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REMARKS ON NORMALIZED GROUND STATES OF SCHRÖDINGER EQUATION WITH AT LEAST MASS CRITICAL NONLINEARITY*

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Abstract We are concerned with the nonlinear Schrödinger equation

$$-\Delta u + \lambda u = g(u) \text{ in } \mathbb{R}^N,\ \lambda \in \mathbb{R},$$

with prescribed L^2-norm $\int_{\mathbb{R}^N} u^2 dx = \rho^2$. Under general assumptions about the nonlinearity which allows at least mass critical growth, we prove the existence of a ground state solution to the problem via a clear constrained minimization method.

Keywords Schrödinger equation, normalized ground states, general nonlinearities, minimization methods.

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1. Introduction

In this paper, we consider the following nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + \lambda u = g(u) \ \text{ in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2\, dx = \rho^2,\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\rho > 0$ is a prescribed constant, $N \geq 3$ and $\lambda \in \mathbb{R}$ will appear as Lagrange multiplier.

Such problems are motivated in particular by searching for solitary waves or stationary states in nonlinear equations of the Schrödinger or Klein-Gordon type. For physical reasons, it is natural to study the existence of solutions with prescribed L^2-norm.

Let

$$S := S_\rho = \{u \in H^1(\mathbb{R}^N) : |u|_2 = \rho\} \text{ and } D_\rho = \{u \in H^1(\mathbb{R}^N) : |u|_2 \leq \rho\}$$

where $H^1(\mathbb{R}^N)$ is endowed with the usual norm $\|u\| = (|\nabla u|_2^2 + |u|_2^2)^{1/2}$ and $|\cdot|_q$ stands for the L^q-norm. Under suitable assumptions provided below, solutions to

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(1.1) are critical points of $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} G(u) dx,$$

where $G(u) = \int_0^u g(s)ds$, on the constraint S with $\lambda \in \mathbb{R}$ being the Lagrange multiplier. Set

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^N)\backslash\{0\} : M(u) = 0\}$$

where

$$M(u) := \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{N}{2}\int_{\mathbb{R}^N} H(u) dx$$

with $H(u) := g(u)u - 2G(u)$. Let $h(u) := H'(u)$. It is known that, thanks to the Pohozaev identity in [4], any solution to (1.1) stays in $\mathcal{M}$. For f_1, $f_2 : \mathbb{R} \to \mathbb{R}$, as in [5, Lemma 2.1], we write $f_1(s) \preceq f_2(s)$ for $s \in \mathbb{R}$ if $f_1(s) \le f_2(s)$ and for any $\gamma > 0$ there is $|s| < \gamma$ such that $f_1(s) < f_2(s)$.

In this paper, we make the following assumptions.

(G_0) $g \in C^1(\mathbb{R})$ and there exists $C > 0$ such that $g(u)u \ge 0$ and

$$|g'(u)| \le C(1 + |u|^{2^*-2}), \quad \text{for } u \in \mathbb{R},$$

where $2^* = (2N)/(N-2)$.

(G_1) $\eta := \limsup_{|u|\to 0} g(u)u/|u|^{2+4/N} < +\infty$.

(G_2) $\lim_{|u|\to +\infty} g(u)u/|u|^{2+4/N} = +\infty$.

(G_3) $\lim_{|u|\to +\infty} g(u)u/|u|^{2^*} = 0$.

(G_4) $(2 + 4/N)H(u) \le h(u)u$, for $u \in \mathbb{R}$.

(G_5) $(2 + 4/N)G(u) \preceq g(u)u$, for $u \in \mathbb{R}$.

(G_6) $-\Delta u - g(u) = 0$ has no solutions in $D_\rho\backslash\{0\}$.

