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TRAVELING WAVES OF THE KDV-NKDV EQUATION*

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Abstract In this paper, we use the dynamical system method to investigate the wave solutions of the KdV-nKdV equation. We prove Wazwaz's proposal that the KdV-nKdV equation has continuous periodic wave solutions and give their exact expressions by elliptic integral theory. We confirm that the KdV-nKdV equation has no classical solitary wave solution although it can be regarded as a fusion of the KdV equation with classical solitary wave and the nKdV equation. In addition, we obtain some novel traveling wave solutions of it including trapezoidal wave, inverted 'N' wave, and blow-up wave solutions.

Keywords KdV-nKdV equation, bifurcation, dynamical system, trapezoidal wave, inverted 'N' wave.

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1. Introduction

The KdV equation is a distinguished partial differential equation, which was discovered by Korteweg and Vries when they studied the small-amplitude long-wave motion in shallow water in 1895 [11]. Then, Kruskal [12] found the soliton solution of the KdV equation by studying Fermi-Passa-Ulam problem. It is well known that the KdV equation has infinite conservation laws which makes it be widely used to describe wave phenomena in conservative systems, such as magnetohydrodynamic waves in cold plasmas, ion acoustic waves in plasmas, nonlinear shallow water waves with weak restoring forces, oceanic internal waves in density stratification, sound waves in crystal lattices, nonlinear waves in liquid-bubble mixtures, etc [9].

Particularly, in 1977, Olver [18] proposed a general method for finding evolution equations with infinitely many symmetries and presented the recursive operator of the KdV equation. Meanwhile, he derived a series of increasing negative order equations by using the negative order recursive operator. By studying symmetries and negative powers of recursion operator of the KdV equation, Verosky [29] gave the negative-order KdV (nKdV) equation in 1991. Subsequently, Qiao [19–26] researched the Hamiltonian structures, Lax pairs, conservation laws, and explicit

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multisoliton and multikink wave solutions of the nKdV equation through bilinear Bäcklund transformations. Chen [2] investigated the residual symmetry of the nKdV equation and got the explicit soliton-cnoidal wave interaction solutions. These works show that the nKdV equation has many properties similar to the KdV equation, but there are obvious differences. Rodriguez [27] has pointed out that the nKdV equation has no classical solitary traveling wave solution, i.e., a single wave solution [6] whose energy is concentrated at the wave peak and both sides approaching a constant.

In 2018, by using both the KdV recursive operator and the inverse KdV recursive operator, Wazwaz [30] constructed a new equation

$$v_t + 6vv_x + v_{xxx} + v_{xxt} + 4vv_t + 3v_x\partial_x^{-1}(v_t) = 0,$$

which is called the KdV-negative-order KdV (KdV-nKdV) equation. Wazwaz introduced the potential $v(x,t) = u_x(x,t)$ to eliminate the integral operator ∂_x^{-1}, and then the above equation becomes the form

$$u_{xt} + 6u_xu_{xx} + u_{xxxx} + u_{xxxt} + 4u_xu_{xt} + 2u_{xx}u_t = 0. \tag{1.1}$$

As a combination of the KdV equation and the negative order KdV equation, does equation (1.1) still has classical solitary wave solutions or has more wave phenomena? These points arouse people great interest.

In 2018, Wazwaz [30] showed that the KdV-nKdV equation was integrable since it passed the Painlevé test well, and gave the multiple soliton solutions and kink wave solution. In his work, he also proposed a proposal that the equation (1.1) has continuous periodic solutions. Later, Cheng [3] supplemented the integrability of the equation by giving its Lax pair. Then, Kumar [13] applied the Lie symmetry method to study the equation (1.1) and obtained a traveling wave solution set including multiple soliton solutions. In 2020, Hu [8] applied the Painlevé truncation method to study the nonlocal symmetry and similarity reduction of the KdV-nKdV equation, and derived explicit solutions of equation (1.1). Ekici [5] used the double (G′/G, 1/G)-expansion method to seek the rational solutions, hyperbolic function solutions and trigonometric function solutions of the KdV-nKdV equation.

Although so many wave solutions of the KdV-nKdV equation have been given, we note that the continuous periodic solutions and the classical solitary wave solutions are not reported yet. We guess that some wave solutions may still be lost. So, in this paper, we try to solve these problems: (i) Answer whether equation (1.1) has the continuous periodic solutions and the classical solitary wave solutions. If they exist, we will give their explicit expressions. (ii) Find all possible traveling wave solutions of equation (1.1) as many as possible. In fact, many scholars have proved that dynamical the system approach [4] is a powerful method to study wave solutions of PDEs. Specially for the integrable systems, one can often obtain all single wave solutions by this method. In 2012, Li [16] obtained more than 26 exact explicit traveling wave solutions of kudryashov-sinelshchikov equation by this method. In 2013, Samanta [14] established the existence of solitary wave solutions and periodic traveling wave solutions by applying this method to the Zakharov-Kuznetsov equation. In 2015, Zhang [33] used this method to derive two families of solitary wave solutions and two families of periodic wave solutions of the fifth-order Kaup-Kuperschmidt equation. In 2018, Shi [28] uniformly constructed solitary wave solutions, periodic wave solutions, compactions and kink wave solutions of the Fujimoto-Watanabe equation by this method. In 2021, Han [7] obtained Jacobian elliptic and hyperbolic function solutions of the time-space coupled fractional nonlinear Schrödinger equation by this method. For application of this method in other PDEs, one can see the KdV-Burgers-Kuramoto equation [36], the Green-Naghdi equation [31], a class of third-order MKdV equations [17], the Ginzburg-Landau equation [32] and the generalized Burgers-$\alpha\beta$ equation [37], etc.

