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THE LIE SYMMETRY ANALYSIS, OPTIMAL SYSTEM, EXACT SOLUTIONS AND CONSERVATION LAWS OF THE (2+1)-DIMENSIONAL VARIABLE COEFFICIENTS DISPERSIVE LONG WAVE EQUATIONS*

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Abstract In this article, the (2+1)-dimensional variable coefficients dispersive long wave equations (vcDLWs) are studied by the Lie symmetry analysis method. The infinitesimal generators and geometric vector fields are given. Optimal system of the (2+1)-dimensional vcDLWs are analyzed by Olver's method. Based on the optimal system, the (2+1)-dimensional vcDLW equations are reduced to (1+1)-dimensional equations. A number of new exact solutions of vcDLW equations are derived. Some kink solutions and 2-soliton solutions are obtained by using $(1/G')$-expansion method and (G'/G)-expansion method. Many different types of exact solutions can be obtained by changing the coefficient functions. By exploring the evolution of the solutions with function of the coefficients and time t, the dynamic behaviors of the solutions are analysed. At last, the conservation laws of the (2+1)-dimensional vcDLWs are derived based on the nonlinear self-adjointness.

Keywords The (2+1)-dimensional vcDLWs, the Lie symmetry analysis, conservation laws, $(1/G')$-expansion method, (G'/G)-expansion method.

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1. Introduction

Nonlinear partial differential equations (NLPDEs) are important models used to describe nonlinear phenomena in chemistry, biology, physics, etc [5,20]. With the rapid development of society, the use of NLPDEs is becoming more and more widespread and intensive. Therefore, the study of the exact solutions of the NLPDEs is of

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great theoretical importance and research value [1, 4, 26, 33]. Moreover, scholars have proposed various methods to solve the NLPDEs, such as the Hirota bilinear method [3, 16], the Lie symmetry analysis [6, 12, 13, 21], the Darboux transformation [8, 17], the inverse scattering method and the Bäcklund transformation [10,14,30]. The Lie symmetry analysis method uses the symmetry of differential equations and continuous infinitesimal transformations to study the equations. It is a common method for solving partial differential equations. Due to the limitations of constant coefficient equations in describing physical phenomena, the study of variable coefficient equations that introduce some arbitrary functions has become a hot issue [9].

The (2+1)-dimensional dispersive long wave equations (DLWs) are NLPDEs [11]. The form of the (2+1)-dimensional DLWs are as follows

$$\begin{aligned} &u_{yt}+v_{xx}+u_y u_x+u u_{xy}=0, \\ &v_t+u_x+u_x v+u v_x+u_{xxy}=0, \end{aligned} \tag{1.1}$$

where u, v are two functions of x, y, t.

In reference [29], Kajal et al. used the similar transformation method obtain some solutions of physical significance for the (2+1)-dimensional DLWs. In reference [35], Xia et al. derive the Bäcklund transform and residual symmetry of the (2+1)-dimensional DLWs from the standard truncated Painlevé expansion. At the same time Xia succeeded in reducing the residual symmetry to a Lie point symmetry by introducing an appropriate auxiliary dependent variable. In this article, we will study the DLW equation using the Lie symmetry analysis. A crucial variety of exact solutions can be obtained. The dynamical behaviour of the solutions are also analysed by drawing figures of the solutions [34].

The (2+1)-dimensional vcDLWs are the following form

$$\begin{aligned} &F_1=u_{yt}+\mathrm{a}\left(t\right)v_{xx}+b\left(t\right)\left(u_y u_x+u u_{xy}\right)=0, \\ &F_2=v_t+c\left(t\right)\left(u_x+u_x v+u v_x\right)+d\left(t\right)u_{xxy}=0, \end{aligned} \tag{1.2}$$

where $a\left(t\right), b\left(t\right), c\left(t\right), d\left(t\right)$ are four arbitrary functions. When $a\left(t\right)=b\left(t\right)=c\left(t\right)=d\left(t\right)=1$, Eqs.(1.2) is converted to Eqs.(1.1).

This paper is organised as follows: Infinitely generated small elements of the independent variables are obtained by constructing third-order extensions and Lie symmetry analysis method in Section 2. The Lie exchange table and the Lie accompanying table are derived in Section 3. The invariants are derived [22, 23]. The optimal system is determined by discussing the invariants. Based on the optimal system, the reduced equations of Eqs.(1.2) are obtained in Section 4. Various types of exact solutions, which include kink solutions, periodic solutions and 2-soliton solutions, are obtained using the (G'/G)-expansion and $(1/G')$-expansion method [7, 18, 27, 37]. In addition, the relevant figures of the solution are given for discussion in section 5. The nonlinear self-adjointness of (2+1)-dimensional vcDLWs is given. Based on nonlinear self-adjointness, conservation laws are derived in section 6. The conclusions of this paper are given in section 7.

2. Lie symmetry analysis

In this section, the symmetry Lie group of Eqs.(1.2) is determined. First, a set of one-parameter transformations is assumed as follows

$$\begin{aligned}
\tilde{x} &= x + \tilde{\varepsilon}\widehat{\xi} + o\left(\tilde{\varepsilon}^2\right),\\
\tilde{y} &= y + \tilde{\varepsilon}\widehat{\eta} + o\left(\tilde{\varepsilon}^2\right),\\
\tilde{t} &= t + \tilde{\varepsilon}\widehat{\tau} + o\left(\tilde{\varepsilon}^2\right),\\
\tilde{u} &= u + \tilde{\varepsilon}\widehat{\varphi} + o\left(\tilde{\varepsilon}^2\right),\\
\tilde{v} &= v + \tilde{\varepsilon}\widehat{\psi} + o\left(\tilde{\varepsilon}^2\right),
\end{aligned} \tag{2.1}$$

where $\tilde{\varepsilon}$ is an infinitesimal parameter and $\widehat{\xi}, \widehat{\tau}, \widehat{\eta}, \widehat{\varphi}, \widehat{\psi}$ are the infinitesimal generator elements. $\widehat{\xi}, \widehat{\tau}, \widehat{\eta}, \widehat{\varphi}, \widehat{\psi}$ relate to x, y, t, u, v. The vector field $\hat{V}$ associated with Eqs.(1.2) is given as

$$\hat{V} = \widehat{\xi}\frac{\partial}{\partial x} + \widehat{\tau}\frac{\partial}{\partial t} + \widehat{\eta}\frac{\partial}{\partial y} + \widehat{\varphi}\frac{\partial}{\partial u} + \widehat{\psi}\frac{\partial}{\partial v}. \tag{2.2}$$

Subsequently, the Lie symmetry of Eqs.(1.2) will be generated by Eq.(2.1). Also, we can obtain $\hat{V}$ of the third-order prolongation

$$\begin{aligned}
\mathrm{Pr}^{(3)}\hat{V} = {} & \widehat{\xi}\frac{\partial}{\partial x} + \widehat{\tau}\frac{\partial}{\partial t} + \widehat{\eta}\frac{\partial}{\partial y} + \widehat{\varphi}\frac{\partial}{\partial u} + \widehat{\psi}\frac{\partial}{\partial v} + \widehat{\varphi}^{x}\frac{\partial}{\partial u_x} + \widehat{\varphi}^{y}\frac{\partial}{\partial u_y} + \widehat{\varphi}^{xy}\frac{\partial}{\partial u_{xy}}\\
& + \widehat{\varphi}^{yt}\frac{\partial}{\partial u_{yt}} + \widehat{\varphi}^{xxy}\frac{\partial}{\partial u_{xxy}} + \widehat{\psi}^{x}\frac{\partial}{\partial v_x} + \widehat{\psi}^{t}\frac{\partial}{\partial v_t} + \widehat{\psi}^{xx}\frac{\partial}{\partial v_{xx}},
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
\widehat{\varphi}^{x} &= D_x\left(\widehat{\varphi} - \widehat{\xi}u_x - \widehat{\eta}u_y - \widehat{\tau}u_t\right) + \widehat{\xi}u_{xx} + \widehat{\eta}u_{yx} + \widehat{\tau}u_{tx},\\
\widehat{\varphi}^{y} &= D_y\left(\widehat{\varphi} - \widehat{\xi}u_x - \widehat{\eta}u_y - \widehat{\tau}u_t\right) + \widehat{\xi}u_{xy} + \widehat{\eta}u_{yy} + \widehat{\tau}u_{ty},\\
\widehat{\varphi}^{xy} &= D_{xy}\left(\widehat{\varphi} - \widehat{\xi}u_x - \widehat{\eta}u_y - \widehat{\tau}u_t\right) + \widehat{\xi}u_{xxy} + \widehat{\eta}u_{yxy} + \widehat{\tau}u_{txy},\\
\widehat{\varphi}^{yt} &= D_{yt}\left(\widehat{\varphi} - \widehat{\xi}u_x - \widehat{\eta}u_y - \widehat{\tau}u_t\right) + \widehat{\xi}u_{xyt} + \widehat{\eta}u_{yyt} + \widehat{\tau}u_{tyt},\\
\widehat{\varphi}^{xxy} &= D_{xxy}\left(\widehat{\varphi} - \widehat{\xi}u_x - \widehat{\eta}u_y - \widehat{\tau}u_t\right) + \widehat{\xi}u_{xxxy} + \widehat{\eta}u_{yxxy} + \widehat{\tau}u_{txxy},\\
\widehat{\psi}^{x} &= D_x\left(\widehat{\psi} - \widehat{\xi}v_x - \widehat{\eta}v_y - \widehat{\tau}v_t\right) + \widehat{\xi}v_{xx} + \widehat{\eta}v_{xy} + \widehat{\tau}v_{xt},\\
\widehat{\psi}^{t} &= D_t\left(\widehat{\psi} - \widehat{\xi}v_x - \widehat{\eta}v_y - \widehat{\tau}v_t\right) + \widehat{\xi}v_{xt} + \widehat{\eta}v_{yt} + \widehat{\tau}v_{tt},\\
\widehat{\psi}^{xx} &= D_{xx}\left(\widehat{\psi} - \widehat{\xi}v_x - \widehat{\eta}v_y - \widehat{\tau}v_t\right) + \widehat{\xi}v_{xxx} + \widehat{\eta}v_{yxx} + \widehat{\tau}v_{txx},
\end{aligned} \tag{2.4}$$

where D_x, D_y, D_t are full differential of x, y, t.