Let $C_{N,p}$ be the optimal constant of the Gagliardo-Nirenberg inequality

$$|u|_p \le C_{N,p}|\nabla u|_2^\delta |u|_2^{1-\delta} \quad \text{for } u \in H^1(\mathbb{R}^N),$$

where $\delta = N(\frac{1}{2} - \frac{1}{p})$. Since G can have L^2-critical growth at the origin by (G_1), we need the assumption

(P_0) $\frac{N}{2}\eta C_{N,2+4/N}^{2+4/N}\rho^{\frac{4}{N}} < 1$,

which implies that ρ or η is small.

In recent years, the existence of normalized solutions for nonlinear Schrödinger equations has been studied widely under variant assumptions about g for instance in [1, 2, 5, 7–9, 11–13, 15, 17, 18] and the references therein. Let $2_N = 2 + \frac{4}{N}$. In the L^2-subcritical case, i.e. $G(u) \sim |u|^p$ with $2 < p < 2_N$, one can obtain the existence of a global minimizer of J directly on S, see [16]. In the L^2-supercritical and Sobolev subcritical ($2_N < p < 2^*$) case, the energy functional J is unbounded from above and from below and minimization does not work. For this case, using a mountain-pass argument developed on S, Jeanjean [12] showed the existence of one normalized solution. A different mini-max approach based on the σ-homotopy stable family of compact subsets of $\mathcal{M}$ has been applied in [2, 3]. Note that in [2, 3], 12] the nonlinearity was assumed to satisfy the following condition of Ambrosetti-Rabinowitz type: there exist $2_N < \alpha \leq \beta < 2^*$ such that

$$0 < \alpha G(u) \leq g(u)u \leq \beta G(u), \quad \text{for} \quad u \in \mathbb{R} \backslash \{0\}.$$

Recently, Bieganowski and Mederski in [5] considered general growth conditions on G in the spirit of Berestycki and Lions [4] and obtained ground states by a direct minimization method. The delicate approach in [5] consists of minimizing J on the constraint $D_\rho \cap \mathcal{M}$. Among other results, they obtained a normalized ground state solution for (1.1) under assumptions $(G_0) - (G_5)$, (P_0) and

$(\widetilde{G}_6)$ $\quad g(u)u \preceq 2^* G(u)$ for $u \in \mathbb{R}$.

Moreover, if $N \in \{3, 4\}$, g is odd, it is sufficient to assume that $g(u)u \leq 2^* G(u)$ for $u \in \mathbb{R}$.

In this paper, we investigate the existence of ground state solutions for (1.1) and deal with general nonlinearities in a version slightly different from the ones in [5]. We weaken some assumptions about g and refine some key ingredients of the arguments in [5], such as the determination of the sign of the corresponding Lagrange multipliers and the nonexistence of nontrivial solutions for the associated elliptic equation. Precisely, we prove the following theorem.

Theorem 1.1. *Assume $(G_0) - (G_6)$ and (P_0) hold. Then (1.1) has a normalized ground state solution $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ with $\lambda > 0$.*

Remark 1.1. (G_6) is an abstract assumption. As shown in [5], (G_6) holds under the assumptions $(G_0) - (G_5)$ and $(\widetilde{G}_6)$ for $u \in \mathbb{R}$. In fact, if $\widetilde{u} \in D_\rho \backslash \{0\}$ is a weak solution to $-\Delta \widetilde{u} = g(\widetilde{u})$, then by regularity, $\widetilde{u}$ is continuous. Moreover,

$$\int_{\mathbb{R}^N} |\nabla \widetilde{u}|^2 dx = \int_{\mathbb{R}^N} g(\widetilde{u})\widetilde{u} dx \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla \widetilde{u}|^2 dx = 2^* \int_{\mathbb{R}^N} G(\widetilde{u}) dx.$$

Then it follows that

$$\int_{\mathbb{R}^N} \left(g(\widetilde{u})\widetilde{u} - 2^* G(\widetilde{u}) \right) dx = 0,$$

which implies

$$g(\widetilde{u}(x))\widetilde{u}(x) - 2^* G(\widetilde{u}(x)) = 0 \quad \text{for} \quad x \in \mathbb{R}^N.$$

