FRACTIONAL LANGEVIN EQUATIONS WITH INFINITE-POINT BOUNDARY CONDITION: APPLICATION TO FRACTIONAL HARMONIC OSCILLATOR

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Abstract The current study is concerned with the existence and uniqueness of the solution to the Langevin equation of two separate fractional orders. With the infinite-point boundary condition, the boundary value problem is studied. The Banach contraction principle, Leray-nonlinear Schauder's alternative, and Leray-Schauder degree theorems are all implemented. A numerical example is presented to demonstrate the accuracy of our results. In addition, as an application of our results, the mean and variance of a fractional harmonic oscillator with the undamped angular frequency of the oscillator under the effect of a random force described as Gaussian colored noise are calculated.

Keywords Fractional Langevin equations, fixed point theorem, existence and uniqueness, infinite-point boundary condition, mean, and variance.

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1. Introduction

Given that it lacks a clear geometrical explanation, the concept of fractional calculus is not entirely apparent. In the realm of [6], a need for order has emerged as a result of the appearance of several unique forms. Even more problematic is the wide range of possible implementations. The benefits that fractional derivatives might add to the model must be carefully considered. Typically, fractional derivatives are used to represent mass transport, optical, diffusion, and other processes [8, 11, 24, 27]. Fractional-order models' benefits in simulating supercapacitor capacitances [15], temperature controllers [7], DC motors [34], and RC, LC, and RLC electric circuits [1] have been detailed with the addition of these derivatives.

When the random fluctuation force is assumed to be white noise, the Brownian motion drags the Langevin equation through to an extreme degree. The generalized Langevin equation, [28], describes the motion of the item if the random oscillation force is not white noise. Overall, the fractional order differential equation models are widely utilized nowadays as an alternative to conventional differential equations

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because they can more accurately represent experimental data and area measurement [22,30,33,36]. A crucial differential equation in applied mathematics, physics, and other branches of science and engineering is the generalized Langevin equation. Mainardi and Pironi [18] have created and presented it. The significance of their technique is that it models the Brownian motion more accurately than the traditional one based on the classical Langevin equation, because it includes the retarding effects owing to hydrodynamic back-flow, i.e. the additional mass and the Basset memory drag. The two fluctuation-dissipation theorems and fractional calculus methods were used to produce analytical formulas for the correlation functions (both for the random force and the particle velocity).

The contributions [2, 25, 26, 31] and else have studied several properties and results to the solution of the fractional Langevin equation using multi-point and multi-strip boundary conditions. The uniqueness of solution and other features for boundary value problems of generalized Langevin equation have attracted a great deal of attention from many writers over the last several decades, as evidenced by epitome [23, 29, 32, 35, 37] and the extensive list of references offered therein.

The generalized Langevin equation GLE can be used to analyze anomalous diffusive phenomena connected with physical or biological processes. Some recent articles on anomalous diffusion may be found in the literature [4] and a huge number of references given therein. We recall that anomalous diffusion is the phenomena that occurs most commonly in disordered or fractal media and in which the mean squared displacement (the variance) is proportional to a power of fractional order rather than linear in time (as in standard diffusion) [17].

Inspired of the previous studies, the following nonlinear fractional Langevin equation of two fractional orders is considered

$${}^{c}D^{\gamma}({}^{c}D^{\alpha}+\lambda)u(t) = f(t,u(t)), \qquad t \in [0,1]$$
(1.1)

supplemented with the infinite-point boundary conditions

$$u(0) = 0,^{c} D^{\alpha} u(0) = 0, u(1) - \sum_{i=1}^{\infty} \beta_{i} u(\xi_{i}) = u_{0}$$
(1.2)

where u(t) represents the position of a particle of mass m = 1 at time $t \in [0, 1], \lambda \in \mathbb{R}$ is frictional memory kernel, $^{c}D^{\alpha}$ and $^{c}D^{\gamma}$ are the Caputo's fractional derivatives of orders $0 < \alpha \leq 1$ and $1 < \gamma \leq 2, 0 < \xi_1 < \xi_2 < \cdots < \xi_i < \cdots < 1; i \in \mathbb{N}$ and $f: [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Although numerous research have been conducted on fractional Langevin equations under multi-point boundary conditions, as far as we are aware, just Li et al [16] has investigated it under an infinite-point boundary conditions and provided various novel existence results of solution utilizing Leray-nonlinear Schauder's alternative and Leray-Schauder degree theory. However, it has been proved that their given outcomes count on solution form incorporates boundary values. It suggests that their method need more conclusive results in order to be more useful.

In more details, we note that their unique solution of the linear boundary value problem of fractional Langevin differential equation

$${}^{c}D^{\gamma}({}^{c}D^{\alpha}+\lambda)u(t)=h(t), \qquad t\in[0,1]$$

subject to the infinite-point boundary conditions (1.2) was given as

$$u(t) = \int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} h(s) ds - \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds$$

$$+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)\left(1-\sum_{i=1}^{\infty}\beta_{i}\xi_{i}\right)}\left(\sum_{i=1}^{\infty}\beta_{i}\int_{0}^{\xi_{i}}\frac{(\xi_{i}-s)^{\gamma-1}}{\Gamma(\gamma)}h(s)ds\right.\\\left.-\int_{0}^{1}\frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)}h(s)ds+\lambda u(1)-\lambda\sum_{i=1}^{\infty}\beta_{i}u(\xi_{i})\right)$$

with $\lambda > 0, \beta_i > 0, i \in \mathbb{N}$ and $1 - \sum_{i=1}^{\infty} \beta_i \xi_i > 0$, which contains the boundary values u(1) and $u(\xi_i), i \in \mathbb{N}$, even though we can insert the values of these boundary values after obtaining the form of u(t). This means that the solution above is not in the final form.

To be out of these criticisms, we resolve the boundary value problem (1.1)-(1.2) without the appearance of the boundary values u(1) and $u(\xi_i), i \in \mathbb{N}$ in the unique solution u(t). Also, we extend some restrictions on $\lambda, \beta_i, i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \beta_i \xi_i$.

Our analysis is carried out using three important fixed point theorems: the Banach contraction principle, Leray-nonlinear Schauder's alternative, and the Leray-Schauder degree theorems.

Furthermore, as an application, the fractional harmonic oscillator with the undamped angular frequency of the oscillator under the influence of a random force described as Gaussian colored noise was examined. The mean and variance, the two most often used statistical measures of transportation, are evaluated and plotted.