In this paper, by discussing bifurcation of traveling wave solutions, we explore wave phenomena of system (1.1). Wave velocity c is selected as an important bifurcation parameter, which allows one to clearly see how waveforms of traveling waves change with different wave speeds. Due to existence of an integrable 2-dimensional invariant manifold, we obtain all single wave solutions of system (1.1). These conclusions enable us to answer all questions raised above in the last section.

2. Bifurcation analysis of traveling wave system

First, we make the following traveling wave transformation

$$U(x,t) = u(x-ct) = u(\xi)$$

to convert equation (1.1) into the form

$$(1-c)u^{''''} - cu^{''} + 6(1-c)u^{'}u^{''} = 0, \tag{2.1}$$

where $c \neq 0$ is the wave velocity and $'$ represents $d/d\xi$. Integrating (2.1) once, we have

$$(1-c)u^{'''} - cu^{'} + 3(1-c)(u^{'})^2 = e, \tag{2.2}$$

where parameter e is an integral constant. Further, we transform equation (2.2) into a 3-dimensional system

$$\begin{cases} u^{'} = v, \\ v^{'} = y, \\ y^{'} = -3v^2 + \frac{c}{1-c}v + \frac{e}{1-c}. \end{cases} \tag{2.3}$$

It is not difficult to see that

$$\begin{cases} v^{'} = y, \\ y^{'} = -3v^2 + \frac{c}{1-c}v + \frac{e}{1-c} \end{cases} \tag{2.4}$$

defines a 2-dimensional invariant manifold in $\mathbb{R}^3$. One can check that system (2.4) has the energy function

$$H(v,y) = \frac{1}{2}y^2 + v^3 - \frac{1}{2}\frac{c}{1-c}v^2 - \frac{e}{1-c}v. \tag{2.5}$$

Next, we give a theorem to discuss the parameter bifurcation sets of system (2.4) and its phase portraits and types of equilibria under different parameter bifurcation sets.

Theorem 2.1. *For system* (2.4)*, we have the parameter bifurcation sets* $(I)\{(c,e)|c < 6e-\sqrt{36e^2-12e}$ *or* $c > 6e+\sqrt{36e^2-12e}\}$*,* $(II)\{(c,e)|c = 6e-\sqrt{36e^2-12e}$ *or* $c = 6e+\sqrt{36e^2-12e}\}$ *and* $(III)\{(c,e)|6e-\sqrt{36e^2-12e} < c < 6e+\sqrt{36e^2-12e}\}$*, where* $e < 0$ *or* $e > \frac{1}{3}$*.*

If $c < 6e-\sqrt{36e^2-12e}$ *or* $c > 6e+\sqrt{36e^2-12e}$*, system* (2.4) *has a center* $B_1(\frac{c+\sqrt{c^2-12ec+12e}}{6(1-c)},0)$ *and a saddle* $B_2(\frac{c-\sqrt{c^2-12ec+12e}}{6(1-c)},0)$*. System* (2.4) *has a homoclinic orbit* γ *connecting saddle* B_2*. In the homoclinic loop* $\gamma \cup B_2$ *consisting of the saddle* B_2 *and homoclinic orbit* γ *, periodic orbit families* $\Gamma(h)$ *around the center* B_1 *forms compact region, where* $\Gamma(h)$ *is defined by* $H(v,y)=h, h\in(h(B_1),h(B_2))$*, and* $h(B_i)$ $(i=1,2)$ *denote the energy at the equilibria* B_i $(i=1,2)$*, respectively. In addition, the rest of the orbits in Fig. 2(a) are unbounded.*

If $c = 6e-\sqrt{36e^2-12e}$ *or* $c = 6e+\sqrt{36e^2-12e}$*, system* (2.4) *has a unique cusp* $B_3(\frac{c}{6(1-c)},0)$*. Every orbit of system* (2.4) *are not bounded. In this case, the saddle in the first case degenerates into a cusp* B_3*, but its stable manifold* L^2 *and unstable manifold* L_2 *are preserved, as shown in Fig. 2(b).*

If $6e-\sqrt{36e^2-12e} < c < 6e+\sqrt{36e^2-12e}$*, system* (2.4) *has no equilibrium. System* (2.4) *has only one kind of unbounded orbit, as shown in Fig. 2(c).*

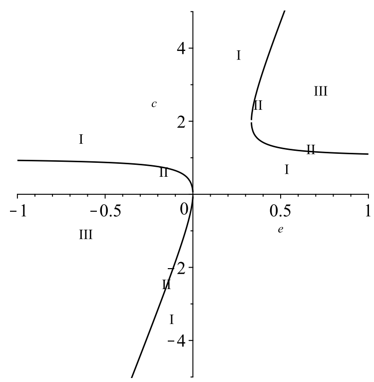


Figure 1. Transition boundaries on $(e-c)$ plane.