Since $\widetilde{u} \in H^1(\mathbb{R}^N)$, there is an open interval $I \subset \mathbb{R}$ such that $0 \in \overline{I}$ and $g(u)u - 2^* G(u) = 0$ for $u \in I$. Then we deduce that $G(u) = C|u|^{2^*}$ for some $C > 0$ and $u \in \overline{I}$, contradicting the definition of $\preceq$. As an immediate corollary of this observation, we obtain a normalized ground state solution of (1.1) under assumptions $(G_0) - (G_5)$, (P_0) and $(\widetilde{G}_6)$. Therefore, Theorem 1.1 can be regarded as a generalization of the existence result in [5, Theorem 1.1]. Moreover, inspired by [10], we know in advance the sign of the corresponding Lagrange multiplier $\lambda > 0$ by Clark's Theorem [6, 10].

Under $(G_0) - (G_5)$ and (P_0), we can prove (G_6) if $N \in \{3, 4\}$ and g is odd. Therefore, the following corollary is a generalization of the result in [5, Theorem 1.1(b)] where additional assumption $g(u)u \leq 2^* G(u)$ for $u \in \mathbb{R}$ was assumed.

Corollary 1.1. *Suppose $N \in \{3, 4\}$ and g is odd. Then under assumptions $(G_0) - (G_5)$ and (P_0), (1.1) has a normalized ground state solution $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ with $u > 0$ and $\lambda > 0$.*

Next, we replace (G_6) by the following assumption which is simpler to check.

(G_7) There exists $C_0 > 0$ such that

$$g(u)u - (2 + 4/N)G(u) \leq C_0|u|^{2+4/N}, \quad \text{for} \quad u \in \mathbb{R}.$$

We introduce the following additional assumption about ρ.

(P_1) $\frac{N^2}{4} C_0 C_{N,2+4/N}^{2+4/N} \rho^{\frac{4}{N}} < 1$ with C_0 as in (G_7).

As a corollary of Theorem 1.1, we obtain

Corollary 1.2. *Assume* $(G_0) - (G_5)$, (G_7) *and* $(P_0) - (P_1)$ *hold. Then* (1.1) *has a normalized ground state solution* $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ *with* $\lambda > 0$.

Remark 1.2. An example of such a nonlinearity satisfying $(G_0) - (G_5)$ and (G_7) is given by

$$G(u) = |u|^{2+\frac{4}{N}} \log(1 + |u|).$$

Then g satisfies (G_7) with $C_0 = 1$, (G_1) with $\eta = 0$ and $(G_2) - (G_5)$. Therefore,

$$\begin{cases} -\Delta u + \lambda u = (2 + \frac{4}{N})|u|^{\frac{4}{N}} u \log(1 + |u|) + \frac{|u|^{1+\frac{4}{N}} u}{1+|u|} & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \, dx = \rho^2, \end{cases}$$

has a normalized ground state solution if $\rho < \rho^*$ with $\rho^* = \left(\frac{4}{N^2 C_{N,2+4/N}^{2+4/N}} \right)^{\frac{N}{4}}$.

Here and in the sequel, C denotes a generic positive constant which may vary from one equation to another. In the next section, we give the proof of the main results.

2. Proof of main results

First, we recall the property on the notion $f_1 \preceq f_2$ in [5, Lemma 2.1].

Lemma 2.1. *Let* $f_1, f_2 \in C(\mathbb{R})$ *and assume there exists* $C > 0$ *such that* $|f_1(u)| + |f_2(u)| \leq C(|u|^2 + |u|^{2^*})$ *for every* $u \in \mathbb{R}$. *Then* $f_1 \preceq f_2$ *if and only if* $f_1 \leq f_2$ *and*

$$\int_{\mathbb{R}^N} f_1(u) - f_2(u) dx < 0, \quad \text{for every } u \in H^1(\mathbb{R}^N).$$

