quad (\phi_p(x'(\bar{t})) + a \cdot (\phi_q(x'(\bar{t}))' = 0, \\ \phi_p(x'(\underline{t})) + a \cdot \phi_q(x'(\underline{t})) \le 0 \quad &\text{and} \quad (\phi_p(x'(\underline{t})) + a \cdot \phi_q(x'(\underline{t}))' = 0. \end{aligned} \tag{2.9}$$

Applying (2.9), we deduce

$$g(x(\bar{t}))x'(\bar{t}) + \sigma(\bar{t})x^v(\bar{t}) - \frac{\xi(\bar{t})}{x^\mu(\bar{t})} + \frac{\zeta(\bar{t})}{x^m(\bar{t})} = 0,$$
$$g(x(\underline{t}))x'(\underline{t}) + \sigma(\underline{t})x^v(\underline{t}) - \frac{\xi(\underline{t})}{x^\mu(\underline{t})} + \frac{\zeta(\underline{t})}{x^m(\underline{t})} = 0.$$

By condition (A_1), we suppose that $g(x) > 0$, for any $x \in \mathbb{R}$. By (2.9), we obtain $x'(\bar{t}) \geq 0$ and $x'(\underline{t}) \leq 0$. It follows that

$$\sigma(\bar{t})x^{v+\mu}(\bar{t}) + \zeta(\bar{t})x^{\mu-m}(\bar{t}) \leq \xi(\bar{t}), \tag{2.10}$$

and

$$\sigma(\underline{t})x^{v+\mu}(\underline{t}) + \zeta(\underline{t})x^{\mu-m}(\underline{t}) \geq \xi(\underline{t}). \tag{2.11}$$

It means that

$$\sigma_* x^{v+\mu}(\bar{t}) + \zeta_* x^{\mu-m}(\bar{t}) < \xi^*,$$

and

$$x^{\mu-m}(\bar{t})(\sigma_* x^{m+v}(\bar{t}) + \zeta_*) < \xi^*.$$

So, by (2.10), there is a constant η_1,

$$x(\eta_1) < \min\left\{1, \left(\frac{\xi^* - \zeta_*}{\sigma_*}\right)^{\frac{1}{m+v}}\right\} := A_1.$$

Similarly, using (2.11), there is a constant η_0

$$x(\eta_0) < \max\left\{1, \left(\frac{\xi_* - \zeta^*}{\sigma^*}\right)^{\frac{1}{m+\mu}}\right\} := A_0.$$

Then (2.8) is proved. □

Lemma 2.4. *Assume (A_1), (A_2) hold, suppose that*

$$B_0 := \inf_{x \in [A_0, +\infty)} [G(x) - T\bar{\sigma}x^v] > -\infty \tag{2.12}$$

and

$$\lim_{s \to 0^+} \left(G(s) + \frac{T\overline{\xi_+}}{s^m} + \frac{T\overline{\zeta_+}}{s^\mu} \right) < B_0 \tag{2.13}$$

hold, where $G(x)$, A_0, μ are defined by (2.7), (2.8) and (1.2). Then there exists a positive constant r_0, satisfied

$$\min_{t \in [0,T]} x(t) \geq r_0, \text{ uniformly for } x \in E.$$

Proof. Let $x \in E$, then $x(t)$ satisfies

$$(\phi_p(x'(t)))' + (a \cdot \phi_q(x'(t)))' + \lambda g(x(t))x'(t) + \lambda\sigma(t)x^v(t) - \lambda\frac{\xi(t)}{x^\mu(t)}$$
$$+ \lambda\frac{\zeta(t)}{x^m(t)} = 0, \quad \lambda \in (0, 1].$$

Since $x \in E$, there exist $t_1, t_2 \in \mathbb{R}$ with $0 < t_2 - t_1 < T$, satisfied

$$x(t_1) = \max_{t\in[0,T]} x(t)$$

and

$$x(t_2) = \min_{t\in[0,T]} x(t).$$

Due to Lemma 2.3,

$$A_0 \le x(t_1) < +\infty,$$

by assumption (2.12) yields

$$G(x(t_1)) - T\bar{\sigma}x^v(t_1) \ge \inf_{A_0\le s<+\infty}[G(s) - T\bar{\sigma}x^v(s)] := C_0 > -\infty. \tag{2.14}$$

Integrating (2.5) over $[t_1, t_2]$, we obtain

$$\int_{t_1}^{t_2} (\phi_p(x'(t)))'\,dt + \int_{t_1}^{t_2} a\,(\phi_q(x'(t)))'\,dt + \lambda\int_{t_1}^{t_2} g(x(t))x'(t)dt + \lambda\int_{t_1}^{t_2} \sigma(t)x^v(t)dt$$
$$- \lambda\int_{t_1}^{t_2} \frac{\xi(t)}{x^\mu(t)}dt + \lambda\int_{t_1}^{t_2} \frac{\zeta(t)}{x^m(t)}dt = 0.$$