Applying $\mathrm{Pr}^{(3)}\hat{V}$ to Eqs.(1.2) and expanding gives

$$\begin{aligned}&\widehat{\varphi}^{yt}+a\left(t\right)\widehat{\psi}^{xx}+b\left(t\right)\left[\widehat{\varphi}^{y}u_{x}+u_{y}\widehat{\varphi}^{x}+\widehat{\varphi}u_{yx}+u\widehat{\varphi}^{yx}\right]=0,\\&\widehat{\psi}^{t}+c\left(t\right)\left[\widehat{\varphi}^{x}+\widehat{\varphi}^{x}v+u_{x}\widehat{\psi}+\widehat{\varphi}v_{x}+u\widehat{\psi}^{x}\right]+d\left(t\right)\widehat{\varphi}^{xxy}=0.\end{aligned}\tag{2.5}$$

Substituting Eqs.(2.4) into Eqs.(2.5) and extracting the derivative coefficients of each order, a set of decision equations is obtained. Solving it can derive

$$\begin{aligned}&\widehat{\xi}=-\left(\bar{c}_{1}+\bar{c}_{2}\right)x+\bar{c}_{3},\ \widehat{\eta}=\bar{c}_{2}y+\bar{c}_{5},\ \widehat{\tau}=\frac{\int\left(-2\left(\bar{c}_{1}+\bar{c}_{2}\right)a\left(t\right)\right)dt+\bar{c}_{4}}{a\left(t\right)},\\&\widehat{\varphi}=\left(\bar{c}_{1}+\bar{c}_{2}\right)u,\widehat{\psi}=\left(\bar{c}_{1}+1\right)v,\end{aligned}\tag{2.6}$$

where $\bar{c}_j(j=1,2,3,4,5)$ are arbitrary constants. Moreover, $a\left(t\right),b\left(t\right),c\left(t\right),d\left(t\right)$ must satisfy

$$\begin{aligned}&2\left(H+L\right)a\left(t\right)+\left(\frac{d}{dt}a\left(t\right)\right)\widehat{\tau}+\left(\frac{\partial}{\partial t}\widehat{\tau}\right)a\left(t\right)=0,\\&2\left(H+L\right)b\left(t\right)+\left(\frac{d}{dt}b\left(t\right)\right)\widehat{\tau}+\left(\frac{\partial}{\partial t}\widehat{\tau}\right)b\left(t\right)=0,\\&2\left(H+L\right)c\left(t\right)+\left(\frac{d}{dt}c\left(t\right)\right)\widehat{\tau}+\left(\frac{\partial}{\partial t}\widehat{\tau}\right)c\left(t\right)=0,\\&2\left(H+L\right)d\left(t\right)+\left(\frac{d}{dt}d\left(t\right)\right)\widehat{\tau}+\left(\frac{\partial}{\partial t}\widehat{\tau}\right)d\left(t\right)=0.\end{aligned}\tag{2.7}$$

Assigning different values to $\bar{c}_j(j=1,2,3,4,5)$, the vector field can be obtained as follows

$$\begin{aligned}&\hat{V}_1=-x\frac{\partial}{\partial x}-\frac{2\int a\left(t\right)dt}{a\left(t\right)}\frac{\partial}{\partial t}+u\frac{\partial}{\partial u}+\left(v+1\right)\frac{\partial}{\partial v},\\&\hat{V}_2=-x\frac{\partial}{\partial x}-\frac{2\int a\left(t\right)dt}{a\left(t\right)}\frac{\partial}{\partial t}+y\frac{\partial}{\partial y}+u\frac{\partial}{\partial u},\\&\hat{V}_3=\frac{\partial}{\partial x},\ \hat{V}_4=\frac{1}{a\left(t\right)}\frac{\partial}{\partial t},\ \hat{V}_5=\frac{\partial}{\partial y}.\end{aligned}\tag{2.8}$$

These vector fields $\hat{V}_i\left(1\le i\le 5\right)$ can be expressed as $\hat{V}=\bar{c}_1\hat{V}_1+\bar{c}_2\hat{V}_2+\bar{c}_3\hat{V}_3+\bar{c}_4\hat{V}_4+\bar{c}_5\hat{V}_5$.

Besides, the Lie exchange relation of Eqs.(1.2) is given in **Table 1** by using vector fields $\hat{V}_i\left(1\le i\le 5\right)$.

3. Optimal system of vcDLWs

In this section, the one-dimensional optimal system of Eqs.(1.2) is derived based on the Lie exchange relation and the Lie accompanying relation [19, 24, 28, 31, 32].

First, we can easily verify that the vector fields $\hat{V}_i\left(1\le i\le 5\right)$ is closed in the Lie bracket operation $\left[\hat{V}_\beta,\hat{V}_\gamma\right]=\hat{V}_\beta\hat{V}_\gamma-\hat{V}_\gamma\hat{V}_\beta$. Based on the exchange relations of the vector fields, we can find that the transformed Lie groups constitute an infinite

Table 1. Commutator table.

$\left[\hat{V}_\beta, \hat{V}_\gamma\right]$	$\hat{V}_1$	$\hat{V}_2$	$\hat{V}_3$	$\hat{V}_4$	$\hat{V}_5$
$\hat{V}_1$	0	0	$\hat{V}_3$	$2\hat{V}_4$	0
$\hat{V}_2$	0	0	$\hat{V}_3$	$2\hat{V}_4$	$-\hat{V}_5$
$\hat{V}_3$	$-\hat{V}_3$	$-\hat{V}_3$	0	0	0
$\hat{V}_4$	$-2\hat{V}_4$	$-2\hat{V}_4$	0	0	0
$\hat{V}_5$	0	$\hat{V}_5$	0	0	0

dimensional Lie algebra. Meanwhile, Lie accompanying **Table 2** can be obtained from **Table 1**. The Lie series for determining the Lie algebraic representation is

$$Ad\left(\exp\left(\tilde{\varepsilon}\right)\hat{V}_\beta\right)\hat{V}_\gamma = \hat{V}_\gamma - \tilde{\varepsilon}\left[\hat{V}_\beta, \hat{V}_\gamma\right] + \frac{1}{2}\tilde{\varepsilon}^2\left[\hat{V}_\beta, \left[\hat{V}_\beta, \hat{V}_\gamma\right]\right] - \dots. \tag{3.1}$$

Table 2. Adjoint representation table.

Ad	$\hat{V}_1$	$\hat{V}_2$	$\hat{V}_3$	$\hat{V}_4$	$\hat{V}_5$
$\hat{V}_1$	$\hat{V}_1$	$\hat{V}_2$	$\hat{V}_3 e^{-\tilde{\varepsilon}}$	$\hat{V}_4 e^{-2\tilde{\varepsilon}}$	$\hat{V}_5$
$\hat{V}_2$	$\hat{V}_1$	$\hat{V}_2$	$\hat{V}_3 e^{-\tilde{\varepsilon}}$	$\hat{V}_4 e^{-2\tilde{\varepsilon}}$	$\hat{V}_5 e^{\tilde{\varepsilon}}$
$\hat{V}_3$	$\hat{V}_1 + \tilde{\varepsilon}\hat{V}_3$	$\hat{V}_2 + \tilde{\varepsilon}\hat{V}_3$	$\hat{V}_3$	$\hat{V}_4$	$\hat{V}_5$
$\hat{V}_4$	$\hat{V}_1 + 2\tilde{\varepsilon}\hat{V}_4$	$\hat{V}_2 + 2\tilde{\varepsilon}\hat{V}_4$	$\hat{V}_3$	$\hat{V}_4$	$\hat{V}_5$
$\hat{V}_5$	$\hat{V}_1$	$\hat{V}_2 - \tilde{\varepsilon}\hat{V}_5$	$\hat{V}_3$	$\hat{V}_4$	$\hat{V}_5$

For vector $\hat{V} = \sum\limits_{\beta=1}^{5} \breve{g}_\beta \hat{V}_\beta$, an arbitrary vector $\hat{W} = \sum\limits_{\gamma=1}^{5} \breve{h}_\gamma \hat{V}_\gamma$ is chosen and its accompanying expression is given

$$\begin{aligned}
& Ad_{\exp\left(\tilde{\varepsilon}\hat{W}\right)}\left(\hat{V}\right) \\
&= e^{-\tilde{\varepsilon}\hat{W}}\hat{V}e^{\tilde{\varepsilon}\hat{W}} \\
&= \hat{V} - \tilde{\varepsilon}\left[\hat{W}, \hat{V}\right] + \frac{\tilde{\varepsilon}^2}{2!}\left[\hat{W}, \left[\hat{W}, \hat{V}\right]\right] - \dots \\
&= \left(\breve{g}_1\hat{V}_1 + \dots + \breve{g}_6\hat{V}_6\right) - \tilde{\varepsilon}\left[\breve{h}_1\hat{V}_1 + \dots + \breve{h}_6\hat{V}_6, \breve{g}_1\hat{V}_1 + \dots + \breve{g}_6\hat{V}\right] + o\left(\tilde{\varepsilon}^2\right) \\
&= \left(\breve{g}_1\hat{V}_1 + \dots + \breve{g}_6\hat{V}_6\right) - \tilde{\varepsilon}\left(\tilde{\rho}_1\hat{V}_1 + \dots + \tilde{\rho}_6\hat{V}_6\right) + o\left(\tilde{\varepsilon}^2\right),
\end{aligned} \tag{3.2}$$

where $\tilde{\rho}_i\,(i = 1, 2, 3, 4, 5)$ is decided by **Table 1** and

$$\begin{aligned}
\tilde{\rho}_1 &= 0, \\
\tilde{\rho}_2 &= 0, \\
\tilde{\rho}_3 &= \breve{h}_1\breve{g}_3 + \breve{h}_2\breve{g}_3 - \breve{h}_3\breve{g}_1 - \breve{h}_3\breve{g}_2,
\end{aligned}$$

$$\tilde{\rho}_4 = 2\breve{h}_1\breve{g}_4 + 2\breve{h}_2\breve{g}_4 - 2\breve{h}_4\breve{g}_1 - 2\breve{h}_4\breve{g}_2,$$
$$\tilde{\rho}_5 = -\breve{h}_2\breve{g}_5 + \breve{h}_5\breve{g}_2, \tag{3.3}$$

where $\tilde{\rho}_i$ $(i = 1, 2, 3, 4, 5)$ must satisfy

$$\tilde{\rho}_1\frac{\partial\tilde{\phi}}{\partial\breve{g}_1} + \tilde{\rho}_2\frac{\partial\tilde{\phi}}{\partial\breve{g}_2} + \tilde{\rho}_3\frac{\partial\tilde{\phi}}{\partial\breve{g}_3} + \tilde{\rho}_4\frac{\partial\tilde{\phi}}{\partial\breve{g}_4} + \tilde{\rho}_5\frac{\partial\tilde{\phi}}{\partial\breve{g}_5} = 0. \tag{3.4}$$