Lemma 2.2. *Assume* (G_0), (G_1), (G_3), (G_5) *and* (P_0) *hold. Then*

$$\inf_{u \in D_\rho \cap \mathcal{M}} |\nabla u|_2 > 0.$$

Proof. By (G_0), (G_1) and (G_3), for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$g(u)u \leq \varepsilon |u|^{2^*} \text{ for } |u| \geq R_\varepsilon \text{ and } g(u)u \leq (\varepsilon + \eta)|u|^{2+\frac{4}{N}} \text{ for } |u| \leq R_\varepsilon^{-1}. \tag{2.1}$$

By (2.1), we deduce that for some $p \in (2 + 4/N, 2^*)$, there exists $c_\varepsilon > 0$ such that

$$H(u) \leq g(u)u \leq \varepsilon |u|^{2^*} + c_\varepsilon |u|^p + (\varepsilon + \eta)|u|^{2+\frac{4}{N}} \text{ for } u \in \mathbb{R}.$$

This implies that for $u \in D_\rho \cap \mathcal{M}$,

$$|\nabla u|_2^2 = \frac{N}{2}\int_{\mathbb{R}^N} H(u)dx \le \frac{N}{2}\int_{\mathbb{R}^N} \varepsilon|u|^{2^*} + c_\varepsilon|u|^p + (\varepsilon+\eta)|u|^{2+\frac{4}{N}}dx.$$

Then the Gagliardo-Nirenberg inequality and the Sobolev embedding inequality imply that

$$|\nabla u|_2^2 \le \varepsilon C|\nabla u|_2^{2^*} + \varepsilon C|\nabla u|_2^2 + c_\varepsilon C|\nabla u|_2^{\delta p} + \tfrac{N}{2}\eta C_{N,2+4/N}^{2+4/N}\rho^{\frac{4}{N}}|\nabla u|_2^2 \tag{2.2}$$

where $\delta p := N(p-2)/2 > 2$ and C is a positive constant. Therefore, we see that $|\nabla u|_2^2$ stays away from 0 on $D_\rho \cap \mathcal{M}$ if $\frac{N}{2}\eta C_{N,2+4/N}^{2+4/N}\rho^{\frac{4}{N}} < 1$. □

Next, let $\lambda > 0$ and consider the function

$$\varphi(\lambda) := J(\lambda^{\frac{N}{2}}u(\lambda\cdot)).$$

Note that (G_1), (G_3) and (G_5) yield that for any $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that

$$g(u)u \le (\varepsilon+\eta)|u|^{2+\frac{4}{N}} + c_\varepsilon|u|^{2^*},$$

and

$$G(u) \le \frac{N}{2N+4}\big((\varepsilon+\eta)|u|^{2+\frac{4}{N}} + c_\varepsilon|u|^{2^*}\big). \tag{2.3}$$

For $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ satisfying

$$\eta C_{N,2+\frac{4}{N}}^{2+\frac{4}{N}}|u|_2^{\frac{4}{N}} < 1 + \frac{2}{N}, \tag{2.4}$$

we have the following result by using of a slight modification of the proof of [5, Lemma 2.3].

Lemma 2.3. *Assume $(G_0)-(G_5)$ hold. Then for $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ satisfying (2.4), there is an interval $[a,b] \subset (0,+\infty)$ such that each $\lambda \in [a,b]$ is a global maximizer for φ and φ is increasing on $(0,a)$ and decreasing on (b,∞). Moreover, $M(\lambda^{\frac{N}{2}}u(\lambda\cdot)) = 0$ if and only if $\lambda \in [a,b]$, $M(\lambda^{\frac{N}{2}}u(\lambda\cdot)) > 0$ if and only if $\lambda \in (0,a)$, $M(\lambda^{\frac{N}{2}}u(\lambda\cdot)) < 0$ if and only if $\lambda > b$.*

Remark 2.1. We observe that for $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ satisfying (P_0), $\varphi'(\lambda_0) = 0$ means $\lambda_0 * u := \lambda_0^{\frac{N}{2}}u(\lambda_0\cdot) \in \mathcal{M}$. Then from (2.3) and Lemma 2.3, a similar argument to [5, Lemma 2.5] leads to

$$c_\rho := \inf_{u\in D_\rho\cap\mathcal{M}} J(u) > 0. \tag{2.5}$$

By using similar arguments as in the proof of [5, Lemma 2.4], we deduce that any minimizing sequence $\{u_n\} \subset D_\rho \cap \mathcal{M}$ for c_ρ is bounded. Moreover, along the lines of the proof of [5, Lemma 2.7], one can easily establish that $J|_{D_\rho\cap\mathcal{M}}$ attains its infimum c_ρ at some point $u_0 \in D_\rho \cap \mathcal{M}$.

