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NONTRIVIAL GENERALIZED SOLUTION OF SCHRÖDINGER-POISSON SYSTEM IN $\mathbb{R}^3$ WITH ZERO MASS AND PERIODIC POTENTIAL*

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Abstract In this paper, we are concerned with a class of Schrödinger-Poisson systems in $\mathbb{R}^3$ with zero mass and periodic potential. Under some 3-superlinear assumptions on the nonlinearity, one nontrivial generalized solution is obtained by a combination of variational methods and perturbation method.

Keywords Zero mass, perturbation method, nontrivial generalized solution, Pohožaev-Palais-Smale sequence.

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1. Introduction

In this paper we consider the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + \phi u = K(x)f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

We assume that

(f_1) f is a continuous function defined on $\mathbb{R}$ and satisfies

$$\lim_{t\to 0} \frac{f(t)}{t^5} = \lim_{t\to\infty} \frac{f(t)}{t^5} = 0.$$

(f_2) There exist $\alpha \in (3,6)$ and $R > 0$ such that

$$\inf_{|t|\geq R} F(t) > 0, \ f(t)t \geq \alpha F(t), \ \text{for } t \in \mathbb{R},$$

where $F(t) = \int_0^t f(\xi)d\xi$.

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(K) For the case $\alpha \in (3,4)$, K is a positive constant function; for the case $\alpha \in [4,6)$, $K \in C(\mathbb{R}^3,(0,\infty))$ is a 1-periodic potential function, that is,

$$K(x+y) = K(x), \text{ for every } x \in \mathbb{R}^3 \text{ and } y \in \mathbb{Z}^3.$$

For simplicity, we assume $\max_{x\in\mathbb{R}^3} K(x) = 1$ for all $\alpha \in (3,6)$.

Schrödinger-Poisson system arises in many mathematical physics contexts, such as in quantum electrodynamics, to describe the interaction between a charge particle interacting with the electromagnetic field. In the mathematical literatures, [6] proposed an abstract framework to deal with it via variational methods. Since then, it has gradually become a hot spot for constant attention. A great deal of research results have been obtained, for example, [2,3,6,8–13,17,20,24,26].

Formally, the main difference between system (1.1) and the classical Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \phi u = g(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

is that there is no linear term of u in the first equation of system (1.2). In fact, system (1.1) is a class of zero mass problem since (f_1) implies that $f'(0) = 0$. The studies on problems with zero mass have been concerned in [1,4,5,7,14,18,19,21,25] and the references therein. In these aspects, Berestycki and Lions [7] established the well-known Berestycki-Lions condition which is an almost optimal condition for the existence of nontrivial solutions to Schrödinger equation with zero mass. Alves etc [1] obtained one positive solution for elliptic equations in $\mathbb{R}^N$ with zero mass and potentials of some integrability or asymptotically periodic assumptions. Azzollini etc [4] studied a class of Klein-Gordon-Maxwell system with zero mass and 4-superlinear nonlinearities by a perturbation method. Schrödinger-Poisson system with zero mass was first noticed by Ruiz [21]. By introducing a new class of Sobolev type space involving Coulomb energy, one positive solution was got by the compactness concentration principle method in [21] for the following nonlocal problem with zero mass

$$-\Delta u + \left(u^2 * \frac{1}{|x|}\right) u = |u|^{p-2}u, \ x \in \mathbb{R}^3, \tag{1.3}$$

where $p \in (\frac{18}{7}, 3)$. Later, the ground and bound states were obtained in [14] for equation (1.3) with $p \in (3,6)$. The existence of radial solutions for equation (1.3) with $p = 3$ was also got in [14]. Furthermore, it was pointed out in [14] that $p = 3$ is "critical" for equation (1.3). Mercuri etc [19] studied a general type of Schrödinger-Poisson-Slater system with Riesz potential. Under the variational setting of [21], Yang and Liu [25] obtained infinitely many solutions for a class of Schrödinger-Poisson-Slater system with a combination of sublinear and Sobolev critical terms by a truncation technique and Krasnoselskii genus theory. Recently, Liu and Moroz [18] studied the asymptotic profile of ground states for a class of Schrödinger-Poisson-Slater system for the case $p \in (3,6)$.

Before stating our main result, we give several notations. For any $q \in [1,+\infty]$, we denote by $|\cdot|_q$ the norm of the space $L^q(\mathbb{R}^3)$. $D^{1,2}(\mathbb{R}^3)$ is the space defined as the completion of the functions $C_0^\infty(\mathbb{R}^3)$ with respect to the L^2 norm of the gradient.

For every $\varepsilon \in (0,1]$, $H^1_\varepsilon(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \ |\nabla u| \in L^2(\mathbb{R}^3)\}$ is a Hilbert space equipped with the following norm and inner product

$$\|u\|_\varepsilon = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + \varepsilon u^2)\ dx\right)^{\frac{1}{2}},\ (u,v)_\varepsilon = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \varepsilon uv)dx.$$

As shown in [4], a generalized solution for system (1.1) is a pair $(u,\phi) \in D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ such that

$$\begin{cases} \int_{\mathbb{R}^3} (\nabla u \cdot \nabla \psi + \phi u \psi)dx = \int_{\mathbb{R}^3} K(x)f(u)\psi dx, & \psi \in C_0^\infty(\mathbb{R}^3), \\ \int_{\mathbb{R}^3} \nabla \phi \cdot \nabla \varphi dx = \int_{\mathbb{R}^3} u^2 \varphi dx, & \varphi \in C_0^\infty(\mathbb{R}^3). \end{cases}$$

The solution defined above may not be the usual weak solution since the test functions are in $C_0^\infty(\mathbb{R}^3)$ instead of $D^{1,2}(\mathbb{R}^3)$. It is an interesting problem whether the generalized solution is also a weak solution.

