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SOLVABILITY OF QUASILINEAR MAXWELL EQUATIONS IN EXTERIOR DOMAINS*

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Abstract In this paper we consider some q-curl-curl equations with lack of compactness. Our analysis is developed in the abstract setting of exterior domains. We first recall a decomposition of curl-free space based on L^r-Helmholtz-Weyl decomposition in exterior domains. Then by reducing the original system into a div-curl system and a p-Laplacian equation with Neumann boundary condition, we obtain the solvability of solutions for the q-curl-curl equation.

Keywords Helmholtz-Weyl decomposition, solvability, Maxwell equation, exterior domains.

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1. Introduction and main results

Let G be an exterior domain in $\mathbb{R}^3$ and satisfy some topological hypothesis (O_1)-(O_2), i.e., its complement $\tilde{G} = \mathbb{R}^3 \backslash G$ is a bounded domain with smooth boundary, first Betti number n and second Betti number zero, see section 2.2 below.

We are interested in the solvability of solutions for the semilinear Maxwell equation

$$\begin{cases} \operatorname{curl}\left[a(x) \operatorname{curl} \mathbf{u}\right] = F & \text{in } G, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } G, \\ \mathbf{u}_T = \mathbf{u}_T^0 & \text{on } \partial G, \end{cases} \tag{1.1}$$

and also the quasilinear equation

$$\begin{cases} \operatorname{curl}\left[|\operatorname{curl} \mathbf{u}|^{q-2} \operatorname{curl} \mathbf{u}\right] = F & \text{in } G, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } G, \\ \mathbf{u}_T = \mathbf{u}_T^0 & \text{on } \partial G, \end{cases} \tag{1.2}$$

where $\mathbf{u}_T = (\nu \times \mathbf{u}) \times \nu$ is the tangential component of $\mathbf{u}$ and ν is the unit outer normal vector of ∂G, $\mathbf{u}_T^0$ is a given tangential vector field on ∂G.

For the problems set on bounded domains, the q-curl-curl equation has attracted a great deal of attention in theoretical and numerical astrophysics over the past years. It arises in Born-Infeld model for nonlinear electrodynamics [6], and also in

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Bean's critical state model of type II superconductors [8]. For the latter model, we may refer the readers to [19].

Maxwell's equations [15] unifying electricity and magnetism, which were formulated more than 150 years ago, are now at the core of many of the most fundamental advances in theoretical physics and the various far-reaching technological advances. In the words of Albert Einstein, "One scientific epoch ended and another began with James Clerk Maxwell". For the foreseeable future, the significance of Maxwell's equations appear likely to dominate physics and technology.

Despite these numerous advances, there are few results on the Maxwell equation set in unbounded domains. To the best of our knowledge, Peter [21] studied the static magnetic and semi-static problem in exterior domains. The div-curl system theory in exterior domains has been established by Neudert and Wahl, see [16].

For the Maxwell equation set in exterior domains, there are not so many results. The problem becomes much more interesting since the exterior domain is a multiply connected domain with second Betti number large than zero strictly, see Section 2.2. The Maxwell equation set on multiply connected domain has been considered by Pan in [18,20], there the author pointed out that the structure of topology determines the assumption of the boundary conditions in equation. If the nonhomogeneous term is nonlinearly depends on the vector $\mathbf{u}$, then a scalar potential ∇p should be added to balance the equation in mathematical sense. Compared to the bounded domain case, the exterior domain is a multiply connected domain with more complicated topological properties. And the decomposition of vector field space differs from the bounded domain case greatly, the main feature is that the harmonic field does not always exist in the appropriate Sobolev space, For example (see [22])

$$\begin{cases} \Delta \mathbf{h} = 0 & \text{in } \ G, \\ \mathbf{h} = 0 & \text{on } \ \partial G. \end{cases} \tag{1.3}$$

It is easy to check that $\nabla \mathbf{h} \in L^s$ for $s > \frac{3}{2}$, but $\mathbf{h} \notin L^r$ for $1 < r < \infty$, which means the classical Sobolev space is not enough to discuss our problem in exterior domains. Thus, an appropriate Sobolev space $\hat{H}^{1,r}_{\bullet}(G)$ or a Beppo-Levi space (L^2 framework) is proposed, for more details, see [12,14]. On the other hand, the dimensions of the harmonic filed follows from variance of the integrality of vector filed. Hieber [11] pointed out that one critical index $r = 2$ arises in two dimension exterior domains and two thresholds $r = \frac{3}{2}$, $r = 3$ happen in the three-dimensional exterior domains.

To study the quasilinear Maxwell equation in exterior domains, we need to discuss some properties of the vector-valued function space related to the operator curl and div. Inspired by the decomposition of divergence-free space in exterior domains, see Kozono [12], we establish a decomposition of the curl-free space, see Propositions 2.1. Follow the decomposition theory of curl kernel space under L^2 and L^p framework, see [9] and [3] respectively, we prove the corresponding results for the decomposition in exterior domains. Meanwhile, we prove the equivalent relationship of divergence-free space and image space of the operator curl with some special boundary characterizations, see Proposition 2.2.

As it was shown by Pan in [17], the operators curl and div are crucial to carry out the reduction process. Having these results of decomposition in hand, we start the reduction process and prove the result of solvability to equation (1.1) and (1.2). The main results are stated in the following theorems.

Theorem 1.1. *Let G be an exterior domain in $\mathbb{R}^3$ and satisfy some topological hypothesis (O_1)-(O_2). Suppose $F \in \mathcal{W}^{2,\Gamma}(G, \mathrm{div}\, 0)$ is a given vector-valued function, and $\mathbf{u}_T^0 \in TW^{1/2,2}(\partial G, \mathbb{R}^3)$ with $\nu \cdot \mathrm{curl}\mathbf{u}_{\mathrm{T}}^0 = 0$ is a given function on ∂G. If $a(x)$ is a real, symmetric matrix-valued function on G with its reverse function $c(x)$ satisfying*

$$c_0|\xi|^2 \leq [c(x)\xi] \cdot \xi \leq c_1|\xi|^2 \quad \text{for all} \quad \xi \in G, \tag{1.4}$$

where $0 < c_0 < c_1 < \infty$. Then (1.1) has a weak solution $\mathbf{u} \in \dot{W}^{1,2}(G, \mathrm{div}\, 0)$.

