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FINITE-TIME BLOW UP OF SOLUTIONS FOR A FOURTH-ORDER VISCOELASTIC WAVE EQUATION WITH DAMPING TERMS

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Abstract In this paper, a class of fourth-order viscoelastic wave equations with damping terms is studied. First, the local existence and uniqueness of weak solutions for the proposed problem are proved by the linear approximation and the Faedo-Galerkin method. Next, a special case of the original problem is considered. Then, under some suitablely sufficient conditions on the relaxation functions and by using contrary arguments, we show that the corresponding problem in this case does not admit any global solutions. Ultimately, we prove the finite-time blow up of solutions in case of negative initial energy.

Keywords Fourth-order wave equation, Faedo-Galerkin method, finite-time blow up, linear approximation, damping term.

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1. Introduction

In this paper, we consider the following initial-boundary value problem for a fourth-order viscoelastic equation

$$u_{tt} + u_{xxxx} - u_{txx} - \frac{\partial}{\partial x}\left[\mu\left(x,t,u,u_x,\|u_x(t)\|^2\right)\right] + \int_0^t g(t-s)u_{xx}(s)ds$$

$$= f\left(x,t,u,u_t,u_x\right),\ 0<x<1,\ 0<t<T, \tag{1.1}$$

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

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where μ, f, g, $\tilde{u}_0$, $\tilde{u}_1$, are given functions satisfying conditions specified later. In Eq. $(1.1)_1$, the nonlinear term $\mu\left(x,t,u,u_x,\|u_x(t)\|^2\right)$ depends on the integral $\|u_x(t)\|^2=\int_0^1 u_x^2\,(x,t)\,dx$.

In view of structure, Eq. (1.1) is a very complex model and can be considered as a generalized one-dimensional model of fourth-order viscoelastic wave equations with Kirchhoff term. It is clear that the model in hand does not exist in the first place, however, of which more special forms have been mentioned in literature describing a variety of important physical processes. So we will introduce its development and evolution to show its background by listing several related models, For example, it concerns in some extensible beam models describing evolution of transverse deflection of an extensible beam obeying continuous dynamics, of which the original equation was proposed by Woinowsky-Krieger [28], and given by

$$u_{tt}+\alpha u_{xxxx}+\left(\beta+k\int_0^L u_x^2dx\right)u_{xx}=0, \tag{1.4}$$

where β represents an initial axial displacement measured from the unstressed state and $u(x,t)$ represents the transverse deflection of an extensible beam of natural length L whose ends are held a fixed distance apart. In [1], An studied a model related to elastoplastic-microstructure flows to explore the immediate post-critical behavior of the solutions

$$u_{tt}+\gamma u_{xxxx}=\beta\left(u_x^2\right)_x,\ x\in(0,1),\ t>0,\ \gamma>0,\ \beta\neq 0. \tag{1.5}$$

For considering other wide applications in connection physics and mechanics related to some special cases of Eq. (1.1), we refer to nonlinear resonance of rectangular plates [7], or time-periodic transverse oscillations of a rod under external forces [8], or viscous flows in materials with memory [23] and the references therein.

After its appearance, numerous extensively interesting mathematical results of Eq. (1.5) such as local existence, global existence, asymptotic behavior and blow up in finite time of solutions have been investigated. As $N=1$ (one-dimensional case), with substituting $\sigma\left(u_x\right)_x$ for $\beta\left(u_x^2\right)_x$ and adding the dissipative term u_t, Yang [34] considered Eq. (1.5) in the form

$$u_{tt}+u_{xxxx}+\lambda u_t=\sigma\left(u_x\right)_x,\ x\in(0,1),\ t>0, \tag{1.6}$$

where $\sigma(s)$ is a given nonlinear function and $\lambda\geq 0$ is a real number. Then, if the initial energy is positive and suitably small, the global existence and asymptotic behavior of weak solutions are proved. Moreover, under some sufficient conditions on initial data and with the negative initial energy, the solution of Eq. (1.6) blows up in finite time. In [4], Chen and Lu considered Eq. (1.6) with replacing the weak dissipative term u_t by the strong dissipation $-u_{xxt}$ and with the various boundary conditions. In the case that the nonlinear source f is a $C^1(\mathbb{R})$ function and $f'(s)$ is bounded below, they proved the existence and uniqueness of the global generalized solution and the global classical solution, and also considered the finite time blow up with negative initial energy and the exponential decay with $F(s)=\int_0^s f(\tau)d\tau\geq 0$. Later, by using potential well method, Xu et al. [30] have extended and obtained the same results in [4] of finite time blow up with the initial energy $E(0)$ satisfying $0 < E(0) < d$ (the depth of potential well). However, in their paper, the cases $E(0) = d$ and $E(0) > d$ were still an unsolved problem. In the frame of potential well theory, it seems that the initial energy must be less than the depth of potential well, so the high arbitrary initial energy case without such restriction becomes a very interesting and important problem. Subsequently, this unsolved problem has been solved and showed in some recent works, see for example [9] , [27], [35] (in one-dimensional case), in which the authors established the finite time blow up of the solutions for the corresponding problem with arbitrary positive energy and suitable initial data.

For multi-dimensional forms, the following model of fourth-order equations

$$u_{tt} - M\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u + \Delta^2 u - \alpha\Delta u_t + g(u_t) = f(u), \tag{1.7}$$