2. Preliminaries and relevant lemmas

In this part, we introduce certain fractional calculus notations and terminology, as well as preliminary findings that will be used later in our proofs. We are grateful for the terminology utilized in the books [13,20].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ for a continuous function f is defined as

$$I^{\alpha}f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds$$

provided that the right-hand-side integral exists, where $\Gamma(\alpha)$ denotes the Gamma function is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha - 1} e^{-t} dt, \qquad \alpha > 0.$$

Definition 2.2. Let $n \in \mathbb{N}$ be a positive integer and α be a positive real such that $n-1 < \alpha \leq n$, then the fractional derivative of a function f in the Caputo sense is defined as

$${}^{c}D^{\alpha}f(t) = \int_{0}^{t} \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(s) ds$$

provided that the right-hand-side integral exists and is finite. We notice that the Caputo derivative of a constant is zero.

Lemma 2.1. Let α and β be positive reals. If f is a continuous function, then we have

$$I^{\alpha}I^{\beta}f(t) = I^{\alpha+\beta}f(t),$$

$${}^{c}D^{\alpha}I^{\beta}f(t) = I^{\beta-\alpha}f(t), \qquad \beta \ge \alpha$$

Lemma 2.2. Let α be positive real. Then we have

$$I^{\alpha}t^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+\alpha+1)}t^{\rho+\alpha}, \qquad \rho > -1,$$

$${}^{c}D^{\alpha}t^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(\rho-\alpha+1)}t^{\rho-\alpha}, \qquad \rho > -1, \quad \rho \neq m \in \mathbb{N}_{0}, \ m < n,$$

$${}^{c}D^{\alpha}t^{m} = 0, \qquad m \in \mathbb{N}_{0}, \ m < n.$$

Lemma 2.3. Let $n \in \mathbb{N}$ and $n-1 < \alpha \leq n$. If u is a continuous function, then we have

$$I^{\alpha \ c}D^{\alpha}u(t) = u(t) + c_0 + c_1t + \dots + c_{n-1}t^{n-1}$$

Let us now consider the nonlinear fractional Langevin differential equation (1.1) supplemented with the infinite-point boundary conditions (1.2), then we can state the following lemma:

Lemma 2.4. The unique representation of the solution of the boundary value problem (1.1) and (1.2) is given by

$$u(t) = g(u,t) + \frac{t^{\alpha+1}}{\Delta} \left(u_0 - g(u,1) + \sum_{i=1}^{\infty} \beta_i g(u,\xi_i) \right)$$

where

$$g(u,t) = \int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} f(s,u(s)) ds - \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds,$$

$$\Delta = 1 - \sum_{i=1}^\infty \beta_i \xi_i^{\alpha+1} \neq 0.$$

Proof. From Lemmas 2.1, 2.2 and 2.3 and the Definition 2.1, it follows that

$${}^{c}D^{\alpha}u(t) = \int_{0}^{t} \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} h(s)ds + c_{0} + c_{1}t - \lambda u(t)$$
(2.1)

followed by operating I^{α} on both sides, we find that

$$u(t) = g(u,t) + \frac{c_0}{\Gamma(\alpha+1)}t^{\alpha} + \frac{c_1}{\Gamma(\alpha+2)}t^{\alpha+1} + c_2.$$
 (2.2)

By inserting the boundary condition u(0) = 0 in (2.2) gives $c_2 = 0$ and also by inserting the boundary condition $^{c}Du(0) = 0$ in (2.1) gives $c_0 = 0$. Using the third boundary condition in (1.2), gives

$$\frac{c_1}{\Gamma(\alpha+2)} = \frac{1}{\Delta} \left(u_0 - g(u,1) + \sum_{i=1}^{\infty} \beta_i g(u,\xi_i) \right).$$

Substituting the above values in (2.2) to obtain the desired results. Conversely, inserting the formula of u(t) into the left hand side of (1.1) with using the second relation of Lemma 2.1 and the second and third relations of Lemma 2.2 to obtain the right hand side of (1.1). Also, it is not difficult to see that the solution u(t) satisfies all conditions of (1.2).

In the proofs of our main results for problem (1.1)-(1.2), we use the Banach contraction principle for providing sufficient conditions to the uniqueness of solution and Leray-Schauder degree theorem and nonlinear alternative Leray-Schauder theorem for providing sufficient conditions to the existence of solution.

Definition 2.3. Let (\mathbb{E}, d) be a Banach space. Then a map $\mathbf{T} \colon \mathbb{E} \to \mathbb{E}$ is called a contraction mapping on \mathbb{E} if there exists $r \in [0, 1)$ such that $d(\mathbf{T}(t), \mathbf{T}(s)) \leq rd(t, s)$ for all $t, s \in \mathbb{E}$.

Lemma 2.5 (Banach contraction principle [10]). Let (\mathbb{E}, d) be a non-empty complete metric space with a contraction mapping $\mathbf{T} \colon \mathbb{E} \to \mathbb{E}$. Then \mathbf{T} admits a unique fixed-point t^* in \mathbb{E} (i.e. $\mathbf{T}(t^*) = t^*$).

Lemma 2.6 (Nonlinear alternative Leray-Schauder theorem [9]). Let \mathbb{E} be a Banach space, C be a closed and convex subset of \mathbb{E} , U be an open subset of C and $0 \in U$. Suppose that the operator $\mathbf{T} \colon \overline{U} \to C$ is a continuous and compact map (that is, $\mathbf{T}(\overline{U})$ is a relatively compact subset of C). Then either

(i) **T** has a fixed point in $x^* \in \overline{U}$, or

(ii) there is $x \in \partial U$ (boundary of U in C) and $\delta \in (0,1)$ such that $\delta \mathbf{T}(x) = x$.

Lemma 2.7 (Leray-Schauder degree theorem [5, 19]). Assume that \mathbb{E} is a real Banach space, Ω is a bounded, open subset of \mathbb{E} and $\Phi: [a, b] \times \overline{\Omega} \to \mathbb{E}$ is given by $\Phi(\lambda, u) = u - \mathbf{T}(\lambda, u)$ with \mathbf{T} a compact map. Suppose also that

$$\Phi(\lambda, u) = u - \mathbf{T}(\lambda, u) \neq 0, \qquad \forall \lambda, u \in [a, b] \times \partial \Omega.$$

If $deg(\Phi(a, \cdot), \Omega, 0) \neq 0$, then the equation $\Phi(\lambda, u) = u - \mathbf{T}(\lambda, u) = 0$ has a solution in Ω for every $a \leq \lambda \leq b$.

3. Existence and uniqueness results

Let $\mathbb{E} = C([0, 1], \mathbb{R})$ be the Banach space of all continuous functions from $[0, 1] \longrightarrow \mathbb{R}$ endowed the norm defined by

$$||u|| = \sup\{|u(t)|, t \in [0,1]\}.$$

Before stating and proving the main results, we introduce the following hypotheses: Assume that

 (\mathcal{H}_1) The function $f: [0,1] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous.