Theorem 1.2. *Let G be an exterior domain in $\mathbb{R}^3$ and satisfy some topological hypothesis (O_1)-(O_2). Suppose $F \in \mathcal{W}^{q,\Gamma}(G, \mathrm{div}\, 0)$ is a given vector-valued function and $\mathbf{u}_T^0 \in TW^{1/q,q}(\partial G, \mathbb{R}^3)$ with $\nu \cdot \mathrm{curl}\mathbf{u}_{\mathrm{T}}^0 = 0$ is a given function on ∂G. If $\frac{\sqrt{5}+1}{2} \leq q \leq 2$, then (1.2) has a weak solution $\mathbf{u} \in \dot{W}^{1,q}(G, \mathrm{div}\, 0)$.*

By a reduction process, it follows that the q-curl-curl equation shares the existence of p-Laplacian systems with some variance in index. So the range of q for the solvability of equation (1.2) follows the requirement to p for the existence of p-Laplacian equation. And for equation (1.1), it can be regarded as a special case of equation (1.2) when $p = 2$.

The paper is organized as follows. In Section 2, we give some preliminaries about exterior domains especially the characteristic of new defined Sobolev space and harmonic function, to get the kernel space decomposition in exterior domains, we list some vector function space both in bounded domains and exterior domains. Simultaneously, L^r-Helmholtz-Weyl decomposition theory are recalled. We also obtain the kernel space decomposition of the operator curl and div in exterior domains. In essence speaking, they are associate with the problem of existence of some div-curl systems in exterior domains. In next subsection, we modify the reduction process in exterior domains. In Section 4, we discuss the existence of solutions for the p-Laplacian equation arises in reduction process, and armed with these results we complete the proof of Theorems 1.1 and 1.2.

2. Preliminary results

We start by introducing some notations about the Sobolev space in exterior domains.

2.1. Sobolev space in exterior domains

Hinted by example (1.3) in exterior domains, we may work in a appropriate Sobolev space defined as follow, see [22, Dedinition1.4].

$$\begin{aligned}\hat{W}_{\bullet}^{1,p}(G) :=\Big\{ & u : G \to R : u \text{ measurable}, u \in L^p(G_R) \text{ for each } R > 0, \\ & \nabla u \in L^p(G) \text{ and for each } \eta \in C_0^\infty(\mathbb{R}^n) \text{ holds } \eta u \in H_0^{1,p}(G)\Big\},\end{aligned}$$

where $G_R = G \cap B_R$ with $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$. If G is an exterior domain with ∂G Lipschitz, then it can be easily seen

$$\hat{W}_{\bullet}^{1,p}(G) :=\Big\{ u : G \to R : u \text{ measurable}, u \in L^p(G_R) \text{ for each } R > 0.$$

$$\nabla u \in L^p(G)^n,\ \text{trace } u|_{\partial G} = 0\Big\},$$

it is equivalent to the function spaces proposed in [12, p4]

$$\dot{W}_0^{1,p}(G) = \Big\{u \in \dot{W}^{1,p}; u|_{\partial G} = 0\Big\},$$

where $\dot{W}^{1,p}(G) = \Big\{[u]; u \in L^p_{loc}(\bar{G}), \nabla G \in L^p(G)\Big\}$ and $[u]$ denotes the equivalent class consisting of all u such that $v \in [u]$ implies that $u - v = const$ in G. This means $\dot{W}_0^{1,p}(G) = \hat{W}_\bullet^{1,p}(G)$, and we discuss $\dot{W}_0^{1,p}(G)$ hereafter.

To study the decomposition of vector function space, the following characteristic of Sobolev space $\dot{W}_0^{1,p}(G)$ plays an important role.

Lemma 2.1 ([12], Proposition 3.1)**.** *Let G be an exterior domain.*

(i) Let $1 < p < 3$, $p^* = \frac{3p}{3-p}$. Then

$$\hat{W}_0^{1,p}(G) = \Big\{u \in \dot{W}_0^{1,p}(G); u \in L^{p^*}(G)\Big\},$$

where $\hat{W}_0^{1,p}(G)$ is the completion of $C_0^\infty(G)$ with respect to the norm $||\nabla u||_{L^p}$.

(ii) For $\frac{3}{2} < p < 3$, we have

$$\dot{W}_0^{1,p}(G) = \hat{W}_0^{1,p}(G) \oplus \{\lambda q_0; \lambda \in \mathbb{R}\}, \quad \text{(direct sum)}$$

where q_0 is the harmonic function in $\dot{W}_0^{1,p}(G)$ with some asymptotic behavior, i.e., $q_0(x) \to 1$ as $|x| \to \infty$.

(iii) Let $3 \le p < \infty$. Then $\hat{W}_0^{1,p}(G) = \dot{W}_0^{1,p}(G)$. Moreover, there is a closed subspace $\tilde{W}_0^{1,p}(G)$ of $\dot{W}_0^{1,p}(G)$ such that

$$\dot{W}_0^{1,p}(G) = \tilde{W}_0^{1,p}(G) \oplus \{\lambda q_0; \lambda \in \mathbb{R}\}.$$

In exterior domains, harmonic function does not always exist in arbitrarily Sobolev space.

Lemma 2.2 ([22], Theorem 6.2)**.** *Let $G \subset \mathbb{R}^3$ be an exterior domain with $\partial G \in C^1$. For $1 < p < \infty$, let*

$$N^p(G) := \Big\{u \in \dot{W}_0^{1,p}(G) : \triangle u = 0,\ \text{in } G\Big\},$$

then $N^p(G) = \{0\}$ for $1 < p < \frac{3}{2}$.

2.2. Note on vector space

We introduce some notations of vector space in exterior domains, mainly in L^p framework. Most of the materials in this subsection can be found in [9, p.217]. Generally, we assume that the exterior domain G has the following connected properties.

(O_1): G is an exterior domain in $\mathbb{R}^3$ with $C^{2,\alpha}$ boundary ∂G. G is locally situated on one side of ∂G, and ∂G has a finite number of connected components $\Gamma_1, \cdots, \Gamma_{m+1}$, where $m \ge 0$ and Γ_{m+1} denoting the boundary of the bounded connected component of the set $\mathbb{R}^3 \backslash G$.

(O_2): There exist n manifolds of dimension 2 and of class C^r denoted by $\Sigma_1,\cdots,\Sigma_n$, $n\geq 0$, which are non-tangential to ∂G and $\Sigma_i\cap\Sigma_j=\emptyset$ for $i=j$, such that $\dot{G}=G\backslash(\Sigma_{j=1}^n\Sigma_j)$ is simply connected and Lipschitzian.