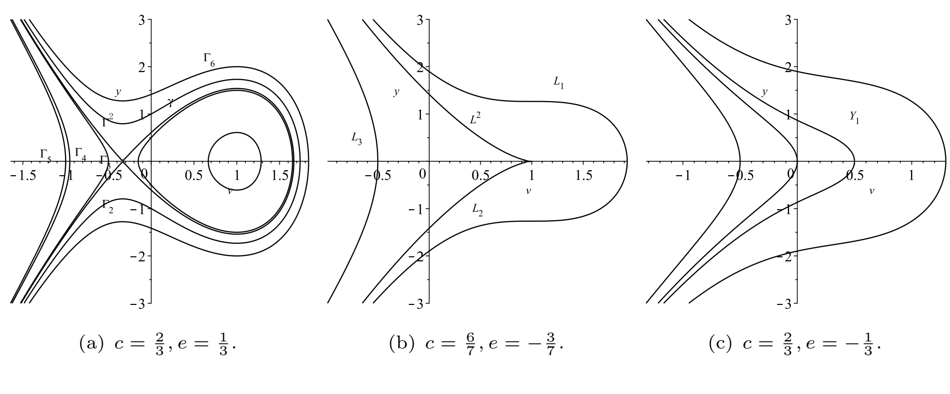


Figure 2. Different phase portraits of system (2.4) .

Proof. By the theory of dynamic system, we find the Jacobi matrix $M(B)$ at the equilibrium B of system (2.4)

$$M(B) = \begin{bmatrix} 0 & 1 \\ -6v + \frac{c}{1-c} & 0 \end{bmatrix},$$

since the trace of Jacobi matrix $M(B)$ is zero, the type of equilibrium only needs to consider the sign of determinant $M(B)$.

When $c < 6e - \sqrt{36e^2 - 12e}$ or $c > 6e + \sqrt{36e^2 - 12e}$, system (2.4) has two equilibria. A direct calculation shows that the $M(B_1) > 0$, $M(B_2) < 0$, which implies that B_1 is a center and B_2 is a saddle.

When $c = 6e - \sqrt{36e^2 - 12e}$ or $c = 6e + \sqrt{36e^2 - 12e}$,

$$M(B_3) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$M(B_3)$ is a degenerate Jacobi matrix. At this time, the two eigenvalues of the corresponding linear equations are zero, but the coefficients of the linear terms are not all zero. To determine the equilibrium B_3 type, we use the transformation

$$\phi = v - \frac{c}{6(1-c)}, \psi = y,$$

to convert system (2.4) into its normal form

$$\begin{cases} \phi^{'} = \psi, \\ \psi^{'} = a_k\phi^k[1 + h(\phi)] + b_n\phi^n[1 + g(\phi)] + \psi^2 p(\phi, \psi) = -3\phi^2 + \frac{c^2-12ce+12e}{12}. \end{cases}$$

According to the corresponding theory in [34], the type of equilibrium B_3 are determined by the odevity of k and n and the signs of a_k and b_n. From the fact that $k = 2$, $a_k = -3$, $b_n = 0$, B_3 is a cusp. □

3. Exact solutions of system (2.4) and traveling wave solutions of equation (1.1)

In the second section, we have obtained three phase portraits of system (2.4) by bifurcation analysis. In fact, each orbit in the phase portrait corresponds to a solution of system (2.4). Therefore, by elliptic integral theory, we first calculate various solutions of system (2.4), and then get the final traveling wave solutions of system (1.1).

Before the calculation, we give an illustration of the elliptic integral of the second kind and three Jacobian basic elliptic functions [1]: $E(u)$ is the Legendre's incomplete elliptic integral of the second kind, $sn(u)$ is the sine amplitude u, $cn(u) = \sqrt{1 - sn^2(u)}$, $dn(u) = \sqrt{1 - k^2 sn^2(u)}$. The other nine Jacobian auxiliary elliptic functions can be derived from the three Jacobian basic elliptic functions.

3.1. Bounded solutions of system (2.4) and traveling wave solutions of corresponding system (1.1)

For the parameter bifurcation set (I), the system (2.4) has bounded orbits, which are homoclinic orbits γ and periodic orbits $\Gamma(h)$, as shown in phase portrait Fig. 2(a), we first consider its periodic orbits.