 (\mathcal{H}_2) The function f satisfies

$$|f(t,u) - f(t,v)| \le \mathcal{L}|u-v|, \qquad \forall t \in [0,1], u, v \in \mathbb{R}$$

where \mathcal{L} is the Lipschitz constant.

(\mathcal{H}_3) There exists a positive function $\omega \in C([0,1], \mathbb{R}_+)$ and a nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t,u)| \le \omega(t)\varphi(\|u\|), \qquad \forall (t,u) \in ([0,1],\mathbb{R}).$$

 (\mathcal{H}_4) There exist two positive constants η and L such that

$$|f(t,u)| \le \eta |u| + L, \qquad \forall (t,u) \in ([0,1],\mathbb{R}).$$

For computational convenience, we set

$$\mathbf{A} = S_{\gamma} \left(1 + \frac{1}{|\Delta|} \right) + S_{\gamma}', \tag{3.1}$$

$$\mathbf{B} = |\lambda|S_0\left(1 + \frac{1}{|\Delta|}\right) + |\lambda|S_0' \tag{3.2}$$

where

$$S_{\gamma} = \frac{1}{\Gamma(\alpha + \gamma + 1)}, \qquad S'_{\gamma} = \frac{S_{\gamma}}{|\Delta|} \sum_{i=1}^{\infty} |\beta_i| \xi_i^{\alpha + \gamma}.$$

Lemma 3.1. Under the assumption (\mathcal{H}_1) , the function $g(\cdot, \cdot)$ satisfies the following

$$|g(u,t)| \le ||f||S_{\gamma}t^{\alpha+\gamma} + |\lambda|||u||S_0t^{\alpha}, |g(u,t_2) - g(u,t_1)| \le ||f||S_{\gamma}(t_2^{\alpha+\gamma} - t_1^{\alpha+\gamma}) + 2|\lambda|||u||S_0(t_2 - t_1)^{\alpha}$$

for all $u \in \mathbb{E}$ and $t, t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$. Furthermore, under the assumptions (\mathcal{H}_1) and (\mathcal{H}_2) , it satisfies

$$|g(u,t) - g(v,t)| \le \left(\mathcal{L}S_{\gamma}t^{\alpha+\gamma} + |\lambda|S_0t^{\alpha}\right) ||u-v||$$

for all $u, v \in \mathbb{E}$ and $t \in [0, 1]$.

Proof. From the definition of the function g in Lemma 2.4, we have

$$\begin{split} |g(u,t)| &\leq \int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} |f(s,u(s))| ds + |\lambda| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |u(s)| ds \\ &\leq \|f\| \int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} ds + |\lambda| \|u\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &\leq \|f\| \frac{t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + |\lambda| \|u\| \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &= \|f\| S_\gamma t^{\alpha+\gamma} + |\lambda| \|u\| S_0 t^{\alpha}. \end{split}$$

Suppose that $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$, we get

$$\begin{split} &|g(u,t_{2}) - g(u,t_{1})| \\ \leq & \int_{0}^{t_{1}} \frac{(t_{2}-s)^{\alpha+\gamma-1} - (t_{1}-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} |f(s,u(s))| ds \\ &+ \int_{t_{1}}^{t_{2}} \frac{(t_{2}-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} |f(s,u(s))| ds \\ &+ |\lambda| \int_{0}^{t_{1}} \frac{(t_{1}-s)^{\alpha-1} - (t_{2}-s)^{\alpha-1}}{\Gamma(\alpha)} |u(s)| ds + |\lambda| \int_{t_{1}}^{t_{2}} \frac{(t_{2}-s)^{\alpha-1}}{\Gamma(\alpha)} |u(s)| ds \end{split}$$

$$\leq \frac{\|f\|}{\Gamma(\alpha+\gamma)} \left[\int_0^{t_1} \left((t_2-s)^{\alpha+\gamma-1} - (t_1-s)^{\alpha+\gamma-1} ds \right) + \int_{t_1}^{t_2} (t_2-s)^{\alpha+\gamma-1} ds \right] \\ + \frac{|\lambda| \|u\|}{\Gamma(\alpha)} \left[\int_0^{t_1} \left((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1} \right) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \right] \\ = \frac{\|f\|}{\Gamma(\alpha+\gamma+1)} \left[t_2^{\alpha+\gamma} - t_2^{\alpha+\gamma} \right] + \frac{2|\lambda| \|u\|}{\Gamma(\alpha+1)} (t_2-t_1)^{\alpha} \\ = \|f\| S_{\gamma} \left(t_2^{\alpha+\gamma} - t_1^{\alpha+\gamma} \right) + 2|\lambda| \|u\| S_0 (t_2-t_1)^{\alpha}.$$

Also, by using assumption (\mathcal{H}_2) with letting $u, v \in \mathbb{E}$ and $t \in [0, 1]$, we can deduce that

$$\begin{aligned} &|g(u,t) - g(v,t)| \\ &\leq \int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} |f(s,u(s)) - f(s,v(s))| ds + |\lambda| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |u(s) - v(s)| ds \\ &\leq \mathcal{L} \|u - v\| \int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} ds + |\lambda| \|u - v\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &= \left(\mathcal{L}S_\gamma t^{\alpha+\gamma} + |\lambda|S_0 t^\alpha\right) \|u - v\| \end{aligned}$$

which completes the proof.

In view of Lemma 2.4, we transform problem (1.1)-(1.2) as

$$u = T(u) \tag{3.3}$$

where the operator $T: \mathbb{E} \longrightarrow \mathbb{E}$ is defined by

$$(Tu)(t) = g(u,t) + \frac{t^{\alpha+1}}{\Delta} \left(u_0 - g(u,1) + \sum_{i=1}^{\infty} \beta_i g(u,\xi_i) \right)$$

where $g(\cdot, \cdot)$ is defined in Lemma 2.4.

The following theorem is devoted to provide the conditions that satisfy the assumptions of Banach contraction mapping principle to give a unique solution of the boundary value problem (1.1)-(1.2).

Theorem 3.1. Assume that the assumptions (\mathcal{H}_1) and (\mathcal{H}_2) hold. Then the boundary value problem (1.1)-(1.2) has a unique solution if $\mathbf{Q} < 1$, where $\mathbf{Q} = \mathcal{L}\mathbf{A} + \mathbf{B}$ and \mathbf{A} and \mathbf{B} are given by (3.1) and (3.2), respectively.