The numbers n and m in (O_2) and (O_1) are the first Betti number and second Betti number respectively, namely the number of singularity cohomology group. The first Betti number n of a domain $\tilde{G}\subset R^3$ is understood as the number of handles, i.e., the number of equivalence classes of simply closed curves in $\tilde{G}$ which are not null homotopic in $\tilde{G}$. From Alexander's duality theorem, we know that the number of handles of $\tilde{G}$ is equal to the number of handles of $G=\mathbb{R}^3\backslash\tilde{G}$. The second Betti number m of $\tilde{G}$ is, in our terminology, the number of bounded, connected components of the complementary domain $G=\mathbb{R}^3\backslash\tilde{G}$.

We shall list some vector space involving the operator curl and div in $L^p(G,\mathbb{R}^3)$ framework in exterior domains, see [11, 12].

$$\begin{aligned}&\mathcal{W}^p(G,\mathrm{div})=\{\mathbf{u}\in L^p(G,\mathbb{R}^3):\mathrm{div}\,\mathbf{u}\in L^p(G)\},\\&\mathcal{W}^p(G,\mathrm{curl})=\{\mathbf{u}\in L^p(G,\mathbb{R}^3):\mathrm{curl}\,\mathbf{u}\in L^p(G,\mathbb{R}^3)\},\\&\mathcal{W}^p(G,\mathrm{curl},\mathrm{div})=\mathcal{W}^p(G,\mathrm{curl})\cap\mathcal{W}^p(G,\mathrm{div}).\end{aligned}$$

For the bounded domain case in L^2 framework, it has been given in [9, 18, 20], see also [3, 14] for the bounded domain case in L^p framework. Similarly, we denote divergence-free space and curl-free space by

$$\begin{aligned}&\mathcal{W}^p(G,\mathrm{div}\,0)=\{\mathbf{u}\in\mathcal{W}^p(G,\mathrm{div}):\mathrm{div}\,\mathbf{u}=0\},\\&\mathcal{W}^p(G,\mathrm{curl}\,0)=\{\mathbf{u}\in\mathcal{W}^p(G,\mathrm{curl}\,):\mathrm{curl}\,\mathbf{u}=0\},\\&\mathcal{W}^p(G,\mathrm{curl}\,,\mathrm{div}\,0)=\mathcal{W}^p(G,\mathrm{curl}\,\mathbf{u})\cap\mathcal{W}^p(G,\mathrm{div}\,0)\}.\end{aligned}$$

As the boundary condition are given, these vector spaces are denoted by

$$\begin{aligned}&\mathcal{W}^p_{n0}(G,\mathrm{div}\,0)=\{\mathbf{u}\in\mathcal{W}^p(G,\mathrm{div}\,0):\mathbf{u}\cdot\nu=0\text{ on }\partial G\},\\&\mathcal{W}^p_{t0}(G,\mathrm{curl}\,0)=\{\mathbf{u}\in\mathcal{W}^p(G,\mathrm{curl}\,0):\mathbf{u}_T=0\text{ on }\partial G\},\\&TW^{-1/p,p}(\partial G,\mathbb{R}^3)=\{\mathbf{u}\in W^{1/p,p}(\partial G,\mathbb{R}^3):\nu\cdot\mathbf{u}=0\}.\end{aligned}$$

For more topological properties, we need the following divergence lemma. For $\phi\in L^p(G)$ and $\mathbf{u}\in L^p(G,\mathbb{R}^3)$, we have

$$\langle 1,\mathrm{div}\,(\phi\mathbf{u})\rangle_G=\langle\nabla\mathbf{u},\phi\rangle_G+\langle\ \mathrm{div}\,\phi,\mathbf{u}\rangle_G=\langle\phi,\mathbf{u}\cdot\nu\rangle_{\partial G},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of vectors in $\mathbb{R}^3$. And for $\phi,\mathbf{u}\in L^p(G,\mathbb{R}^3)$, we have

$$\langle 1,\mathrm{div}\,(\phi\times\mathbf{u})\rangle_G=\langle\phi,\nabla\times\mathbf{u}\rangle_G+\langle\ \nabla\times\mathbf{u},\phi\rangle_G=\langle\phi,\mathbf{u}\times\nu\rangle_{\partial G}.$$

Particularly,

$$\langle\phi,\eta\rangle_{\partial G,1/p}=\langle\phi,\eta\rangle_{W^{-1/p,p}(\partial G),W^{1/p,p}(\partial G)}.$$

We also need the following vector space with boundary condition on connected component:

$$\mathcal{W}^{p,\Gamma}(G,\mathrm{div}\,0)=\{\mathbf{u}\in\mathcal{W}^p(G,\mathrm{div}\,0):\langle\nu\cdot\mathbf{u},1\rangle_{W^{-1/p,p}(\Gamma_j),W^{1/p,p}(\Gamma_j)}=0\},$$

$$\mathcal{W}^{p,\Sigma}_{n0}(G,\operatorname{div}0)=\{\mathbf{u}\in\mathcal{W}^{p}_{n0}(G,\operatorname{div}0):\langle\mathbf{u}\cdot\nu,1\rangle_{W^{-1/p,p}(\Sigma_i),W^{1/p,p}(\Sigma_i)}=0\}.$$

where $j=1,\cdots,m+1; i=1,\cdots,N$. The definitions of Dirichlet and Neumann harmonic fields are given by

$$\mathbb{H}^p_{har,1}(G,\mathbb{R}^3)=\{\mathbf{u}\in L^p(G,\mathbb{R}^3):\operatorname{curl}\mathbf{u}=0\text{ and }\operatorname{div}\mathbf{u}=0\text{ in }G,\nu\cdot\mathbf{u}=0\text{ on }\partial G\},$$
$$\mathbb{H}^p_{har,2}(G,\mathbb{R}^3)=\{\mathbf{u}\in L^p(G,\mathbb{R}^3):\operatorname{curl}\mathbf{u}=0\text{ and }\operatorname{div}\mathbf{u}=0\text{ in }G,\mathbf{u}_T=0\text{ on }\partial G\}.$$