Formally, the working space associated to system (1.1) is $D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$. Due to the fact that the Poisson equation in system (1.1) may not have a solution for some given $u \in D^{1,2}(\mathbb{R}^3)$, then the conventional reduction method [6, 9] is not valid for system (1.1) directly. In order to obtain a nontrivial generalized solution of system (1.1), we firstly consider a perturbed system

$$\begin{cases} -\Delta u + \varepsilon u + \phi u = K(x)f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\varepsilon \in (0,1]$. For every fixed ε, one nontrivial weak solution $(u_\varepsilon,\ \phi_{u_\varepsilon}) \in H^1_\varepsilon(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ for system (1.4) can be obtained by the general minimax principle [22] and Lions vanishing lemma [16]. Then let $\varepsilon \to 0^+$, a nontrivial generalized solution of system (1.1) can be got by a version of Lions vanishing lemma in the space $D^{1,2}(\mathbb{R}^3)$ (see Lemma 2.2 of [1]).

Our main result is as follows.

Theorem 1.1. *Under the assumptions* (K), (f_1) *and* (f_2), *system* (1.1) *has at least one nontrivial generalized solution* $(u_0,\phi_0) \in D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.

Remark 1.1. Under assumption (f_1), f can not be homogeneous. It enlightens that system (1.1) may enjoy nontrivial solutions. To overcome the difficulty of nature of zero mass, we use the perturbation method and borrow some ideas from [4], which is quite different from [21]. Contrast with [4], our study can contain the case $\alpha \in (3,4]$ due to the positive homogeneity property of the solution for Poisson equation in system (1.4). Following Jeanjean [15], we can get a Pohožaev-Palais-Smale sequence by the general minimax principle [22] for the case $\alpha \in (3,4)$ which can be turned out to be bounded. Then one nontrivial weak solution for perturbed system (1.4) is obtained by Lions vanishing lemma. Some boundedness estimates of the weak solution for perturbed system (1.4) with respect to ε are also got which play an important role in the process of finding a nontrivial generalized solution for system (1.1).

Throughout the paper, S denotes the optimal constant in the Sobolev inequality, that is, $S = \inf_{v \in D^{1,2}(\mathbb{R}^3))\setminus\{0\}} \frac{|\nabla v|_2^2}{|v|_6^2}$. The rest of the paper is organized as follows. One nontrivial weak solution for perturbed system (1.4) is obtained in Section 2. The proof of Theorem 1.1 is given in Section 3.

2. Nontrivial weak solution for system (1.4)

By the standard reduction procedure [6, 9], system (1.4) can be reduced to a Schrödinger equation with one nonlocal term ϕ_u,

$$-\Delta u + \varepsilon u + \phi_u u = K(x)f(u),\ x \in \mathbb{R}^3, \tag{2.1}$$

where ϕ_u is the uniqueness solution of the second equation in system (1.4) for every fixed $u \in H_\varepsilon^1(\mathbb{R}^3)$. If $u_\varepsilon \in H_\varepsilon^1(\mathbb{R}^3)$ is a solution of equation (2.1), it gives rise to a solution $(u_\varepsilon, \phi_{u_\varepsilon})$ of system (1.4).

The variational functional associated to equation (2.1) is given by

$$J_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} K(x)F(u)dx,\ u \in H_\varepsilon^1(\mathbb{R}^3). \tag{2.2}$$

Under our assumptions, by Proposition 4.1 in [9], $J_\varepsilon \in C^1(H_\varepsilon^1(\mathbb{R}^3), \mathbb{R})$ and its Fréchet derivative at u is

$$J_\varepsilon'(u)v = \int_{\mathbb{R}^3} (\nabla u \nabla v + \varepsilon uv + \phi_u uv)dx - \int_{\mathbb{R}^3} K(x)f(u)vdx,\ v \in H_\varepsilon^1(\mathbb{R}^3).$$

We collect some properties on the nonlocal term ϕ_u (see, for instance [2, 10, 26]).

Lemma 2.1. *The nonlocal term enjoys the following properties.*

(*i*) *If $u_n \rightharpoonup u$ in $H_\varepsilon^1(\mathbb{R}^3)$, then, up to a subsequence, $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ and*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \geq \int_{\mathbb{R}^3} \phi_u u^2 dx.$$

(*ii*) $\phi_{u(\cdot+y)}(\cdot) = \phi_u(\cdot + y)$, $y \in \mathbb{R}^3$.

(*iii*) *For any $a, b, t > 0$, let $u_t(\cdot) = t^a u(t^b \cdot)$, then $\phi_{u_t}(\cdot) = t^{2(a-b)}\phi_u(t^b \cdot)$.*

The following result which resembles the classical vanishing lemma of Lions [16] plays an important role in obtaining a nontrivial generalized solution of system (1.1).

Lemma 2.2. *(Lemma 2.2 of [1]) Let $\{u_n\}$ be a bounded sequence in $D^{1,2}(\mathbb{R}^3)$, then either*

(*i*) *there exist $R, \eta > 0$ and $y_n \in \mathbb{R}^3$ such that $\int_{B_R(y_n)} |u_n|^2 dx > \eta$; or*

(*ii*) $\int_{\{x\in\mathbb{R}^3:\ |u_n(x)|\geq\tau\}} |u_n|^q dx \to 0$, *for every $q \in (2, 6)$ and $\tau > 0$.*

First, we prove the functional J_ε enjoys the mountain pass geometry structure.