By the definition of t_1, t_2, we obtain

$$\int_{t_1}^{t_2} (\phi_p(x'(t)))'\,dt + \int_{t_1}^{t_2} a\,(\phi_q(x'(t)))'\,dt = 0.$$

Therefore,

$$\int_{t_1}^{t_2} g(x(t))x'(t)dt + \int_{t_1}^{t_2} \sigma(t)x^v(t)dt - \int_{t_1}^{t_2} \frac{\xi(t)}{x^\mu(t)}dt + \int_{t_1}^{t_2} \frac{\zeta(t)}{x^m(t)}dt = 0. \tag{2.15}$$

It follows from (2.14) and (2.15)

$$\begin{aligned}
G(x(t_2)) &= G(x(t_1)) - \int_{t_1}^{t_2} \sigma(t)x^v(t)dt + \int_{t_1}^{t_2} \frac{\xi(t)}{x^\mu(t)}dt - \int_{t_1}^{t_2} \frac{\zeta(t)}{x^m(t)}dt \\
&\ge G(x(t_1)) - \int_0^T \sigma(t)x^v(t)dt - \int_0^T \frac{\xi_+(t)}{x^\mu(t)}dt - \int_0^T \frac{\zeta_+(t)}{x^m(t)}dt \\
&\ge G(x(t_1)) - T\bar{\sigma}x^v(t_1) - \int_0^T \frac{\xi_+(t)}{x^\mu(t)}dt - \int_0^T \frac{\zeta_+(t)}{x^m(t)}dt \\
&\ge C_0 - \frac{T\overline{\xi_+}}{x^\mu(t_2)} - \frac{T\overline{\zeta_+}}{x^m(t_2)},
\end{aligned}$$

and then

$$G(x(t_2)) + \frac{T\overline{\xi_+}}{x^\mu(t_2)} + \frac{T\overline{\zeta_+}}{x^m(t_2)} \ge C_0. \tag{2.16}$$

Combining with (2.13), there is a constant $r_0 > 0$, satisfied

$$G(s) + \frac{T\overline{\xi_+}}{s^\mu} + \frac{T\overline{\zeta_+}}{s^m} < C_0, \text{ for all } s \in (0, r_0). \tag{2.17}$$

Due to (2.16) and (2.17), we get

$$\min_{t\in[0,T]} x(t) = x(t_2) \ge r_0.$$

□

Lemma 2.5. *Suppose that* (A_1), (A_2) *hold, and* $\bar{\zeta} > 0$ *for a.e.* $t \in [0,T]$. *The following hypotheses*

$$C_0 := \sup_{x\in[A_0,+\infty)} [G(x) + T\overline{\sigma}x^v] < +\infty, \tag{2.18}$$

and

$$\lim_{s\to 0^+} \left(G(s) - \frac{T\overline{\xi_+}}{s^\mu} - \frac{T\overline{\zeta_-}}{s^m} \right) > C_0, \text{ for all } s \in (0, r_1) \tag{2.19}$$

hold. Then there is a positive constant r_1, *satisfied*

$$\min_{t\in[0,T]} x(t) \geq r_1, \text{ uniformly for } x \in E.$$

Proof. Let $x(t) \in E$, then $x(t)$ satisfies (2.5). By the definition of t_1 and t_2 in Lemma 2.4, from (2.8), we get

$$A_0 \leq x(t_1) < +\infty,$$

combined with the assumption (2.18), yields that

$$G(x(t_1)) + T\bar{\sigma}x^v(t_1) \leq \sup_{A_0\leq s<+\infty} [G(s) + T\bar{\sigma}s^v] := B_0. \tag{2.20}$$

Similarly with Lemma 2.4, we obtain

$$\int_{t_1}^{t_2} (\phi_p(x'(t)))'\, dt + \int_{t_1}^{t_2} (a\phi_q(x'(t)))'\, dt = 0.$$

Integrating (2.5) over $[t_1, t_2]$, we obtain

$$G(x(t_2)) - G(x(t_1)) = -\int_{t_1}^{t_2} \sigma(t)x^v(t)dt + \int_{t_1}^{t_2} \frac{\xi(t)}{x^\mu(t)}dt - \int_{t_1}^{t_2} \frac{\zeta(t)}{x^m(t)}dt.$$