Expanding Eq.(3.4) and extracting extracting the coefficient of $\breve{g}_i$ $(i=1, 2, 3, 4, 5)$ can obtain

$$\begin{aligned}
&\breve{h}_1 : \breve{g}_3\frac{\partial\tilde{\phi}}{\partial\breve{g}_3} + 2\breve{g}_4\frac{\partial\tilde{\phi}}{\partial\breve{g}_4} = 0,\\
&\breve{h}_2 : \breve{g}_3\frac{\partial\tilde{\phi}}{\partial g_3} + 2\breve{g}_4\frac{\partial\tilde{\phi}}{\partial\breve{g}_4} - \breve{g}_5\frac{\partial\tilde{\phi}}{\partial\breve{g}_5} = 0,\\
&\breve{h}_3 : -\breve{g}_1\frac{\partial\tilde{\phi}}{\partial\breve{g}_3} - \breve{g}_2\frac{\partial\tilde{\phi}}{\partial\breve{g}_3} = 0,\\
&\breve{h}_4 : -2\breve{g}_1\frac{\partial\tilde{\phi}}{\partial\breve{g}_4} - 2\breve{g}_2\frac{\partial\tilde{\phi}}{\partial\breve{g}_4} = 0,\\
&\breve{h}_5 : \breve{g}_2\frac{\partial\tilde{\phi}}{\partial\breve{g}_5} = 0.
\end{aligned} \tag{3.5}$$

Solving Eqs.(3.5), we can get the invariant function

$$\tilde{\phi}\left(\breve{g}_1, \cdots, \breve{g}_5\right) = \hat{F}\left(\breve{g}_1, \breve{g}_2\right), \tag{3.6}$$

where $\hat{F}$ is a free function about $\breve{g}_1, \breve{g}_2$.

Based on **Table 2**, the accompanying transformation matrix can be constructed. Applying the accompanying actions of $\hat{W}$ to $\tilde{V} = \sum_{\beta=1}^{5} g_\beta\tilde{V}_\beta$, we can obtain

$$\begin{aligned}
Ad_{\exp\left(\tilde{\varepsilon}_1\hat{V}_1\right)}\left(\hat{V}\right) &= \breve{g}_1 Ad_{\exp\left(\tilde{\varepsilon}_1\hat{V}_1\right)}\left(\hat{V}_1\right) + \cdots + \breve{g}_5 Ad_{\exp\left(\tilde{\varepsilon}_1\hat{V}_1\right)}\left(\hat{V}_5\right)\\
&= \breve{g}_1\hat{V}_1 + \breve{g}_2\hat{V}_2 + \breve{g}_3 e^{-\tilde{\varepsilon}}\hat{V}_3 + \breve{g}_4 e^{-2\tilde{\varepsilon}}\hat{V}_4 + \breve{g}_5\hat{V}_5\\
&= \left(\breve{g}_1, \breve{g}_2, \breve{g}_3, \breve{g}_4, \breve{g}_5\right) A_1\left(\hat{V}_1, \hat{V}_2, \hat{V}_3, \hat{V}_4, \hat{V}_5\right)^T.
\end{aligned} \tag{3.7}$$

So

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & e^{-\tilde{\varepsilon}_1} & 0 & 0\\ 0 & 0 & 0 & e^{-2\tilde{\varepsilon}_1} & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.8}$$

Similarly, we can obtain

$$
A_2=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&e^{-\tilde{\varepsilon}_2}&0&0\\0&0&0&e^{-2\tilde{\varepsilon}_2}&0\\0&0&0&0&e^{\tilde{\varepsilon}_2}\end{pmatrix},\ A_3=\begin{pmatrix}1&0&\tilde{\varepsilon}_3&0&0\\0&1&\tilde{\varepsilon}_3&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},
$$
$$
A_4=\begin{pmatrix}1&0&0&2\tilde{\varepsilon}_4&0\\0&1&0&2\tilde{\varepsilon}_4&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\qquad A_5=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&-\tilde{\varepsilon}_5\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}. \tag{3.9}
$$

Thus the general accompanying matrix is $A=A_1A_2A_3A_4A_5$. Substituting Eq.(3.8) and Eqs.(3.9) into A, we get

$$
A=\begin{pmatrix}1&0&\tilde{\varepsilon}_3&2\tilde{\varepsilon}_4&0\\0&1&\tilde{\varepsilon}_3&2\tilde{\varepsilon}_4&-\tilde{\varepsilon}_5\\0&0&e^{-\tilde{\varepsilon}_1}e^{-\tilde{\varepsilon}_2}&0&0\\0&0&0&e^{-2\tilde{\varepsilon}_1}e^{-2\tilde{\varepsilon}_2}&0\\0&0&0&0&e^{\tilde{\varepsilon}_2}\end{pmatrix}. \tag{3.10}
$$

The accompanying transformation equation of Eqs.(1.2) can be expressed as

$$
\begin{pmatrix}\hat{g}_1 & \hat{g}_2 & \hat{g}_3 & \hat{g}_4 & \hat{g}_5\end{pmatrix}=\begin{pmatrix}\breve{g}_1 & \breve{g}_2 & \breve{g}_3 & \breve{g}_4 & \breve{g}_5\end{pmatrix}A. \tag{3.11}
$$

Substituting Eq.(3.10) into Eq.(3.11) obtains

$$
\begin{aligned}
\hat{g}_1 &= \breve{g}_1,\\
\hat{g}_2 &= \breve{g}_2,\\
\hat{g}_3 &= \breve{g}_1\tilde{\varepsilon}_3+\breve{g}_2\tilde{\varepsilon}_3+\breve{g}_3e^{-\tilde{\varepsilon}_1}e^{-\tilde{\varepsilon}_2},\\
\hat{g}_4 &= 2\breve{g}_1\tilde{\varepsilon}_4+2\breve{g}_2\tilde{\varepsilon}_4+\breve{g}_4e^{-2\tilde{\varepsilon}_1}e^{-2\tilde{\varepsilon}_2},\\
\hat{g}_5 &= -\breve{g}_2\tilde{\varepsilon}_5+\breve{g}_5e^{\tilde{\varepsilon}_2}.
\end{aligned} \tag{3.12}
$$

The optimal system of Eqs.(1.2) is assumed as follows

$$
\bar{V}=\hat{g}_1\hat{V}_1+\hat{g}_2\hat{V}_2+\hat{g}_3\hat{V}_3+\hat{g}_4\hat{V}_4+\hat{g}_5\hat{V}_5. \tag{3.13}
$$

Finally, the several cases of one-dimensional optimal systems are discussed based on invariant functions.

Case 1. $\breve{g}_1 \neq 0, \breve{g}_2 \neq 0$

Supposing $\breve{g}_i = 1\,(i=1,2)\,,\hat{g}_j = 0\,(j=3,4,5)$ and substituting them into Eq.(3.13), we can obtain $\tilde{\varepsilon}_3 = \dfrac{-\breve{g}_3 e^{-\tilde{\varepsilon}_1} e^{-\tilde{\varepsilon}_2}}{\breve{g}_1 + \breve{g}_2}$, $\tilde{\varepsilon}_4 = \dfrac{-\breve{g}_4 e^{-2\tilde{\varepsilon}_1} e^{-2\tilde{\varepsilon}_2}}{2\left(\breve{g}_1 + \breve{g}_2\right)}$, $\tilde{\varepsilon}_5 = \dfrac{\breve{g}_5 e^{\tilde{\varepsilon}_2}}{\breve{g}_2}$. The representative element of the one-dimensional subalgebraic optimal system is $\bar{V} = \hat{V}_1 + \hat{V}_2$.

Case 2. $\breve{g}_1 \neq 0, \breve{g}_2 = 0$

Making $\breve{g}_i = 1\,(i=1)\,,\hat{g}_j = 0\,(j=2,3,4)\,,\hat{g}_5 = 1$, we can get $\tilde{\varepsilon}_2 = \ln \dfrac{1}{\breve{g}_5}$, $\tilde{\varepsilon}_3 = \dfrac{-\breve{g}_3 e^{-\tilde{\varepsilon}_1} e^{-\tilde{\varepsilon}_2}}{\breve{g}_1}, \tilde{\varepsilon}_4 = \dfrac{-\breve{g}_4 e^{-2\tilde{\varepsilon}_1} e^{-2\tilde{\varepsilon}_2}}{2\breve{g}_1}$. The optimal system is $\bar{V} = \hat{V}_1 + \hat{V}_5$.

Case 3. $\breve{g}_1 = 0, \breve{g}_2 \neq 0$

Letting $\breve{g}_i = 1\,(i=2)\,,\hat{g}_j = 0\,(j=1,3,4,5)$ and substituting into Eq.(3.13), we can obtain $\tilde{\varepsilon}_3 = \dfrac{-\breve{g}_3 e^{-\tilde{\varepsilon}_1} e^{-\tilde{\varepsilon}_2}}{\breve{g}_2}$, $\tilde{\varepsilon}_4 = \dfrac{-\breve{g}_4 e^{-2\tilde{\varepsilon}_1} e^{-2\tilde{\varepsilon}_2}}{2\breve{g}_2}$, $\tilde{\varepsilon}_5 = \dfrac{\breve{g}_5 e^{\tilde{\varepsilon}_2}}{\breve{g}_2}$. The representative element of the one-dimensional subalgebraic optimal system is $\bar{V} = \hat{V}_2$.

Case 4. $\breve{g}_1 = 0, \breve{g}_2 = 0$

In this case, new invariants $\tilde{\phi}\left(\breve{g}_1, \cdots, \breve{g}_5\right) = \hat{F}\left(\dfrac{\breve{g}_4}{\breve{g}_3{}^2}\right)$ have to be found. Obviously the new invariants are $\breve{g}_3, \breve{g}_4$. Next we will discuss the optimal system according to the new invariants.

Case 4.1. $\breve{g}_3 \neq 0, \breve{g}_4 \neq 0$

Supposing $\breve{g}_i = 1\,(i=3,4)\,,\hat{g}_j = 0\,(j=1,2)\,,\hat{g}_5 = 1$, the optimal system is $\bar{V} = \hat{V}_3 + \hat{V}_4 + \hat{V}_5$.