For the Sobolev spaces, we denote $\dot{W}^{k,p}(G)$ by the scalar functions space and $\dot{W}^{k,p}(G,\mathbb{R}^3)$ by the vector valued function space. We use the same notation for the norm of scalar functions and for vector fields. For instance, the norms of the functions in $\dot{W}^{k,p}(G)$, and that of the vector fields in $\dot{W}^{k,p}(G,\mathbb{R}^3)$ are both denoted by $||\cdot||_{\dot{W}^{k,p}(G)}$. Moreover, we define the following vector valued function space

$$\begin{aligned}
&\dot{W}^{1,p}(G,\operatorname{div}0)=\{\mathbf{u}\in\dot{W}^{1,p}(G,\mathbb{R}^3):\operatorname{div}\mathbf{u}=0\},\\
&\dot{W}^{1,p}(G,\operatorname{curl}0)=\{\mathbf{u}\in\dot{W}^{1,p}(G,\mathbb{R}^3):\operatorname{curl}\mathbf{u}=0\},\\
&\dot{W}^{1,p}_{n0}(G,\operatorname{div}0)=\{\mathbf{u}\in\dot{W}^{1,p}(G,\operatorname{div}0):\mathbf{u}\cdot\nu=0\text{ on }\partial G\},\\
&\dot{W}^{1,p}_{t0}(G,\operatorname{div}0)=\{\mathbf{u}\in\dot{W}^{1,p}(G,\operatorname{div}0):\mathbf{u}_T=0\text{ on }\partial G\},\\
&\dot{W}^{1,p}_{n0}(G,\operatorname{curl}0)=\{\mathbf{u}\in\dot{W}^{1,p}(G,\operatorname{curl}0):\mathbf{u}\cdot\nu=0\text{ on }\partial G\},\\
&\dot{W}^{1,p}_{t0}(G,\operatorname{curl}0)=\{\mathbf{u}\in\dot{W}^{1,p}(G,\operatorname{curl}0):\mathbf{u}_T=0\text{ on }\partial G\}.
\end{aligned}$$

2.3. Decomposition of curl-free space

Similar to the decomposition of vector valued function space in a bounded domain, we have the following decomposition in exterior domains:

$$L^p(G,\mathbb{R}^3)=\mathcal{W}^p_{n0}(G,\operatorname{div}0)\oplus\operatorname{grad}\dot{W}^{1,p}(G), \tag{2.1}$$
$$L^p(G,\mathbb{R}^3)=\mathcal{W}^p_{t0}(G,\operatorname{div}0)\oplus\operatorname{grad}\dot{W}^{1,p}_0(G), \tag{2.2}$$
$$L^p(G,\mathbb{R}^3)=\mathcal{W}^p_{t0}(G,\operatorname{curl}0)\oplus\operatorname{curl}\dot{W}^{1,p}_{n0}(G,\operatorname{div}0), \tag{2.3}$$
$$L^p(G,\mathbb{R}^3)=\mathcal{W}^p_{n0}(G,\operatorname{curl}0)\oplus\operatorname{curl}\dot{W}^{1,p}_{t0}(G,\operatorname{div}0). \tag{2.4}$$

As for the decomposition of divergence-free space in first and second line of (2.1), Konozo [12, Theorem 2.3] provided the following Helmholtz-Weyl decomposition

$$\begin{aligned}
&L^p(G,\mathbb{R}^3)=\operatorname{grad}\dot{W}^{1,p}(G)\oplus\mathbb{H}^p_{har,1}(G,\mathbb{R}^3)\oplus\operatorname{curl}\dot{W}^{1,p}_{t0}(G,\operatorname{div}0),\\
&L^p(G,\mathbb{R}^3)=\operatorname{grad}\dot{W}^{1,p}_0(G)\oplus\mathbb{H}^p_{har,2}(G,\mathbb{R}^3)\oplus\operatorname{curl}\dot{W}^{1,p}_{n0}(G,\operatorname{div}0), 1<p\le\frac{3}{2}, p\ge 3,\\
&L^p(G,\mathbb{R}^3)=\operatorname{grad}\hat{W}^{1,p}_0(G)\oplus\mathbb{H}^p_{har,2}(G,\mathbb{R}^3)\oplus\operatorname{curl}\dot{W}^{1,p}_{n0}(G,\operatorname{div}0), \frac{3}{2}<p<3.
\end{aligned} \tag{2.5}$$

The authors also remarked that

$$dim\mathbb{H}^p_{har,1}(G,\mathbb{R}^3)=n,$$
$$dim\mathbb{H}^{p_1}_{har,2}(G,\mathbb{R}^3)=dim\mathbb{H}^{p_2}_{har,2}(G,\mathbb{R}^3)-1=m-1,\quad 1<p_1<\frac{3}{2}<p_2<\infty,$$

where m and n are the numbers given in (O_1) and (O_2). Based on these identities, one can easily derive the decomposition of curl$-$free space as follow. For readers convenient, we repeat the part proofs of Konozo [12].

Proposition 2.1. *(i) For* $\mathbf{u} \in L^p(G, \mathbb{R}^3)$ *with* $\operatorname{curl} \mathbf{u} = 0$ *in* G *and* $\nu \cdot \mathbf{u} = 0$ *on* ∂G, *there exists* $p \in \dot{W}^{1,p}(G)$ *and* $\mathbf{h_1} \in \mathbb{H}^p_{har,1}(G, \mathbb{R}^3)$ *such that* $\mathbf{u} = \nabla p + \mathbf{h_1}$, *namely*

$$\mathcal{W}^p_{n0}(G, \operatorname{curl} 0) = \operatorname{grad} \dot{W}^{1,p}(G) \oplus \mathbb{H}^p_{har,1}(G, \mathbb{R}^3).$$

(ii) For $\mathbf{u} \in L^p(G, \mathbb{R}^3)$ *with* $\operatorname{curl} \mathbf{u} = 0$ *in* G *and* $\mathbf{u}_T = 0$ *on* ∂G, *there exist* $p \in \dot{W}^{1,p}_0(G)$ *and* $\mathbf{h_2} \in \mathbb{H}^p_{har,2}(G, \mathbb{R}^3)$ *with* $1 < p < 3$ *such that* $\mathbf{u} = \nabla p + \mathbf{h_2}$, *namely*

$$\mathcal{W}^p_{t0}(G, \operatorname{curl} 0) = \operatorname{grad} \dot{W}^{1,p}_0(G) \oplus \mathbb{H}^p_{har,2}(G, \mathbb{R}^3), 1 < p \leq \frac{3}{2} \text{ or } p > 3,$$

$$\mathcal{W}^p_{t0}(G, \operatorname{curl} 0) = \operatorname{grad} \hat{W}^{1,p}_0(G) \oplus \mathbb{H}^p_{har,2}(G, \mathbb{R}^3), \frac{3}{2} < p < 3.$$