Lemma 2.3. *Under the assumptions* (K), (f_1) *and* (f_2), *there exist* $\rho_0 > 0$ *and* $e_0 \in H^1_\varepsilon(\mathbb{R}^3)$ *such that* $\|e_0\|_\varepsilon > \rho_0$ *and*

$$\inf_{u\in H^1_\varepsilon(\mathbb{R}^3),\ \|u\|_\varepsilon=\rho_0} J_\varepsilon(u) > J_\varepsilon(0) = 0 > J_\varepsilon(e_0),\ \textit{for } \varepsilon \in (0,1].$$

Proof. On the one hand, it follows from (f_1) that there exists $C > 0$ such that

$$|f(t)| \leq C|t|^5 \text{ and } |F(t)| \leq \frac{C}{6}|t|^6,\ \text{for } t \in \mathbb{R}. \tag{2.3}$$

Then (2.2) and (2.3) imply that

$$J_\varepsilon(u) = \frac{1}{2}\|u\|^2_\varepsilon + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} K(x)F(u)dx \geq \frac{1}{2}\|u\|^2_\varepsilon - \frac{C}{6S^3}\|u\|^6_\varepsilon.$$

We conclude that there exists $\rho_0 > 0$ small enough such that for any $u \in H^1_\varepsilon(\mathbb{R}^3)$ with $0 < \|u\|_\varepsilon \leq \rho_0$, it results that $J_\varepsilon(u) > 0$. In particular, we have

$$J_\varepsilon(u) \geq \frac{1}{2}\rho_0^2 - \frac{C}{6S^3}\rho_0^6 > 0,$$

for any $u \in H^1_\varepsilon(\mathbb{R}^3)$ with $\|u\|_\varepsilon = \rho_0$.

On the other hand, by (f_1) and (f_2), there exist $a_1, a_2 > 0$ such that

$$F(t) \geq a_1|t|^\alpha - a_2t^2,\ \text{for } t \in \mathbb{R}. \tag{2.4}$$

Choosing $\overline{u} \in H^1_1(\mathbb{R}^3) \setminus \{0\}$ with $|\nabla\overline{u}|_2 = 1$, by *(iii)* of Lemma 2.1, we have

$$\phi_{\overline{u}_t}(x) = t^2\phi_{\overline{u}}(tx),\ x \in \mathbb{R}^3,$$

where $\overline{u}_t(\cdot) = t^2\overline{u}(t\cdot)$. By (2.4),

$$\begin{aligned} J_\varepsilon(\overline{u}_t) \leq &\frac{t^3}{2}\int_{\mathbb{R}^3} |\nabla\overline{u}|^2dx + \frac{(1+2a_2)t}{2}\int_{\mathbb{R}^3} \overline{u}^2dx + \frac{t^3}{4}\int_{\mathbb{R}^3} \phi_{\overline{u}}\overline{u}^2dx \\ &- a_1\min_{x\in\mathbb{R}^3} K(x)t^{2\alpha-3}\int_{\mathbb{R}^3} |\overline{u}|^\alpha dx. \end{aligned}$$

Since $2\alpha - 3 > 3$, we have that $J_\varepsilon(\overline{u}_t) \to -\infty$, as $t \to +\infty$. Thus, by choosing $t_0 > \rho_0$ large enough, we can get $J_\varepsilon(\overline{u}_{t_0}) < 0$ and $\|\overline{u}_{t_0}\|_\varepsilon > \rho_0$ for every $\varepsilon > 0$, so we can choose $e_0 = \overline{u}_{t_0}$. □

Then by Lemma 2.3, we can define the level such that

$$c_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon}\max_{t\in[0,1]} J_\varepsilon(\gamma(t)),$$

where $\Gamma_\varepsilon := \{\gamma \in C([0,1], H^1_\varepsilon(\mathbb{R}^3)) : \gamma(0) = 0, J_\varepsilon(\gamma(1)) < 0\}$. By the argument of Lemma 2.3 again,

$$c_\varepsilon \geq \frac{1}{2}\rho_0^2 - \frac{C}{6S^3}\rho_0^6 > 0,\ \text{for every } \varepsilon \in (0,1]. \tag{2.5}$$

Furthermore, by the definition of J_ε, we can get that $c_\varepsilon \leq c_1$, for every $\varepsilon \in (0,1]$. In fact, for every $\gamma \in \Gamma_1$, since $J_\varepsilon(\gamma(1)) \leq J_1(\gamma(1)) < 0$ and $\|\cdot\|_\varepsilon \leq \|\cdot\|_1$, then $\gamma \in \Gamma_\varepsilon$ for every $\varepsilon \in (0,1]$. Thus,

$$c_\varepsilon \leq \max_{t\in[0,1]} J_\varepsilon(\gamma(t)) \leq \max_{t\in[0,1]} J_1(\gamma(t)).$$

The arbitrariness of γ leads to $c_\varepsilon \le c_1$, for every $\varepsilon \in (0,1]$.

For the case $\alpha \in (3,4)$, following Jeanjean [15], we define a continuous map $\Phi : \mathbb{R} \times H^1_\varepsilon(\mathbb{R}^3) \to H^1_\varepsilon(\mathbb{R}^3)$ for $\sigma \in \mathbb{R}$ and $v \in H^1_\varepsilon(\mathbb{R}^3)$ by

$$\Phi(\sigma, v)(x) = v(e^{-\sigma}x),\ x \in \mathbb{R}^3.$$

For every $\sigma \in \mathbb{R}$ and $v \in H^1_\varepsilon(\mathbb{R}^3)$, the functional $J_\varepsilon \circ \Phi$ is computed as

$$(J_\varepsilon \circ \Phi)(\sigma, v) = J_\varepsilon(\Phi(\sigma, v)) = \frac{e^\sigma}{2}|\nabla v|_2^2 + \frac{\varepsilon e^{3\sigma}}{2}|v|_2^2 + \frac{e^{5\sigma}}{4}\int_{\mathbb{R}^3}\phi_v v^2 dx - e^{3\sigma}\int_{\mathbb{R}^3}F(v)dx.$$

It follows from (f_1) that $J_\varepsilon \circ \Phi$ is continuously Fréchet-differentiable on $\mathbb{R} \times H^1_\varepsilon(\mathbb{R}^3)$. We define a family of paths

$$\widetilde{\Gamma}_\varepsilon = \{\widetilde{\gamma} \in C([0,1], \mathbb{R} \times H^1_\varepsilon(\mathbb{R}^3)) : \widetilde{\gamma}(0) = (0,0) \text{ and } (J_\varepsilon \circ \Phi)(\widetilde{\gamma}(1)) < 0\}.$$