Case 4.2. $\breve{g}_3 \neq 0, \breve{g}_4 = 0$

Making $\hat{g}_i = 0\,(i=1,2,4)\,,\hat{g}_3 = \alpha_1, \hat{g}_5 = \alpha_2$, the optimal system is $\bar{V} = \alpha_1 \hat{V}_3 + \alpha_2 \hat{V}_5$.

Case 4.3. $\breve{g}_3 = 0, \breve{g}_4 \neq 0$

Letting $\hat{g}_i = 0\,(i=1,2,3)\,,\hat{g}_j = 1\,(j=4,5)$, the optimal system is $\bar{V} = \hat{V}_4 + \hat{V}_5$.

Case 4.4. $\breve{g}_3 = 0, \breve{g}_4 = 0$

Supposing $\hat{g}_i = 0\,(i=1,2,3,4)\,,\hat{g}_j = 1\,(j=5)$, the representative element of the one-dimensional subalgebraic optimal system is $\bar{V} = \hat{V}_5$.

In summary, the optimal system of Eqs.(1.2) should be the following

$$\left\{\hat{V}_1 + \hat{V}_2, \hat{V}_1 + \hat{V}_5, \hat{V}_2, \hat{V}_3 + \hat{V}_4 + \hat{V}_5, \alpha_1\hat{V}_3 + \alpha_2\hat{V}_5, \hat{V}_4 + \hat{V}_5, \hat{V}_5\right\}. \tag{3.14}$$

4. Similarity reduction of the vcDLW

In this section, Eqs.(1.2) are reduced based on the optimal system. The **Table 3** shows the reduced equations and **Table 4** shows the expressions for the coefficient functions where α_1, α_2 are none zero constants. Obviously the reduced partial differential equations only relate to κ and ς.

Table 3. Similarity variables.

Case	Similarity variables.	Reduced PDEs
$(1): \quad \hat{V}_1+\hat{V}_2$	$\kappa = xe^{2\left(\int \frac{1}{\widehat{\tau}(t)}dt\right)}, \varsigma = ye^{-\left(\int \frac{1}{\widehat{\tau}(t)}dt\right)}, u = E(\kappa,\varsigma)e^{\int \frac{2}{\widehat{\tau}(t)}dt}, v = -1+F(\kappa,\varsigma)e^{\int \frac{1}{\widehat{\tau}(t)}dt}.$	$E_\kappa E_\varsigma + EE_{\kappa\varsigma} + F_{\kappa\kappa} + 2\kappa E_{\kappa\varsigma} - \varsigma E_{\varsigma\varsigma} + E_\varsigma = 0,$ $F_\kappa E + FE_\kappa + E_{\kappa\kappa\varsigma} + 2\kappa F_\kappa - \varsigma F_\varsigma + F = 0.$
$(2): \quad \hat{V}_1+\hat{V}_5$	$\kappa = xe^{\int \frac{1}{\widehat{\tau}(t)}dt}, \varsigma = y - \int \frac{1}{\widehat{\tau}(t)}dt, u = E(\kappa,\varsigma)e^{\int \frac{1}{\widehat{\tau}(t)}dt}, v = -1+F(\kappa,\varsigma)e^{\int \frac{1}{\widehat{\tau}(t)}dt}.$	$E_\kappa E_\varsigma + EE_{\kappa\varsigma} + F_{\kappa\kappa} + \kappa E_{\kappa\varsigma} + E_{\varsigma\varsigma} + E_\varsigma = 0,$ $F_\kappa E + FE_\kappa + E_{\kappa\kappa\varsigma} + \varsigma F_\varsigma + F - F_\varsigma = 0.$
$(3): \quad \hat{V}_2$	$\kappa = xe^{\int \frac{1}{\widehat{\tau}(t)}dt}, \varsigma = ye^{-\left(\int \frac{1}{\widehat{\tau}(t)}dt\right)}, u = E(\kappa,\varsigma)e^{\int \frac{1}{\widehat{\tau}(t)}dt}, v = F(\kappa,\varsigma).$	$E_\kappa E_\varsigma + EE_{\kappa\varsigma} + F_{\kappa\kappa} + \kappa E_{\kappa\varsigma} + E_{\varsigma\varsigma}\varsigma = 0,$ $F_\kappa E + FE_\kappa + E_{\kappa\kappa\varsigma} + \kappa F_\kappa + E_\kappa - F_\varsigma\varsigma = 0.$
$(4): \quad \hat{V}_3+\hat{V}_4+\hat{V}_5$	$\kappa = x - \int \frac{1}{\widehat{\tau}(t)}dt, \varsigma = y - \int \frac{1}{\widehat{\tau}(t)}dt, u = E(\kappa,\varsigma), v = F(\kappa,\varsigma).$	$E_\kappa E_\varsigma + EE_{\kappa\varsigma} + F_{\kappa\kappa} - E_{\kappa\varsigma} - E_{\varsigma\varsigma} = 0,$ $F_\kappa E + FE_\kappa + E_{\kappa\kappa\varsigma} - F_\kappa + E_\kappa - F_\varsigma = 0.$
$(5): \quad \hat{V}_4+\hat{V}_5$	$\kappa = x, \varsigma = y - \int \frac{1}{\widehat{\tau}(t)}dt, u = E(\kappa,\varsigma), v = F(\kappa,\varsigma).$	$E_\kappa E_\varsigma + EE_{\kappa\varsigma} + F_{\kappa\kappa} - E_{\varsigma\varsigma} = 0,$ $F_\kappa E + FE_\kappa + E_{\kappa\kappa\varsigma} + E_\kappa - F_\varsigma = 0.$
$(6): \quad \alpha_1\hat{V}_3+\alpha_2\hat{V}_5$	$\kappa = x, \varsigma = \frac{-t\alpha_2 + y\alpha_1}{\alpha_1}, u = E(\kappa,\varsigma), v = F(\kappa,\varsigma).$	$-\alpha_1\alpha_2 EE_{\varsigma\varsigma} - \alpha_1\alpha_2 {E_\varsigma}^2 + F_{\varsigma\varsigma}{\alpha_2}^2 + E_{\kappa\varsigma}{\alpha_1}^2 = 0,$ $-\alpha_1\alpha_2 FE_\varsigma - \alpha_1\alpha_2 EF_\varsigma + {\alpha_2}^2 E_{\varsigma\varsigma\varsigma} - \alpha_1\alpha_2 E_\varsigma + {\alpha_1}^2 E_\kappa = 0.$
$(7): \quad \hat{V}_5$	$\kappa = x, \varsigma = y, u = E(\kappa,\varsigma), v = F(\kappa,\varsigma).$	$F_{\varsigma\varsigma} = 0,$ $F_\varsigma E + FE_\varsigma + E_\varsigma - F_\kappa = 0.$

Table 4. The expressions of the coefficient functions.

Case	The forms of corresponding coefficient functions
$(1): \quad \hat{V}_1+\hat{V}_2$	$a(t) = \frac{\left(e^{\left(\int \frac{1}{\widehat{\tau}(t)}\right)}\right)^{-4}}{\widehat{\tau}(t)}, b(t) = \frac{\left(e^{\left(\int \frac{1}{\widehat{\tau}(t)}\right)}\right)^{-4}}{\widehat{\tau}(t)}, c(t) = \frac{\left(e^{\left(\int \frac{1}{\widehat{\tau}(t)}\right)}\right)^{-4}}{\widehat{\tau}(t)}, d(t) = \frac{\left(e^{\left(\int \frac{1}{\widehat{\tau}(t)}\right)}\right)^{-4}}{\widehat{\tau}(t)}$
$(2): \quad \hat{V}_1+\hat{V}_5$	$a(t) = \frac{\left(e^{\left(\int \frac{1}{\widehat{\tau}(t)}\right)}\right)^{-2}}{\widehat{\tau}(t)}, b(t) = \frac{\left(e^{\left(\int \frac{1}{\widehat{\tau}(t)}\right)}\right)^{-2}}{\widehat{\tau}(t)}, c(t) = \frac{\left(e^{\left(\int \frac{1}{\widehat{\tau}(t)}\right)}\right)^{-2}}{\widehat{\tau}(t)}, d(t) = \frac{\left(e^{\left(\int \frac{1}{\widehat{\tau}(t)}\right)}\right)^{-2}}{\widehat{\tau}(t)}.$
$(3): \quad \hat{V}_2$	$a(t) = \frac{\left(e^{\left(\int \frac{1}{\widehat{\tau}(t)}\right)}\right)^{-2}}{\widehat{\tau}(t)}, b(t) = \frac{\left(e^{\left(\int \frac{1}{\widehat{\tau}(t)}\right)}\right)^{-2}}{\widehat{\tau}(t)}, c(t) = \frac{\left(e^{\left(\int \frac{1}{\widehat{\tau}(t)}\right)}\right)^{-2}}{\widehat{\tau}(t)}, d(t) = \frac{\left(e^{\left(\int \frac{1}{\widehat{\tau}(t)}\right)}\right)^{-2}}{\widehat{\tau}(t)}.$
$(4): \quad \hat{V}_3+\hat{V}_4+\hat{V}_5$	$a(t) = \frac{1}{\widehat{\tau}(t)}, b(t) = \frac{1}{\widehat{\tau}(t)}, c(t) = \frac{1}{\widehat{\tau}(t)}, d(t) = \frac{1}{\widehat{\tau}(t)}.$
$(5): \quad \hat{V}_4+\hat{V}_5$	$a(t) = \frac{1}{\widehat{\tau}(t)}, b(t) = \frac{1}{\widehat{\tau}(t)}, c(t) = \frac{1}{\widehat{\tau}(t)}, d(t) = \frac{1}{\widehat{\tau}(t)}.$
$(6): \quad \alpha_1\hat{V}_3+\alpha_2\hat{V}_5$	$a(t), b(t), c(t), d(t)$ are arbitrary functions.
$(7): \quad \hat{V}_5$	$a(t), b(t), c(t), d(t)$ are arbitrary functions.

5. Exact solutions of the (2+1)-dimensional vcDLWs

In this section, the solutions of Eqs.(1.2) are solved by using $(1/G')$-expansion method and (G'/G)-expansion method.

Firstly, the solutions of **Case 4.1** reduced equations in the optimal system are solved by the $(1/G')$-expansion method.