By $\bar{\zeta} > 0$ and (2.20), we obtain

$$\begin{aligned} G(x(t_2)) &\leq G(x(t_1)) - \int_0^T \sigma(t)x^v(t)dt + \int_0^T \frac{\xi_+(t)}{x^\mu(t)}dt + \int_0^T \frac{\zeta_-(t)}{x^m(t)}dt \\ &\leq G(x(t_1)) + T\overline{\sigma}x^v(t_1) + \frac{T\overline{\xi_+}}{x^\mu(t)} + \frac{T\overline{\zeta_-}}{x^m(t)} \\ &\leq B_0 + \frac{T\overline{\xi_+}}{x^\mu(t)} + \frac{T\overline{\zeta_-}}{x^m(t)}, \end{aligned}$$

i.e.,

$$G(x(t_2)) - \frac{T\overline{\xi_+}}{x^\mu(t_2)} - \frac{T\overline{\zeta_-}}{x^m(t_2)} \leq B_0. \tag{2.21}$$

From assumption (2.19), there is a positive constant $r_1 > 0$, satisfied

$$G(s) - \frac{T\overline{\xi_+}}{s^\mu} - \frac{T\overline{\zeta_-}}{s^m} > B_0, \text{ for all } s \in (0, r_1),$$

due to (2.21), we get

$$\min_{t\in[0,T]} x(t) = x(t_2) \geq r_1.$$

□

3. Existence results

Theorem 3.1. *Suppose $p \geq q > 1,\ a \geq 0,\ \sigma > 0$, and the hypotheses (A_1), (A_2), (2.12) and (2.13) hold. Then for any $\mu > m > 0$ and $v > 0$, Problem (1.2) has at least one positive T-periodic solution.*

Proof. By Lemma 2.4, there exists a positive constant $r_0 > 0$, satisfied

$$\min_{t\in[0,T]} x(t) \geq r_0,\ \text{uniformly for } x \in E, \tag{3.1}$$

where E is defined by (2.6). Then, we will prove that there are constants $M_1, M_2 > 0$, satisfied

$$\max_{t\in[0,T]} x(t) \leq M_1,\quad \max_{t\in[0,T]} |x'(t)| \leq M_2,\ \text{uniformly for } x \in E.$$

Since $x \in E$, we get

$$\begin{aligned}&(\phi_p(x'(t)))' + (a\cdot\phi_q(x'(t)))' + \lambda g(x(t))x'(t) + \lambda\sigma(t)x^v(t)\\ &-\lambda\frac{\xi(t)}{x^\mu(t)} + \lambda\frac{\zeta(t)}{x^m(t)} = 0,\ \lambda\in(0,1].\end{aligned} \tag{3.2}$$

Integrating it on $[0,T]$, we get

$$\int_0^T \sigma(t)x^v(t)dt = \int_0^T \frac{\xi(t)}{x^\mu(t)}dt - \int_0^T \frac{\zeta(t)}{x^m(t)}dt. \tag{3.3}$$

From Lemma 2.3, there exists $\eta_1 \in [0,T]$, satisfied

$$x(\eta_1) \leq A_1,$$

which together with Lemma 2.2, we get

$$|x|_\infty \leq A_1 + \left(\frac{T}{2}\right)^{\frac{p-1}{p}}\left(\int_0^T |x'(t)|^p\,dt\right)^{\frac{1}{p}}. \tag{3.4}$$

Multiplying (3.2) with $x(t)$, and integrating on $[0,T]$, we obtain

$$\begin{aligned}&-\int_0^T |x'(t)|^p dt - a\int_0^T |x'(t)|^q dt\\ =&-\lambda\int_0^T \sigma(t)x^{v+1}(t)dt + \int_0^T \frac{\xi(t)}{x^\mu(t)}x(t)dt - \lambda\int_0^T \frac{\zeta(t)}{x^m(t)}x(t)dt.\end{aligned}$$

From $\sigma(t) \geq 0$ for a.e. $t \in [0,T]$, and by (3.3) we have

$$\begin{aligned}
\int_0^T |x'(t)|^p dt &\leq \int_0^T \frac{\zeta(t)}{x^m(t)} x(t)dt - \int_0^T \frac{\xi(t)}{x^\mu(t)} x(t)dt + \int_0^T \sigma(t)x^{v+1}(t)dt \\
&\leq \int_0^T \frac{\zeta_+(t)}{x^m(t)} x(t)dt + \int_0^T \frac{\xi_+(t)}{x^\mu(t)} x(t)dt + \int_0^T \sigma(t)x^{v+1}(t)dt \\
&\leq |x|_\infty \left(\int_0^T \frac{\zeta_+(t)}{x^m(t)} dt + \int_0^T \frac{\xi_+(t)}{x^\mu(t)} dt + \int_0^T \sigma(t)x^v(t)dt \right) \\
&= |x|_\infty \left(\int_0^T \frac{\zeta_+(t)}{x^m(t)} dt + \int_0^T \frac{\xi_+(t)}{x^\mu(t)} dt + \int_0^T \frac{\xi(t)}{x^\mu(t)} dt - \lambda \int_0^T \frac{\zeta(t)}{x^m(t)dt} \right) \\
&\leq |x|_\infty \left(\int_0^T \frac{2\xi_+(t)}{x^\mu(t)} dt + \int_0^T \frac{\zeta_-(t)}{x^m(t)} dt \right),
\end{aligned}$$

with (3.1) and $\mu > m > 0$, yields that

$$\int_0^T |x'(t)|^p dt \leq \frac{T(\overline{\zeta_-} + 2\overline{\xi_+})}{r_0^m} |x|_\infty.$$