Proof. Indeed, for $\mathbf{h}_1 = \nabla q \in \mathbb{H}^{1,p}_{har,1}$, there exists $q \in \dot{W}^{1,p}(\Gamma_{m+1})$ or $q \in \dot{W}^{1,p}_0(\Gamma_{m+1})$ such that

$$\begin{cases} \Delta q = 0 & \text{in } \Gamma_{m+1}, \\ \dfrac{\partial q}{\partial n} = 0 & \text{on } \Gamma, \\ [q]_{\Sigma_i} = constant & i = 1 \text{ to } N, \\ [\dfrac{\partial q}{\partial n}]_{\Sigma_i} = 0, & i = 1 \text{ to } N, \end{cases} \tag{2.6}$$

see the same analysis in [9, P. 219]. This implies that the spaces $\mathbb{H}^p_{har,2}(G, \mathbb{R}^3)$ and $\operatorname{grad} \dot{W}^{1,p}_0(G)$ are orthogonal in $\mathcal{W}^p_{t0}(G, \operatorname{curl} 0)$. As for the threshold value $p = \frac{3}{2}$ and $p = 3$ arisen in case (ii), it can be seen from the characteristic of Sobolev space and harmonic field, see more details in Lemma 2.1 and Lemma 2.2. □

The following proposition characterize the image space of the operator curl and the kernel space of the operator div.

Proposition 2.2. *For* $\mathbf{u} \in L^p(G, \mathbb{R}^3)$ *with* $\operatorname{div} \mathbf{u} = 0$, $\int_{\Gamma_i} \mathbf{u} \cdot n \, d\Gamma = 0, i = 0$ *to* m, *there exists* $\mathbf{w} \in \dot{W}^{1,p}_{n0}(G, \operatorname{div} 0)$, *or* $\mathbf{w} \in \dot{W}^{1,p}(G, \mathbb{R}^3)$ *such that* $\mathbf{u} = \operatorname{curl} \mathbf{w}$, *namely*

$$\mathcal{W}^{p,\Gamma}(G, \operatorname{div} 0) = \operatorname{curl} \dot{W}^{1,p}(G, \mathbb{R}^3) = \operatorname{curl} \dot{W}^{1,p}_{n0}(G, \operatorname{div} 0).$$

Proof. We prove the second identity firstly. For $\mathbf{v} \in \dot{W}^{1,p}(G, \mathbb{R}^3)$, we have $\Phi \in \dot{W}^{1,p}(G)$ (see [12, P.36, line 5]) satisfies the following elliptic equation with Neumann boundary condition, i.e.,

$$\begin{cases} \Delta \Phi = \operatorname{div} \mathbf{v} & \text{in } G, \\ \dfrac{\partial \Phi}{\partial \nu} = \mathbf{v} \cdot n & \text{on } \partial G. \end{cases}$$

Then from the regularity estimate of Φ in [12, P.36], we have $\Phi \in \dot{W}^{2,p}(G, \mathbb{R}^3)$. Setting $\mathbf{w} = \mathbf{v} - \nabla \Phi$, we can easily check that $\operatorname{div} \mathbf{w} = 0$ in G and $\mathbf{w} \cdot n = 0$ on ∂G. This implies $\operatorname{curl} \dot{W}^{1,p}(G, \mathbb{R}^3) = \operatorname{curl} \dot{W}^{1,p}_{n0}(G, \operatorname{div} 0)$.

Next, we prove the first identity. Follow the idea of extend method from [2, Theorem 4.1], there exists an auxiliary function $\tilde{\mathbf{u}} \in L^p(\mathbb{R}^3, \mathbb{R}^3)$ satisfying

$$\tilde{\mathbf{u}} = \begin{cases} \mathbf{u} & \text{in } G, \\ \nabla \chi_i & \text{in } G_i \\ 0 & \text{in } \mathbb{R}^3 \setminus \Gamma_{m+1}, \end{cases}$$

where $\chi_{\mathbf{i}} \in \dot{W}^{1,p}(G), i = 0, 1, ..., n$ satisfies

$$-\triangle \chi_{\mathbf{i}} = 0 \ \text{ in } G, \quad \partial_n \chi_{\mathbf{i}} = \mathbf{u} \cdot n \ \text{ on } \ \Gamma_i,$$

and specially, $\partial \chi_{\mathbf{0}} = 0$ on $\partial \Gamma_{m+1}$ with the estimate of $\|\chi_i\|_{\dot{W}^{1,p}(G)} \leq \|\mathbf{u}\|_{L^p(G)}$. one can easily check that $\tilde{\mathbf{u}} \in \mathcal{W}^p(\operatorname{div}, \mathbb{R}^3)$ with free divergence, namely $\tilde{\mathbf{u}} \in \mathcal{W}^{p,\Gamma}(\operatorname{div} 0, \mathbb{R}^3)$. Taking the function $\psi_0 = \operatorname{curl}(E * \tilde{\mathbf{u}})$ satisfies

$$\operatorname{curl} \psi_0 = \tilde{\mathbf{u}}, \qquad \operatorname{div} \psi_0 = 0 \ \text{ in } \ \mathbb{R}^3,$$

and applying the Calderón-Zygmund inequality, we obtain the following estimate

$$\|\nabla \psi_0\|_{L^p(\mathbb{R}^3)} \leq \ C\|\triangle(E * \tilde{\mathbf{u}})\|_{L^p(\mathbb{R}^3)} \leq \ C\|\tilde{\mathbf{u}}\|_{L^p(\mathbb{R}^3)} \leq \ C\|\mathbf{u}\|_{L^p(G)},$$

where E is the fundamental solution of the Laplacian. Due to Proposition 2.10 of [1], we have $\psi_0|_G$ belongs to $\dot{W}^{1,p}(G)$.