Case 4.1. $\hat{V}_3+\hat{V}_4+\hat{V}_5$

At this point, the reduced equations are

$$\begin{aligned} &E_\kappa E_\varsigma + EE_{\kappa\varsigma} + F_{\kappa\kappa} - E_{\kappa\varsigma} - E_{\varsigma\varsigma} = 0, \\ &F_\kappa E + FE_\kappa + E_{\kappa\kappa\varsigma} - F_\kappa + E_\kappa - F_\varsigma = 0. \end{aligned} \tag{5.1}$$

At first, the travelling wave transform is assumed to be as follows

$$E\left(\kappa,\varsigma\right)=R\left(\hat{\sigma}\right),\ F\left(\kappa,\varsigma\right)=S\left(\hat{\sigma}\right),\ \hat{\sigma}=\kappa-\upsilon\varsigma, \tag{5.2}$$

where υ is the travelling wave speed. Substituting Eqs.(5.2) into Eqs.(5.1) obtains the following ordinary differential equations (ODEs)

$$\begin{aligned}&-EE''\upsilon-\left(E'\right)^2\upsilon+F''+E''\upsilon-E''\upsilon^2=0,\\&FE'+EF'+E'-E'''\upsilon-F'+F'\upsilon=0.\end{aligned} \tag{5.3}$$

The traveling wave solution of Eqs.(5.3) is assumed to be

$$E\left(\hat{\sigma}\right)=a_0+\sum_{m=1}^{\infty}a_m\left(\frac{1}{G'}\right)^m,\ F\left(\hat{\sigma}\right)=b_0+\sum_{n=1}^{\infty}b_n\left(\frac{1}{G'}\right)^n, \tag{5.4}$$

where $G=G\left(\hat{\sigma}\right)$ satisfies

$$G''+kG'+\mu=0. \tag{5.5}$$

From the homogeneous balance of Eqs.(5.3)£¬ we can get that the value of m is 1 and the value of n from 1 to 2. So Eqs.(5.4) are

$$E=a_0+\frac{a_1}{G'},\ F=b_0+\frac{b_1}{G'}+\frac{b_2}{\left(G'\right)^2}. \tag{5.6}$$

Substituting Eq.(5.5) and Eqs.(5.6) into Eqs.(5.3) and extracting the coefficients of all terms of the same power of G', a set of polynomials is obtained. Letting these polynomials equal zero, a set of associative algebraic equations for $\upsilon, a_0, a_1, b_0, b_1, b_2$ is obtained as follows

$$\begin{aligned}&-2\upsilon\mu^2+b_2=0,\\&-\upsilon{a_1}^2+2a_2=0,\\&a_1\upsilon\left(1-\upsilon-a_0\right)+b_1=0,\\&a_1\left(1+b_0-\upsilon k^2\right)-b_1\left(1-a_0-\upsilon\right)=0,\\&\upsilon a_1\left(-2\upsilon\mu-5ka_1-2\mu a_0+2\mu\right)+10kb_2+2\mu b_1=0,\\&\upsilon a_1\left(3\upsilon\mu-2ka_1-3\mu a_0+3\mu\right)+k\left(4b_2+3\mu b_1\right)=0,\\&2\mu\left(\upsilon b_2+a_0b_2+a_1b_1-b_2-6\upsilon k\mu a_1\right)+3ka_1b_2=0,\\&a_1\left(\mu+\mu b_0+2kb_1-7\upsilon k^2\mu\right)+\left(b_1\mu+2kb_2\right)\left(\upsilon+a_0-1\right)=0.\end{aligned} \tag{5.7}$$

Solving Eqs.(5.7) can get

$$\begin{aligned}&\upsilon=k-a_0+1, a_1=2\mu, b_0=-1,\\&b_1=2\left(k-a_0+1\right)k\mu,\end{aligned} \tag{5.8}$$

$$b_2 = 2\,(k - a_0 + 1)\,\mu^2,$$

where a_0, μ, k are arbitrary constants.

Substituting Eqs.(5.8) into Eqs.(5.6), we can get the solutions of Eqs.(5.3) as follow

$$\begin{aligned} E &= a_0 + \frac{2\mu}{G'}, \\ F &= -1 + \frac{2\,(k - a_0 + 1)\,\mu}{G'}\left(k + \frac{\mu}{G'}\right), \end{aligned} \tag{5.9}$$

where $\upsilon = k - a_0 + 1$ and $G = G\,(\hat{\sigma})$ satisfies Eq.(5.5).

Finally, substituting the general solution of Eq.(5.5) into Eqs.(5.9), the traveling wave solutions of Eqs.(1.2) are obtained

$$\begin{aligned} F &= -\frac{2\hat{c}_1\mu\left(k^4 - k^3 a_0 + k + k^3\right)(\sinh(k\hat{\sigma}) - \cosh(k\hat{\sigma})) + \hat{c}_1^2 k^2 (\sinh(k\hat{\sigma}) - \cosh(k\hat{\sigma}))^2 + \mu^2}{(\sinh(k\hat{\sigma})\,\hat{c}_1 k - \cosh(k\hat{\sigma})\,\hat{c}_1 k + \mu)^2}, \\ E &= \frac{\sinh(k\hat{\sigma})\,\hat{c}_1 k a_0 - \cosh(k\hat{\sigma})\,\hat{c}_1 k a_0 - 2k\mu + \mu a_0}{\sinh(k\hat{\sigma})\,\hat{c}_1 k - \cosh(k\hat{\sigma})\,\hat{c}_1 k + \mu}. \end{aligned} \tag{5.10}$$

Figure 1 is evolution of the kink solutions and the 2-soliton solutions determined by Eqs.(5.10) at $t = 1$. When $\widehat{\tau}\,(t)$ is respectively taken as $\widehat{\tau}\,(t) = 1, \widehat{\tau}\,(t) = \sinh(t), \widehat{\tau}\,(t) = \cosh(t)$, (a), (b) and (c) are 3D plots for $k = 1, \mu = 1, a_0 = 2, \hat{c}_1 = 1$. The corresponding coefficient functions are $a\,(t) = \dfrac{1}{\widehat{\tau}\,(t)}, b\,(t) = \dfrac{1}{\widehat{\tau}\,(t)}, c\,(t) = \dfrac{1}{\widehat{\tau}\,(t)}, d\,(t) = \dfrac{1}{\widehat{\tau}\,(t)}$.

Secondly, the solutions of **Case 4.3** reduced equations in the optimal system are solved by the (G'/G)-expansion method.

Case 4.3. $\hat{V}_4 + \hat{V}_5$

At this point, the reduced equations are

$$\begin{aligned} &E_\kappa E_\varsigma + E E_{\kappa\varsigma} + F_{\kappa\kappa} - E_{\varsigma\varsigma} = 0, \\ &F_\kappa E + F E_\kappa + E_{\kappa\kappa\varsigma} + E_\kappa - F_\varsigma = 0. \end{aligned} \tag{5.11}$$

First of all, the traveling wave transform is assumed the same form of Eqs.(5.2). Substituting Eqs.(5.2) into Eqs.(5.11) can get ODEs as

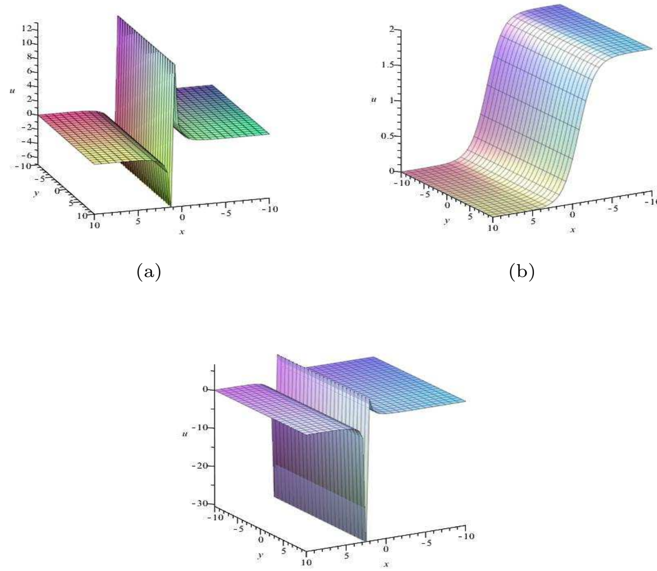
$$\begin{aligned} &-(E')^2\upsilon - EE''\upsilon + F'' - E''\upsilon^2 = 0, \\ &FE' + EF' + E' - E'''\upsilon + F'\upsilon = 0. \end{aligned} \tag{5.12}$$

The traveling wave solutions of Eqs.(5.12) are assumed to be

$$E\,(\hat{\sigma}) = \sum_{m=0}^{\infty} c_m \left(\frac{G'}{G}\right)^m,\; F\,(\hat{\sigma}) = \sum_{n=0}^{\infty} d_n \left(\frac{G'}{G}\right)^n, \tag{5.13}$$

where $G = G\,(\hat{\sigma})$ satisfies

$$G'' + \lambda G' + \mu G = 0. \tag{5.14}$$



(a) (b)

(c)

Figure 1. Evolution of kink solutions and 2-soliton solutions when (a) is $\widehat{\tau}\,(t) = 1$, (b) is $\widehat{\tau}\,(t) = \sinh(t)$ and (c) is $\widehat{\tau}\,(t) = \cosh(t)$.