Combined with (3.4), we have

$$\int_0^T |x'(t)|^p \, dt \leq \frac{T(\overline{\zeta_-} + 2\overline{\xi_+})}{r_0^m} \left(A_1 + \left(\frac{T}{2}\right)^{\frac{p-1}{p}} \left(\int_0^T |x'(t)|^p \, dt \right)^{\frac{1}{p}} \right).$$

Using the inequality $X^2 - AX - B < 0$, there exists a constant $\rho > 0$ satisfied

$$\left(\int_0^T |x'(t)|^p \, ds \right)^{\frac{1}{p}} \leq \rho. \tag{3.5}$$

Obviously, in (3.5), $\rho > 0$ is independent of $x \in E$. Using (3.4), we get

$$\max_{t\in[0,T]} x(t) < A_0 + \left(\frac{T}{2}\right)^{\frac{p-1}{p}} \rho \ := \rho_0, \text{ uniformly for } x \in E. \tag{3.6}$$

Then, we will prove the boundedness of $|x'(t)|$. By the definition of t_1 in Lemma 2.4, we have

$$x'(t_1) = 0.$$

Due to (3.2),

$$\begin{aligned}
&\int_t^{t_1} (\phi_p(x'(t)))' \, dt + \int_t^{t_1} (a \cdot \phi_q(x'(t)))' \, dt + \lambda \int_t^{t_1} g(x(t))x'(t)dt \\
&+ \lambda \int_t^{t_1} \sigma(t)x^v(t)dt - \lambda \int_t^{t_1} \frac{\xi(t)}{x^\mu(t)} dt + \lambda \int_t^{t_1} \frac{\zeta(t)}{x^m(t)} dt = 0.
\end{aligned} \tag{3.7}$$

Since

$$\int_t^{t_1} (\phi_p(x'(t)))' \, dt + \int_t^{t_1} (a \cdot \phi_q(x'(t)))' \, dt = - |x'(t)|^{p-2} x'(t) - a|x'(t)|^{q-2}x'(t),$$

from (3.7) and (3.5), if $G' = g$, we have

$$
\begin{aligned}
|x'(t)|^{p-1} =& \left| a|x'(t)|^{q-2}x'(t) - \lambda \int_t^{t_1} g(x(t))x'(t)dt \right. \\
& \left. +\lambda \int_t^{t_1} \sigma(t)x^v(t)dt + \lambda \int_t^{t_1} \frac{\xi(t)}{x^\mu(t)}dt - \lambda \int_t^{t_1} \frac{\zeta(t)}{x^m(t)}dt \right| \\
\le& \lambda a \left|x'(t)\right|^{q-1} + \lambda \left| \int_t^{t_1} g(x(t))x'(t)dt \right| + \lambda \left| \int_t^{t_1} \sigma(t)x^v(t)dt \right| \\
& + \lambda \left| \int_t^{t_1} \frac{\xi(t)}{x^\mu(t)}dt \right| + \lambda \left| \int_t^{t_1} \frac{\zeta(t)}{x^m(t)}dt \right| \\
\le& a \left|x'(t)\right|^{q-1} + |G(x(t_1)) - G\left(x\left(t\right)\right)| + \lambda \left| \int_0^t \sigma(t)x^v(t)dt \right| \\
& + \int_0^T \left| \frac{\xi(t)}{x^\mu(t)} \right| dt + \int_0^T \left| \frac{\zeta(t)}{x^m(t)} \right| dt \\
\le& a \left|x'(t)\right|^{q-1} + 2 \max_{r_0<t<\rho_0} |G(x)| + T\bar{\sigma}\rho^v + \frac{T\bar{\xi}}{r_0^\mu} + \frac{T\bar{\zeta}}{r_0^m}.
\end{aligned} \tag{3.8}
$$