By a direct calculation, we can get that $\Gamma_\varepsilon = \{\Phi \circ \widetilde{\gamma} : \widetilde{\gamma} \in \widetilde{\Gamma}_\varepsilon\}$. In fact, on the one hand, for every $\gamma \in \Gamma_\varepsilon$, setting $\widetilde{\gamma}(\cdot) = (0, \gamma(\cdot))$, the definition of Φ leads to $\Phi(\widetilde{\gamma}(\cdot)) = \gamma(\cdot)$. Since

$$\widetilde{\gamma}(0) = (0,0) \text{ and } (J_\varepsilon \circ \Phi)(\widetilde{\gamma}(1)) = J_\varepsilon(\Phi(\widetilde{\gamma}(1))) = J_\varepsilon(\gamma(1)) < 0,$$

then $\widetilde{\gamma} \in \widetilde{\Gamma}_\varepsilon$, which implies that $\widetilde{\Gamma}_\varepsilon \neq \emptyset$ and $\Gamma_\varepsilon \subset \{\Phi \circ \widetilde{\gamma} : \widetilde{\gamma} \in \widetilde{\Gamma}_\varepsilon\}$. On the other hand, for every $\widetilde{\gamma} \in \widetilde{\Gamma}_\varepsilon$, setting $\gamma = \Phi \circ \widetilde{\gamma}$, by the definition of Φ,

$$\gamma(0) = (\Phi \circ \widetilde{\gamma})(0) = \Phi(\widetilde{\gamma}(0)) = \Phi(0,0) = 0$$

and

$$J_\varepsilon(\gamma(1)) = J_\varepsilon((\Phi \circ \widetilde{\gamma})(1)) = (J_\varepsilon \circ \Phi)(\widetilde{\gamma}(1)) < 0.$$

Thus, $\{\Phi \circ \widetilde{\gamma} : \widetilde{\gamma} \in \widetilde{\Gamma}_\varepsilon\} \subset \Gamma_\varepsilon$. Therefore, the mountain pass levels of J_ε and $J_\varepsilon \circ \Phi$ coincide:

$$c_\varepsilon = \inf_{\widetilde{\gamma} \in \widetilde{\Gamma}_\varepsilon} \max_{t \in [0,1]} (J_\varepsilon \circ \Phi)(\widetilde{\gamma}(t)).$$

By the general minimax principle (see Theorem 2.8 in [22]), there exists a sequence $\{(\sigma_n, v_n)\} \subset \mathbb{R} \times H^1_\varepsilon(\mathbb{R}^3)$ such that

$$\sigma_n \to 0,\ (J_\varepsilon \circ \Phi)(\sigma_n, v_n) \to c_\varepsilon,\ (J_\varepsilon \circ \Phi)'(\sigma_n, v_n) \to 0, \text{ as } n \to \infty.$$

Since for every $(h, w) \in \mathbb{R} \times H^1_\varepsilon(\mathbb{R}^3)$,

$$(J_\varepsilon \circ \Phi)'(\sigma_n, v_n)(h, w) = P_\varepsilon(\Phi(\sigma_n, v_n))h + J'_\varepsilon(\Phi(\sigma_n, v_n))\Phi(\sigma_n, w),$$

where P_ε is the Pohožaev functional

$$P_\varepsilon(u) = \frac{1}{2}|\nabla u|_2^2 + \frac{3\varepsilon}{2}|u|_2^2 + \frac{5}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx - 3\int_{\mathbb{R}^3}F(u)dx.$$

By taking $u_n = \Phi(\sigma_n, v_n)$, we get a Pohožaev-Palais-Smale sequence $\{u_n\}$ of J_ε at level c_ε ($(\mathrm{PPS})_{c_\varepsilon}$ sequence for short) which satisfies

$$J_\varepsilon(u_n) \to c_\varepsilon,\ J'_\varepsilon(u_n) \to 0,\ P_\varepsilon(u_n) \to 0, \text{ as } n \to \infty.$$

For the case $\alpha \in [4,6)$, by Theorem 2.10 of [22], there also exists a Palais-Smale sequence $\{u_n\}$ of J_ε at level c_ε ($(\mathrm{PS})_{c_\varepsilon}$ sequence for short).

Lemma 2.4. *Under the assumptions* (K), (f_1) *and* (f_2), *for every given* $\varepsilon \in (0,1]$,

(*i*) *for the case* $\alpha \in (3,4)$, *every* $(PPS)_{c_\varepsilon}$ *sequence of* J_ε *is bounded in* $H^1_\varepsilon(\mathbb{R}^3)$;

(*ii*) *for the case* $\alpha \in [4,6)$, *every* $(PS)_{c_\varepsilon}$ *sequence of* J_ε *is bounded in* $H^1_\varepsilon(\mathbb{R}^3)$.

Proof. (*i*) For the case $\alpha \in (3,4)$, by calculating $\frac{2}{\alpha} \times J'_\varepsilon(u_n)u_n - \frac{1}{3} \times P_\varepsilon(u_n)$, it follows from (f_2) that

$$\int_{\mathbb{R}^3} F(u_n)dx \leq \frac{12-\alpha}{6\alpha}|\nabla u_n|_2^2 + \frac{4-\alpha}{2\alpha}\varepsilon|u_n|_2^2 + \left(\frac{2}{\alpha} - \frac{5}{12}\right)\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx + o_n(1).$$

Then substituting the last inequality into $J_\varepsilon(u_n)$, we have

$$J_\varepsilon(u_n) \geq \left(\frac{2}{3} - \frac{2}{\alpha}\right)|\nabla u_n|_2^2 + \left(1 - \frac{2}{\alpha}\right)\varepsilon|u_n|_2^2 + \left(\frac{2}{3} - \frac{2}{\alpha}\right)\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx + o_n(1).$$

Since $\alpha \in (3,4)$, all the coefficients in the previous inequality are positive. It follows from $J_\varepsilon(u_n) \to c_\varepsilon$ as $n \to \infty$ that for $n \in \mathbb{N}$ large enough,

$$c_\varepsilon + 1 \geq \left(\frac{2}{3} - \frac{2}{\alpha}\right)\|u_n\|_\varepsilon^2.$$

Therefore, $\{u_n\}$ is bounded in $H^1_\varepsilon(\mathbb{R}^3)$.