On the contrary, for a cut-off function $\mu_i \in C^\infty(G)$ which equals to 1 in a neighborhood of Γ_i and vanishes in a neighborhood of Γ^k for $0 < k < i$, we have

$$\begin{cases} \operatorname{div}(\operatorname{curl} \psi_0) = 0 \ \text{ in } \ G, \\ \langle \mathbf{u} \cdot n, 1 \rangle_{\Gamma_j} = \langle \operatorname{curl}(\mu_i \psi_0) \cdot n, 1 \rangle_{\partial G} = \displaystyle\int_G \operatorname{div}(\operatorname{curl}(\mu_i \psi_0)) dx = 0 \qquad \text{on } \ \partial G. \end{cases}$$

This means that any elements in $\operatorname{curl} \dot{W}^{1,p}(G, \mathbb{R}^3)$ are divergence free, and hence belong to $\mathcal{W}^{p,\Gamma}(G, \operatorname{div} 0)$. □

Remark 2.1. Similarly, we have

$$\mathcal{W}^{p,\Sigma}_{n0}(G, \operatorname{div} 0) = \operatorname{curl} \dot{W}^{1,p}_{t0}(G, \operatorname{div} 0).$$

Proof. From the first line of equations (2.1) and the first line of (2.5), we have

$$\mathcal{W}^p_{n0}(G, \operatorname{div} 0) = \mathbb{H}^p_{har,1}(G, \mathbb{R}^3) \oplus \operatorname{curl} \dot{W}^{1,p}_{t0}(G, \operatorname{div} 0). \tag{2.7}$$

On the other hand, by applying Green's formula that the orthogonal complement of the space $\mathbb{H}^p_{har,1}(G, \mathbb{R}^3)$ in $\mathcal{W}^p_{n0}(G, \operatorname{div} 0)$ is the space of elements $\mathbf{u} \in \mathcal{W}^p_{n0}(G, \operatorname{div} 0)$ satisfying $\int_{\Sigma_i} \mathbf{u} \cdot \nu = 0$, $i = 1, ..., n$, that is $\mathcal{W}^{p,\Sigma}_{n0}(G, \operatorname{div} 0)$. Then, we get our conclusion. □

3. Reduction process in exterior domains

3.1. Reduction process

Based on the reduction process on bounded domains, see the work of Pan [17], we provide a modified version in exterior domains and consider the following model problem

$$\begin{cases} \operatorname{curl}[\mathcal{H}(x, \operatorname{curl} \mathbf{u})] = F & \text{in } \ G, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \ G, \\ \mathbf{u}_T = \mathbf{u}^0_T & \text{on } \ \partial G. \end{cases} \tag{3.1}$$

We shall suppose that the nonhomogeneous $F \in \mathcal{W}^{q,\Gamma}(G, \operatorname{div} 0)$ is a vector-valued function, and let the structure function satisfy a reverse function condition

$$\mathbf{w} = \mathcal{H}(x, \mathbf{z}) \quad \text{if and only if} \quad \mathbf{z} = \mathcal{B}(x, \mathbf{w}) \quad \text{for all } x \in G, \tag{3.2}$$

moreover, there exists a function $h(x,t)$ such that

$$\nu \cdot \mathcal{H}(x, \mathbf{z}) = h(x, \nu \cdot \mathbf{z}) \quad \text{for all } x \in \partial G. \tag{3.3}$$

Then by Proposition 2.2, one can find a vector function $\mathbf{Q} \in \dot{W}^{1,q}_{n0}(G, \mathbb{R}^3)$ such that

$$\begin{cases} \operatorname{curl} \mathbf{Q} = F & \text{in } G, \\ \operatorname{div} \mathbf{Q} = 0 & \text{in } G, \\ \nu \cdot \mathbf{Q} = 0 & \text{on } \partial G. \end{cases} \tag{3.4}$$

Combining equation (3.1) and (3.4), we deduce that

$$\operatorname{curl} [\mathcal{H}(x, \operatorname{curl} \mathbf{u}) - \mathbf{Q}] = 0.$$

By the decomposition of curl-free space in case (i) of Proposition 2.1, one can find $p \in \dot{W}^{1,q}(G)$ and $\mathbf{h_1} \in \mathbb{H}^q_{har,1}(G, \mathbb{R}^3)$ such that

$$[\mathcal{H}(x, \operatorname{curl} \mathbf{u}) - \mathbf{Q}] = \nabla p + \mathbf{h_1},$$

where the harmonic function $\mathbf{h_1}$ satisfies the homogeneous div-curl equation

$$\begin{cases} \operatorname{curl} \mathbf{h_1} = 0 & \text{in } G, \\ \operatorname{div} \mathbf{h_1} = 0 & \text{in } G, \\ \nu \cdot \mathbf{h_1} = 0 & \text{on } \partial G. \end{cases} \tag{3.5}$$

Then, by reverse function condition assumption (3.2) and (3.3), one has

$$\begin{cases} \operatorname{curl} \mathbf{u} = \mathcal{B}(x, \nabla p + \mathbf{h_1} + \mathbf{Q}) & \text{in } G, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } G, \\ \mathbf{u}_T = \mathbf{u}^0_T & \text{on } \partial G. \end{cases} \tag{3.6}$$

Using the fact $\operatorname{div}(\operatorname{curl} \mathbf{u}) = 0$, we can take divergence on both sides of the first equation of (3.6), and get an elliptic equation with translate term $\mathbf{h_1} + \mathbf{Q}$

$$\begin{cases} \operatorname{div}\ \mathcal{B}(x, \nabla p + \mathbf{h_1} + \mathbf{Q}) = 0 & \text{in } G, \\ \dfrac{\partial p}{\partial \nu} = h(x, \nu \cdot \operatorname{curl} \mathbf{u}^0_T) & \text{on } \partial G. \end{cases} \tag{3.7}$$

Then equation (3.6) has a weak solution if and only if equation (3.7) has a weak solution p satisfies the following orthogonality condition

$$\int_G \langle \mathcal{B}(x, \nabla p + \mathbf{h_1} + \mathbf{Q}), \mathbf{h}_1 \rangle dx = 0, \tag{3.8}$$

see [20, Lemma 4.5].

To get the solvability of $\mathbf{u}$ in (3.6), we shall firstly get the solvability of $\mathbf{Q}$ in nonhomogeneous div-curl equation (3.4), and the same as $\mathbf{h_1}$ in homogeneous div-curl equation (3.5), we can also get the existence of scalar potential p in equation (3.7) in divergence form, see [19, Lemma A.1], [10, p.1110] and [7]. For the weak solution p of equation (3.7) satisfies equation (3.8), we just need to check that $\mathcal{B}(x, \mathbf{z})$ satisfying conditions $(B1)$ and $(B2)$ in [20], see [20, Proposition 4.7]. Lastly, for $\mathbf{Q}, \mathbf{h}$ and p given, we can obtain the solvability of $\mathbf{u}$ in nonhomogeneous div-curl equation (3.6).