Using contradiction, we assume that for each constant K, there is $t_0 \in [0, T]$, such that $|x'(t_0)| > K > 0$. Without loss of generality, we fixed

$$
A := 2 \max_{r_0<t<\rho_0} |G(x)| + T\bar{\sigma}\rho^v + \frac{T\bar{r}}{r_0^\mu} + \frac{T\bar{\zeta}}{r_0^m},
$$

and

$$
K = \max\left\{1, \left(\frac{1-a}{A}\right)^{\frac{1}{1-p}}\right\}.
$$

From (3.8) and the definition of K , we obtain

$$
|x'(t_0)|^{p-q} \le a + \frac{1}{|x'(t_0)|^{q-1}} \left(2 \max_{r_0<t<\rho_0} |G(x)| + T\bar{\sigma}\rho^v + \frac{T\bar{r}}{r_0^\mu} + \frac{T\bar{\zeta}}{r_0^m}\right),
$$

which means

$$
K^{p-q} - AK^{1-q} \le a.
$$

However, by the definition of K, we have $K^{p-q} - AK^{1-q} > a$, which is a contradiction. Therefore, we obtain

$$
|x'(t)| \le K, \text{ uniformly for } x \in E. \tag{3.9}
$$

Let $K_0 = \min\{E_1, r_0\}$, and $K_1 = \max\{\rho_0, E_2\}$, and E_1, E_2 are constants defined by Remark 2.1, then using hypotheses (3.6) and (3.9), any possible positive T-periodic solution x to (2.5) satisfies

$$
K_0 < x(t) < K_1, \ \ |x'(t)| < K_2, \text{ for all } t \in [0, T].
$$

So it satisfies (H_1). Using Remark 2.1 we can obtain

$$
\frac{\bar{\xi}}{x^\mu} - \frac{\bar{\zeta}}{x^m} - \bar{\sigma}x^v > 0 \text{ for all } x \in (0, K_1),
$$

and
$$\frac{\bar{\xi}}{x^{\mu}}-\frac{\bar{\zeta}}{x^{m}}-\bar{\sigma} x^{v}<0 \text{ for all } x \in\left(K_{2} .+\infty\right),$$
Therefore, (H_2) in Lemma 2.1 is satisfied. Also, we obtain
$$\left(\frac{\bar{\xi}}{K_{0}^{\mu}}-\frac{\bar{\zeta}}{K_{0}^{m}}-\bar{\sigma} K_{0}^{v}\right)\left(\frac{\bar{\xi}}{K_{1}^{\mu}}-\frac{\bar{\zeta}}{K_{1}^{m}}-\bar{\sigma} K_{1}^{v}\right)<0.$$
So (H_3) in Lemma 2.1 is satisfied. Applying Lemma 2.1, Problem (1.2) has at least one positive T-periodic solution. □

Theorem 3.2. *Suppose that $\bar{\zeta}>0,\ \sigma(t)>0$ for a.e. $t\in[0,T]$, $p\geq q>1,\ a\geq 0$ holds, (A_1), (A_2), (2.18) and (2.19) hold. Then, for any $\mu-m>0,\ v>0$, (1.2) has at least one positive T-periodic solution.*

Theorem 3.3. *Assume that $\overline{\zeta}>0$, $\sigma(t)>0$ for a.e. $t\in[0,T]$, $p\geq q>1, a\geq 0$, and (A_1), (A_2) hold. Suppose that*
$$\lim_{x\rightarrow+\infty}\left(H(x)-2T\bar{\sigma}x^{v+\mu+m}\right)=+\infty \tag{3.10}$$
and
$$\lim_{s\rightarrow 0^{+}}H(s)<\delta_0-2T\overline{\xi_{+}}-2T\overline{\zeta_{-}} \tag{3.11}$$
hold, where $H(x)$ is defined by (2.5), and
$$\delta_0:=\inf_{x\in[A_0,+\infty)}\left(H(x)-2T\bar{\sigma}x^{v+\mu+m}\right). \tag{3.12}$$
Then, there exist two constants $\rho_2>r_2>0$, satisfied
$$\min_{t\in[0,T]}x(t)\geq r_2,\ \textit{uniformly for } x\in E$$
and
$$\max_{t\in[0,T]}x(t)\leq\rho_2,\ \textit{uniformly for } x\in E.$$
Then Eq.(1.2) has at least one T-periodic solution.

Proof. Let $x\in E$, satisfied (2.5). By the definition of t_1 and t_2 in Lemma 2.4, with $0<t_2-t_1\leq T,\ x(t_1)=\max_{t\in[0,T]}x(t)$ and $x(t_2)=\min_{t\in[0,T]}x(t)$. Taking $t_3=t_1-T$, then $x(t_3)=\max_{t\in[0,T]}x(t)$ and $T<t_2-t_3<2T$ hold. Furthermore, by $\zeta(t)>0$ for a.e. $t\in[0,T]$ and (2.8), we get
$$A_0\leq x(t_3)<+\infty,$$
which together with (3.12), yields that
$$H\left(x\left(t_{3}\right)\right)-2T\overline{\sigma}x^{\mu+v+m}\left(t_{3}\right)\geq\inf_{A_0\leq s<+\infty}\left(H(s)-2T\overline{\sigma}s^{\mu+v+m}\right)=\delta_0.$$
Multiplying (2.5) with $x^{\mu+m}(t)$, and then integrating it over the interval $[t_2,t_3]$, we have
$$\int_{t_3}^{t_2}\left(\phi_p(x'(t))\right)'x^{\mu+m}(t)dt+\int_{t_3}^{t_2}\left(a\cdot\phi_q(x'(t))\right)'x^{\mu+m}(t)dt+\lambda\int_{t_3}^{t_2}g(x(t))x'(t)x^{\mu+m}(t)dt$$
$$+\lambda\int_{t_3}^{t_2}\sigma(t)x^{v+\mu+m}(t)dt-\lambda\int_{t_3}^{t_2}\xi(t)dt+\lambda\int_{t_3}^{t_2}\zeta(t)dt=0.$$