(*ii*) For the case $\alpha \in [4,6)$, let $\{u_n\}$ be a $(\mathrm{PS})_{c_\varepsilon}$ sequence of J_ε. That is,

$$J_\varepsilon(u_n) \to c_\varepsilon,\ J'_\varepsilon(u_n) \to 0,\ \text{as } n \to \infty.$$

Then, by (f_2), for $n \in \mathbb{N}$ large enough,

$$c_\varepsilon + 1 + \|u_n\|_\varepsilon \geq J_\varepsilon(u_n) - \frac{1}{\alpha}\langle {J_\varepsilon}'(u_n), u_n\rangle \geq \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_n\|_\varepsilon^2.$$

Therefore, $\{u_n\}$ is bounded in $H^1_\varepsilon(\mathbb{R}^3)$. □

Lemma 2.5. *Under the assumptions* (K), (f_1) *and* (f_2), *for every given* $\varepsilon \in (0,1]$, *system* (1.4) *has at least one nontrivial weak solution* $(\widetilde{u}_\varepsilon, \phi_{\widetilde{u}_\varepsilon}) \in H^1_\varepsilon(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.

Proof. For the $(\mathrm{PPS})_{c_\varepsilon}$ sequence (or $(\mathrm{PS})_{c_\varepsilon}$ sequence) $\{u_n\}$ obtained above, Lemma 2.4 indicates that $\{u_n\}$ is bounded in $H^1_\varepsilon(\mathbb{R}^3)$. It is clear that $\{u_n\}$ is either

(*i*) *vanishing*: for each $r > 0$, $\lim\limits_{n\to\infty} \sup\limits_{y\in\mathbb{R}^3} \int_{B(y,r)} u_n^2 dx = 0$, or

(*ii*) *non-vanishing*: there exist $r, \eta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\limsup_{n\to\infty} \int_{B(y_n,r)} u_n^2 dx \geq \eta.$$

Suppose case (*ii*) holds and let $\widetilde{u}_n(x) := u_n(x + y_n)$. Without loss of generality, we can assume $y_n \in \mathbb{Z}^3$. The periodic assumption of K and (*ii*) of Lemma 2.1 imply that

$$J_\varepsilon(\widetilde{u}_n) = J_\varepsilon(u_n) \to c_\varepsilon \text{ and } \|J'_\varepsilon(\widetilde{u}_n)\| = \|J'_\varepsilon(u_n)\| \to 0,\ \text{as } n \to \infty. \tag{2.6}$$

Furthermore,

$$P_\varepsilon(u_n) = P_\varepsilon(\widetilde{u}_n) \to 0, \text{ as } n \to \infty, \text{ if } \{u_n\} \text{ is a } (\mathrm{PPS})_{c_\varepsilon} \text{ sequence.}$$

That is, $\{\widetilde{u}_n\}$ is also a $(\mathrm{PPS})_{c_\varepsilon}$ (or $(\mathrm{PS})_{c_\varepsilon}$) sequence of J_ε. Since $\{\widetilde{u}_n\}$ is also bounded in $H^1_\varepsilon(\mathbb{R}^3)$, there exists $\widetilde{u}_\varepsilon \in H^1_\varepsilon(\mathbb{R}^3)$, which is nonzero due to the fact that $\limsup\limits_{n\to\infty} \int_{B(0,r)} \widetilde{u}_n^2 dx \geq \eta$, such that $\widetilde{u}_n \rightharpoonup \widetilde{u}_\varepsilon$ in $H^1_\varepsilon(\mathbb{R}^3)$, after passing to a subsequence. A direct calculation shows that $J'_\varepsilon(\widetilde{u}_\varepsilon) = 0$. In fact, for every $v \in H^1_\varepsilon(\mathbb{R}^3)$,

$$o_n(1) = \langle J'_\varepsilon(\widetilde{u}_n), v\rangle = \int_{\mathbb{R}^3} (\nabla \widetilde{u}_n \nabla v + \varepsilon \widetilde{u}_n v + \phi_{\widetilde{u}_n} \widetilde{u}_n v - K(x) f(\widetilde{u}_n) v) dx.$$

The weak convergence in $H^1_\varepsilon(\mathbb{R}^3)$ leads to

$$\int_{\mathbb{R}^3} (\nabla \widetilde{u}_n \nabla v + \varepsilon \widetilde{u}_n v) dx \to \int_{\mathbb{R}^3} (\nabla \widetilde{u}_\varepsilon \nabla v + \varepsilon \widetilde{u}_\varepsilon v) dx, \text{ as } n \to \infty.$$

By (i) of Lemma 2.1, $\phi_{\widetilde{u}_n} \rightharpoonup \phi_{\widetilde{u}_\varepsilon}$ in $D^{1,2}(\mathbb{R}^3)$. Then $\phi_{\widetilde{u}_n} \rightharpoonup \phi_{\widetilde{u}_\varepsilon}$ in $L^6(\mathbb{R}^3)$. Since $\widetilde{u}_n v \to \widetilde{u}_\varepsilon v$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$ due to [23, Proposition 5.4.7], then

$$\int_{\mathbb{R}^3} \phi_{\widetilde{u}_n} \widetilde{u}_n v dx \to \int_{\mathbb{R}^3} \phi_{\widetilde{u}_\varepsilon} \widetilde{u}_\varepsilon v dx, \text{ as } n \to \infty.$$

By [23, Proposition 5.4.7] again, it follows from (2.3) that $f(\widetilde{u}_n) \rightharpoonup f(\widetilde{u}_\varepsilon)$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$. Furthermore, since $Kv \in L^6(\mathbb{R}^3)$, by the definition of weak convergence in $L^{\frac{6}{5}}(\mathbb{R}^3)$, we can get that

$$\int_{\mathbb{R}^3} K(x) f(\widetilde{u}_n) v dx \to \int_{\mathbb{R}^3} K(x) f(\widetilde{u}_\varepsilon) v dx, \text{ as } n \to \infty.$$

Thus, $\langle J'_\varepsilon(\widetilde{u}_\varepsilon), v\rangle = 0$. That is, $\widetilde{u}_\varepsilon$ is a nontrivial weak solution of equation (2.1) which gives rise to one nontrivial solution of system (1.4).