Lemma 2.4. *Assume $(G_0)-(G_6)$ hold. Then $S_\rho \cap \mathcal{M}$ contains all minimizers of J on $D_\rho \cap \mathcal{M}$.*

Proof. We modify the proof of [5, Lemma 2.8]. Suppose on the contrary that $\widetilde{u} \neq 0$ is a minimizer of J on $D_\rho \cap \mathcal{M}$ with $\widetilde{u} \in D_\rho \backslash S_\rho$, which implies that $\widetilde{u}$ is a minimizer of J on $(D_\rho \backslash S_\rho) \cap \mathcal{M}$. Therefore there is a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$J'(\widetilde{u})v + \mu \int_{\mathbb{R}^N} \left(\nabla \widetilde{u} \nabla v - \frac{N}{4} h(\widetilde{u}) v \right) dx = 0$$

for every $v \in H^1(\mathbb{R}^N)$. Then $\widetilde{u}$ solves

$$-(1+\mu)\Delta \widetilde{u} = g(\widetilde{u}) + \frac{N}{4}\mu h(\widetilde{u}). \tag{2.6}$$

Next, we distinguish three cases to deduce contradictions.

Case 1 ($\mu = -1$). Then we have

$$\int_{\mathbb{R}^N} g(\widetilde{u})\widetilde{u} - \frac{N}{4} h(\widetilde{u})\widetilde{u} dx = 0. \tag{2.7}$$

On the other hand, it follows from (G_5) and Lemma 2.1 that

$$\int_{\mathbb{R}^N} g(\widetilde{u})\widetilde{u} - (2 + \frac{4}{N}) G(\widetilde{u}) dx > 0.$$

This together with (G_4), we have

$$\int_{\mathbb{R}^N} g(\widetilde{u})\widetilde{u} dx < (1 + \frac{N}{2}) \int_{\mathbb{R}^N} H(\widetilde{u}) dx \leq \int_{\mathbb{R}^N} \frac{N}{4} h(\widetilde{u})\widetilde{u} dx,$$

which contradicts (2.7).

Case 2 ($\mu = 0$). Then $\widetilde{u}$ is a nontrivial weak solution to

$$-\Delta \widetilde{u} = g(\widetilde{u}),$$

which contradicts (G_6).

Case 3 ($\mu \neq -1$ and $\mu \neq 0$). Then we have the Nehari-type identity

$$(1+\mu) \int_{\mathbb{R}^N} |\nabla \widetilde{u}|^2 dx = \int_{\mathbb{R}^N} g(\widetilde{u})\widetilde{u} + \frac{N}{4}\mu h(\widetilde{u})\widetilde{u} dx$$

and the Pohožaev-type identity

$$\frac{N-2}{2}(1+\mu) \int_{\mathbb{R}^N} |\nabla \widetilde{u}|^2 dx = N \int_{\mathbb{R}^N} G(\widetilde{u}) + \frac{N}{4}\mu H(\widetilde{u}) dx.$$

Combining the above equalities with $\widetilde{u} \in \mathcal{M}$ yields

$$(1+\mu)\frac{N}{2} \int_{\mathbb{R}^N} H(\widetilde{u}) dx = \frac{N}{2} \int_{\mathbb{R}^N} H(\widetilde{u}) + \frac{N}{4}\mu (h(\widetilde{u})\widetilde{u} - 2H(\widetilde{u})) dx,$$

that is,

$$\mu \int_{\mathbb{R}^N} H(\widetilde{u}) dx = \frac{N}{4}\mu \int_{\mathbb{R}^N} h(\widetilde{u})\widetilde{u} - 2H(\widetilde{u}) dx.$$