Due to the definition of t_1, t_2, we obtain

$$\begin{aligned}
&\int_{t_3}^{t_2} \left(\phi_p(x'(t))\right)' x^{\mu+m}(t)dt + \int_{t_3}^{t_2} \left(a \cdot \phi_q(x'(t))\right)' x^{\mu+m}(t)dt \\
=& -(\mu+m)\int_{t_3}^{t_2} |x'(t)|^p x^{m+\mu-1}(t)dt - \lambda a(\mu+m)\int_{t_3}^{t_2} |x'(t)|^q x^{m+\mu-1}(t)dt \\
\leq& \ 0.
\end{aligned}$$

Therefore, it follows that

$$H\left(x\left(t_2\right)\right) - H\left(x\left(t_3\right)\right) + \int_{t_3}^{t_2} \sigma(t)x^{v+\mu+m}(t)dt + \int_{t_3}^{t_2} \zeta(t)dt \geq \int_{t_3}^{t_2} \xi(t)dt,$$

with $T < t_2 - t_3 \leq 2T$, leads to

$$\begin{aligned}
H\left(x\left(t_2\right)\right) \geq & H\left(x\left(t_3\right)\right) - \int_{t_3}^{t_2} \sigma(t)x^{v+\mu+m}(t)dt - \int_{t_3}^{t_2} \zeta(t)dt + \int_{t_3}^{t_2} \xi(t)dt \\
\geq & H\left(x\left(t_3\right)\right) - \int_0^{2T} \sigma(t)x^{v+\mu+m}(t)dt \\
& - \int_0^{2T} \zeta_-(t)dt + \int_0^{2T} \xi(t)dt \\
\geq & H\left(x\left(t_3\right)\right) - 2T\bar{\sigma}x^{v+\mu+m}(t_3) - 2T\overline{\zeta_-} - 2T\overline{\xi_+}.
\end{aligned} \tag{3.13}$$

By using (3.12), we get

$$H\left(x\left(t_2\right)\right) \geq \delta_0 - 2T\overline{\zeta_-} - 2T\overline{\xi_+}. \tag{3.14}$$

From assumption (3.11), there exists a constant $r_2 > 0$, satisfied

$$H(s) < \delta_0 - 2T\overline{\zeta_-} - 2T\overline{\xi_+}, \text{ for all } s \in \left(0, r_2\right), \tag{3.15}$$

by (3.14) and (3.15), we get

$$\min_{t\in[0,T]} x(t) = x\left(t_2\right) \geq r_2. \tag{3.16}$$

Therefore, by (2.8), we obtain

$$r_2 \leq x\left(t_2\right) \leq A_1.$$

Using (3.13) again, we obtain

$$\begin{aligned}
H\left(x\left(t_3\right)\right) - 2T\bar{\sigma}x^{v+\mu+m} &\leq H\left(x\left(t_2\right)\right) + 2T\overline{\zeta_-} + 2T\overline{\xi_+} \\
&\leq \max_{r_2\leq x\leq A_1} H(x) + 2T\overline{\zeta_-} + 2T\overline{\xi_+}.
\end{aligned}$$

By the assumption (3.10), there exists a constant $\rho_2 > r_2$, satisfied

$$H(s) - 2T\bar{\sigma}s^{v+\mu+m} > \max_{r_2\leq x\leq A_1} H(x) + 2T\overline{\zeta_-} + 2T\overline{\xi_+}, \text{ for all } s \in \left(\rho_2, +\infty\right).$$

Therefore, we can imply

$$\max_{t\in[0,T]} x(t) = x\left(t_3\right) \leq \rho_2. \tag{3.17}$$

From (3.16) and (3.17), we see that $x(t)$ has maximum and minimum. The rest of the proof is similar to the Theorem 3.1. $\square$

Theorem 3.4. *Assume $\sigma(t) > 0$ for a.e. $t \in [0,T]$, and $p \geq q > 1, a \geq 0$, satisfied hypotheses (A_1) and (A_2), suppose that*

$$\lim_{x\to+\infty}\left[H\left(x\left(t\right)\right)+2T\bar{\sigma}x^{\mu+v+m}\left(t\right)\right]=-\infty \tag{3.18}$$

and

$$\lim_{s\to 0^+} H(s) > \delta_1 + 2T\overline{\xi_+} + 2T\overline{\zeta_+}, \tag{3.19}$$

hold, where $H(x)$ is defined in (2.5), and

$$\delta_1 := \sup_{x\in[A_0,+\infty]}\left[H(x)+2T\bar{\sigma}x^{\mu+v+m}\right].$$

Then, there exist two constants $\rho_3 > r_3$, satisfied

$$\min_{t\in[0,T]} x(t) \geq r_3,\ \textit{uniformly for } x\in E$$

and

$$\max_{t\in[0,T]} x(t) \leq \rho_3,\ \textit{uniformly for } x\in E.$$

Then, Problem (1.2) has at least one T-periodic solution.