Therefore, it remains to show that the vanishing case can not occur. On the contrary, if $\{u_n\}$ is vanishing, then it follows from Lemma I.1 in [16] that $u_n \to 0$ in $L^r(\mathbb{R}^3)$ whenever $2 < r < 6$. By (f_1), for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$|f(t)t| \leq \delta t^6 + C_\delta |t|^q, \quad \text{for some } q \in (2, 6).$$

Then

$$0 \leq \int_{\mathbb{R}^3} |f(u_n) u_n| dx \leq \left(\delta \int_{\mathbb{R}^3} u_n^6 dx + C_\delta \int_{\mathbb{R}^3} |u_n|^q dx\right),$$

which implies that

$$\int_{\mathbb{R}^3} K(x) f(u_n) u_n dx \to 0, \text{ as } n \to \infty.$$

It follows from $u_n \to 0$ in $L^{\frac{12}{5}}(\mathbb{R}^3)$ that

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \to 0, \text{ as } n \to \infty.$$

Then, by $\langle J'_\varepsilon(u_n), u_n\rangle \to 0$, we can get $u_n \to 0$ in $H^1_\varepsilon(\mathbb{R}^3)$, which leads to a contradiction with (2.5). Therefore, $\{u_n\}$ is non-vanishing. □

At last, we give some uniform boundedness estimates on the families of the weak solutions $\{(\widetilde{u}_\varepsilon, \phi_{\widetilde{u}_\varepsilon})\}$ for system (1.4).

Lemma 2.6. *Under the assumptions* (K), (f_1) *and* (f_2),

(*i*) *for the case* $\alpha \in (3,4)$, *there exist* $b_1, b_2 > 0$ *such that*

$$b_1 \le \|\widetilde{u}_\varepsilon\|_{D^{1,2}(\mathbb{R}^3)} \le b_2,\ \|\phi_{\widetilde{u}_\varepsilon}\|_{D^{1,2}(\mathbb{R}^3)} \le b_2 \text{ and } \frac{2\alpha-6}{3\alpha} b_1^2 < J_\varepsilon(\widetilde{u}_\varepsilon),$$

for every $\varepsilon \in (0,1]$;

(*ii*) *for the case* $\alpha \in [4,6)$, *there exist* $b_3, b_4 > 0$ *such that*

$$b_1 \le \|\widetilde{u}_\varepsilon\|_{D^{1,2}(\mathbb{R}^3)} \le b_3,\ \|\phi_{\widetilde{u}_\varepsilon}\|_{D^{1,2}(\mathbb{R}^3)} \le b_4 \text{ and } \frac{\alpha-2}{2\alpha} b_1^2 < J_\varepsilon(\widetilde{u}_\varepsilon) \le c_\varepsilon,$$

for every $\varepsilon \in (0,1]$.

Proof. On the one hand, it follows from $\langle J'_\varepsilon(\widetilde{u}_\varepsilon), \widetilde{u}_\varepsilon\rangle = 0$ and (2.3) that

$$\|\widetilde{u}_\varepsilon\|^2_{D^{1,2}(\mathbb{R}^3)} \le \frac{C}{S^3}\|\widetilde{u}_\varepsilon\|^6_{D^{1,2}(\mathbb{R}^3)}.$$

Then

$$\|\widetilde{u}_\varepsilon\|_{D^{1,2}(\mathbb{R}^3)} \ge \frac{S^{\frac{3}{4}}}{C^{\frac{1}{4}}} := b_1. \tag{2.7}$$

On the other hand, for the case $\alpha \in (3,4)$, similar to the argument in Lemma 2.4, by Fatou lemma we have

$$\begin{aligned} c_1 \ge c_\varepsilon &= \lim_{n\to\infty} J_\varepsilon(\widetilde{u}_n) \\ &\ge \liminf_{n\to\infty}\left[\left(\frac{2}{3}-\frac{2}{\alpha}\right)|\nabla \widetilde{u}_n|_2^2 + \left(1-\frac{2}{\alpha}\right)\varepsilon|\widetilde{u}_n|_2^2 + \left(\frac{2}{3}-\frac{2}{\alpha}\right)\int_{\mathbb{R}^3}\phi_{\widetilde{u}_n}\widetilde{u}_n^2dx\right] \\ &\ge \left(\frac{2}{3}-\frac{2}{\alpha}\right)|\nabla \widetilde{u}_\varepsilon|_2^2 + \left(1-\frac{2}{\alpha}\right)\varepsilon|\widetilde{u}_\varepsilon|_2^2 + \left(\frac{2}{3}-\frac{2}{\alpha}\right)\int_{\mathbb{R}^3}\phi_{\widetilde{u}_\varepsilon}\widetilde{u}_\varepsilon^2dx, \end{aligned}$$

where $\{\widetilde{u}_n\}$ is the $(\mathrm{PPS})_{c_\varepsilon}$ sequence of J_ε which has been obtained in (2.6). Thus,

$$\|\widetilde{u}_\varepsilon\|_{D^{1,2}(\mathbb{R}^3)} \le \left(\frac{3\alpha c_1}{2\alpha-6}\right)^{\frac{1}{2}} := b_2,\ \|\phi_{\widetilde{u}_\varepsilon}\|_{D^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}\phi_{\widetilde{u}_\varepsilon}\widetilde{u}_\varepsilon^2dx\right)^{\frac{1}{2}} \le b_2.$$

Furthermore, by (2.7) and Pohožaev type identity satisfied by $\widetilde{u}_\varepsilon$, we can get

$$0 < \left(\frac{2}{3}-\frac{2}{\alpha}\right)b_1^2 \le \left(\frac{2}{3}-\frac{2}{\alpha}\right)|\nabla\widetilde{u}_\varepsilon|_2^2 < J_\varepsilon(\widetilde{u}_\varepsilon).$$

Thus, (i) is true.