named Kirchhoff plate equations and considered as a generalization of the Woinowsky-Krieger model to describe the large deflection of the plate [3], attract a lot of interest and are studied in various directions like local existence, global existence, nonexistence, stability and asymptotic behavior of solutions [11], [12], [15], [18], [30], [29], [33]. Indeed, when $\alpha = 0,\ g = \varepsilon\,|u_t|^{m-2}\, u_t,\ f = f(x,t)$, Lan et. al. [11] applied the Galerkin method and the weak compact method to Eq. (1.7) to obtain the existence and uniqueness of global solutions. Afterward, the same results of [11] have been extended by Long and Thuyet [18] in which the weakly nonlinear damping $\varepsilon\,|u_t|^{\alpha-1}\, u_t$ was replacing by the continuous and non-decreasing function g of Nemytsky-type operator. For Eq. (1.7) with weakly nonlinear damping and source, i.e. $\alpha = 0,\ M = 1,\ g = |u_t|^{m-2}\, u_t,\ f = |u|^{r-2}\, u$, Ouaoua et al. [22] considered a nonlinear equation of Timoshenko type and then proved the local existence giving by the Faedo-Galerkin method, the global existence under suitable assumptions with positive initial energy, and the algebra stability of solution based on Komornik's integral inequality. Once again, the blow-up phenomena of solutions at arbitrary positive energy has been attracted a great deal of attention. In [29] , Wu and Tsai studied Eq. (1.7) in case of the absence of dispersions, i.e., $\alpha = 0$ and $g = 0$, and established the blow-up properties with small positive initial energy after verifying the local existence by using the contraction mapping principle, and the global existence under some restrictions on the initial data. In the framework of potential well, considering Eq. (1.7) with nonlinear weak damping, linear strong damping and nonlinear exponential source, precisely as $\alpha = 1,\ g = |u_t|^{r-1}\, u_t,\ f = |u|^{p-1}\, u$, very recent Yang et. al. [33] have proved the blow-up phenomena of solutions with the high arbitrarily initial energy $E(0) > 0$ and the linear weak damping ($r = 1$). In addition, the local existence by using the contraction mapping principle, and the results of global existence, nonexistence and asymptotic behavior of solution for both subcritical initial energy level and critical initial energy level have also been discussed. Meanwhile, by exploiting the properties of the Nehari manifold and by constructing the appropriate and relaxed sufficient conditions, Liu et al. [15] used directly the relationship between the energy functionals associated with Eq. (1.7) in two-dimensional case and $\alpha = 0,\ g = u_t,\ f = |u|^{p-1}\, u$ to obtain the global existence and finite-time blow up of solutions without the aid of d.

It is worth mentioning to the following abstract model of fourth-order equations including nonlinear strain and dissipative terms in the form

$$u_{tt} - \beta\Delta u + \Delta^2 u - \alpha\Delta u_t + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) + g(u_t) = f(u), \tag{1.8}$$

in which a large amount of attention in various directions like well-posedness of weak solution, global existence, asymptotic analysis and blow up in finite time have been investigated. Actually, Esquivel-Avila [6] studied Eq. (1.8) with $\alpha = 0,\ \sigma_i = |x_i|^{m-2} x_i,\ g = 0,\ f = |u|^{r-2} u$, and established some sufficient conditions on the initial data to obtain the global solution and the finite time blow-up solution when $E(0) \leq d$. Before that, Yang [34] has considered Eq. (1.8) with $\alpha = 0,\ g = u_t,\ f = 0$ and the strain term in more general form $\sigma_i = \sigma_i(x_i)$, including the global existence and exponential decay with large initial data and small initial energy, and the finite time blow up of solutions in one-dimensional case $N = 1$. Recently, the results in [6], [16], [17], [34] were extended to that given in [12], [32], [33]. Precisely, also in the structure frame of the potential well theory, the results of global existence, asymptotic behavior and blow up of solutions for both subcritical initial energy level and critical initial energy level have been proved; moreover, the finite-time blow up of solutions at arbitrarily positive initial energy has been discussed. There have been a large number of published papers on fourth-order equations that it is difficult to list all results; thus, we refer here more some recent models of fourth-order equations with various characteristic term such as [5] with nonlinear boundary damping and interior source, [15] with weak damping term and exponential source, [14] with Hardy-Hénon potential and polynomial nonlinearity, [21] with variable-exponents, [31] with nonlinear damping and nonlinear source.

Considering that the above mentioned papers are devoted to the models with the single linear/nonlinear weak damping term or the single strong damping term sometimes the combinations of two of them; meanwhile few viscoelastic versions of the problem with a memory term, see [2], [19], [25], [26] and the references herein as examples; in which the authors got the existence of global attractor and asymptotic stability. Normally, the presence of memory term is one of factors causing decay property of solution energy for corresponding system; thus, problems with memory term seem difficult to obtain results of blow-up phenomena of soutions; in our observation, there seem have been no published results of finite-time blow up of solutions for fourth-order equations. In the present paper, we shall consider below a more general case with both the weak damping term and the strong damping term and the memory term, in which the finite-time blow up of solutions for the problem (1.1) in the case $f = -\lambda u_t + f(u, u_x)$ and $\mu = \mu\left(\|u_x(t)\|^2\right) u_x + \mu_1(u, u_x)$ shall be studied. Precisely, we consider the initial-boundary value problem as follows

$$\begin{cases} u_{tt} + \lambda u_t + \Delta^2 u - \Delta u_t - \dfrac{\partial}{\partial x}\left[\mu\left(\|u_x(t)\|^2\right) u_x + \mu_1(u, u_x)\right] \\ -\displaystyle\int_0^t g(t-s) u_{xx}(x,s)ds = f(u, u_x),\ 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ \ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.9}$$

where $\lambda > 0$ is a given constant and $\tilde{u}_0$, $\tilde{u}_1$, μ, μ_1, g, f are given functions. Then, under suitable assumptions on initial data and some contrary arguments, we show that Prob. (1.9) does not admit any global solutions. Finally, by constructing the appropriate energy functionals and using arguments of continuity, we prove that the solution of Prob. (1.9) blows up in finite time. To best our knowledge, there seems no results of finite-time blow up of solutions for fourth-order viscoelastic problems such as Prob. (1.9) that includes both the weak damping term u_t and the strong damping term $-u_{xxt}$ and the viscoelastic term $\int_0^t g(t-s)u_{xx}(x,s)ds$. Thus, our paper tries to answer this issue and also mentions to the solvability and the finite-time blow up of solutions for Prob. (1.1)-(1.3). In order to obtain the expected results above, we study Prob. (1.1)–(1.3) according to the following structure. In Section 2, some required preliminaries are introduced. In Section 3, the local existence and uniqueness of solutions for Prob. (1.1)-(1.3) are proved by the linear approximation and the Faedo-Galerkin method. In Section 4, we consider a special case of Prob. (1.1)-(1.3) provided by Prob. (1.9). Then, under some suitablely sufficient conditions on the relaxation functions and by constructing appropriate energy functionals together with contrary arguments, we show that Prob. (1.9) does not admit any global solutions and its solution for the negative initial energy case blows up in finite time.