Since $\mu \neq 0$, we have

$$\int_{\mathbb{R}^N} h(\widetilde{u})\widetilde{u} - (2 + \frac{4}{N}) H(\widetilde{u}) dx = 0.$$

Notice that, by regularity, any weak solution of (2.6) is continuous. Then by (G_4) we have

$$h(\widetilde{u}(x))\widetilde{u}(x) - (2 + \frac{4}{N})H(\widetilde{u}(x)) = 0, \quad \text{for } x \in \mathbb{R}^N.$$

Since $\widetilde{u} \in H^1(\mathbb{R}^N)$, there is an open interval $I \subset \mathbb{R}$ such that $0 \in \overline{I}$ and $h(u)u - (2 + \frac{4}{N})H(u) = 0$ for $u \in I$. Thus from (G_0) we deduce that

$$H(u) = C|u|^{2+\frac{4}{N}} \text{ and } G(u) = C|u|^{2+\frac{4}{N}}$$

for some $C > 0$ and $u \in \overline{I}$, in contradiction with (G_5). □

If u_0 is a minimizer of J on $D_\rho \cap \mathcal{M}$. Then under the assumptions of Lemma 2.4, every minimizer $u \in D_\rho \cap \mathcal{M}$ of $J|_{D_\rho \cap \mathcal{M}}$ is a minimizer of $J|_{S_\rho \cap \mathcal{M}}$. Then $u_0 \in S_\rho \cap \mathcal{M}$ and $J(u_0) = \inf_{u \in S_\rho \cap \mathcal{M}} J(u)$. Therefore, by the Lagrange multiplier rule, there exists $\lambda, \mu \in \mathbb{R}$ such that

$$-\Delta u_0 - g(u_0) + \lambda u_0 + \mu(-\Delta u_0 - \frac{N}{4}h(u_0)) = 0. \tag{2.8}$$

We need the following proposition which is related with Clarke's [6, Theorem 1]. The proof can be found in [10, Proposition A.1].

Proposition 2.1. *Let $\mathcal{H}$ be a real Hilbert space and $f, \phi_i, \psi_j \in \mathcal{C}^1(\mathcal{H})$, $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$. Suppose that for every*

$$x \in \bigcap_{i=1}^{m} \phi_i^{-1}(0) \cap \bigcap_{j=1}^{n} \psi_j^{-1}(0)$$

the differential

$$\left(\phi_i'(x), \psi_j'(x)\right)_{1 \le i \le m, 1 \le j \le n} : \mathcal{H} \to \mathbb{R}^{m+n}$$

is surjective. If $\bar{x} \in \mathcal{H}$ minimizes f over

$$\{x \in \mathcal{H} : \phi_i(x) \le 0 \text{ for every } i = 1, \dots, m \text{ and } \psi_j(x) = 0 \text{ for every } j = 1, \dots, n\},$$

then there exist $(\lambda_i)_{i=1}^m \in [0, \infty)^m$ and $(\sigma_j)_{j=1}^n \in \mathbb{R}^n$ such that

$$f'(\bar{x}) + \sum_{i=1}^{m} \lambda_i \phi_i'(\bar{x}) + \sum_{j=1}^{n} \sigma_i \psi_j'(\bar{x}) = 0.$$

Using Proposition 2.1, we obtain the following estimate on the sign of λ, which plays an important role in the proceeding arguments.