For the case $\alpha \in [4,6)$, by Fatou lemma and (f_2), we have

$$\begin{aligned} c_1 \ge c_\varepsilon &= \lim_{n\to\infty}\left(J_\varepsilon(\widetilde{u}_n) - \frac{1}{\alpha}\langle J'_\varepsilon(\widetilde{u}_n), \widetilde{u}_n\rangle\right) \\ &\ge \frac{\alpha-2}{2\alpha}\|\widetilde{u}_\varepsilon\|_\varepsilon^2 + \frac{\alpha-4}{4\alpha}\int_{\mathbb{R}^3}\phi_{\widetilde{u}_\varepsilon}\widetilde{u}_\varepsilon^2dx + \int_{\mathbb{R}^3}K(x)\left(\frac{1}{\alpha}f(\widetilde{u}_\varepsilon)\widetilde{u}_\varepsilon - F(\widetilde{u}_\varepsilon)\right)dx \\ &= J_\varepsilon(\widetilde{u}_\varepsilon) - \frac{1}{\alpha}\langle J'_\varepsilon(\widetilde{u}_\varepsilon), \widetilde{u}_\varepsilon\rangle \\ &= J_\varepsilon(\widetilde{u}_\varepsilon), \end{aligned}$$

where $\{\widetilde{u}_n\}$ is the $(\mathrm{PS})_{c_\varepsilon}$ sequence of J_ε which has been obtained in (2.6). Thus,

$$\|\widetilde{u}_\varepsilon\|_{D^{1,2}(\mathbb{R}^3)} \leq \|\widetilde{u}_\varepsilon\|_\varepsilon \leq \left(\frac{2\alpha c_1}{\alpha-2}\right)^{\frac{1}{2}} := b_3.$$

Furthermore, by (2.7), we can get that

$$0 < \frac{\alpha-2}{2\alpha} b_1^2 < J_\varepsilon(\widetilde{u}_\varepsilon) \leq c_\varepsilon \leq c_1.$$

$\langle J_\varepsilon'(\widetilde{u}_\varepsilon), \widetilde{u}_\varepsilon\rangle = 0$ and (2.3) also imply that

$$\int_{\mathbb{R}^3} \phi_{\widetilde{u}_\varepsilon} \widetilde{u}_\varepsilon^2 dx \leq C|\widetilde{u}_\varepsilon|_6^6 \leq \frac{C}{S^3}\|\widetilde{u}_\varepsilon\|_{D^{1,2}(\mathbb{R}^3)}^6 \leq \frac{C}{S^3}\|\widetilde{u}_\varepsilon\|_\varepsilon^6.$$

Thus,

$$\|\phi_{\widetilde{u}_\varepsilon}\|_{D^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} \phi_{\widetilde{u}_\varepsilon} \widetilde{u}_\varepsilon^2 dx\right)^{\frac{1}{2}} \leq \frac{C^{\frac{1}{2}}}{S^{\frac{3}{2}}}\|\widetilde{u}_\varepsilon\|_\varepsilon^3 \leq \frac{C^{\frac{1}{2}}}{S^{\frac{3}{2}}} b_3^3 := b_4.$$

Therefore, (ii) is also right. The proof is completed. □

3. Proof of Theorem 1.1

In Section 2, one nontrivial weak solution $(\widetilde{u}_\varepsilon, \phi_{\widetilde{u}_\varepsilon})$ of system (1.4) has been obtained for every $\varepsilon \in (0,1]$. We can get one nontrivial generalized solution of system (1.1) by letting $\varepsilon \to 0^+$.

Proof of Theorem 1.1. It follows from Lemma 2.6 that both of $\{\widetilde{u}_\varepsilon\}$ and $\{\phi_{\widetilde{u}_\varepsilon}\}$ are bounded in $D^{1,2}(\mathbb{R}^3)$. Apply Lemma 2.2 to $\{\widetilde{u}_\varepsilon\}$, if case (ii) of Lemma 2.2 holds we can get that $J_\varepsilon(\widetilde{u}_\varepsilon) \to 0,\ \varepsilon \to 0^+$ which contradicts with the fact that

$$J_\varepsilon(\widetilde{u}_\varepsilon) > \begin{cases} (\frac{2}{3} - \frac{2}{\alpha})b_1^2, & \text{if } \alpha \in (3,4); \\ \left(\frac{1}{2} - \frac{1}{\alpha}\right) b_1^2, & \text{if } \alpha \in [4,6). \end{cases}$$

In fact, for some given $q \in (2,6)$, by (f_1), we can get for every $\delta > 0$ there exist τ_δ and C_δ such that

$$0 \leq \int_{\mathbb{R}^3} |f(\widetilde{u}_\varepsilon)\widetilde{u}_\varepsilon| dx \leq \left(\delta \int_{\mathbb{R}^3} \widetilde{u}_\varepsilon^6 dx + C_\delta \int_{\{x:|\widetilde{u}_\varepsilon(x)| \geq \tau_\delta\}} |\widetilde{u}_\varepsilon|^q dx\right). \tag{3.1}$$

Under case (ii) of Lemma 2.2, (3.1) and (K) lead to

$$\int_{\mathbb{R}^3} K(x) f(\widetilde{u}_\varepsilon)\widetilde{u}_\varepsilon dx \to 0, \text{ as } \varepsilon \to 0^+.$$

Together with the fact that $\langle J_\varepsilon'(\widetilde{u}_\varepsilon), \widetilde{u}_\varepsilon\rangle = 0$ and $\phi_{\widetilde{u}_\varepsilon}$ is nonnegative, we can get

$$\|\widetilde{u}_\varepsilon\|_\varepsilon \to 0, \int_{\mathbb{R}^3} \phi_{\widetilde{u}_\varepsilon} \widetilde{u}_\varepsilon^2 dx \to 0, \text{ as } \varepsilon \to 0^+.$$