2. Preliminaries

Put $\Omega = (0,1)$. Throughout this paper, we use the following notations: $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{H^m} = \|\cdot\|_{H^m(\Omega)}$ (m is a natural number) and denote $\|\cdot\|_X$ to be a norm in a Banach space X. Also, we use notation $\langle u,v\rangle = \int_0^1 u(x)v(x)dx$ being a scalar product in L^2 or a dual pair of a linear continuous functional and an element of a function space.

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, $u_{xxx}(t) = \Delta u_x(t)$, $u_{xxxx}(t) = \Delta^2 u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, $\frac{\partial^3 u}{\partial x^3}(x,t)$, $\frac{\partial^4 u}{\partial x^4}(x,t)$, respectively.

Let $T^* > 0$, with $f \in C^k([0,1]\times[0,T^*]\times\mathbb{R}^3)$, $f = f(x,t,y_1,\cdots,y_3)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_{i+2} f = \frac{\partial f}{\partial y_i}$ with $i = 1,2,3$, and $D^\alpha f = D_1^{\alpha_1}\cdots D_8^{\alpha_8} f$, $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{Z}_+^3$, $|\alpha| = \alpha_1+\alpha_2+\alpha_3 = k$, $D^{(0,\cdots,0)}f = f$.

On $H^1,\cdots,H^4$, we use the following norms

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{\frac{1}{2}}, \tag{2.1}$$

and

$$\|v\|_{H^i} = \left(\|v\|_{H^{i-1}}^2 + \left\|\frac{\partial^i u}{\partial x^i}\right\|^2\right)^{\frac{1}{2}},\ i = 2,3,4, \tag{2.2}$$

respectively.

Put

$$H_0^1 = \{v \in H^1 : v(0) = v(1) = 0\}, \tag{2.3}$$

$$\begin{aligned}
&H^2\cap H_0^1=\{v\in H^2: v(0)=v(1)=0\},\\
&H_{\#}^4=\{v\in H^4: v(0)=v(1)=\Delta v(0)=\Delta v(1)=0\},\\
&a_1(u,\varphi)=\int_0^1 \Delta u(x)\Delta\varphi(x)dx,\ \forall u,\varphi\in H^2\cap H_0^1,\\
&a_2(u,\varphi)=\int_0^1 \Delta^2 u(x)\Delta^2\varphi(x)dx,\ \forall u,\varphi\in H_{\#}^4.
\end{aligned}$$

Then, it is not difficult to prove the following lemmas (see [13]), hence the proofs of which are omitted the details.

Lemma 2.1. *The imbedding $H^1\hookrightarrow C^0\left(\overline{\Omega}\right)$ is compact and $\|v\|_{C^0\left(\overline{\Omega}\right)}\leq\sqrt{2}\,\|v\|_{H^1}$, for all $v\in H^1$.*

Lemma 2.2. *The imbedding $H_0^1\hookrightarrow C^0\left(\overline{\Omega}\right)$ is compact and*

$$\|v\|_{C^0\left(\overline{\Omega}\right)}\leq\|v_x\|,\quad \frac{1}{\sqrt{2}}\|v\|_{H^1}\leq\|v_x\|\leq\|v\|_{H^1},\ \textit{for all } v\in H_0^1.$$

H_0^1 is a closed subspace of H^1 and, on H_0^1, two norms $v\longmapsto\|v\|_{H^1}$ and $v\longmapsto\|v_x\|$ are equivalent norms.

$H^2\cap H_0^1$ is a closed subspace of H^2 and, on $H^2\cap H_0^1$, two norms $v\longmapsto\|v\|_{H^2}$ and $v\longmapsto\|\Delta v\|$ are equivalent norms and

$$\|\Delta v\|\leq\|v\|_{H^2}\leq\sqrt{3}\,\|\Delta v\|,\ \forall v\in H^2\cap H_0^1.$$

$H_{\#}^4$ is a closed subspace of H^4 and, on $H_{\#}^4$, two norms $v\longmapsto\|v\|_{H^4}$ and $v\longmapsto\left\|\Delta^2 v\right\|$ are equivalent norms and

$$\left\|\Delta^2 v\right\|\leq\|v\|_{H^4}\leq\sqrt{5}\left\|\Delta^2 v\right\|,\ \forall v\in H_{\#}^4.$$

Lemma 2.3. *The symmetric bilinear form $a_1(\cdot,\cdot)$ defined by (2.3)$_4$ is continuous on $(H^2\cap H_0^1)\times(H^2\cap H_0^1)$ and coercive on $H^2\cap H_0^1$. Furthermore,*

$$\begin{aligned}
&\textit{(i)}\quad |a_1(u,v)|\leq\sqrt{3}\,\|\Delta u\|\,\|\Delta v\|,\ \forall u,v\in H^2\cap H_0^1,\\
&\textit{(ii)}\quad a_1(v,v)\geq\|\Delta v\|^2,\ \forall v\in H^2\cap H_0^1.
\end{aligned}$$

Similarly, we also have the following lemma.

Lemma 2.4. *The symmetric bilinear form $a_2(\cdot,\cdot)$ defined by (2.3)$_5$ is continuous on $H_{\#}^4\times H_{\#}^4$ and coercive on $H_{\#}^4$. Furthermore,*

$$\begin{aligned}
&\textit{(i)}\quad |a_2(u,v)|\leq\sqrt{5}\left\|\Delta^2 u\right\|\left\|\Delta^2 v\right\|,\ \forall u,v\in H_{\#}^4,\\
&\textit{(ii)}\quad a_2(v,v)\geq\left\|\Delta^2 v\right\|^2,\ \forall v\in H_{\#}^4.
\end{aligned}$$

Lemma 2.5. *There is a Hilbert orthonormal base $\{w_j\}_{j=1}^{\infty}$ of L^2 including the eigenfunctions corresponding to the eigenvalues λ_j of the problem $-\Delta w_j=\lambda_j w_j$ in $(0,1)$, $w_j(0)=w_j(1)=0$, and satisfies*

$$\begin{aligned}
&\textit{(i)}\quad 0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_j\leq\cdots,\quad \lim_{j\to+\infty}\lambda_j=+\infty,\\
&\textit{(ii)}\quad \langle w_{jx},v_x\rangle=\lambda_j\langle w_j,v\rangle,\ \forall v\in H_0^1,\ j=1,2,\cdots.
\end{aligned}$$

Moreover, $\{w_j/\sqrt{\lambda_j}\}_{j=1}^{\infty}$ *is also a Hilbert orthonormal base of* H_0^1 *with respect to the scalar product* $\langle u_x, v_x\rangle$.