B1. For the phase portrait (a), by reference [35], any closed orbit in the periodic orbit family $\Gamma(h)$ has the form

$$y = \pm\sqrt{2}\sqrt{(v-v_1)(v-v_2)(v_3-v)}, \tag{3.1}$$

where the constraint condition $v_1 < v_2 < v < v_3$ holds. If the period is set to $2T$ and $v(0) = v_2$ is chosen as the initial value, we get

$$\int_{v_2}^{v} \frac{dv}{\sqrt{2}\sqrt{(v-v_1)(v-v_2)(v_3-v)}} = |\xi|, \quad -T < \xi < T.$$

We can simplify the above equation by the following elliptic integral formula

$$\int_{v_2}^{v} \frac{dv}{\sqrt{(v-v_1)(v-v_2)(v_3-v)}} = g \cdot sn^{-1}\left(\sqrt{\frac{(v_3-v_1)(v-v_2)}{(v_3-v_2)(v-v_1)}}, k\right),$$

where $g = \frac{2}{\sqrt{v_3-v_1}}$, $k^2 = \frac{v_3-v_2}{v_3-v_1}$. Then the periodic solution of system (2.4) is expressed by

$$v_{b1}(\xi) = v_1 + \frac{(v_2-v_1)(v_3-v_1)}{(v_3-v_1)-(v_3-v_2)sn^2(\sqrt{\frac{v_3-v_1}{2}}\xi)}, \quad -T < \xi < T.$$

Noting that

$$\int \frac{d\eta}{1 \pm k \cdot sn(\eta)} = \frac{1}{k'^2}[E(\eta) + k(1 \mp k \cdot sn(\eta))cd(\eta)],$$

where $k' = \sqrt{1-k^2}$. By using the above elliptic integral formula, the first kind of traveling wave solution of (1.1) is derived

$$\begin{aligned} u_1(\xi) &= \int v_{b1}(\xi) d\xi \\ &= v_1 \cdot \xi + \sqrt{2(v_3-v_1)}\left[E\left(\sqrt{\frac{v_3-v_1}{2}}\xi\right) + \sqrt{\frac{v_3-v_2}{v_3-v_1}} \cdot cd\left(\sqrt{\frac{v_3-v_1}{2}}\xi\right)\right] + C_1, \end{aligned}$$

where $-T < \xi < T$ and C_1 is a constant.

B2. Similarly, using the method of reference [35], the homoclinic orbit γ has the following forms

$$y = \pm\sqrt{2}\sqrt{(v-v_4)^2(v_5-v)}, \tag{3.2}$$

where $v_4 = \frac{c-\sqrt{c^2-12ec+12e}}{6(1-c)}$ and $v_5 = \frac{c+2\sqrt{c^2-12ec+12e}}{6(1-c)}$ satisfy the condition $v_4 < v < v_5$, we select the initial value $v(0) = v_5$, and then get

$$\int_{v_5}^{v} \frac{dv}{\sqrt{2}(v-v_4)\sqrt{(v_5-v)}} = -|\xi|, \quad -\infty < \xi < \infty.$$

Noting that

$$\int_{v_5}^{v} \frac{dv}{(v-v_4)\sqrt{(v_5-v)}} = \frac{1}{\sqrt{v_5-v_4}} \ln \frac{\sqrt{v_5-v_4}-\sqrt{v_5-v}}{\sqrt{v_5-v_4}+\sqrt{v_5-v}}.$$

Using the above two formulas, we obtain corresponding solution of system (2.4), which is looked like a bell

$$v_{b2}(\xi) = v_5 - \frac{(v_5-v_4)(1-\exp(\sqrt{2(v_5-v_4)}\xi))^2}{(1+\exp(\sqrt{2(v_5-v_4)}\xi))^2}, \quad -\infty < \xi < \infty.$$

By direct computation, the second kind of traveling wave solutions of the equation (1.1) is obtained

$$\begin{aligned} u_2(\xi) &= \int v_{b2}(\xi) d\xi \\ &= v_5 \cdot \xi - \sqrt{\frac{v_5-v_4}{2}} [\frac{4+(1+\exp(\sqrt{2(v_5-v_4)}\xi)\sqrt{2(v_5-v_4)}\xi}{1+\exp(\sqrt{2(v_5-v_4)}\xi)}] + C_2, \end{aligned}$$

where $-\infty < \xi < \infty$ and C_2 is a constant.

3.2. Unbounded solutions of system (2.4) and traveling wave solutions of corresponding system (1.1)

In this section, we will give the traveling wave solutions corresponding to all unbounded orbits of system (2.4) .

I. Firstly, we consider all unbounded orbits in phase portrait (a). Actually, these orbits can be divided into five types, i.e., the first kind of unbounded orbits are equal to the energy of the saddle, such as Γ^2 and Γ_2; the second kind of unbounded orbits are higher than the energy of the center but lower than the energy of the saddle, such as Γ_3; the third kind of unbounded orbit is equal to the energy of the center, such as Γ_4; the fourth kind of unbounded orbits are lower than the energy of center, such as Γ_5; the fifth kind of unbounded orbits are higher than the energy of the saddle, such as Γ_6.