By (f_1) and (K), we can also obtain

$$\int_{\mathbb{R}^3} K(x)F(\widetilde{u}_\varepsilon)dx \to 0, \text{ as } \varepsilon \to 0^+.$$

Then $J_\varepsilon(\widetilde{u}_\varepsilon) \to 0$, as $\varepsilon \to 0^+$. Thus, there exist $R, \eta > 0$ and $y_\varepsilon \in \mathbb{Z}^3$ such that

$$\int_{B_R(y_\varepsilon)} |\widetilde{u}_\varepsilon|^2 dx > \eta.$$

Set $\widehat{u}_\varepsilon(\cdot) = \widetilde{u}_\varepsilon(\cdot + y_\varepsilon)$, then $\phi_{\widehat{u}_\varepsilon}(\cdot) = \phi_{\widetilde{u}_\varepsilon}(\cdot + y_\varepsilon)$ due to (ii) of Lemma 2.1. The translation invariance results that both of $\{\widehat{u}_\varepsilon\}$ and $\{\phi_{\widehat{u}_\varepsilon}\}$ are bounded in $D^{1,2}(\mathbb{R}^3)$ and $(\widehat{u}_\varepsilon, \phi_{\widehat{u}_\varepsilon})$ is also a solution of system (1.4). Thus, up to a subsequence, there exist $u_0 \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}$ and $\phi_0 \in D^{1,2}(\mathbb{R}^3)$ such that

$$\widehat{u}_\varepsilon \rightharpoonup u_0,\ \phi_{\widehat{u}_\varepsilon} \rightharpoonup \phi_0, \text{ in } D^{1,2}(\mathbb{R}^3), \text{ as } \varepsilon \to 0^+.$$

Next, we show that (u_0, ϕ_0) is a nontrivial generalized solution of system (1.1). For every $\psi,\ \varphi \in C_0^\infty(\mathbb{R}^3)$,

$$0 = \langle J'_\varepsilon(\widehat{u}_\varepsilon), \psi\rangle = \int_{\mathbb{R}^3} (\nabla \widehat{u}_\varepsilon \cdot \nabla\psi + \varepsilon \widehat{u}_\varepsilon \psi + \phi_{\widehat{u}_\varepsilon} \widehat{u}_\varepsilon \psi - K(x)f(\widehat{u}_\varepsilon)\psi)dx, \tag{3.2}$$

$$\int_{\mathbb{R}^3} \nabla\phi_{\widehat{u}_\varepsilon} \cdot \nabla\varphi dx = \int_{\mathbb{R}^3} \widehat{u}_\varepsilon^2 \varphi dx. \tag{3.3}$$

Since $\widehat{u}_\varepsilon \rightharpoonup u_0$ in $D^{1,2}(\mathbb{R}^3)$, it is easy to see that

$$\int_{\mathbb{R}^3} \nabla \widehat{u}_\varepsilon \cdot \nabla\psi dx \to \int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla \psi dx, \text{ as } \varepsilon \to 0^+.$$

By Hölder inequality, one can get that

$$\varepsilon \int_{\mathbb{R}^3} \widehat{u}_\varepsilon \psi dx \to 0, \text{ as } \varepsilon \to 0^+.$$

Since $\psi \in C_0^\infty(\mathbb{R}^3)$, by the local compact embedding theorem and Hölder inequality, we can get

$$|(\widehat{u}_\varepsilon - u_0)\psi|_{\frac{6}{5}} \to 0 \text{ and } \int_{\mathbb{R}^3} |\phi_{\widehat{u}_\varepsilon} - \phi_0||u_0\psi|dx \to 0, \text{ as } \varepsilon \to 0^+.$$

Thus, by Hölder inequality, we deduce

$$\begin{aligned}\left|\int_{\mathbb{R}^3} (\phi_{\widehat{u}_\varepsilon} \widehat{u}_\varepsilon - \phi_0 u_0)\psi dx\right| &\le \int_{\mathbb{R}^3} |\phi_{\widehat{u}_\varepsilon} \widehat{u}_\varepsilon - \phi_{\widehat{u}_\varepsilon} u_0||\psi|dx + \int_{\mathbb{R}^3} |\phi_{\widehat{u}_\varepsilon} u_0 - \phi_0 u_0||\psi|dx \\ &\le |\phi_{\widehat{u}_\varepsilon}|_6 |(\widehat{u}_\varepsilon - u_0)\psi|_{\frac{6}{5}} + \int_{\mathbb{R}^3} |\phi_{\widehat{u}_\varepsilon} - \phi_0||u_0\psi|dx \\ &\to 0, \text{ as } \varepsilon \to 0^+.\end{aligned}$$

Since $K\psi \in C_0^\infty(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, the fact that $f(\widehat{u}_\varepsilon) \rightharpoonup f(u_0)$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$ leads to

$$\int_{\mathbb{R}^3} K(x)f(\widehat{u}_\varepsilon)\psi dx \to \int_{\mathbb{R}^3} K(x)f(u_0)\psi dx, \text{ as } \varepsilon \to 0^+.$$

Thus, by taking limits as $\varepsilon \to 0^+$ on both sides of (3.2), it reaches that

$$\int_{\mathbb{R}^3} (\nabla u_0 \cdot \nabla \psi + \phi_0 u_0 \psi) dx = \int_{\mathbb{R}^3} K(x) f(u_0) \psi dx. \tag{3.4}$$

Similarly, by taking limits as $\varepsilon \to 0^+$ on both sides of (3.3), we have

$$\int_{\mathbb{R}^3} \nabla \phi_0 \cdot \nabla \varphi dx = \int_{\mathbb{R}^3} u_0^2 \varphi dx. \tag{3.5}$$

Since $u_0 \neq 0$, (3.5) implies that $\phi_0 \neq 0$. Therefore, (3.4) and (3.5) indicate that (u_0, ϕ_0) is a nontrivial generalized solution of system (1.1). The proof is completed.

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