The proof of Lemma 2.5 can be found in [[24], p.87, Theorem 7.7], with $H = L^2$, $V = H_0^1$ and $\langle u_x, v_x\rangle$. Further, the Hilbert orthonormal base $\{w_j\}$ given in Lemma 2.5 can be explicitly determined by

$$w_j(x) = \sqrt{2}\sin(j\pi x), \ \lambda_j = (j\pi)^2, j = 1, 2, \cdots. \tag{2.4}$$

Lemma 2.6. *(i) The sequence* $\{w_j/\lambda_j\}$ *is a Hilbert orthonormal base of* $H^2 \cap H_0^1$ *with respect to the scalar product* $a_1(\cdot,\cdot)$.

(ii) The sequence $\{w_j/\lambda_j^2\}$ *is also a Hilbert orthonormal base of* $H_{\#}^4$ *with respect to the scalar product* $a_2(\cdot,\cdot)$.

On the other hand, w_j *satisfies the following boundary value problem:*

$$\begin{cases} \Delta^2 w_j = \lambda_j^2 w_j \ \textit{in} \ (0,1), \\ w_j(0) = w_j(1) = \Delta w_j(0) = \Delta w_j(1) = 0, \quad w_j \in C^{\infty}\left(\bar{\Omega}\right). \end{cases}$$

3. Local existence and uniqueness

For a fixed $T^* > 0$, we make the following assumptions:

(A_1) $(\tilde{u}_0, \tilde{u}_1) \in H_{\#}^4 \times (H^2 \cap H_0^1)$;

(A_2) $g \in C^0([0, T^*])$;

(A_3) $\mu \in C^2([0,1] \times [0,T^*] \times \mathbb{R}^2 \times \mathbb{R}_+)$;

(A_4) $f \in C^2([0,1] \times [0,T^*] \times \mathbb{R}^3)$ such that

$f(0,t,0,0,y_3) = f(1,t,0,0,y_3) = 0$, $\forall t \in [0,T^*]$, $\forall y_3 \in \mathbb{R}$.

Definition 3.1. A function $u = u(x,t)$ is a weak solution of (1.1) if $u \in \tilde{V}_T = \{u \in L^{\infty}(0,T; H_{\#}^4) : u' \in L^{\infty}(0,T; H^2 \cap H_0^1) \cap L^2\left(0,T; H^3 \cap H_0^1\right), \ u'' \in L^{\infty}(0,T; L^2)\}$, and satisfies

$$\begin{aligned} &\langle u''(t), w\rangle + \langle u_x'(t), w_x\rangle + \langle \Delta u(t), \Delta w\rangle + \langle \mu[u](t), w_x\rangle \\ &= \int_0^t g(t-s)\langle u_x(s), w_x\rangle ds + \langle f[u](t), w\rangle, \end{aligned} \tag{3.1}$$

for all $w \in H^2 \cap H_0^1$, a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0, \ u'(0) = \tilde{u}_1, \tag{3.2}$$

where

$$\begin{cases} \mu[u](x,t) = \mu\left(x,t,u(x,t),u_x(x,t),\|u_x(t)\|^2\right), \\ f[u](x,t) = f\left(x,t,u(x,t),u'(x,t),u_x(x,t)\right). \end{cases} \tag{3.3}$$

For each $M > 0$, we put

$$K_M \equiv K_M(f) = \|f\|_{C^2\left(\bar{A}_M\right)} = \sum_{|\alpha|\le 2} \|D^{\alpha} f\|_{C^0\left(\bar{A}_M\right)}, \tag{3.4}$$

$$\bar{K}_M \equiv \bar{K}_M(\mu) = \|\mu\|_{C^2(\hat{A}_M)} = \sum_{|\alpha|\le 2} \|D^\alpha \mu\|_{C^0(\hat{A}_M)},$$

where

$$\begin{cases} \|f\|_{C^0(\bar{A}_M)} = \sup_{(x,t,y_1,\cdots,y_3)\in \bar{A}_M} |f(x,t,y_1,\cdots,y_3)|, \\ \|\mu\|_{C^0(\hat{A}_M)} = \sup_{(x,t,y_1,\cdots,y_3)\in \hat{A}_M} |\mu(x,t,y_1,\cdots,y_3)|, \\ \bar{A}_M = [0,1]\times[0,T^*]\times[-M,M]^2\times[-\sqrt{2}M,\sqrt{2}M], \\ \hat{A}_M = [0,1]\times[0,T^*]\times[-M,M]\times[-\sqrt{2}M,\sqrt{2}M]\times[0,M^2]. \end{cases}$$

For every $T \in (0,T^*]$, we put

$$\begin{aligned} V_T = &\{v \in L^\infty(0,T;H^4_\#) : v' \in L^\infty(0,T;H^2\cap H^1_0)\cap L^2\left(0,T;H^3\cap H^1_0\right), \\ &v'' \in L^2(Q_T)\}, \end{aligned} \tag{3.5}$$

which is a Banach space (see Lions [13]) with respect to the following norm

$$\|v\|_{V_T} = \max\left\{\|v\|_{L^\infty(0,T;H^4_\#)},\ \|v'\|_{L^\infty(0,T;H^2\cap H^1_0)\cap L^2(0,T;H^3\cap H^1_0)},\ \|v''\|_{L^2(Q_T)}\right\}. \tag{3.6}$$

We also introduce the Banach space

$$W_1(T) = \{v \in C^0([0,T];H^2\cap H^1_0)\cap C^1([0,T];L^2) : v' \in L^2(0,T;H^1_0)\}, \tag{3.7}$$

with the corresponding norm

$$\begin{aligned} \|v\|_{W_1(T)} &= \|v\|_{C^0([0,T];H^2\cap H^1_0)} + \|v\|_{C^1([0,T];L^2)} + \|v'\|_{L^2(0,T;H^1_0)} \\ &= \|\Delta v\|_{C^0([0,T];L^2)} + \|v'\|_{C^0([0,T];L^2)} + \|v'\|_{L^2(0,T;H^1_0)}. \end{aligned} \tag{3.8}$$