4. Solvability of Maxwell equation

Following the reduction process in exterior domains, we start to prove the solvability of equation (1.1).

4.1. Proof of Theorem 1.1

Proof. Set $c(x) = a(x)^{-1}$, then (3.6) is transformed into

$$\begin{cases} \operatorname{curl} \mathbf{u} = c(x)(\nabla p + \mathbf{h_1} + \mathbf{Q}) & \text{in } G, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } G, \\ u_T = \mathbf{u}_T^0 & \text{on } \partial G, \end{cases} \tag{4.1}$$

where $\operatorname{div} \mathbf{Q} = 0$ and $\operatorname{div} \mathbf{h_1} = 0$. Similarly, we transform (3.7) into

$$\begin{cases} \operatorname{div} [c(x)\nabla p] = -\nabla c(x) \cdot \mathbf{Q} & \text{in } G, \\ c(x)\dfrac{\partial p}{\partial \nu} = \nu \cdot \operatorname{curl} u_T^0 & \text{on } \partial G. \end{cases} \tag{4.2}$$

Easily check that $\mathcal{B}(x, \mathbf{z}) = c(x)(\nabla p + \mathbf{h_1} + \mathbf{Q})$ satisfies $(B1)$ and $(B2)$ in [20], then one may observe that the existence of (4.1) is equivalent to the existence of equation (4.2) cause we already have $\mathbf{Q} \in \dot{W}^{2,\Gamma}(G, \operatorname{div} 0)$ as well as $\mathbf{h_1} \in \mathbb{H}^2_{har,1}(G, \mathbb{R}^3) \subset \dot{W}^{2,\Gamma}(G, \operatorname{div} 0)$. Indeed, the existence of equation (4.2) in exterior regions has been studied by Auchmuty and Han in [4]. Set $f(x) = -\nabla c(x) \cdot \mathbf{Q}$, and $g(x) = \nu \cdot \operatorname{curl} u_T^0 = 0$ on ∂G, then easily check that $f(x) \in L^{\frac{6}{5}}(G)$, where $\frac{6}{5} = \frac{2N}{N+2} = p_s^*$, we also have $g(x) \in L^{p_T^*}(\partial G, d\sigma)$ satisfies condition (A3) in [4].

Finally, combining the existence of $\mathbf{Q}, \mathbf{h_1}, p$, we have $\mathbf{u} \in W^{1,2}(G, \operatorname{div} 0)$ is a weak solution of equation (1.1), means Theorem 1.1 holds. □

For the quasilinear equation (1.2), it quietly relays on the solvability of the p-Laplacian equation.

4.2. Proof of Theorem 1.2

Proof. Firstly, we rewrite equation (3.6) and (3.7) as above. Similarly, there exist $\mathbf{Q} \in \dot{W}^{1,q}_{n0}(G, \mathbb{R}^3)$, $\mathbf{h_1} \in \mathbb{H}^q_{har,1}(G, \mathbb{R}^3)$ and $p \in \dot{W}^{1,q}(G)$ such that

$$|\operatorname{curl} \mathbf{u}|^{q-2} \operatorname{curl} u = \mathbf{Q} + \mathbf{h_1} + \nabla p.$$

By a reformulation, we have

$$\begin{cases} \operatorname{curl} \mathbf{u} = |\mathbf{Q} + \mathbf{h_1} + \nabla p|^{\frac{2-q}{q-1}}(\mathbf{Q} + \mathbf{h_1} + \nabla p) & \text{in } G, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } G, \\ \mathbf{u}_T = \mathbf{u}_T^0 & \text{on } \partial G. \end{cases} \tag{4.3}$$

Easily check that $\mathcal{B}(x, \mathbf{z}) = \mathcal{B}(\mathbf{z}) = |\mathbf{Q} + \mathbf{h_1} + \nabla p|^{\frac{2-q}{q-1}}(\mathbf{Q} + \mathbf{h_1} + \nabla p)$ satisfies $(B1)$ and $(B2)$ in [20], and by repeating the identity $\operatorname{div}(\operatorname{curl} u) = 0$, we have

$$\begin{cases} \operatorname{div}\left[|\mathbf{Q} + \mathbf{h_1} + \nabla p|^{p-2}(\mathbf{Q} + \mathbf{h_1} + \nabla p)\right] = 0 & \text{in } G, \\ |\nabla p + \mathbf{h_1} + \mathbf{Q}|^{p-2}\dfrac{\partial p}{\partial \nu} = \nu \cdot \operatorname{curl} u_T^0 & \text{on } \partial G, \end{cases}$$

where $p = \frac{q}{q-1}$. And since $\nu \cdot \operatorname{curl} u_T^0 = 0$ on ∂G, we have

$$\begin{cases} \operatorname{div}\left[|\mathbf{Q} + \mathbf{h_1} + \nabla p|^{p-2}(\mathbf{Q} + \mathbf{h_1} + \nabla p)\right] = 0 & \text{in } G, \\ \dfrac{\partial p}{\partial \nu} = 0 & \text{on } \partial G. \end{cases} \tag{4.4}$$

Secondly, we discuss the existence of p-Laplacian equation (4.4) in exterior domains. Actually, this similar work is well done by Auchmuty and Han in [5]. We can observe that equation (4.4) is not the standard p-Laplacian equation, then, we need reformulate it as the following equation and verify that the p-Laplacian equation satisfies condition (a) which is required in the work of [5, Theorem 6.1]

$$\begin{cases} -\operatorname{div}(|\nabla p|^{p-2}\nabla p) = \operatorname{div}(|A|^{p-2}A - |\nabla p|^{p-2}\nabla p) & \text{in } G, \\ \dfrac{\partial p}{\partial \nu} = 0 & \text{on } \partial G, \end{cases} \tag{4.5}$$

where $A = \mathbf{Q} + \mathbf{h_1} + \nabla p$. We shall discuss the p-Laplacian equation on $\dot{W}^{1,p}(G)$ with $1 < p < 3$, which is required in [5]. Moreover, since solution $p \in \dot{W}^{1,p}(G)$ of equation (4.5) also belongs to $\dot{W}^{1,q}(G)$, we hence have $\frac{p}{p-1} = q \leq p$, correspondingly, $2 \leq p < 3$. Following the strategy of [19, Lemma 4.3], we consider (4.5) with the nonhomogeneous term $f(x,p) = \operatorname{div}\Phi$ in the case of $2 \leq p < 3$, where $\Phi = |A|^{p-2}A - |\nabla p|^{p-2}\nabla p$.