(U1) For the first kind of unbounded orbits , according to reference [35], it can be directly expressed as the following form

$$y = \pm\sqrt{2}\sqrt{(v_4-v)^2(v_5-v)}, \tag{3.3}$$

where $-\infty < v < v_4 < v_5$. In this subcase, we only consider Γ^2 since the calculation is similar for Γ_2. For a prescribed initial value $v(0) = -\infty$, we have

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{2}(v_4-v)\sqrt{v_5-v}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_{-\infty}^{v} \frac{dv}{(v_4-v)\sqrt{v_5-v}} = -\frac{1}{\sqrt{v_5-v_4}} \ln \frac{\sqrt{v_5-v}-\sqrt{v_5-v_4}}{\sqrt{v_5-v}+\sqrt{v_5-v_4}}.$$

Similar to the calculation of B2, the first kind of unbounded solution of system (2.4) is deduced

$$v_{u1}(\xi) = v_5 - \frac{(v_5 - v_4)(1 + \exp(\sqrt{2(v_5 - v_4)}\xi))^2}{(1 - \exp(\sqrt{2(v_5 - v_4)}\xi))^2}, \quad \xi > 0.$$

By direct calculation, it is easy to see the third kind of traveling wave solutions of equation (1.1)

$$\begin{aligned} u_3(\xi) &= \int v_{u1}(\xi) d\xi \\ &= v_5 \cdot \xi - \sqrt{\frac{v_5 - v_4}{2}} [\frac{4 + (1 - \exp(\sqrt{2(v_5 - v_4)}\xi) \sqrt{2(v_5 - v_4)}\xi}{1 - \exp(\sqrt{2(v_5 - v_4)}\xi)}] + C_3, \end{aligned}$$

where $\xi > 0$ and C_3 is a constant.

(U2) For the second kind of unbounded orbits, it can be written as

$$y = \pm\sqrt{2}\sqrt{(v_6 - v)(v_7 - v)(v_8 - v)}, \tag{3.4}$$

where v_6, v_7, v_8 are reals and relationship $-\infty < v < v_6 < v_7 < v_8$ holds. Referring to (U1), we only discuss the upper branch of Γ_3. Let $v(0) = -\infty$, we have

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{(v_6 - v)(v_7 - v)(v_8 - v))}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

With elliptic integral formula

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{(v_6 - v)(v_7 - v)(v_8 - v))}} = g \cdot sn^{-1}(\sqrt{\frac{v_8 - v_6}{v_8 - v}}, k),$$

where $g = \frac{2}{\sqrt{v_8 - v_6}}, k^2 = \frac{v_8 - v_7}{v_8 - v_6}$, similar calculation shows that the second kind of unbounded solution of system (2.4) has the form

$$v_{u2}(\xi) = v_8 - \frac{v_8 - v_6}{sn^2(\sqrt{\frac{v_8 - v_6}{2}}\xi)}, \quad 0 < \xi < \xi_1,$$

where $\xi_1 = \frac{2\sqrt{2}}{\sqrt{v_8 - v_6}} \cdot \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{v_8 - v_7}{v_8 - v_6} \cdot \sin^2\theta}}$, from the fact that

$$\int \frac{d\eta}{sn^2(\eta)} = \int ns^2(\eta) du = \eta - E(\eta) - dn(\eta) \cdot cs(\eta),$$

by the above two formulas, we obtain the fourth kind of traveling wave solution of equation (1.1)

$$u_4(\xi) = v_6 \cdot \xi + \sqrt{2(v_8 - v_6)}[E(\sqrt{\frac{v_8 - v_6}{2}}\xi) + dn(\sqrt{\frac{v_8 - v_6}{2}}\xi) \cdot cs(\sqrt{\frac{v_8 - v_6}{2}}\xi)] + C_4,$$

where $0 < \xi < \xi_1$ and C_4 is a constant.

(U3) For the third kind of unbounded orbits, it can be written as

$$y = \pm\sqrt{2}\sqrt{(v_{10} - v)^2(v_9 - v)}, \tag{3.5}$$

where $v_9 = \frac{c-2\sqrt{c^2-12ec+12e}}{6(1-c)}$ and $v_{10} = \frac{c+\sqrt{c^2-12ec+12e}}{6(1-c)}$ are reals and relationship $-\infty < v_9 < v_{10}$ holds. Likewise, we only analyze the upper branch of Γ_4 and give the initial condition $v(0) = -\infty$

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{2}(v_{10}-v)\sqrt{v_9-v}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_{-\infty}^{v} \frac{dv}{(v_{10}-v)\sqrt{v_9-v}} = \frac{1}{\sqrt{v_{10}-v_9}}(\pi - 2\arctan\sqrt{\frac{v_9-v}{v_{10}-v_9}}),$$

thus we find the third kind of unbounded solution of system (2.4)

$$v_{u3} = v_9 - (v_{10}-v_9)\cot^2(\sqrt{\frac{v_{10}-v_9}{2}}\xi), \quad 0 < \xi < \xi_2,$$

where $\xi_2 = \frac{\pi}{\sqrt{2(v_{10}-v_9)}}$. Integrating the above formula directly, we can easily achieve the fifth kind of traveling wave solution of equation (1.1)

$$\begin{aligned} u_5(\xi) &= \int v_{u3}(\xi)d\xi \\ &= v_9\cdot\xi - \sqrt{2(v_{10}-v_9)}(-cot(\sqrt{\frac{(v_{10}-v_9)}{2}}\xi) + \frac{\pi}{2} - \sqrt{\frac{(v_{10}-v_9)}{2}}\xi) + C_5, \end{aligned}$$

where $0 < \xi < \xi_2$ and C_5 is a constant.