For every $M > 0$, we consider two sets

$$\begin{aligned} &W(M,T) = \{v \in V_T : \|v\|_{V_T} \le M\}, \\ &W_1(M,T) = \{v \in V_T : v'' \in L^\infty(0,T;L^2)\}. \end{aligned} \tag{3.9}$$

We construct a recurrent sequence $\{u_m\}$ defined by choosing the first term $u_0 \equiv \tilde{u}_0$, and suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.10}$$

then we find $u_m \in W_1(M,T)$ $(m \ge 1)$ such that u_m satisfies the following problem

$$\begin{cases} \quad \langle u''_m(t), w\rangle + \langle u'_{mx}(t), w_x\rangle + \langle \Delta u_m(t), \Delta w\rangle \\ = \int_0^t g(t-s)\langle u_{mx}(s), w_x\rangle ds \\ \quad + \langle F_m(t), w\rangle - \langle \mu_m(t), w_x\rangle,\ \forall w \in H^2\cap H^1_0, \\ u_m(0) = \tilde{u}_0,\ u'_m(0) = \tilde{u}_1, \end{cases} \tag{3.11}$$

where

$$\begin{aligned}
&\mu_m(x,t) = \mu[u_{m-1}](x,t) = \mu\left(x,t,u_{m-1}(x,t),\nabla u_{m-1}(x,t),\|\nabla u_{m-1}(t)\|^2\right), \\
&F_m(x,t) = f[u_{m-1}](x,t) = f\left(x,t,u_{m-1}(x,t),u'_{m-1}(x,t),\nabla u_{m-1}(x,t)\right). \qquad (3.12)
\end{aligned}$$

The existence of u_m is given by the following theorem.

Theorem 3.1. *Let $(A_1)-(A_4)$ hold, then there are positive constants M and T such that, for $u_0 \equiv \tilde{u}_0$, there exists a recurrent sequence $\{u_m\} \subset W_1(M,T)$ defined by (3.10)-(3.12).*

Proof. The proof of Theorem 3.1 spends several steps as follows.

Step 1. *Faedo-Galerkin approximation.* Consider the basis $\{w_j\}$ for L^2 as in (2.4). Put

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \qquad (3.13)$$

where $c_{mj}^{(k)}$ are determined by the following system of linear integro-differential equations

$$\begin{cases}
\quad \langle \ddot{u}_m^{(k)}(t), w_j\rangle + \langle \dot{u}_{mx}^{(k)}(t), w_{jx}\rangle + \langle \Delta u_m^{(k)}(t), \Delta w_j\rangle \\
= \int_0^t g(t-s)\langle u_{mx}^{(k)}(s), w_{jx}\rangle ds + \langle F_m(t), w_j\rangle - \langle \mu_m(t), w_{jx}\rangle, \ 1 \le j \le k, \\
u_m^{(k)}(0) = \tilde{u}_{0k}, \ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k},
\end{cases} \qquad (3.14)$$

in which

$$\begin{aligned}
\tilde{u}_{0k} &= \sum_{j=1}^{k} \alpha_j^{(k)} w_j \to \tilde{u}_0 \text{ strongly in } H_{\#}^4, \qquad (3.15) \\
\tilde{u}_{1k} &= \sum_{j=1}^{k} \beta_j^{(k)} w_j \to \tilde{u}_1 \text{ strongly in } H^2 \cap H_0^1.
\end{aligned}$$

The system (3.14)-(3.15) is equivalent to a system of linear intergal equations that can be rewritten in the following form

$$c_m^{(k)} = U[c_m^{(k)}], \qquad (3.16)$$

where

$$\begin{aligned}
&c_m^{(k)} = (c_{m1}^{(k)}, \cdots, c_{mk}^{(k)}), \\
&U[c_m^{(k)}] = (U_1[c_m^{(k)}], \cdots, U_k[c_m^{(k)}]), \\
&U_j[c_m^{(k)}](t) = G_{mj}^{(k)}(t) + F_j^{(k)}[c_m^{(k)}](t), \\
&F_j^{(k)}[c_m^{(k)}](t) = \int_0^t \left(\int_s^t \left(1 - e^{-\lambda_j(t-r)}\right) g(r-s)dr\right) c_{mj}^{(k)}(s)\, ds \\
&\qquad\qquad - \lambda_j \int_0^t \left(1 - e^{-\lambda_j(t-s)}\right) c_{mj}^{(k)}(s) ds,
\end{aligned}$$

$$G_{mj}^{(k)}(t) = \alpha_j^{(k)} + \frac{\beta_j^{(k)}}{\lambda_j}\left(1 - e^{-\lambda_j t}\right) + \frac{1}{\lambda_j}\int_0^t \left(1 - e^{-\lambda_j (t-s)}\right)\left\langle \bar{F}_m(s), w_j\right\rangle ds,$$
$$\left\langle \bar{F}_m(t), w_j\right\rangle = \langle F_m(t), w_j\rangle - \langle \mu_m(t), w_{jx}\rangle,\ 1 \le j \le k,\ 0 \le t \le T.$$

Using Banach's contraction principle, it is not difficult to prove that the fixed point equation (3.16) admits a unique solution $c_m^{(k)} = (c_{m1}^{(k)}, \cdots, c_{mk}^{(k)}) \in C([0,T]\,;\mathbb{R}^k)$, thus we omit the details of the proof.