Owing to homogeneous balance, the values of m of Eqs.(5.13) is from 0 to 1 and the values of n of Eqs.(5.13) is from 0 to 2. So Eqs.(5.13) become

$$\begin{aligned} E &= c_0 + c_1 \frac{G'}{G}, \\ F &= d_0 + d_1 \frac{G'}{G} + d_2 \left(\frac{G'}{G}\right)^2. \end{aligned} \tag{5.15}$$

Substituting Eq.(5.14) and Eqs.(5.15) into Eqs.(5.12) and extracting the coefficients of all terms of the same power of (G'/G), a set of polynomials is obtained. Letting these polynomials equal zero, a set of associative algebraic equations for $\upsilon, c_0, c_1, d_0, d_1, d_2$ is obtained as follows

$$\begin{aligned}
&3\upsilon - \mathrm{d}_2 = 0, \\
&-\upsilon {c_1}^2 + 3d_2 = 0, \\
&3\lambda c_1 \left(4\upsilon - d_2\right) - 2d_2 \left(\upsilon + c_0\right) - 2c_1 d_1 = 0, \\
&\upsilon c_1 \left(\upsilon\lambda - \lambda c_0 - \mu c_1\right) + \lambda d_1 + 2\mu d_2 = 0, \\
&\upsilon c_1 \left(-5\upsilon\lambda c_1 - 2\upsilon - 2c_0\right) + 10\lambda d_2 + 2d_1 = 0, \\
&c_1 \left(\upsilon\lambda^2 + 2\upsilon\mu - 1\right) - d_1 \left(\upsilon + c_0\right) - c_1 d_0 = 0, \\
&\upsilon\lambda c_1 \left(-2\lambda c_1 - 3\upsilon - 3c_0\right) + 4\mu \left(2d_2 - \upsilon {c_1}^2\right) + \lambda \left(4\lambda d_2 + 3d_1\right) = 0, \\
&\upsilon\lambda c_1 \left(-\upsilon\lambda - \lambda c_0 - 3\mu c_1\right) - 2\upsilon\mu c_1 \left(\upsilon + c_1\right) + d_1 \left(\lambda^2 + 2\mu\right) + 6\lambda\mu d_2 = 0, \\
&\upsilon \left(\lambda^3 c_1 + 8\lambda\mu c_1 - \lambda d_1 - 2\mu d_2\right) - c_0 \left(\lambda d_1 + 2\mu d_2\right) - c_1 \left(\lambda d_0 + 2\mu d_1 + \lambda\right) = 0,
\end{aligned}$$

$$\lambda\left(7\upsilon\lambda c_1-2\upsilon d_2-2c_0d_2-2c_1d_1\right)+\mu\left(8\upsilon c_1-3c_1d_2\right)-\upsilon d_1-c_0d_1-c_1d_0-c_1=0. \tag{5.16}$$

Solving Eqs.(5.16) can obtain

$$\begin{aligned} &c_0=-\upsilon-\lambda, c_1=-2, d_0=2\upsilon\mu-1,\\ &d_1=2\upsilon\lambda, d_2=2\upsilon, \end{aligned} \tag{5.17}$$

where υ,λ,μ are arbitrary constants. Substituting Eqs.(5.12) into Eqs.(5.15), the solutions of Eqs.(5.12) is derived as follow

$$\begin{aligned} &E=-\upsilon-\lambda-\frac{2G'}{G},\\ &F=2\upsilon\mu-1+\frac{2\upsilon\lambda G'}{G}+\frac{2\upsilon(G')^2}{G}, \end{aligned} \tag{5.18}$$

where $\hat{\sigma}=\kappa-\upsilon\varsigma$ and $G=G\left(\hat{\sigma}\right)$ satisfies Eq.(5.14).

At last, the three traveling wave solutions of Eqs.(1.2) are obtained by substituting the general solution of Eq.(5.14) into Eqs.(5.18).

When $\lambda^2-4\mu>0$,

$$\begin{aligned} E&=-\frac{\sqrt{\lambda^2-4\mu}\left(C_1\sinh\left(\frac{1}{2}\hat{\sigma}\sqrt{\lambda^2-4\mu}\right)+C_2\cosh\left(\frac{1}{2}\hat{\sigma}\sqrt{\lambda^2-4\mu}\right)\right)}{C_1\cosh\left(\frac{1}{2}\hat{\sigma}\sqrt{\lambda^2-4\mu}\right)+C_2\sinh\left(\frac{1}{2}\hat{\sigma}\sqrt{\lambda^2-4\mu}\right)}-\upsilon,\\ F&=\frac{\upsilon}{2}\frac{\left(\lambda^2-4\mu\right)\left(C_1\sinh\left(\frac{1}{2}\hat{\sigma}\sqrt{\lambda^2-4\mu}\right)+C_2\cosh\left(\frac{1}{2}\hat{\sigma}\sqrt{\lambda^2-4\mu}\right)\right)}{C_1\cosh\left(\frac{1}{2}\hat{\sigma}\sqrt{\lambda^2-4\mu}\right)+C_2\sinh\left(\frac{1}{2}\hat{\sigma}\sqrt{\lambda^2-4\mu}\right)}\\ &\quad-\frac{\upsilon\lambda^2}{2}+2\upsilon\mu-1, \end{aligned} \tag{5.19}$$

where $\hat{\sigma}=\kappa-\upsilon\varsigma$ and $C_1,C_2,b_0,\lambda,\mu,\upsilon$ are arbitrary constants.

Figure 2 is evolution of the periodic solutions determined by Eqs.(5.19) at $t=2$. (a) and (d) are 3D plots, (b) and (e) are 2D plots and (c) and (f) are dimensional plots for $\lambda=3.2055,\mu=2,\upsilon=2I,\widehat{\tau}\left(t\right)=\sin\left(t\right),C_1=2,C_2=1,b_0=1$. The coefficient corresponding functions are $a\left(t\right)=\dfrac{1}{\widehat{\tau}\left(t\right)},b\left(t\right)=\dfrac{1}{\widehat{\tau}\left(t\right)},c\left(t\right)=\dfrac{1}{\widehat{\tau}\left(t\right)},d\left(t\right)=\dfrac{1}{\widehat{\tau}\left(t\right)}$.

When $\lambda^2-4\mu<0$,

$$\begin{aligned} E&=-\frac{\sqrt{4\mu-\lambda^2}\left(-C_1\sin\left(\frac{1}{2}\hat{\sigma}\sqrt{4\mu-\lambda^2}\right)+C_2\cos\left(\frac{1}{2}\hat{\sigma}\sqrt{4\mu-\lambda^2}\right)\right)}{C_1\cos\left(\frac{1}{2}\hat{\sigma}\sqrt{4\mu-\lambda^2}\right)+C_2\sin\left(\frac{1}{2}\hat{\sigma}\sqrt{4\mu-\lambda^2}\right)}-\upsilon,\\ F&=\frac{\upsilon}{2}\frac{\left(4\mu-\lambda^2\right)\left(-C_1\sin\left(\frac{1}{2}\hat{\sigma}\sqrt{4\mu-\lambda^2}\right)+C_2\cos\left(\frac{1}{2}\hat{\sigma}\sqrt{4\mu-\lambda^2}\right)\right)}{C_1\cos\left(\frac{1}{2}\hat{\sigma}\sqrt{4\mu-\lambda^2}\right)+C_2\sin\left(\frac{1}{2}\hat{\sigma}\sqrt{4\mu-\lambda^2}\right)} \end{aligned}$$

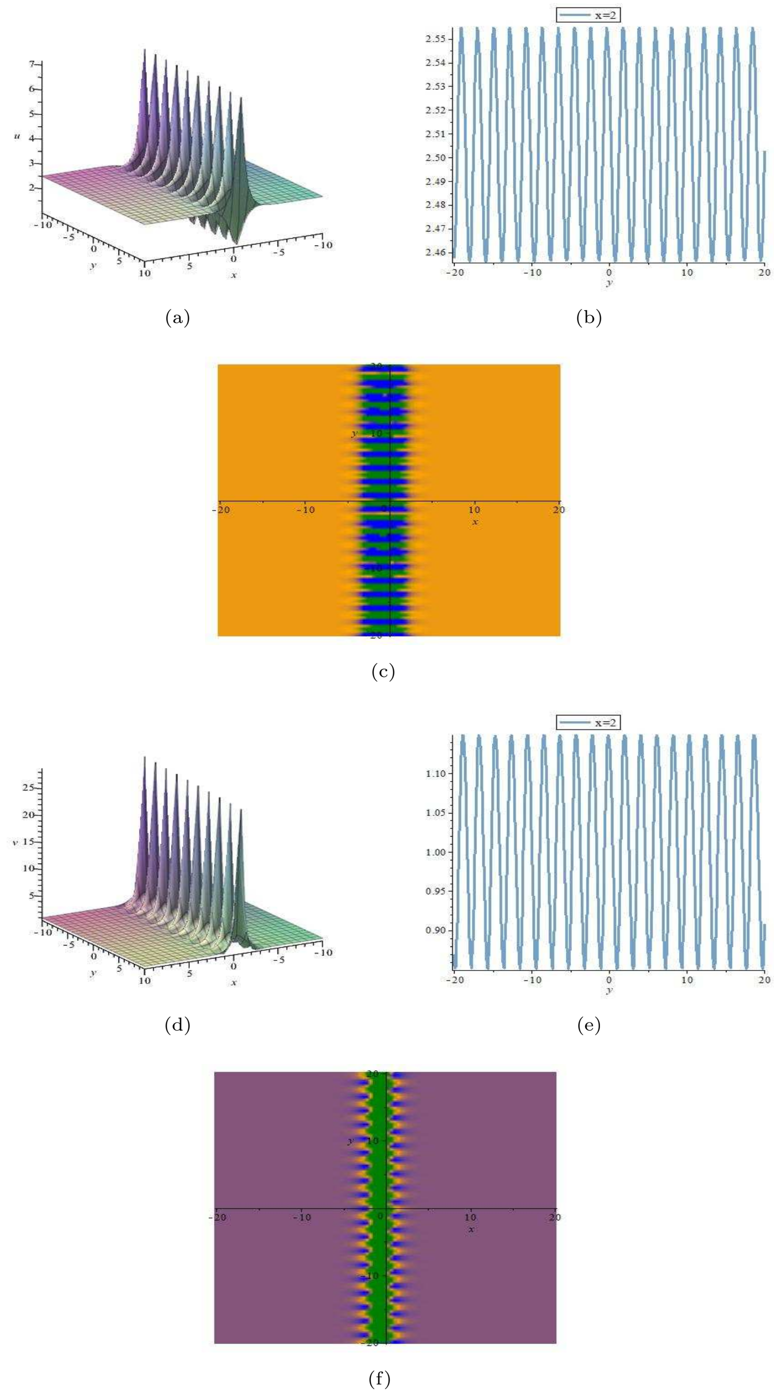


Figure 2. Evolution of the periodic solutions.

$$-\frac{\upsilon\lambda^2}{2}+2\upsilon\mu-1, \tag{5.20}$$

where $\hat{\sigma}=\kappa-\upsilon\varsigma$ and $C_1, C_2, b_0, \lambda, \mu, \upsilon$ are arbitrary constants.

Figure 3 is evolution of the kink solutions and the soliton solutions determined by Eqs.(5.20) at $t=1$. (a) and (d) are 3D plots, (b) and (e) are 2D plots and (c) and (f) are dimensional plots for $b_0=2I, \lambda=3, \mu=2, \upsilon=1, \widehat{\tau}(t)=\tanh(t), C_1=-2I, C_2=1$. The coefficient corresponding functions are $a(t)=\dfrac{1}{\widehat{\tau}(t)}, b(t)=\dfrac{1}{\widehat{\tau}(t)}, c(t)=\dfrac{1}{\widehat{\tau}(t)}, d(t)=\dfrac{1}{\widehat{\tau}(t)}$.