Proof. Let $x \in E$. If set $t_4 = t_2 - T$, then $0 < t_1 - t_4 \leq T$ and $x(t_4) = \min_{t\in[0,T]} x(t)$. Multiplying (2.5) with $x^{\mu+m}(t)$ and integrating it on $[t_4, t_1]$, we obtain

$$\begin{aligned}&\int_{t_4}^{t_1}\left(\phi_p(x'(t))\right)' x^{\mu+m}(t)dt+\int_{t_4}^{t_1}\left(a\cdot\phi_q(x'(t))\right)' x^{\mu+m}(t)dt\\&+\lambda\int_{t_4}^{t_1}\sigma(t)x^{m+\mu+v}(t)dt+\lambda\int_{t_4}^{t_1}g(x(t))x'(t)x^{m+\mu}(t)dt\\&-\lambda\int_{t_4}^{t_1}\xi(t)dt+\lambda\int_{t_4}^{t_1}\zeta(t)dt=0.\end{aligned}$$

By the definition of t_1, t_4, we obtain

$$\begin{aligned}&\int_{t_4}^{t_1}\left(\phi_p(x'(t))\right)' x^{\mu+m}(t)dt+\lambda\int_{t_4}^{t_1}\left(a\cdot\phi_q(x'(t))\right)' x^{\mu+m}(t)dt\\=&-\mu\int_{t_4}^{t_1}\left|x'(t)\right|^p x^{m+\mu-1}(t)dt-\lambda(m+\mu)a\int_{t_4}^{t_1}\left|x'(t)\right|^q x^{m+\mu-1}(t)dt\\\leq&\ 0.\end{aligned}$$

Therefore, there follows that

$$H\left(x\left(t_1\right)\right)-H\left(x\left(t_4\right)\right)+\int_{t_4}^{t_1}\sigma(t)x^{\mu+v+m}(t)dt+\int_{t_4}^{t_1}\zeta(t)dt\geq\int_{t_4}^{t_1}\xi(t)dt,$$

which together with the fact $T < t_1 - t_4 \leq 2T$ leads to

$$\begin{aligned}H\left(x\left(t_1\right)\right)&\geq H\left(x\left(t_4\right)\right)-\int_{t_4}^{t_1}\sigma(t)x^{\mu+v+m}(t)dt-\int_{t_4}^{t_1}\zeta(t)dt+\int_{t_4}^{t_1}\xi(t)dt\\&\geq H\left(x\left(t_4\right)\right)-\int_{0}^{2T}\sigma(t)x^{\mu+v+m}(t)dt-\int_{0}^{2T}\zeta(t)dt+\int_{0}^{2T}\xi(t)dt\\&\geq H\left(x\left(t_4\right)\right)-2T\bar{\sigma}x^{\mu+v+m}\left(t_1\right)-2T\overline{\zeta_+}-2T\overline{\xi_+}.\end{aligned}\tag{3.20}$$

Using the definition of δ_1, we obtain

$$\begin{aligned} H\left(x\left(t_4\right)\right) &\leq H\left(x\left(t_1\right)\right)+2T\overline{\zeta_+}+2T\bar{\sigma}x^{\mu+v+m}(t_1)+2T\overline{\xi_+} \\ &\leq \sup_{x\in[A_0,+\infty]}\left[H(x)+2T\bar{\sigma}x^{\mu+v+m}\right]+2T\overline{\zeta_+}+2T\overline{\xi_+} \\ &\leq \delta_1+2T\overline{\zeta_+}+2T\overline{\xi_+}. \end{aligned}$$

By (3.19), there exist a constant $r_3>0$ satisfied

$$\min_{t\in[0,T]} x(t)>r_3. \tag{3.21}$$

Combined with (2.8)

$$r_3 \leq x\left(t_4\right) \leq A_1,$$

by (3.20),

$$\begin{aligned} H\left(x\left(t_1\right)\right)+2T\overline{\sigma}x^{\mu+v+m}\left(t_1\right) &\geq H\left(x\left(t_4\right)\right)-2T\overline{\xi_+}-2T\overline{\zeta_+} \\ &\geq \min_{r_4\leq x\leq A_1} H(x)-2T\overline{\xi_+}-2T\overline{\zeta_+}. \end{aligned}$$