Step 2. *Priori estimates*

Putting

$$\begin{aligned} S_m^{(k)}(t) = {} & \left\|\Delta u_m^{(k)}(t)\right\|^2 + \left\|\Delta u_{mx}^{(k)}(t)\right\|^2 + \left\|\Delta^2 u_m^{(k)}(t)\right\|^2 \\ & + \left\|\dot{u}_m^{(k)}(t)\right\|^2 + \left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|\Delta \dot{u}_m^{(k)}(t)\right\|^2 \\ & + 2\int_0^t \left(\left\|\dot{u}_{mx}^{(k)}(s)\right\|^2 + \left\|\Delta \dot{u}_m^{(k)}(s)\right\|^2 + \left\|\Delta \dot{u}_{mx}^{(k)}(s)\right\|^2 + \left\|\ddot{u}_m^{(k)}(s)\right\|^2\right) ds, \end{aligned} \tag{3.17}$$

then it follows from (3.14) and (3.17) that

$$\begin{aligned} S_m^{(k)}(t) = {} & S_m^{(k)}(0) - 2g(0)\int_0^t \left(\left\|u_{mx}^{(k)}(s)\right\|^2 + \left\|\Delta u_m^{(k)}(s)\right\|^2 + \left\|\Delta u_{mx}^{(k)}(s)\right\|^2\right) ds \\ & + 2\int_0^t g(t-s)\Big[\langle u_{mx}^{(k)}(s), u_{mx}^{(k)}(t)\rangle + \langle \Delta u_m^{(k)}(s), \Delta u_m^{(k)}(t)\rangle \\ & + \langle \Delta u_{mx}^{(k)}(s), \Delta u_{mx}^{(k)}(t)\rangle\Big] ds \\ & - 2\int_0^t dr \int_0^r g'(r-s)\Big[\langle u_{mx}^{(k)}(s), u_{mx}^{(k)}(r)\rangle + \langle \Delta u_m^{(k)}(s), \Delta u_m^{(k)}(r)\rangle \\ & + \langle \Delta u_{mx}^{(k)}(s), \Delta u_{mx}^{(k)}(r)\rangle\Big] ds \\ & + 2\int_0^t \left\langle F_m(s) + \Delta F_m(s), \dot{u}_m^{(k)}(s) - \Delta \dot{u}_m^{(k)}(s)\right\rangle ds \\ & - 2\int_0^t \left\langle \mu_m(s), \dot{u}_{mx}^{(k)}(s) - \Delta \dot{u}_{mx}^{(k)}(s)\right\rangle ds \\ & - 2\int_0^t \left\langle \mu_m(s), \Delta^2 \dot{u}_{mx}^{(k)}(s)\right\rangle ds + 2\int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|^2 ds \\ = {} & S_m^{(k)}(0) + \sum_{j=1}^{7} I_j. \end{aligned} \tag{3.18}$$

Taking the derivative of $F_m(x,t)$ and $\mu_m(x,t)$ (in (3.12)) up to second order and doing some calculations, we get the following lemma.

Lemma 3.1. *If $(A_3)-$ (A_4) hold, then the following estimates are valid*

$$\begin{array}{ll} \text{(i)} & \|F_{mx}(t)\| \leq (1+3M)K_M = F_M^{(1)}, \\ \text{(ii)} & \|\Delta F_m(t)\| \leq \left(1+18\sqrt{2}M+9\sqrt{2}M^2\right)K_M = F_M^{(2)}, \\ \text{(iii)} & \|\mu_{mx}(t)\| \leq (1+2M)\,\bar{K}_M = \mu_M^{(1)}, \\ \text{(iv)} & \|\mu_{mx}'(t)\| \leq \left[1+\left(4+2\sqrt{2}\right)M+\left(2+4\sqrt{2}\right)M^2+4\sqrt{2}M^3\right]\bar{K}_M = \mu_M^{(2)}, \\ \text{(v)} & \|\mu_{mx}(t)\|^2 \leq 2\,\|\mu_{mx}(0)\|^2 + 2T^2\left|\mu_M^{(2)}\right|^2, \end{array} \tag{3.19}$$

where

$$\begin{aligned} F_M^{(1)} &= (1+3M)K_M, \\ F_M^{(2)} &= \left(1+18\sqrt{2}M+9\sqrt{2}M^2\right)K_M, \\ \mu_M^{(1)} &= (1+2M)\,\bar{K}_M, \\ \mu_M^{(2)} &= \left[1+\left(4+2\sqrt{2}\right)M+\left(2+4\sqrt{2}\right)M^2+4\sqrt{2}M^3\right]\bar{K}_M. \end{aligned} \tag{3.20}$$

Using Lemma 3.1, and the following inequalities

$$\begin{aligned} &2ab \leq \gamma a^2 + \frac{1}{\gamma}b^2,\ \forall a,b\in\mathbb{R},\ \forall\gamma>0, \\ &\|v_x\| \leq \|\Delta v\|,\ \forall v\in H^2\cap H_0^1, \end{aligned} \tag{3.21}$$

we can estimate $I_1 - I_5$ on the right-hand side of (3.18) as follows

$$\begin{aligned} I_1 &= -2g(0)\int_0^t \left(\left\|u_{mx}^{(k)}(s)\right\|^2 + \left\|\Delta u_m^{(k)}(s)\right\|^2 + \left\|\Delta u_{mx}^{(k)}(s)\right\|^2\right)ds \\ &\leq 2\,|g(0)|\int_0^t \left(2\left\|\Delta u_m^{(k)}(s)\right\|^2 + \left\|\Delta u_{mx}^{(k)}(s)\right\|^2\right)ds \leq 4\,|g(0)|\int_0^t S_m^{(k)}(s)ds, \\ I_2 &= 2\int_0^t g(t-s)\Big[\langle u_{mx}^{(k)}(s), u_{mx}^{(k)}(t)\rangle ds + \langle \Delta u_m^{(k)}(s), \Delta u_m^{(k)}(t)\rangle \\ &\quad + \langle \Delta u_{mx}^{(k)}(s), \Delta u_{mx}^{(k)}(t)\rangle\Big]ds \\ &\leq 4\int_0^t |g(t-s)|\sqrt{S_m^{(k)}(s)}\sqrt{S_m^{(k)}(t)}ds \\ &\leq \gamma S_m^{(k)}(t) + \frac{4}{\gamma}\left(\int_0^t |g(t-s)|\sqrt{S_m^{(k)}(s)}ds\right)^2 \\ &\leq \gamma S_m^{(k)}(t) + \frac{4}{\gamma}\|g\|_{L^2(0,T^*)}^2\int_0^t S_m^{(k)}(s)ds, \qquad (3.22) \\ I_3 &= -2\int_0^t dr\int_0^r g'(r-s)\Big[\langle u_{mx}^{(k)}(s), u_{mx}^{(k)}(r)\rangle + \langle \Delta u_m^{(k)}(s), \Delta u_m^{(k)}(r)\rangle \\ &\quad + \langle \Delta u_{mx}^{(k)}(s), \Delta u_{mx}^{(k)}(r)\rangle\Big]ds \\ &\leq 4\int_0^t dr\int_0^r |g'(r-s)|\sqrt{S_m^{(k)}(s)}\sqrt{S_m^{(k)}(r)}ds \end{aligned}$$