When $\lambda^2-4\mu=0$,

$$\begin{aligned}
&E=-\upsilon C_4\hat{\sigma}+\upsilon C_3+2C_4/C_4\hat{\sigma}+C_3,\\
&F=-\big(\upsilon {C_4}^2\lambda^2\hat{\sigma}^2+2\upsilon C_3C_4\lambda^2\hat{\sigma}-4\upsilon {C_4}^2\mu\hat{\sigma}^2+\upsilon {C_3}^2\lambda^2-8\upsilon C_3C_4\mu\hat{\sigma}-4\upsilon {C_3}^2\mu\\
&\qquad+2{C_4}^2\hat{\sigma}^2-4\upsilon {C_4}^2+4C_3C_4\hat{\sigma}+2{C_3}^2\big)/2(C_4\hat{\sigma}+C_3)^2,
\end{aligned} \tag{5.21}$$

where $\hat{\sigma}=\kappa-\upsilon\varsigma$ and $C_1, C_2, b_0, \lambda, \mu, \upsilon$ are arbitrary constants.

6. Nonlinear self-adjointness and conservation laws of vcDLW

Conservation laws play an essential role in the studies of mathematical physical equations. Firstly, the conservation laws can reflect the characteristics of the change of motion of the mathematical physical equations. Secondly, having an infinite number of conservation laws is one of the primary indicators of integrability of differential equations. Finally, the conservation laws can also be used to prove the existence and uniqueness of solutions. There are many ways to solve the conservation laws of equations, such as Nöether theorem, eigenvalue method and adjoint equation method, etc [2,15,25,32,36]. In this section, the nonlinear self-adjoint and conservation laws of Eqs.(1.2) are discussed using adjoint equation method.

6.1. Nonlinear self-adjointness

The Lagrange form of Eqs.(1.2) is

$$\begin{aligned}
L=&\bar{u}\left[u_{yt}+a(t)v_{xx}+b(t)(u_yu_x+uu_{xy})\right]\\
&+\bar{v}\left[v_t+c(t)(u_x+u_xv+uv_x)+d(t)u_{xxy}\right],
\end{aligned} \tag{6.1}$$

where $\bar{u}, \bar{v}$ are two new independent variables about x, y, t.

The adjoint equations of Eqs.(1.2) are as follows

$$\begin{aligned}
{f_1}^*=\frac{\delta L}{\delta u}=0,\\
{f_2}^*=\frac{\delta L}{\delta v}=0,
\end{aligned} \tag{6.2}$$

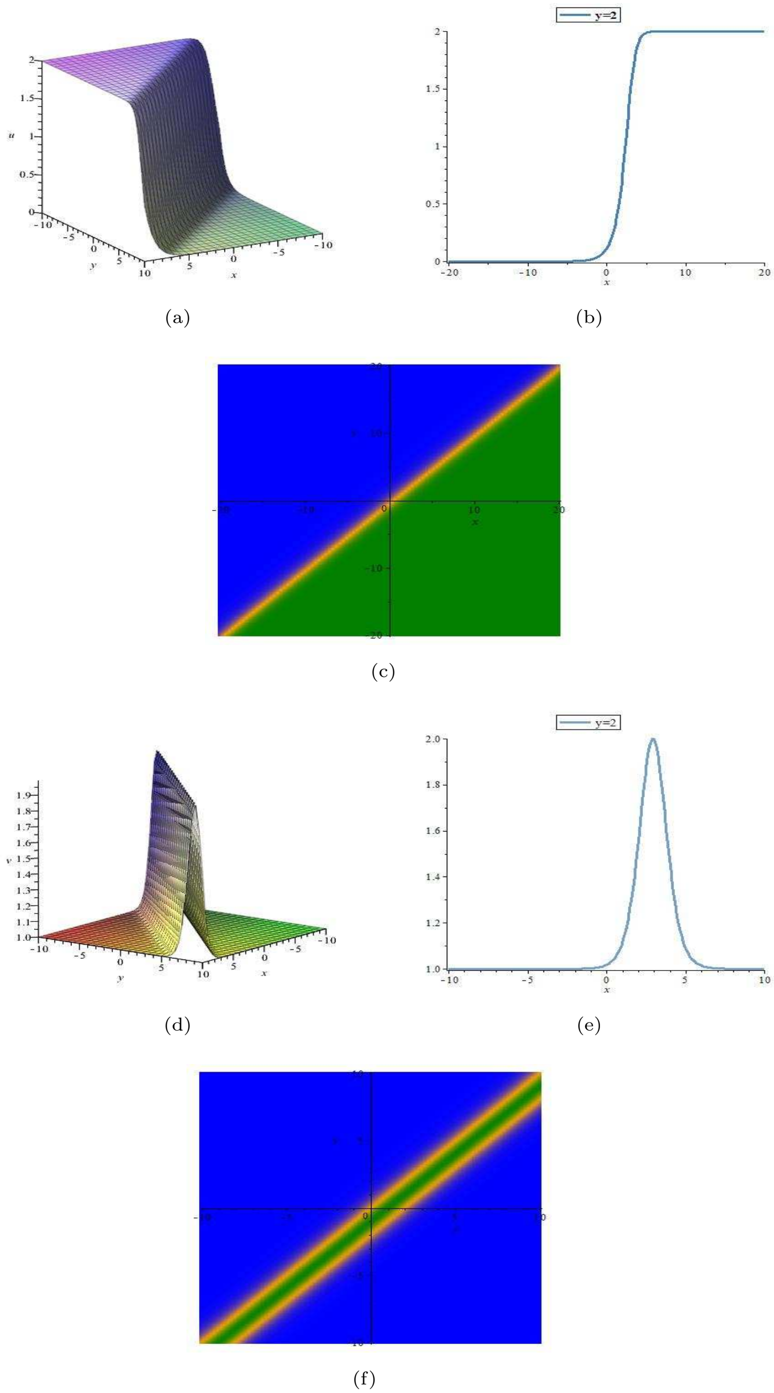


Figure 3. Evolution of the kink solutions and the soliton solutions.

where

$$\begin{aligned}\frac{\delta L}{\delta u}&=\frac{\partial L}{\partial u}-D_x\frac{\partial L}{\partial u_x}-D_y\frac{\partial L}{\partial u_y}+D_{xy}\frac{\partial L}{\partial u_{xy}}+D_{yt}\frac{\partial L}{\partial u_{yt}}-D_{xxy}\frac{\partial L}{\partial u_{xxy}},\\ \frac{\delta L}{\delta v}&=\frac{\partial L}{\partial v}-D_x\frac{\partial L}{\partial v_x}-D_t\frac{\partial L}{\partial v_t}+D_{xx}\frac{\partial L}{\partial v_{xx}}.\end{aligned}\tag{6.3}$$

Substituting Eq.(6.1) and Eqs.(6.3) into Eqs.(6.2), the accompanying system of Eqs.(1.2) can be derived as follows

$$\begin{aligned}&{f_1}^*=\bar{u}_{yt}+b\left(t\right)u\bar{u}_{xy}-c\left(t\right)\left(1+v\right)\bar{v}_x-d\left(t\right)\bar{v}_{xxy},\\ &{f_2}^*=-\bar{v}_t+a\left(t\right)\bar{u}_{xx}-c\left(t\right)\bar{v}_x.\end{aligned}\tag{6.4}$$

The (2+1)-dimensional vcDLWs are said to be a nonlinear self-adjointness system if Eq.(6.4) satisfies the following conditions

$$\begin{aligned}&{f_1}^*\left|_{\bar{u}=\phi_1(x,y,t,u,v),\bar{v}=\phi_2(x,y,t,u,v)}\right.=\bar{\lambda}_{11}F_1+\bar{\lambda}_{12}F_2,\\ &{f_2}^*\left|_{\bar{u}=\phi_1(x,y,t,u,v),\bar{v}=\phi_2(x,y,t,u,v)}\right.=\bar{\lambda}_{21}F_1+\bar{\lambda}_{22}F_2,\end{aligned}\tag{6.5}$$

where $\phi_1\left(x,y,t,u,v\right)\neq 0,\phi_2\left(x,y,t,u,v\right)\neq 0$ and $\bar{\lambda}_{ij}\left(i,j=1,2\right)$ are undetermined coefficients.

Substituting Eqs.(1.2) and Eqs.(6.4) into Eqs.(6.5) and extracting the coefficients of u,v, we can obtain

$$\begin{aligned}&\bar{\lambda}_{11}=\bar{\lambda}_{12}=\bar{\lambda}_{21}=\bar{\lambda}_{22}=0,\\ &\phi_{1u}=0,\phi_{1v}=0,\phi_{2u}=0,\phi_{2v}=0,\\ &\phi_{1yt}+b\left(t\right)u\phi_{1xy}-c\left(t\right)\left(\phi_{2x}+v\phi_{2x}\right)-d\left(t\right)\phi_{2xxy}=0,\\ &-\phi_{2t}+a\left(t\right)\phi_{1x}-c\left(t\right)u\phi_{2x}=0.\end{aligned}\tag{6.6}$$

Solving Eqs.(6.6) can obtain

$$\begin{aligned}&\phi_1\left(x,y,t,u,v\right)=\frac{x^2F_{1t}\left(t\right)}{2a\left(t\right)}+F_3\left(t\right)x+F_5\left(y\right)+F_4\left(t\right),\\ &\phi_2\left(x,y,t,u,v\right)=F_2\left(y\right)+F_1\left(t\right),\end{aligned}\tag{6.7}$$

where $F_1\left(t\right),F_2\left(y\right),F_3\left(t\right),F_4\left(t\right),F_5\left(y\right)$ are arbitrary functions.

Obviously $\phi_1\left(x,y,t,u,v\right)\neq 0,\phi_2\left(x,y,t,u,v\right)\neq 0$, we can obtain that Eqs.(1.2) are nonlinear self-adjointness. We also can learn

$$\begin{aligned}L=&\,\bar{u}\left[u_{yt}+a\left(t\right)v_{xx}+b\left(t\right)\left(u_yu_x+uu_{xy}\right)\right]\\ &+\bar{v}\left[v_t+c\left(t\right)\left(u_x+u_xv+uv_x\right)+d\left(t\right)u_{xxy}\right],\end{aligned}\tag{6.8}$$

where $\bar{u},\bar{v}$ are arbitrary functions.