Proof. Let

$$\mathcal{B}_r = \{ u \in \mathbb{E} : \|u\| \le r \}$$

be a closed ball with the radius $r \ge (\mathcal{M}\mathbf{A} + u_1)/(1 - \mathbf{Q})$ where

$$\mathcal{M} = \sup_{t \in [0,1]} |f(t,0)|$$
 and $u_1 = \frac{|u_0|}{|\Delta|}.$

Then, for $u \in \mathcal{B}_r$, we have

$$||f|| = \sup_{t \in [0,1]} |f(t, u(t))|$$

=
$$\sup_{t \in [0,1]} |f(t, u(t)) - f(t, 0) + f(t, 0)|$$

$$\leq \sup_{t \in [0,1]} |f(t, u(t)) - f(t, 0)| + \sup_{t \in [0,1]} |f(t, 0)|$$

$$\leq \mathcal{L} ||u|| + \mathcal{M}$$

$$\leq \mathcal{L}r + \mathcal{M}.$$

By employing Lemma 3.1, we have

$$\begin{aligned} \|Tu(t)\| &= \sup_{t \in [0,1]} \left| g(u,t) + \frac{t^{\alpha+1}}{\Delta} \left(u_0 - g(u,1) + \sum_{i=1}^{\infty} \beta_i g(u,\xi_i) \right) \right| \\ &\leq \sup_{t \in [0,1]} |g(u,t)| + \frac{1}{|\Delta|} \left(|u_0| + |g(u,1)| + \sum_{i=1}^{\infty} |\beta_i| |g(u,\xi_i)| \right) \sup_{t \in [0,1]} t^{\alpha+1} \\ &\leq \left((\mathcal{L}r + \mathcal{M})S_{\gamma} + |\lambda| rS_0 \right) \left(1 + \frac{1}{|\Delta|} \right) + u_1 + (\mathcal{L}r + \mathcal{M}) S_{\gamma}' + |\lambda| rS_0' \\ &= (\mathcal{L}\mathbf{A} + \mathbf{B})r + \mathcal{M}\mathbf{A} + u_1 \\ &= \mathbf{Q}r + \mathcal{M}\mathbf{A} + u_1 \leq r \end{aligned}$$

which leads to $Tu \subset \mathcal{B}_r$. Now, let $u, v \in \mathcal{B}_r$, then we have

$$\begin{split} \|(Tu)(t) - (Tv)(t)\| \\ &\leq \sup_{t \in [0,1]} \left(|g(u,t) - g(v,t)| \\ &+ \frac{t^{\alpha+1}}{|\Delta|} \left(|g(u,1) - g(v,1)| + \sum_{i=1}^{\infty} \beta_i |g(u,\xi_i) - g(v,\xi_i)| \right) \right) \\ &\leq \left[(\mathcal{L}S_{\gamma} + |\lambda|S_0) \left(1 + \frac{1}{|\Delta|} \right) + \mathcal{L}S_{\gamma}' + |\lambda|S_0' \right] \|u - v\| \\ &= (\mathcal{L}\mathbf{A} + \mathbf{B}) \|u - v\| = \mathbf{Q} \|u - v\|. \end{split}$$

By the hypothesis $\mathbf{Q} < 1$, it follows that the operator T defined in (3.3) is a contraction. Therefore, with Banach contraction mapping principle (see Lemma 2.5), we deduce that the operator T has a fixed point, which equivalently implies that the boundary value problem (1.1)-(1.2) has a unique solution on [0, 1].

Theorem 3.2. Assume that the assumptions (\mathcal{H}_1) and (\mathcal{H}_3) hold. Then the boundary value problem (1.1)-(1.2) has at least one solution if there exists a constant M > 0 such that $\mathbf{K} > 1$ where \mathbf{K} is given

$$\mathbf{K} = \frac{M}{\|\boldsymbol{\omega}\|\varphi(M)\mathbf{A} + M\mathbf{B} + u_1}$$

Proof. The continuity of the function f implies that the operator $T : \mathbb{E} \to \mathbb{E}$ defined by (3.3) is continuous. Assume that $\mathcal{B}_r = \{u \in \mathbb{E} : ||u|| < r\}$ be an open subset of the Banach space \mathbb{E} with radius r > 0. First, we are in a position to prove that the operator $T : \mathbb{E} \to \mathbb{E}$ is completely continuous. Assume that $u \in \mathcal{B}_r$. Then, as in the proof of Theorem 3.1, we have

$$||Tu(t)|| = \sup_{t \in [0,1]} \left| g(u,t) + \frac{t^{\alpha+1}}{\Delta} \left(u_0 - g(u,1) + \sum_{i=1}^{\infty} \beta_i g(u,\xi_i) \right) \right|$$

$$\leq \left(\|\omega\|\varphi(r)S_{\gamma} + |\lambda|rS_{0}\right)\left(1 + \frac{1}{|\Delta|}\right) + u_{1} + \|\omega\|\varphi(r)S_{\gamma}' + |\lambda|rS_{0}'$$
$$= \|\omega\|\varphi(r)\mathbf{A} + r\mathbf{B} + u_{1}$$

which concludes the boundedness of the operator T. Suppose that $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$, it follows that

$$\begin{aligned} |Tu(t_2) - Tu(t_1)| \\ &= \left| g(u, t_2) - g(u, t_1) + \frac{t_2^{\alpha+1} - t_1^{\alpha+1}}{\Delta} \left(u_0 - g(u, 1) + \sum_{i=1}^{\infty} \beta_i g(u, \xi_i) \right) \right. \\ &\leq |g(u, t_2) - g(u, t_1)| + \frac{t_2^{\alpha+1} - t_1^{\alpha+1}}{|\Delta|} \left| u_0 - g(u, 1) + \sum_{i=1}^{\infty} \beta_i g(u, \xi_i) \right| \end{aligned}$$

which implies, by using Lemma 3.1, that

$$\begin{aligned} |Tu(t_2) - Tu(t_1)| &\leq ||\omega||\varphi(r)S_{\gamma}\left(t_2^{\alpha+\gamma} - t_1^{\alpha+\gamma}\right) + 2|\lambda|||u||S_0(t_2 - t_1)^{\alpha} \\ &+ \frac{t_2^{\alpha+1} - t_1^{\alpha+1}}{|\Delta|} \left| u_0 - g(u, 1) + \sum_{i=1}^{\infty} \beta_i g(u, \xi_i) \right|. \end{aligned}$$

It is clear that the right-hand side of the above inequality approaches zero as $t_1 \to t_2$. Since the operator T satisfies the above assumptions, it follows by the Arzela-Ascoli theorem that $T : \mathbb{E} \to \mathbb{E}$ is completely continuous.