By applying (3.18), we obtain

$$H(s)+2T\overline{\sigma}x^{\mu+v+m}\left(t_1\right)<\delta_1-2T\overline{\xi_+}-2T\overline{\zeta_+},\ \text{for all } s\in(\rho_3,+\infty).$$

Therefore, there exists a constant $\rho_3>r_3$ satisfied

$$\max_{t\in[0,T]} x(t)=x\left(t_1\right)<\rho_3. \tag{3.22}$$

From (3.21) and (3.22), we see that $x(t)$ has maximum and minimum. The rest of the proof is similar to the Theorem 3.1. □

4. Some examples

Example 4.1. Consider the following boundary value problem:

$$\begin{cases} \left(\left|x'(t)\right|^5 x'(t)\right)'+5x''(t)+\left(x^2(t)+x^{-4}(t)\right)x'(t)+(\cos 2t+3)x^{\frac{1}{2}}(t)-\frac{2\cos^2 2t+1}{x^3(t)}+\frac{\sin^2 t}{x^2(t)}=0, \\ x(0)=x(T), x'(0)=x'(T). \end{cases} \tag{4.1}$$

Corresponding to Eq.(1.2), $p=7, q=2$, $a=5$, $g(x(t))=x^2(t)+x^{-4}(t)$, $\sigma(t)=\cos 2t+3$, $\xi(t)=2\cos^2 2t+1$, $\zeta(t)=\sin^2 t$, $\mu=3, m=2, v=\frac{1}{2}$, and $T=\pi$. By a simple computation, we obtain $\sigma(t)>0$, and $A_0=1$,

$$\begin{aligned} G(x) &= \int_1^x g(s)ds=\frac{1}{3}x^3-\frac{1}{x^5}+\frac{11}{3}, \\ b_0 &= \sup_{x\in[A_0,\infty)}[G(x)+T\bar{\sigma}x^v]=3-3\pi<+\infty. \end{aligned}$$

Moreover, we have $\overline{\xi_+}=2\pi$, $\overline{\zeta_+}=\pi$, and

$$\lim_{s\to0^+}\left(G(s)+\frac{T\overline{\xi_+}}{s^m}+\frac{T\overline{\zeta_+}}{s^\mu}\right)=\lim_{s\to0^+}\left(\frac{1}{3}x^3-\frac{1}{x^5}+\frac{2\pi}{x^3}+\frac{\pi}{x^2}+\frac{2}{3}\right)\to-\infty.$$

Obviously, Eq.(4.1) satisfies (2.12) and (2.13). Thus, by Theorem 3.1, (4.1) has at least one positive π-periodic solution.

Example 4.2. Consider the following boundary value problem:

$$\begin{cases}(\phi_p(x'(t)))' + \left(\frac{1}{2}\phi_q(x'(t))\right)' + \left(x^7 - \frac{1}{x}\right)x'(t) + (2\sin t + 7)\,x^2 - \frac{\cos t + 1}{x^{\frac{13}{2}}(t)} + \frac{\sin^2 t}{x^{\frac{1}{2}}(t)} = 0,\\ x(0) = x(T), x'(0) = x'(T).\end{cases} \tag{4.2}$$

Corresponding to Eq.(1.2), $p \geq q > 1$, $a = \frac{1}{2}$, $g(x) = x^7 - \frac{1}{x}$, $\sigma(t) = 2\sin t + 7$, $\xi(t) = \cos t + 1$, $\zeta(t) = \sin^2 t$, $\mu = \frac{13}{2}$, $m = \frac{1}{2}$, and $T = 2\pi$. By a simple computation, we obtain $\sigma(t) > 0$, and $A_0 = 1$,

$$H(x) = \int_1^x x^{\mu+m} g(s)ds = \frac{1}{15}x^{15} - \frac{1}{7}x^7 + \frac{8}{105},$$
$$\lim_{x\to+\infty}\left(H(x) - 2T\bar{\sigma}x^{v+\mu+m}\right) \to +\infty.$$

Moreover, we have $\overline{\sigma_-} = 14\pi$, $\overline{\xi_+} = 2\pi$, $\overline{\zeta_-} = 0$, and

$$\delta_0 = \inf_{x\in[A_0,+\infty)}\left(H(x) - 2T\bar{\sigma}x^{v+\mu+m}\right) = \frac{8}{105} - 2\pi,$$
$$\lim_{s\to0^+} H(s) \to -\infty.$$

Obviously, Eq.(4.2) satisfies (3.10) and (3.11). Thus, by Theorem 3.3, (4.2) has at least one positive 2π-periodic solution.

Acknowledgements

This work is sponsored by the National Natural Science Foundation of China (12371173), the Shandong Natural Science Foundation of China (ZR2021MA064) and the Taishan Scholar project of China.

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