Lemma 2.5. *Let (P_0), (G_0)–(G_6) be satisfied, and λ, μ be the Lagrange multipliers associated to (2.8). Then $\lambda > 0$.*

Proof. First observe that for $u \in S_\rho \cap \mathcal{M}$, ϕ and ψ are of class C^1 where

$$\phi(u) := \int_{\mathbb{R}^N} u^2 dx - \rho^2 \text{ and } \psi(u) := |\nabla u|_2^2 - \frac{N}{2} \int_{\mathbb{R}^N} H(u) dx.$$

To apply Proposition 2.1, we claim that $d(\phi, \psi) : H^1(\mathbb{R}^N) \to \mathbb{R}^2$ is surjective. It is sufficient to show that $d\phi(u)$, $d\psi(u)$ are linearly independent to prove the claim. If not, then there exist $\gamma \neq 0$ such that u is a solution to

$$\gamma u = -\Delta u - \frac{N}{4} h(u).$$

Then the Nehari-type identity and the Pohozaev-type identity read

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \gamma \int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} \frac{N}{4} h(u)u dx$$

and

$$(N-2)\int_{\mathbb{R}^N} |\nabla u|^2 dx = \gamma N \int_{\mathbb{R}^N} |u|^2 dx + \frac{N^2}{2}\int_{\mathbb{R}^N} H(u) dx.$$

Combining the above equalities with $u \in \mathcal{M}$ yields

$$\int_{\mathbb{R}^N} h(u)u - (2+\frac{4}{N})H(u)dx = 0.$$

Arguing as in Lemma 2.4, we obtain a contradiction. Then by Proposition A.1 of [10], we get $\lambda \geq 0$. Using again Lemma 2.4 we deduce also that $\lambda = 0$ is impossible. □

Proof. [Proof of Theorem 1.1] Under assumptions $(G_0) - (G_6)$, from Lemma 2.4 and Lemma 2.5 it follows that $u_0 \in S_\rho \cap \mathcal{M}$ is a minimizer of J on $D_\rho \cap \mathcal{M}$ and for some $\lambda > 0$ and $\mu \in \mathbb{R}$,

$$-(1+\mu)\Delta u_0 + \lambda u_0 = g(u_0) + \frac{N}{4}\mu h(u_0). \tag{2.9}$$

Next we claim that $\mu \neq -1$. If not, then we have

$$\lambda\rho^2 = \int_{\mathbb{R}^N} g(u_0)u_0 - \frac{N}{4}h(u_0)u_0 dx.$$

Together with (G_4), we obtain

$$\lambda\rho^2 \leq \frac{N}{2}\int_{\mathbb{R}^N} (2+\frac{4}{N})G(u_0) - g(u_0)u_0 dx.$$

Hence (G_5) and Lemma 2.1 imply that $\lambda < 0$, a contradiction. Then for (2.9), we have

$$(1+\mu)\int_{\mathbb{R}^N} |\nabla u_0|^2 + \lambda|u_0|^2 dx = \int_{\mathbb{R}^N} g(u_0)u_0 + \frac{N}{4}\mu h(u_0)u_0 dx$$

and

$$(1+\mu)\int_{\mathbb{R}^N} |\nabla u_0|^2 + \frac{2^*}{2}\lambda|u_0|^2 dx = 2^*\int_{\mathbb{R}^N} G(u_0) + \frac{N}{4}\mu H(u_0)dx,$$

which imply that

$$(1+\mu)\int_{\mathbb{R}^N} |\nabla u_0|^2 dx = \frac{N}{2}\int_{\mathbb{R}^N} H(u_0) + \frac{N}{4}\mu(h(u_0)u_0 - 2H(u_0))dx.$$

Then it follows from $u_0 \in \mathcal{M}$ that

$$\mu\int_{\mathbb{R}^N} h(u_0)u_0 - (2+\frac{4}{N})H(u_0)dx = 0.$$

Also, arguing as in Lemma 2.4, we deduce that

$$\int_{\mathbb{R}^N} h(u_0)u_0 - (2+\frac{4}{N})H(u_0)dx > 0.$$

This means that $\mu = 0$ and $u_0 \in S_\rho$ is a solution of $-\Delta u_0 + \lambda u_0 = g(u_0)$ with $\lambda > 0$. □

To prove Corollary 1.1, we recall a Liouville type result due to [11, Lemma A.2].