According to the Leray-Schauder nonlinear alternative Lemma 2.6, the result will follow once we prove the boundedness of the set of all solution to equations $u = \delta T u$ for some $\delta \in [0, 1]$. Let u is a solution of the equation $u = \delta T u$ for some $\delta \in [0, 1]$, then for all $t \in [0, 1]$, from the boundedness of the operator T, we have

$$||u|| = \sup_{t \in [0,1]} |u(t)| = \sup_{t \in [0,1]} |\delta(Tu)(t)| \le ||\omega||\varphi(||u||)\mathbf{A} + ||u||\mathbf{B} + u_1$$

which implies that

$$\frac{\|u\|}{\|\omega\|\varphi(\|u\|)\mathbf{A} + \|u\|\mathbf{B} + u_1} \le 1.$$

By the assumption $\mathbf{K} > 1$, then there exists a constant M > 0 such that $||u|| \neq M$. Setting the open set

$$\Omega = \{ u \in \mathbb{E} : \|u\| < M \}.$$

Based on the form of Ω , there is no $u \in \partial \Omega$ such that $u = \delta T(u)$ for some $\delta \in (0, 1)$. Since the operator $T : \overline{\Omega} \to \mathbb{E}$ is continuous and completely continuous, then by the nonlinear alternative of Leray-Schauder type Lemma 3.1, we deduce that T has a fixed point $u \in \overline{\Omega}$ which is a solution of problem (1.1)-(1.2). This ends the proof.

Theorem 3.3. Assume that the assumptions (\mathcal{H}_1) and (\mathcal{H}_4) hold. Then the boundary value problem (1.1)-(1.2) has at least one solution if

$$0 < \eta < (1 - \mathbf{B}) / \mathbf{A}$$

where \mathbf{A} and \mathbf{B} are given by (3.1) and (3.2), respectively.

Proof. Let us define the open ball $\mathcal{B}_r \subset \mathbb{E}$ with radius r > 0 as

$$\mathcal{B}_r = \{ u \in \mathbb{E} : \|u\| < r \},\$$

where r will be determined later. It is adequate to prove that the operator $T: \overline{\mathcal{B}_r} \to \mathbb{E}$ satisfies

$$u \neq \lambda T u, \quad \forall u \in \partial \mathcal{B}_r, \quad \sigma \in [0, 1].$$
 (3.4)

To do this, assume that $u = \sigma T u$ for some $\sigma \in [0, 1]$. Then, as in the preceding results, we have

$$\|u\| \le (\eta \mathbf{A} + \mathbf{B})\|u\| + L\mathbf{A} + u_1$$

which implies that

$$\|u\| \le \frac{L\mathbf{A} + u_1}{1 - (\eta\mathbf{A} + \mathbf{B})}$$

provided that $\eta \mathbf{A} + \mathbf{B} < 1$ which leads to $\eta < (1 - \mathbf{B})/\mathbf{A}$. Now, suppose that there exists $\epsilon > 0$ such that

$$r = \frac{L\mathbf{A} + u_1}{1 - (\eta \mathbf{A} + \mathbf{B})} + \epsilon.$$

By the analysis above, it follows that (3.4) holds. Let us now define the continuous operator

$$h_{\sigma}(u) = u - \sigma T u, \quad u \in \mathbb{E}, \quad \sigma \in [0, 1].$$

In view of the results in Theorems above, it is clear that the operator $h_{\sigma} : \mathbb{E} \to \mathbb{E}$ is completely continuous. By the homotopy invariance of topological degree, it follows that

$$\deg(h_{\sigma}, \mathcal{B}_r, 0) = \deg(h_1, \mathcal{B}_r, 0) = \deg(h_0, \mathcal{B}_r, 0) = \deg(I, \mathcal{B}_r, 0) = 1 \neq 0$$

where I denotes the unit operator. By the nonzero property of Leray–Schauder degree (see Lemma 2.7), the equation $h_1(u) = uTu = 0$ has at least one solution in \mathcal{B}_r , that is, the boundary value problem (1.1)-(1.2) has at least one solution.

4. Fractional harmonic oscillator

In actual oscillators, damping or friction slows the system's motion. The velocity falls in proportion to the frictional force applied. While in a basic undriven harmonic oscillator the only force operating on the mass is the restoring force, in a damped harmonic oscillator there is also a frictional force acting in the opposite direction of the motion. The dynamic of a fractional treatment of a harmonic oscillator [3] with the undamped angular frequency of the oscillator ω under the influence of a random force modeled as Gaussian colored noise, whose corresponding fractional differential equation, associated with the displacement, can be written as

$${}^{c}D^{\gamma}({}^{c}D^{\alpha}+\lambda)u(t)+\omega^{2}u(t)=\rho(t), \quad 0\le t\le 1,$$
(4.1)

where u(t) indicates the location of a particle with mass m = 1 at time $t \in [0, 1]$, $\lambda \in \mathbb{R}$ is frictional memory kernel and the internal noise $\rho(t)$ is a random force satisfying the fluctuation-dissipation theorem of a zero-mean $\langle \rho(t) \rangle = 0$ and with an arbitrary correlation function $C(t'-t) = \langle \rho(t)\rho(t') \rangle$. The correlation function for a free Brownian particle in one dimension can be taken as [14]

$$C(t) = K\delta(t)$$

where K = kT, k is a Boltzmann constant, T is the absolute temperature of the heat bath and $\delta(\cdot)$ is a dirac delta function.

The equation (4.1) is supplemented with the infinite-point conditions (1.2). It is clear that the function $f(t, u(t)) = -\omega^2 u(t) + \rho(t)$ satisfies the assumptions (\mathcal{H}_1) and (\mathcal{H}_2) with $\mathcal{L} = \omega^2$. According to our main results in Theorems 3.1 provided that $\mathbf{Q} < 1$, the problem (4.1) has a unique solution which can be evaluate through applying Laplace transform as follows

$$\begin{split} u(t) &= \sum_{n=0}^{\infty} (-1)^n \omega^{2n} L^{-1} \left\{ \frac{s^{-\gamma n-2}}{(s^\alpha + \lambda)^{n+1}} \right\} C \\ &+ \sum_{n=0}^{\infty} (-1)^n \omega^{2n} L^{-1} \left\{ \frac{s^{-\gamma (n+1)}}{(s^\alpha + \lambda)^{n+1}} \right\} * \rho(t) \\ &= g_{\lambda,\omega}(t) C + h_{\lambda,\omega} * \rho(t) \end{split}$$

which has the mean $\langle x(t) \rangle = g_{\lambda,\omega} C$ where

$$g_{\lambda,\omega}(t) = \sum_{n=0}^{\infty} (-1)^n \omega^{2n} t^{\alpha(n+1)+\gamma n+1} E_{\alpha,\alpha(n+1)+\gamma n+2}^{n+1} (-\lambda t^{\alpha}),$$
$$h_{\lambda,\omega}(t) = \sum_{n=0}^{\infty} (-1)^n \omega^{2n} t^{(\alpha+\gamma)(n+1)-1} E_{\alpha,(\alpha+\gamma)(n+1)}^{n+1} (-\lambda t^{\alpha}).$$