(U4) For the fourth kind of unbounded orbits, we have

$$y = \pm\sqrt{2}\sqrt{(v_{11}-v)[v^2 + (v_{11} - \frac{1}{2}\frac{c}{1-c})v + (v_{11}^2 - \frac{1}{2}\frac{c}{1-c}v_{11} - \frac{e}{1-c})]}, \tag{3.6}$$

where $-\infty < v < v_{11} < v_9$. Taking $v(0) = -\infty$, we have

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{2}\sqrt{(v_{11}-v)[v^2 + (v_{11} - \frac{1}{2}\frac{c}{1-c})v + (v_{11}^2 - \frac{1}{2}\frac{c}{1-c}v_{11} - \frac{e}{1-c})]}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

By calculating the elliptic integral

$$\begin{aligned} &\int_{-\infty}^{v} \frac{dv}{\sqrt{(v_{11}-v)[v^2 + (v_{11} - \frac{1}{2}\frac{c}{1-c})v + (v_{11}^2 - \frac{1}{2}\frac{c}{1-c}v_{11} - \frac{e}{1-c})]}} \\ =&g\cdot cn^{-1}(\frac{v_{11}-B_{11}-v}{v_{11}+B_{11}-v}, k), \end{aligned}$$

where $B_{11}^2 = 3v_{11}^2 - \frac{c}{1-c}v_{11} - \frac{e}{1-c}$, $g = \frac{1}{\sqrt{B_{11}}}$ and $k^2 = \frac{4B_{11}+6v_{11}-\frac{c}{1-c}}{8B_{11}}$. We can get the fourth kind of unbounded solution of system (2.4)

$$v_{u4}(\xi) = v_{11} + \sqrt{B_{11}^2} - \frac{2\sqrt{B_{11}^2}}{1 - cn(\sqrt[4]{4B_{11}^2}\xi)},$$

where $0<\xi<\xi_3$, $\xi_3=\frac{4}{\sqrt[4]{4B_{11}^2}}\cdot\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{4\sqrt{B_{11}^2}+6v_{11}-\frac{c}{1-c}}{8\sqrt{B_{11}^2}}\cdot\sin^2\theta}}$. Noting that

$$\int\frac{d\eta}{1-cn(\eta)}=\eta-E(\eta)-\frac{dn(\eta)\cdot sn(\eta)}{1-cn(\eta)},$$

by direct calculation, the sixth kind of traveling wave solution of equation (1.1) is derived

$$\begin{aligned}u_6(\xi)=&(v_{11}-\sqrt{B_{11}^2})\cdot\xi+\sqrt[4]{4B_{11}^2}\cdot[E(\sqrt[4]{4B_{11}^2}\xi)+\frac{dn(\sqrt[4]{4B_{11}^2}\xi)\cdot sn(\sqrt[4]{4B_{11}^2}\xi)}{1-cn(\sqrt[4]{4B_{11}^2}\xi)}]\\&+C_6,\end{aligned}$$

where $0<\xi<\xi_3$ and C_6 is a constant.

(U5) For the fifth type of unbounded orbits, it can be described as

$$y=\pm\sqrt{2}\sqrt{(v_{12}-v)[v^2+(v_{12}-\frac{1}{2}\frac{c}{1-c})v+(v_{12}^2-\frac{1}{2}\frac{c}{1-c}v_{12}-\frac{e}{1-c})]},\tag{3.7}$$

where $v_{12}>\frac{c+2\sqrt{c^2-12ec+12e}}{6(1-c)}$ and relationship $-\infty<v<v_{12}$ holds. Choosing $v(0)=-\infty$, the calculation process of $u_7(\xi)$ is similar to $u_6(\xi)$.

$$\begin{aligned}u_7(\xi)=&(v_{12}-\sqrt{B_{12}^2})\cdot\xi+\sqrt[4]{4B_{12}^2}\cdot[E(\sqrt[4]{4B_{12}^2}\xi)+\frac{dn(\sqrt[4]{4B_{12}^2}\xi)\cdot sn(\sqrt[4]{4B_{12}^2}\xi)}{1-cn(\sqrt[4]{4B_{12}^2}\xi)}]\\&+C_7,\end{aligned}$$

where $0<\xi<\xi_4$, C_7 is a constant, $B_{12}^2=3v_{12}^2-\frac{c}{1-c}v_{12}-\frac{e}{1-c}$, $g=\frac{1}{\sqrt{B_{12}}}$ and $k^2=\frac{4B_{12}+6v_{12}-\frac{c}{1-c}}{8B_{12}}$.

II. Secondly, we consider the orbits in phase portrait (b), which are unbounded. Based on the energy of the cusp, we divide them into two cases. The first type of orbits has the same energy as the cusp, for example, L^2 and L_2. The remaining orbits fall into the second type, for example, L_1 and L_3.