6.2. Conservation laws

Next we will construct the conservation laws for Eqs.(1.2). For simplicity, we let $\bar{u} = \phi_1 = \gamma_1, \bar{v} = \phi_2 = \gamma_2$, where γ_1, γ_2 are arbitrary constants. At this point Eq.(6.8) becomes

$$\begin{aligned} L = &\gamma_1 \left[u_{yt} + a(t) v_{xx} + b(t)(u_y u_x + u u_{xy})\right] \\ &+\gamma_2 \left[v_t + c(t)(u_x + u_x v + u v_x) + d(t) u_{xxy}\right]. \end{aligned} \tag{6.9}$$

It is well known that the conservation laws are formulated as

$$\frac{\partial}{\partial x}\left(\hat{C}_x\right) + \frac{\partial}{\partial y}\left(\hat{C}_y\right) + \frac{\partial}{\partial t}\left(\hat{C}_t\right) = 0, \tag{6.10}$$

with different $\hat{C}_x, \hat{C}_y, \hat{C}_t$ based on Eqs.(2.8). Depending on the different vector fields and $\hat{C}_x, \hat{C}_y, \hat{C}_t$, we discuss the following:

Case 1. For vector field $\hat{V}_1 = -x\frac{\partial}{\partial x} - \frac{2\int a(t)\,dt}{a(t)}\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + (v+1)\frac{\partial}{\partial v}$, we can get

$$\begin{aligned} \hat{W}_1 &= u + u_x x + 2u_t \frac{\int a(t)\,\mathrm{d}t}{a(t)}, \\ \hat{W}_2 &= v + 1 + v_x x + 2v_t \frac{\int a(t)\,\mathrm{d}t}{a(t)}. \end{aligned} \tag{6.11}$$

In this case, the corresponding conservation vectors are

$$\begin{aligned} \hat{C}_x = &\frac{1}{a(t)}\Big[2a(t)(b(t) u_y u \hat{c}_1 + u c(t) \hat{c}_2 (v+1) + a(t) v_x \hat{c}_1) \\ &- a(t) x (d(t) u_{xxy} \hat{c}_2 + u_{yt}\hat{c}_1 + v_t \hat{c}_2) \\ &+2\int a(t)\,\mathrm{d}t\,(b(t)\hat{c}_1(u_y u_t + u u_{yt}) + c(t)\hat{c}_2(u v_t + v u_t + u_t) + a(t) v_{xt}\hat{c}_1)\Big], \\ \hat{C}_y = &\left(3u_t + u_{xt} x - 2\frac{\int a(t)\,\mathrm{d}t u_t \frac{\mathrm{d}}{\mathrm{d}t} a(t)}{(a(t))^2} + 2\frac{\int a(t)\,\mathrm{d}t u_{tt}}{a(t)}\right)\hat{c}_1, \\ &+\left(3u_{xx} + u_{xxx} x + 2\frac{\int a(t)\,\mathrm{d}t u_{xxt}}{a(t)}\right) d(t)\hat{c}_2, \\ \hat{C}_t = &\frac{1}{a(t)}\Big[2\int a(t)\,\mathrm{d}t\,(b(t)\hat{c}_1(u_y u_x + u u_{xy}) + c(t)\hat{c}_2(u_x v + u v_x + u_x) \\ &+ v_{xx}\hat{c}_1 a(t) + d(t) u_{xxy}\hat{c}_2 + (u_{yt}\hat{c}_1 + 2v_t\hat{c}_2)) + \hat{c}_2 a(t)(v_x x + v + 1)\Big]. \end{aligned} \tag{6.12}$$

Case 2. For vector field $\hat{V}_2 = -x\frac{\partial}{\partial x} - \frac{2\int a(t)\,dt}{a(t)}\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}$, we can obtain

$$\begin{aligned} \hat{W}_1 &= u + u_x x - u_y y + 2u_t \frac{\int a(t)\,\mathrm{d}t}{a(t)}, \\ \hat{W}_2 &= v_x x - y v_y + 2v_t \frac{\int a(t)\,\mathrm{d}t}{a(t)}. \end{aligned} \tag{6.13}$$

In this case, the corresponding conservation vectors are

$$\hat{C}_x = \frac{1}{a(t)}\Big[2a(t)\left(b(t)u_y u\hat{c}_1 + uc(t)v\hat{c}_2 + uc(t)\hat{c}_2 + a(t)v_x\hat{c}_1\right)$$

$$-a(t)x\left(d(t)u_{xxy}\hat{c}_2 + u_{yt}\hat{c}_1 + v_t\hat{c}_2\right)$$

$$+2\int a(t)\,\mathrm{d}t\left(b(t)\hat{c}_1\left(u_y u_t + uu_{yt}\right) + c(t)\hat{c}_2\left(uv_t + vu_t + u_t\right) + a(t)v_{xt}\hat{c}_1\right)\Big],$$

$$\hat{C}_y = y\left(b(t)\hat{c}_1\left(u_y u_x + uu_{xy}\right) + c(t)\hat{c}_2\left(u_x v + uv_x\right) + a(t)v_{xx}\hat{c}_1 + u_x c(t)\hat{c}_2\right.$$

$$\left.+d(t)u_{xxy}\hat{c}_2 + u_{yt}\hat{c}_1 + v_t\hat{c}_2\right)$$

$$+\left(3u_t + u_{xt}x - 2u_t\frac{\int a(t)\,\mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}t}a(t)}{\left(a(t)\right)^2} + 2u_{tt}\frac{\int a(t)\,\mathrm{d}t}{a(t)}\right)\hat{c}_1$$

$$+\left(3u_{xx} + u_{xxx}x + 2u_{xxt}\frac{\int a(t)\,\mathrm{d}t}{a(t)}\right)d(t)\hat{c}_2,$$

$$\hat{C}_t = \frac{1}{a(t)}\left[2\int a(t)\,\mathrm{d}t\left(b(t)\hat{c}_1\left(u_y u_x + uu_{xy}\right) + c(t)\hat{c}_2\left(u_x v + uv_x\right) + v_{xx}\hat{c}_1 a(t)\right.\right.$$

$$\left.\left.+u_x c(t)\hat{c}_2 + d(t)u_{xxy}\hat{c}_2 + \left(u_{yt}\hat{c}_1 + 2v_t\hat{c}_2\right)\right) + \hat{c}_2 a(t)\left(v_x x + v + 1\right)\right]. \tag{6.14}$$

Case 3. For vector field $\hat{V}_3 = \dfrac{\partial}{\partial x}$, we can gain

$$\begin{aligned}\hat{W}_1 &= -u_x,\\ \hat{W}_2 &= -v_x.\end{aligned} \tag{6.15}$$

In this case, the corresponding conservation vectors are

$$\begin{aligned}\hat{C}_x &= \hat{c}_2 d(t)u_{xxy} + \hat{c}_1 u_{yt} + \hat{c}_2 v_t,\\ \hat{C}_y &= -\hat{c}_1 u_{xt} - \hat{c}_2 d(t)u_{xxx},\\ \hat{C}_t &= -\hat{c}_2 v_x.\end{aligned} \tag{6.16}$$

Case 4. For vector field $\hat{V}_4 = \dfrac{1}{a(t)}\dfrac{\partial}{\partial t}$, we can get

$$\begin{aligned}\hat{W}_1 &= -\frac{u_t}{a(t)},\\ \hat{W}_2 &= -\frac{v_t}{a(t)}.\end{aligned} \tag{6.17}$$

In this case, the corresponding conservation vectors are

$$\hat{C}_x = -\frac{\hat{c}_1 b(t)\left(u_y u_t + uu_{yt}\right) + \hat{c}_2 c(t)\left(u_t v + uv_t + u_t\right) + v_{xt}a(t)\hat{c}_1}{a(t)},$$

$$\hat{C}_y = -\frac{u_{xxt} d(t)\, \hat{c}_2 a(t) + \hat{c}_1 u_{tt} a(t) - \hat{c}_1 u_t \frac{\mathrm{d}}{\mathrm{d}t} a(t)}{(a(t))^2},$$
$$\hat{C}_t = \frac{\hat{c}_1 b(t)(u_y u_x + u u_{xy}) + \hat{c}_2 c(t)(u_x v + u v_x + u_x) + a(t) v_{xx} \hat{c}_1 + d(t) u_{xxy} \hat{c}_2 + u_{yt} \hat{c}_1}{a(t)}. \tag{6.18}$$

Case 5. For vector field $\hat{V}_5 = \dfrac{\partial}{\partial y}$, we can obtain

$$\hat{W}_1 = -u_y, \qquad \hat{W}_2 = -v_y. \tag{6.19}$$

In this case, the corresponding conservation vectors are

$$\hat{C}_x = -\hat{c}_1 b(t)\left({u_y}^2 + u_{yy} u\right) - \hat{c}_2 c(t)(u_y v + u_y + v_y u) - v_{xy} a(t)\, \hat{c}_1,$$
$$\hat{C}_y = \hat{c}_1 b(t)(u_y u_x + u u_{xy}) + \hat{c}_2 c(t)(v u_x + u v_x + u_x) + a(t) v_{xx} \hat{c}_1 + v_t \hat{c}_2, \tag{6.20}$$
$$\hat{C}_t = -\hat{c}_2 v_y.$$

Substituting $\hat{C}_x, \hat{C}_y, \hat{C}_t$ in the above five cases into Eq.(6.10), we can discover that the conditions are satisfied. So the conservation laws of vcDLWs are obtained.

7. Conclusion

In this paper, the exact solutions of (2+1)-dimensional vcDLWs have been investigated by using the Lie symmetry analysis method. The infinitesimal generators and vector fields of vcDLWs have been firstly obtained by the Lie symmetry analysis method. Based on the vector fields, the representative elements of the one-dimensional subalgebra of the optimal system have been calculated by Olver's method. And reducing vcDLWs obtained the ODEs. The reduced equations have been solved in different methods to obtain the exact solutions of analysis. Some of these exact solutions have physical significance, such as kink solutions, periodic solutions and 2-soliton solutions. The evolution of the solutions in the figures have been used to illustrate the dynamic behaviors of the solutions. Finally adjoint equation method was used to find that analysis is the nonlinear self-adjointness. The conservation laws of analysis have been obtained.

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