C is a constant can be evaluated easily by using the last condition in (1.2) as

$$C = \frac{u_0}{g_{\lambda,\omega}(1) - \sum_{i=1}^{\infty} \beta_i g_{\lambda,\omega}(\xi_i)}$$

provided that $g_{\lambda,\omega}(1) \neq \sum_{i=1}^{\infty} \beta_i g_{\lambda,\omega}(\xi_i)$, and $E^{\mu}_{\alpha,\gamma}(\cdot)$ is the generalized of the Mittag-Leffler function [12,21]

$$E^{\mu}_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\mu)_k z^k}{k! \Gamma(\alpha k + \gamma)}, \quad \Re(\alpha) > 0, \ \gamma \in \mathbb{C},$$

where $(\mu)_k = \mu(\mu+1)\cdots(\mu+k-1); k \in \mathbb{N}, \ (\mu)_0 = 1.$

The two most commonly used statistical measures of transport are the mean, $\mu(t)$, and the variance, $\sigma^2(t)$, defined as

$$\mu(t) = \langle \delta u \rangle, \qquad \sigma^2(t) = \langle [\delta u - \langle \delta u \rangle]^2 \rangle \quad \text{and} \quad \delta u = u(t) - u(0).$$

Thus, the mean of displacement can be provided as

$$\mu(t) = \langle u(t) \rangle = \frac{u_0}{g_{\lambda,\omega}(1) - \sum_{i=1}^{\infty} \beta_i g_{\lambda,\omega}(\xi_i)} g_{\lambda,\omega}(t)$$

and the displacement can be given as

$$u(t) = \langle u(t) \rangle + h_{\lambda,\omega}(t) * \rho(t) = \langle u(t) \rangle + \int_0^t h_{\lambda,\omega}(t-s)\rho(s)ds.$$

We can also evaluate the mean square displacement which provides an indication of the most likely displacement that one can expect a particle to have in a certain time. In the free particle case, the mean square displacement can be calculate by

$$\begin{split} \langle x^{2}(t) \rangle &= \left(\langle x(t) \rangle + \int_{0}^{t} h_{\lambda,\omega}(t-s)\rho(s)ds \right)^{2} \\ &= \langle x(t) \rangle^{2} + 2\langle x(t) \rangle \int_{0}^{t} h_{\lambda,\omega}(t-s)\langle \rho(s) \rangle ds \\ &+ \int_{0}^{t} \int_{0}^{t} h_{\lambda,\omega}(t-s)h_{\lambda,\omega}(t-r)\langle \rho(s)\rho(r) \rangle ds dr \\ &= \langle x(t) \rangle^{2} + K \int_{0}^{t} h_{\lambda,\omega}(t-s) \left(\int_{0}^{t} h_{\lambda,\omega}(t-r)\delta(r-s) \rangle dr \right) ds \\ &= \langle x(t) \rangle^{2} + K \int_{0}^{t} h_{\lambda,\omega}^{2}(t-s) ds \end{split}$$

which leads to the variance of the process is given as

$$\sigma^{2}(t) = \langle x^{2}(t) \rangle - \langle x(t) \rangle^{2} = K \int_{0}^{t} h_{\lambda,\omega}^{2}(t-s) ds.$$

The fractional derivative results from the shear stress-induced collective behavior of the liquid. Some of the limiting examples can be related to the fractional harmonic oscillator Langevin equation. We have two interesting constraints here. Setting the liquid's shear stress to zero ($\lambda = 0$), we can find the basic harmonic oscillator. If we exclude the oscillator's undamped angular frequency ($\omega = 0$), we get a special super-diffusive instance of the fractional Langevin equation.

Case I: In the case of the absence of frictional memory kernel ($\lambda = 0$), we have

$$g_{0,\omega}(t) = t^{\alpha+1} E_{\alpha+\gamma,\alpha+2} \left(-\omega^2 t^{\alpha+\gamma} \right),$$

$$h_{0,\omega}(t) = t^{\alpha+\gamma-1} E_{\alpha+\gamma,\alpha+\gamma} \left(-\omega^2 t^{\alpha+\gamma} \right)$$

where $E_{\alpha,\gamma}(\cdot) = E_{\alpha,\gamma}^1(\cdot)$ is the Mittag-Leffler function of two parameters. Consider $\alpha = 1/2$, $\gamma = 3/2$. Then, we get

$$g_{0,\omega}(t) = t^{\frac{3}{2}} E_{2,\frac{5}{2}} \left(-\omega^2 t^2\right) = \frac{\sin(\omega t)}{\omega^2} \int_0^t \frac{\cos(\omega s)}{\sqrt{\pi s}} ds - \frac{\cos(\omega t)}{\omega^2} \int_0^t \frac{\sin(\omega s)}{\sqrt{\pi s}} ds,$$
$$h_{0,\omega}(t) = t E_{2,2} \left(-\omega^2 t^2\right) = \frac{1}{\omega} \sin(\omega t)$$

which implies that

$$\sigma^{2}(t) = K \int_{0}^{t} h_{0,\omega}^{2}(t-s)ds = \frac{K}{4\omega^{3}}(2\omega + \sin(2\omega t)).$$



Figure 1. The expected value when $\lambda = 0$.



Figure 2. The variance when $\lambda = 0$.

The second statement comes immediately from properties of Mittag-Leffler function and the first comes by solving the equation

$$g_{0,\omega}''(t) + \omega^2 g_{0,\omega}(t) = \frac{1}{\sqrt{\pi t}}$$

with initial conditions $g_{0,\omega}(0) = g'_{0,\omega}(0) = 0.$

Case II: In the case of the absence of the frequency of the oscillator ($\omega = 0$), we have

$$g_{\lambda,0}(t) = t^{\alpha+1} E_{\alpha,\alpha+2} \left(-\lambda t^{\alpha}\right),$$

$$h_{\lambda,0}(t) = t^{\alpha+\gamma-1} E_{\alpha,\alpha+\gamma} \left(-\lambda t^{\alpha}\right)$$

Consider $\alpha = 1/2$, $\gamma = 3/2$. Then, we get

$$g_{\lambda,0}(t) = t^{\frac{3}{2}} E_{\frac{1}{2},\frac{5}{2}} \left(-\lambda \sqrt{t}\right)$$



Figure 3. The expected value when $\omega = 0$.