We may observe that equation (4.5) can be recognize as the Euler equation for the variational integral functional

$$\begin{aligned} \mathfrak{F}(p) = ||p||^p_{\dot{W}^{1,p}(G)} + \mathfrak{F}_0 &= \int_G |\nabla p|^p dx - \int_G (\operatorname{div}\Phi \cdot p) dx \\ &= \int_G |\nabla p|^p dx + \langle \Phi, \nabla p\rangle_G - \langle \Phi \cdot \nu, p\rangle_{\partial G} \\ &= \int_G |\nabla p|^p dx + \langle \Phi, \nabla p\rangle_G, \end{aligned} \tag{4.6}$$

where $\langle \cdot, \cdot \rangle_G$ represents a pair of vector field. To estimate the term of $\mathfrak{F}_0$, we list some elemental inequalities which can be find in Lemma 5.2 of [10] and (1.8), (1.5) of [13]:

$$||a|^{p-2}a - |b|^{p-2}b| \leq \gamma|a-b|^{p-1}, 1 < p < 2, \gamma = \gamma(p). \tag{4.7}$$

For $2 < p < 3, \varepsilon > 0$,

$$||a-b|^{p-2}(a-b)-|b|^{p-2}b| \leq ((1+\varepsilon)^{p-2}-1)|a|^{p-1}+(1+\frac{1}{\varepsilon})^{p-2}|b|^{p-1}. \tag{4.8}$$

For $2 < p < 3$,

$$\langle |a|^{p-2}a-|b|^{p-2}b, a-b\rangle \geq \frac{1}{2^{p-1}}|a-b|^p. \tag{4.9}$$

Then, by (4.8), we deduce that

$$|\Phi| = |\mathbf{Q}+\mathbf{h_1}+\nabla p|^{p-2}(\mathbf{Q}+\mathbf{h_1}+\nabla p)-|\nabla p|^{p-2}\nabla p| \tag{4.10}$$
$$\leq C_1|\mathbf{Q}+\mathbf{h_1}|^{p-1}+C_2|\nabla p|^{p-1}, \tag{4.11}$$

where $C_1 = [(1+\varepsilon)^{p-1}-1], C_2 = (1+\varepsilon^{-1})^{p-1}$. Remember that $\frac{\sqrt{5}+1}{2} \leq q \leq 2$ and $2 \leq p \leq \frac{\sqrt{5}+3}{2}$, we deduce that

$$\begin{aligned}
\mathfrak{F}_0 &= -\int_G (\operatorname{div}\Phi\cdot p)dx = \langle \Phi, \nabla p\rangle_G \\
&\leq \int_G |\Phi||\nabla p|dx \leq \int_G (C_1|\mathbf{Q}+\mathbf{h_1}|^{p-1}|\nabla p|+C_2|\nabla p|^p)dx \\
&\leq C_3\int_G (|\mathbf{Q}+\mathbf{h_1}|^{p-1})^{\frac{q}{p-1}}dx + C(\varepsilon)\int_G |\nabla p|^{\frac{q}{q-p+1}}dx + C_2\int_G |\nabla p|^p)dx \\
&\leq C_3\int_G |\mathbf{Q}+\mathbf{h_1}|^q dx + C_4\int_G |\nabla p|^p dx,
\end{aligned} \tag{4.12}$$

where $C_3 = C_1+\varepsilon$ and $C_4 = C_2 + C(\varepsilon)$ from Young's inequality. Since $\mathbf{Q} \in L^q(G,\mathbb{R}^3)$, $\mathbf{h_1} \in L^q(G,\mathbb{R}^3)$, we have $\mathfrak{F}_0 \leq C_5(\varepsilon, G) + ||p||^p_{\dot{W}^{1,p}(G)}$. This implies that $\mathfrak{F}_0$ is weakly l.s.c. on $\dot{W}^{1,p}(G)$.

On the other hand, by (4.9), for $2 < p < 3$, we have

$$|\Phi| = |\mathbf{Q}+\mathbf{h_1}+\nabla p|^{p-2}(\mathbf{Q}+\mathbf{h_1}+\nabla p)-|\nabla p|^{p-2}\nabla p| \geq C_6|\mathbf{Q}+\mathbf{h_1}|^{p-1}, \tag{4.13}$$

where $C_6 = 2^{p-1}$. Similarly, we have

$$\begin{aligned}
\mathfrak{F}_0 &= -\int_G (\operatorname{div}\Phi\cdot p)dx = \langle \Phi, \nabla p\rangle_G \\
&\geq C(\theta)\int_G |\Phi||\nabla p|dx \geq C_7\int_G |\mathbf{Q}+\mathbf{h_1}|^{p-1}|\nabla p|dx \\
&\geq C_7(\int_{G(|\mathbf{Q}+\mathbf{h_1}|>|\nabla p|)} |\nabla p|^p dx + \int_{G(|\mathbf{Q}+\mathbf{h_1}|\leq|\nabla p|)} |\mathbf{Q}+\mathbf{h_1}|^p dx) \\
&\geq C_7(\int_{G(|\mathbf{Q}+\mathbf{h_1}|>|\nabla p|)} |\nabla p|^p dx) + C_8(\int_{G(|\mathbf{Q}+\mathbf{h_1}|\leq|\nabla p|)} |\mathbf{Q}+\mathbf{h_1}|^q dx) \\
&\geq C_9 + C_{10}||p||^{p/Ps}_{Ps},
\end{aligned} \tag{4.14}$$

where $C(\theta) \leq cos^{-1}\langle \Phi, \nabla p\rangle$, $C_7 = C(\theta)C_6$, $Ps = \frac{3p}{3-p}$. This implies that $\mathfrak{F} = ||p||^p_{\dot{W}^{1,p}(G)} + \mathfrak{F}_0$ is coercive. Finally, applying the same ideas developed in [5, Theorem 6.1], we obtain the existence of equation (4.5), it means that there exists a weak solution $\mathbf{u} \in \dot{W}^{1,q}(G, \operatorname{div} 0)$ solves equation (4.3), hence, Theorem 1.2 holds. □

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