$$
\begin{aligned}
&\leq 4\sqrt{T^*}\,\|g'\|_{L^2(0,T^*)}\int_0^t S_m^{(k)}(s)ds,\\
I_4 &= 2\int_0^t \left\langle F_m(s)+\Delta F_m(s),\dot{u}_m^{(k)}(s)-\Delta\dot{u}_m^{(k)}(s)\right\rangle ds\\
&\leq 2\int_0^t \left(\|F_m(s)\|+\|\Delta F_m(s)\|\right)\left(\left\|\dot{u}_m^{(k)}(s)\right\|+\left\|\Delta\dot{u}_m^{(k)}(s)\right\|\right)ds\\
&\leq 2TK_M^2\left(2+18\sqrt{2}M+9\sqrt{2}M^2\right)^2+\int_0^t S_m^{(k)}(s)ds,\\
I_5 &= -2\int_0^t \left\langle \mu_m(s),\dot{u}_{mx}^{(k)}(s)-\Delta\dot{u}_{mx}^{(k)}(s)\right\rangle ds\\
&\leq 2\int_0^t \|\mu_m(s)\|\left(\left\|\dot{u}_{mx}^{(k)}(s)\right\|+\left\|\Delta\dot{u}_{mx}^{(k)}(s)\right\|\right)ds\\
&\leq 2\sqrt{2}\bar{K}_M\int_0^t \sqrt{\left\|\dot{u}_{mx}^{(k)}(s)\right\|^2+\left\|\Delta\dot{u}_{mx}^{(k)}(s)\right\|^2}ds\\
&\leq 2\gamma\int_0^t \left(\left\|\dot{u}_{mx}^{(k)}(s)\right\|^2+\left\|\Delta\dot{u}_{mx}^{(k)}(s)\right\|^2\right)ds+\frac{1}{\gamma}T\bar{K}_M^2\\
&\leq \gamma S_m^{(k)}(t)+\frac{1}{\gamma}T\bar{K}_M^2.
\end{aligned}
$$

For $I_6=-2\int_0^t \left\langle \mu_m(s),\Delta^2\dot{u}_{mx}^{(k)}(s)\right\rangle ds$, we use integral by parts and estimate for I_6 as follows

$$
\begin{aligned}
I_6 &= -2\int_0^t \left\langle \mu_m(s),\Delta^2\dot{u}_{mx}^{(k)}(s)\right\rangle ds\\
&= -2\langle\mu_m(t),\Delta^2 u_{mx}^{(k)}(t)\rangle+2\langle\mu_m(0),\Delta^2\tilde{u}_{0kx}\rangle+2\int_0^t \left\langle \mu_m'(s),\Delta^2 u_{mx}^{(k)}(s)\right\rangle ds\\
&= 2\langle\mu_{mx}(t),\Delta^2 u_m^{(k)}(t)\rangle-2\langle\mu_m(0),\Delta^2\tilde{u}_{0kx}\rangle-2\int_0^t \left\langle \mu_{mx}'(s),\Delta^2 u_m^{(k)}(s)\right\rangle ds\\
&\leq 2\|\mu_{mx}(t)\|\left\|\Delta^2 u_m^{(k)}(t)\right\|+\left|2\langle\mu_m(0),\Delta^2\tilde{u}_{0kx}\rangle\right|+2\int_0^t \|\mu_{mx}'(s)\|\left\|\Delta^2 u_m^{(k)}(s)\right\|ds\\
&\leq 2\|\mu_{mx}(t)\|\sqrt{S_m^{(k)}(t)}-2\langle\mu_m(0),\Delta^2\tilde{u}_{0kx}\rangle+2\int_0^t \|\mu_{mx}'(s)\|\sqrt{S_m^{(k)}(s)}ds\\
&\leq \gamma S_m^{(k)}(t)+\frac{1}{\gamma}\|\mu_{mx}(t)\|^2+2\left|\langle\mu_m(0),\Delta^2\tilde{u}_{0kx}\rangle\right|+2\mu_M^{(2)}\int_0^t\sqrt{S_m^{(k)}(s)}ds\\
&\leq \gamma S_m^{(k)}(t)+\frac{1}{\gamma}\left[2\|\mu_{mx}(0)\|^2+2T^2\left|\mu_M^{(2)}\right|^2\right]+2\left|\langle\mu_m(0),\Delta^2\tilde{u}_{0kx}\rangle\right| \qquad (3.23)\\
&\quad +T\left|\mu_M^{(2)}\right|^2+\int_0^t S_m^{(k)}(s)ds\\
&= \gamma S_m^{(k)}(t)+\frac{2}{\gamma}\|\mu_{mx}(0)\|^2+2\left|\langle\mu_m(0),\Delta^2\tilde{u}_{0kx}\rangle\right|+\left(1+\frac{2}{\gamma}T^*\right)T\left|\mu_M^{(2)}\right|^2\\
&\quad +\int_0^t S_m^{(k)}(s)ds.
\end{aligned}
$$

For $I_7 = \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds$, we estimate as follows.