(U6) For the first type of orbits, by reference [35], they have the following forms

$$y=\pm\sqrt{2}(\frac{c}{6(1-c)}-v)\sqrt{\frac{c}{6(1-c)}-v},\tag{3.8}$$

where $-\infty<v<\frac{c}{6(1-c)}$. Similarly, letting $v(0)=-\infty$ leads to

$$\int_{-\infty}^{v}\frac{dv}{\sqrt{2}(\frac{c}{6(1-c)}-v)\sqrt{\frac{c}{6(1-c)}-v}}=\int_0^{\xi}d\xi,\quad\xi>0.$$

Thus the sixth kind of unbounded solutions of system (2.4) has the form

$$v_{u6}(\xi)=\frac{c}{6(1-c)}-\frac{2}{\xi^2},\quad\xi>0,$$

and corresponding the eighth traveling wave solution of equation (1.1) can be written as

$$u_8(\xi) = \int v_{u6}(\xi)d\xi = \int (\frac{c}{6(1-c)} - \frac{2}{\xi^2})d\xi = \frac{c}{6(1-c)} \cdot \xi + \frac{2}{\xi} + C_8,$$

where $\xi > 0$ and C_8 is a constant.

(U7) The remaining orbits have the unified form

$$y = \pm\sqrt{2}\sqrt{(v_{13} - v)[v^2 + (v_{13} - \frac{1}{2}\frac{c}{1-c})v + (v_{13}^2 - \frac{1}{2}\frac{c}{1-c}v_{13} - \frac{e}{1-c})]}, \quad (3.9)$$

where $-\infty < v < v_{13}$ choosing $v(0) = -\infty$, the calculation process of $u_9(\xi)$ is similar to $u_6(\xi)$.

$$\begin{aligned} u_9(\xi) =&(v_{13} - \sqrt{B_{13}^2}) \cdot \xi + \sqrt[4]{4B_{13}^2} \cdot [E(\sqrt[4]{4B_{13}^2}\xi) + \frac{dn(\sqrt[4]{4B_{13}^2}\xi) \cdot sn(\sqrt[4]{4B_{13}^2}\xi)}{1 - cn(\sqrt[4]{4B_{13}^2}\xi)}] \\ &+ C_9, \end{aligned}$$

where $0 < \xi < \xi_5$, C_9 is a constant, $B_{13}^2 = 3v_{13}^2 - \frac{c}{1-c}v_{13} - \frac{e}{1-c}$, $g = \frac{1}{\sqrt{B_{13}}}$ and $k^2 = \frac{4B_{13}+6v_{13}-\frac{c}{1-c}}{8B_{13}}$.

III. Finally, we consider the orbits in phase portrait (c).

(U8) In fact, all orbits have the unified form

$$y = \pm\sqrt{2}\sqrt{(v_{14} - v)[v^2 + (v_{14} - \frac{1}{2}\frac{c}{1-c})v + (v_{14}^2 - \frac{1}{2}\frac{c}{1-c}v_{14} - \frac{e}{1-c})]}, \quad (3.10)$$

where $-\infty < v < v_{14}$, choosing $v(0) = -\infty$, the calculation process of $u_{10}(\xi)$ is similar to $u_6(\xi)$.

$$\begin{aligned} u_{10}(\xi) =&(v_{14} - \sqrt{B_{14}^2}) \cdot \xi + \sqrt[4]{4B_{14}^2} \cdot [E(\sqrt[4]{4B_{14}^2}\xi) + \frac{dn(\sqrt[4]{4B_{14}^2}\xi) \cdot sn(\sqrt[4]{4B_{14}^2}\xi)}{1 - cn(\sqrt[4]{4B_{14}^2}\xi)}] \\ &+ C_{10}, \end{aligned}$$

where $0 < \xi < \xi_6$, C_{10} is a constant, $B_{14}^2 = 3v_{14}^2 - \frac{c}{1-c}v_{14} - \frac{e}{1-c}$, $g = \frac{1}{\sqrt{B_{14}}}$ and $k^2 = \frac{4B_{14}+6v_{14}-\frac{c}{1-c}}{8B_{14}}$.

4. Discussion and conclusion

In this paper, for the KdV-nKdV equation, we obtained all single wave solutions by phase plane analysis method and elliptic integral theory. According to the property of these solutions, we divide them into four types below:

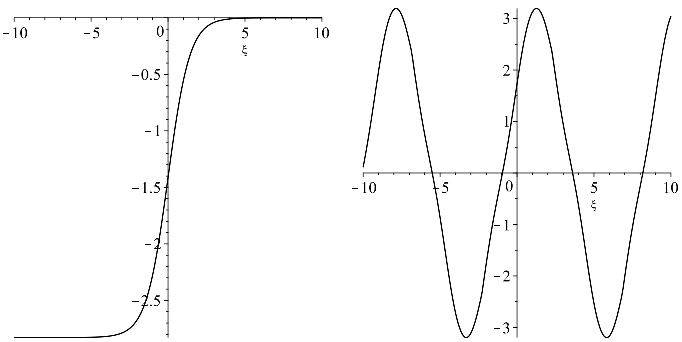
(1) Classical bounded traveling wave solutions

(2) Trapezoidal wave solution

(3) Inverted 'N' wave solution

(4) Blow-up wave solutions

From the facts above, we come to three conclusions:



(a) Kink wave solution. (b) Periodic wave solution.

Figure 3. Two kinds of classical wave solutions.

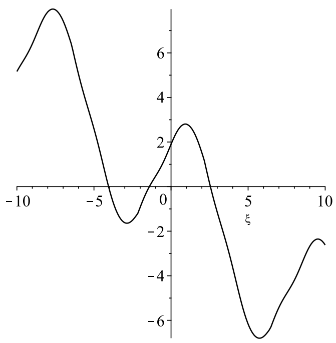


Figure 4. Trapezoidal wave solution.