Figure 4. The variance when $\omega = 0$.

$$= \frac{1}{\lambda^3 \sqrt{\pi}} \left[\sqrt{\pi} e^{\lambda^2 t} \operatorname{erf}(\lambda \sqrt{t}) - 2\lambda \sqrt{t} \right] - \frac{1}{\lambda} \left[e^{\lambda^2 t} - 1 - \lambda^2 t \right],$$
$$h_{\lambda,0}(t) = t E_{\frac{1}{2},2} \left(-\lambda \sqrt{t} \right) = \frac{1}{\lambda^2 \sqrt{\pi}} \left[\sqrt{\pi} e^{\lambda^2 t} \operatorname{erfc}(\lambda \sqrt{t}) - \sqrt{\pi} + 2\lambda \sqrt{t} \right]$$

which implies that

$$\sigma^2(t) = K \int_0^t h_{\lambda,0}^2(t-s) ds.$$

The previous results come from the well-known formula

$$E_{\frac{1}{2},1}(z) = e^{z^2} \operatorname{erfc}(-z)$$

where $\operatorname{erfc}(z) = 1 - \operatorname{erf}(z)$ is the complementary error function with

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

5. Numerical example

Consider the example in the previous section with $\alpha = 1/2$, $\gamma = 3/2$, $\lambda = 1/3$ and $\omega = \sqrt{3}/48$. Note that, we choose those values to simplify our calculations. Also, we take $\xi_i = 1/2$ and $\beta_i = (1/3)^i$ (or $\beta_i = -(5/6)^i$) which imply that $\Delta \sim 0.823223 \bigvee 2.76777$, S(3/2) = 1/2, $S(0) = 2/\sqrt{\pi}$, $S'(3/2) \sim 0.0759211 \bigvee 0.225814$ and $S'(0) \sim 0.906465 \bigvee 1.44139$. These conclude that $\mathbf{A} \sim 1.18329 \bigvee 0.906465$ and $\mathbf{B} \sim 0.994558 \bigvee 0.992484$ which implies that $\mathbf{Q} = \omega^2 \mathbf{A} + \mathbf{B} \sim 0.996099 \bigvee 0.993664 < 1$. According to Theorem 3.1, there exists a unique solution for our problem in both cases. To evaluate the mean and variance, apply Laplace transform to obtain

$$\begin{split} u(t) &= L^{-1} \left\{ \frac{s^{-\frac{1}{2}}}{s^2 + \frac{1}{3}s^{\frac{3}{2}} + \frac{1}{768}} \right\} C + L^{-1} \left\{ \frac{1}{s^2 + \frac{1}{3}s^{\frac{3}{2}} + \frac{1}{768}} \right\} * \rho(t) \\ &\triangleq g(t)C + h(t) * \rho(t). \end{split}$$

These lead us to formulate the mean and the variance of the displacement as follow

$$\mu(t) = \frac{u_0}{g(1) - \sum_{i=1}^{\infty} \beta_i g(\xi_i)} g(t),$$

$$\sigma^2(t) = K \int_0^t h^2(t-s) ds.$$

Because there is no unknown constant in this case, we use a different strategy to determine the exact solution. The denominator can be factorized as

$$s^{2} + \frac{1}{3}s^{\frac{3}{2}} + \frac{1}{768} = \left(\sqrt{s} + \frac{1}{4}\right)^{2}(\sqrt{s} - a_{-})(\sqrt{s} - a_{+})$$

where

$$a_{\pm} = \frac{1}{12}(1 \pm i\sqrt{2}) \triangleq a_1 \pm a_2.$$

Then,

$$g(t) = \frac{8}{3}L^{-1} \left\{ \frac{16s^{-\frac{1}{2}}}{\sqrt{s} + \frac{1}{4}} + \frac{3s^{-\frac{1}{2}}}{(\sqrt{s} + \frac{1}{4})^2} - \frac{cs^{-\frac{1}{2}}}{\sqrt{s} - a_+} - \frac{ds^{-\frac{1}{2}}}{\sqrt{s} - a_-} \right\}$$

where

$$c_{\pm} = 8 \pm 7\sqrt{2}i \triangleq c_1 \pm c_2 i.$$

By applying inverse Laplace transform, we get

$$\begin{split} g(t) = & \frac{8}{3} \left(16E_{\frac{1}{2},1} \left(-\frac{1}{4}\sqrt{t} \right) + 6\sqrt{t}E_{\frac{1}{2},\frac{1}{2}} \left(-\frac{1}{4}\sqrt{t} \right) - c_{+}E_{\frac{1}{2},1} \left(a_{+}\sqrt{t} \right) \\ & -c_{-}E_{\frac{1}{2},1} \left(a_{-}\sqrt{t} \right) \right). \end{split}$$

The second term comes from

$$L^{-1}\left\{\frac{s^{-\frac{1}{2}}}{(s^{\frac{1}{2}}+\frac{1}{4})^2}\right\} = -2L^{-1}\left\{\frac{d}{ds}\left[\frac{1}{s^{\frac{1}{2}}+\frac{1}{4}}\right]\right\}$$

$$= 2tL^{-1} \left\{ \frac{1}{s^{\frac{1}{2}} + \frac{1}{4}} \right\}$$
$$= 2\sqrt{t}E_{\frac{1}{2},\frac{1}{2}} \left(-\frac{1}{4}\sqrt{t} \right).$$

It is easy to see that

$$E_{\frac{1}{2},\frac{1}{2}}\left(\ell\sqrt{t}\right) = \frac{1}{\sqrt{\pi}} + \ell\sqrt{t}E_{\frac{1}{2},1}\left(\ell\sqrt{t}\right) = \frac{1}{\sqrt{\pi}} + \ell\sqrt{t}e^{\ell^{2}t}\operatorname{erfc}\left(-\ell\sqrt{t}\right)$$

which concludes that

$$\begin{split} g(t) &= \left(\frac{128}{3} - 4t\right) e^{\frac{1}{16}t} \mathrm{erfc}\left(\frac{1}{4}\sqrt{t}\right) + \frac{16\sqrt{t}}{\sqrt{\pi}} \\ &\quad -\frac{8}{3}\left(c_{+}e^{a_{+}^{2}t}\mathrm{erfc}(-a_{+}\sqrt{t}) + c_{-}e^{a_{-}^{2}t}\mathrm{erfc}(-a_{-}\sqrt{t})\right) \\ &= \left(\frac{128}{3} - 4t\right) e^{\frac{1}{16}t}\mathrm{erfc}\left(\frac{1}{4}\sqrt{t}\right) + \frac{16\sqrt{t}}{\sqrt{\pi}} \\ &\quad -\frac{8}{3}\left(c_{+}e^{a_{+}^{2}t} + c_{-}e^{a_{-}^{2}t} + c_{+}e^{a_{+}^{2}t}\mathrm{erf}(a_{+}\sqrt{t}) + c_{-}e^{a_{-}^{2}t}\mathrm{erf}(a_{-}\sqrt{t})\right). \end{split}$$

It is easy to see that

$$c_{+}e^{a_{+}^{2}t} + c_{-}e^{a_{-}^{2}t} = e^{(a_{1}^{2}-a_{2}^{2})t}[c_{1}(e^{2ia_{1}a_{2}t} + e^{-2ia_{1}a_{2}t}) + ic_{2}(e^{2ia_{1}a_{2}t} - e^{-2ia_{1}a_{2}t})]$$