Lemma 2.6. *Suppose $0 < p \leq N/(N-2)$ when $N \geq 3$ and $0 < p < \infty$ when $N = 1, 2$. Let $u \in L^p(\mathbb{R}^N)$ be a smooth nonnegative function and satisfy $-\Delta u \geq 0$ in $\mathbb{R}^N$. Then $u \equiv 0$ holds.*

Proof. [Proof of Corollary 1.1] From Lemma 2.2, Lemma 2.3 and Remark 2.1, we deduce the existence of a bounded minimizing sequence $\{u_n\} \subset D_\rho \cap \mathcal{M}$ for c_ρ. We also obtain a point $u_0 \in D_\rho \cap \mathcal{M}$ which is a minimizer of J on $D_\rho \cap \mathcal{M}$. To conclude the proof of Lemma 2.4, it remains to consider *Case* 2 where $\mu = 0$. If $\widetilde{u} \in \mathcal{M}$ is a nontrivial weak solution to $-\Delta \widetilde{u} = g(\widetilde{u})$. Since we suppose that g is odd. Then G and H are even, so that $G(|u|) = G(u)$ and $H(|u|) = H(u)$ for all $u \in H^1(\mathbb{R}^N)$. Moreover,

$$\int_{\mathbb{R}^N} |\nabla |u||^2 dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx,$$

for all $u \in H^1(\mathbb{R}^N)$. Therefore, $|\widetilde{u}|$ is also a minimizer of $J|_{D_\rho \cap \mathcal{M}}$. Then $\widetilde{u}$ can be chosen to be a nonnegative function. Then if $N \in \{3, 4\}$, $2 \leq N/(N-2)$. Since $\widetilde{u} \in L^2(\mathbb{R}^N)$ it follows from Lemma 2.6 that $\widetilde{u} \equiv 0$, a contradiction. Then the existence of the normalized ground state solution can be proved along the lines of the proof of Lemma 2.5 and Theorem 1.1. □

Proof. [Proof of Corollary 1.2] Since G satisfies $(G_0)-(G_5)$, Lemma 2.4 still holds in the case $\mu \neq 0$. For the case $\mu = 0$, it follows from $\widetilde{u} \in \mathcal{M}$ that

$$\int_{\mathbb{R}^N} G(\widetilde{u})\ dx = \frac{1}{2} \int_{\mathbb{R}^N} g(\widetilde{u})\widetilde{u}\ dx - \frac{1}{N} \int_{\mathbb{R}^N} |\nabla \widetilde{u}|^2\ dx. \tag{2.10}$$

Observing that $\int_{\mathbb{R}^N} |\nabla \widetilde{u}|^2 dx = \int_{\mathbb{R}^N} g(\widetilde{u})\widetilde{u} dx$, by (2.10) and direct calculation, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla \widetilde{u}|^2 dx - \frac{N^2}{4} \int_{\mathbb{R}^N} g(\widetilde{u})\widetilde{u} - (2 + \frac{4}{N}) G(\widetilde{u}) dx \\
&= \left(-\tfrac{N}{2}\right) \int_{\mathbb{R}^N} |\nabla \widetilde{u}|^2 - g(\widetilde{u})\widetilde{u} dx \\
&= 0.
\end{aligned}$$

Then together with (G_7) and the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned}
0 &\geq \int_{\mathbb{R}^N} |\nabla \widetilde{u}|^2 dx - \frac{N^2}{4} C_0 \int_{\mathbb{R}^N} |\widetilde{u}|^{2+\frac{4}{N}} dx \\
&\geq (1 - \frac{N^2}{4} C_0 C_{N,2+4/N}^{2+4/N} \rho^{\frac{4}{N}}) \int_{\mathbb{R}^N} |\nabla \widetilde{u}|^2 dx
\end{aligned}$$

which is a contradiction to the fact that $\widetilde{u} \neq 0$ and the assumption (P_1). The rest of the proof proceeds exactly as in Theorem 1.1. □

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Competing interests

The authors have no relevant financial or non-financial interests to disclose.

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