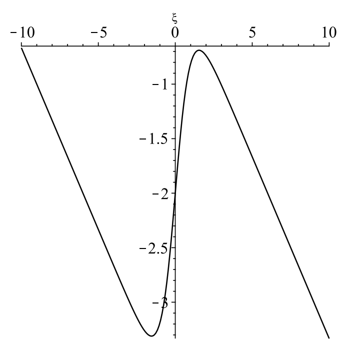


Figure 5. Inverted 'N' wave solution.

1. The KdV-nKdV equation has continuous periodic wave solution as follows

$$u_p(\xi) = \sqrt{2v_3}[E(\sqrt{\frac{v_3}{2}}\xi) + \sqrt{\frac{v_3 - v_2}{v_3}} \cdot cd(\sqrt{\frac{v_3}{2}}\xi)] + C.$$

2. The KdV-nKdV equation has no classical solitary wave solution.
3. The KdV-nKdV equation do possess novel traveling wave solutions, such as the

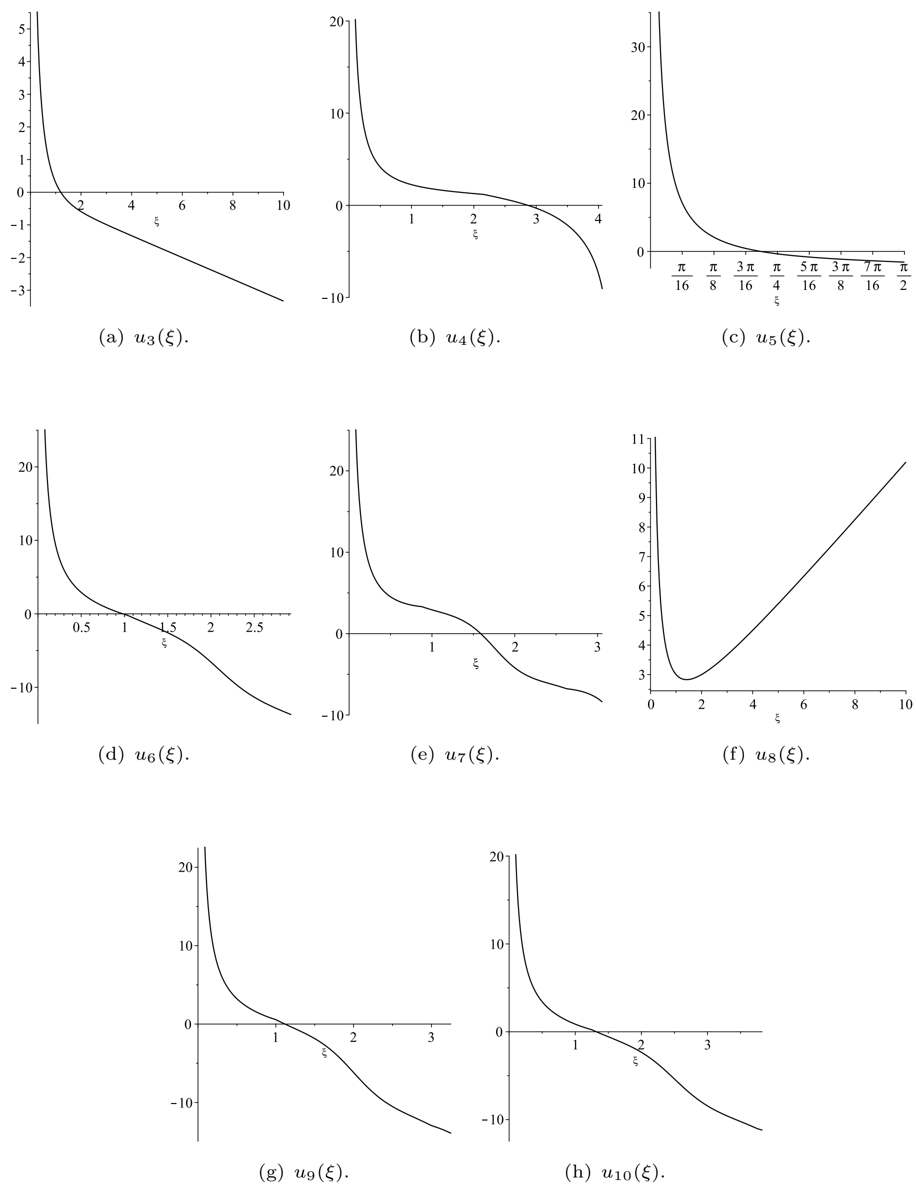


Figure 6. Blow-up wave solutions.

trapezoidal wave solution, inverted 'N' wave solution, and some blow-up wave solutions.

These solutions help us to explore the new phenomena of the KdV-nKdV equation. In addition, the strategy adopted in this paper can also be applied to the traveling wave solutions and dynamic behaviors of other nonlinear equations. Fortunately the KdV-nKdV equation is integrable, otherwise, it will be very complex to find and calculate the exact solution of the system.

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