= $2e^{(a_{1}^{2}-a_{2}^{2})t}[c_{1}\cos(2a_{1}a_{2}t) - c_{2}\sin(2a_{1}a_{2}t)].$

and

$$\begin{split} & c_{+}e^{a_{+}^{2}t}\mathrm{erf}(a_{+}\sqrt{t}) + c_{-}e^{a_{-}^{2}t}\mathrm{erf}(a_{-}\sqrt{t}) \\ &= \frac{2}{\sqrt{\pi}}e^{(a_{1}^{2}-a_{2}^{2})t}\left(c_{+}e^{2ia_{1}a_{2}t}\int_{0}^{a_{+}\sqrt{t}}e^{-s^{2}}ds + c_{-}e^{-2ia_{1}a_{2}t}\int_{0}^{a_{-}\sqrt{t}}e^{-s^{2}}ds\right) \\ &= \frac{2\sqrt{t}}{\sqrt{\pi}}e^{(a_{1}^{2}-a_{2}^{2})t}\left(c_{+}a_{+}e^{2ia_{1}a_{2}t}\int_{0}^{1}e^{-a_{+}^{2}ts^{2}}ds + c_{-}a_{-}e^{-2ia_{1}a_{2}t}\int_{0}^{1}e^{-a_{-}^{2}ts^{2}}ds\right) \\ &= \frac{2\sqrt{t}}{\sqrt{\pi}}e^{(a_{1}^{2}-a_{2}^{2})t}\int_{0}^{1}e^{-(a_{1}^{2}-a_{2}^{2})ts^{2}}\left(c_{+}a_{+}e^{2ia_{1}a_{2}t(1-s^{2})} + c_{-}a_{-}e^{-2ia_{1}a_{2}t(1-s^{2})}\right)ds \\ &= \frac{4\sqrt{t}}{\sqrt{\pi}}e^{(a_{1}^{2}-a_{2}^{2})t}\int_{0}^{1}e^{-(a_{1}^{2}-a_{2}^{2})ts^{2}}\left((c_{1}a_{1}-c_{2}a_{2})\cos(2a_{1}a_{2}t(1-s^{2}))\right) \\ &\quad -(c_{1}a_{2}+c_{2}a_{1})\sin(2a_{1}a_{2}t(1-s^{2}))\right)ds. \end{split}$$

Therefore, we have

$$g(t) = \left(\frac{128}{3} - 4t\right) e^{\frac{1}{16}t} \operatorname{erfc}\left(\frac{1}{4}\sqrt{t}\right) + \frac{16\sqrt{t}}{\sqrt{\pi}} \\ - \frac{16}{3}e^{-\frac{1}{144}t} \left[8\cos\left(\frac{\sqrt{2}}{72}t\right) - 7\sqrt{2}\sin\left(\frac{\sqrt{2}}{72}t\right)\right] \\ + \frac{2\sqrt{t}}{3\sqrt{\pi}}e^{-\frac{1}{144}t} \int_{0}^{1} e^{\frac{1}{144}ts^{2}} \left(2\cos\left(\frac{\sqrt{2}}{72}t(1-s^{2})\right)\right)$$

$$+5\sqrt{2}\sin\left(\frac{\sqrt{2}}{72}t(1-s^2)\right)\right)ds.$$

From the definitions of the functions g(t) and h(t), we find that

$$L\{g(t)\} = s^{-\frac{1}{2}}L\{h(t)\}$$

which implies that $g(t) = I^{\frac{1}{2}}h(t)$ or equivalently to $h(t) = {}^{c}D^{\frac{1}{2}}g(t)$. By using the relation (1.82) in [20], we get

$$\begin{split} h(t) &= \frac{128}{3} t^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}} \left(-\frac{1}{4} \sqrt{t} \right) + \frac{1}{2} t E_{\frac{1}{2},1} \left(-\frac{1}{4} \sqrt{t} \right) + 4 E_{\frac{1}{2},1} \left(-\frac{1}{4} \sqrt{t} \right) - \frac{2\sqrt{t}}{\sqrt{\pi}} \\ &- \frac{8}{3} c_{+} t^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}} \left(a_{+} \sqrt{t} \right) - \frac{8}{3} c_{-} t^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}} \left(a_{-} \sqrt{t} \right) \\ &= \frac{1}{6} (3t - 40) e^{\frac{1}{16} t} \text{erfc} \left(\frac{1}{4} \sqrt{t} \right) - \frac{2\sqrt{t}}{\sqrt{\pi}} - \frac{8}{3} c_{+} t^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}} \left(a_{+} \sqrt{t} \right) \\ &- \frac{8}{3} c_{-} t^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}} \left(a_{-} \sqrt{t} \right) . \end{split}$$

Thus, by using the same method used to construct the function g(t), we may reach at

$$\begin{split} h(t) &= \frac{1}{6} (3t - 40) e^{\frac{1}{16}t} \mathrm{erfc} \left(\frac{1}{4}\sqrt{t}\right) - \frac{2\sqrt{t}}{\sqrt{\pi}} \\ &+ \frac{4}{3} e^{-\frac{1}{144}t} \left(2\cos\left(\frac{\sqrt{2}}{72}t\right) + 5\sqrt{2}\sin\left(\frac{\sqrt{2}}{72}t\right)\right) \\ &+ \frac{2\sqrt{t}}{3\sqrt{\pi}} e^{-\frac{1}{144}t} \int_{0}^{1} e^{\frac{1}{144}ts^{2}} \left(4\cos\left(\frac{\sqrt{2}}{72}t(1 - s^{2})\right) + \sqrt{2}\sin\left(\frac{\sqrt{2}}{72}t(1 - s^{2})\right)\right) \\ &+ \sqrt{2}\sin\left(\frac{\sqrt{2}}{72}t(1 - s^{2})\right) \right) ds. \end{split}$$

6. Conclusion

The existence and uniqueness of solution for nonlinear Langevin equations involving two fractional orders (1.1) with infinite-point boundary condition (1.2) have been discussed. We applied the concepts of fractional calculus together with major fixed point theorems to establish the existence and uniqueness results. To investigate our problem, we used Banach contraction principle, nonlinear alternative Leray-Schauder theorem and Leray-Schauder degree theorem. Our technique was straightforward and applicable to a variety of real world problems. In addition, as an application of our results, the mean and variance of a fractional harmonic oscillator with the undamped angular frequency of the oscillator under the effect of a random force described as Gaussian colored noise were calculated. Two of the limiting examples have been investigated to the fractional harmonic oscillator Langevin equation: We stated the liquid's shear stress to zero ($\lambda = 0$) and found the basic harmonic oscillator. Also, when excluding the oscillator's undamped angular frequency ($\omega = 0$), we got a special super-diffusive instance of the fractional Langevin equation. Finally, a numerical example was offered to demonstrate the fulfillment of our results.

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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