Rewriting $(3.14)_1$ by

$$\begin{aligned}&\langle \ddot{u}_m^{(k)}(t), w_j \rangle - \langle \Delta \dot{u}_m^{(k)}(t), w_j \rangle + \langle \Delta^2 u_m^{(k)}(t), w_j \rangle + \int_0^t g(t-s) \langle \Delta u_m^{(k)}(s), w_j \rangle ds \\ &= \langle F_m(t), w_j \rangle + \langle \mu_{mx}(t), w_j \rangle, \ 1 \leq j \leq k,\end{aligned}$$

replacing w_j with $\ddot{u}_m^{(k)}(t)$ and then integrating, we obtain

$$\begin{aligned}&\left\| \ddot{u}_m^{(k)}(t) \right\|^2 \\ &= \langle \Delta \dot{u}_m^{(k)}(t), \ddot{u}_m^{(k)}(t) \rangle - \langle \Delta^2 u_m^{(k)}(t), \ddot{u}_m^{(k)}(t) \rangle \\ &\quad - \int_0^t g(t-s) \langle \Delta u_m^{(k)}(s), \ddot{u}_m^{(k)}(t) \rangle ds + \left\langle F_m(t), \ddot{u}_m^{(k)}(t) \right\rangle + \left\langle \mu_{mx}(t), \ddot{u}_m^{(k)}(t) \right\rangle \\ &\leq \Big(\left\| \Delta \dot{u}_m^{(k)}(t) \right\| + \left\| \Delta^2 u_m^{(k)}(t) \right\| + \int_0^t |g(t-s)| \left\| \Delta u_m^{(k)}(s) \right\| ds \\ &\quad + \|F_m(t)\| + \|\mu_{mx}(t)\| \Big) \left\| \ddot{u}_m^{(k)}(t) \right\| \\ &\leq \left[\left\| \Delta \dot{u}_m^{(k)}(t) \right\| + \left\| \Delta^2 u_m^{(k)}(t) \right\| + \int_0^t |g(t-s)| \left\| \Delta u_m^{(k)}(s) \right\| ds + K_M + \left| \mu_M^{(1)} \right| \right]^2 \\ &\leq 4 \Bigg[\left\| \Delta \dot{u}_m^{(k)}(t) \right\|^2 + \left\| \Delta^2 u_m^{(k)}(t) \right\|^2 + \left(\int_0^t |g(t-s)| \left\| \Delta u_m^{(k)}(s) \right\| ds \right)^2 \\ &\quad + \left(K_M + \left| \mu_M^{(1)} \right| \right)^2 \Bigg] \\ &\leq 4 \left[S_m^{(k)}(t) + \left(\int_0^t |g(t-s)| \sqrt{S_m^{(k)}(s)} ds \right)^2 + \left(K_M + \left| \mu_M^{(1)} \right| \right)^2 \right] \\ &\leq 4 \left[S_m^{(k)}(t) + \|g\|_{L^2(0,T^*)}^2 \int_0^t S_m^{(k)}(s) ds + \left(K_M + \left| \mu_M^{(1)} \right| \right)^2 \right],\end{aligned}$$

therefore

$$\begin{aligned}I_7 &= 2 \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds \\ &\leq 8 \int_0^t \left(S_m^{(k)}(r) + \|g\|_{L^2(0,T^*)}^2 \int_0^r S_m^{(k)}(s) ds + \left(K_M + \left| \mu_M^{(1)} \right| \right)^2 \right) dr \\ &\leq 8T \left(K_M + \left| \mu_M^{(1)} \right| \right)^2 + 8 \left(1 + T^* \|g\|_{L^2(0,T^*)}^2 \right) \int_0^t S_m^{(k)}(s) ds.\end{aligned} \tag{3.24}$$

Combining the estimates (3.18), (3.22), (3.23), (3.24), and choose $\gamma = \frac{1}{6}$, we get

$$S_m^{(k)}(t) \leq \bar{S}_{0m}^{(k)} + T D_1(M) + D_2(M) \int_0^t S_m^{(k)}(s) ds, \tag{3.25}$$

where

$$\bar{S}_{0m}^{(k)} = 2 S_m^{(k)}(0) + 4 \left| \langle \mu_m(0), \Delta^2 \tilde{u}_{0kx} \rangle \right| + 24 \|\mu_{mx}(0)\|^2, \tag{3.26}$$

$$D_1(M) = 2\Bigg[6\bar{K}_M^2 + 2\left(2 + 18\sqrt{2}M + 9\sqrt{2}M^2\right)^2 K_M^2 + (1+12T^*)\left|\mu_M^{(2)}\right|^2$$
$$+ 8\left(K_M + \left|\mu_M^{(1)}\right|\right)^2\Bigg],$$
$$D_2(M) = 4\left[5 + 2|g(0)| + 4T^*\|g\|^2_{L^2(0,T^*)} + 12\|g\|^2_{L^2(0,T^*)} + 2\sqrt{T^*}\|g'\|_{L^2(0,T^*)}\right].$$

By the convergences given in (3.15), there exists a constant $M > 0$, independent of k and m, such that

$$\bar{S}_{0m}^{(k)} \leq 2\left(\|\Delta\tilde{u}_{0k}\|^2 + \|\Delta\tilde{u}_{0kx}\|^2 + \left\|\Delta^2\tilde{u}_{0k}\right\|^2 + \|\tilde{u}_{1k}\|^2 + \|\tilde{u}_{1kx}\|^2 + \|\Delta\tilde{u}_{1k}\|^2\right)$$
$$+ 4\|\mu[\tilde{u}_{0k}](0)\|\left\|\Delta^2\tilde{u}_{0kx}\right\| \tag{3.27}$$
$$+ 24\|D_1\mu[\tilde{u}_{0k}](0) + D_3\mu[\tilde{u}_{0k}](0)\tilde{u}_{0kx} + D_4\mu[\tilde{u}_{0k}](0)\Delta\tilde{u}_{0k}\|^2$$
$$\leq \frac{M^2}{2}, \text{ for all } m,\ k \in \mathbb{N}.$$

It follows from (3.18), (3.22)-(3.24), (3.27) that

$$S_m^{(k)}(t) \leq \frac{M^2}{2} + TD_1(M) + D_2(M)\int_0^t S_m^{(k)}(s)ds. \tag{3.28}$$

Note that, we can choose $T \in (0, T^*]$ such that

$$\left(\frac{M^2}{2} + TD_1(M)\right)\exp\left(TD_2(M) \leq M^2,\right. \tag{3.29}$$

and

$$k_T = \left(2 + \frac{1}{\sqrt{2}}\right)\sqrt{T\tilde{D}_1(M)\exp(T\tilde{D}_2)} < 1, \tag{3.30}$$

where

$$\tilde{D}_1(M) = 8\left[K_M^2 + 2(1+M)^2\bar{K}_M^2\right], \tag{3.31}$$
$$\tilde{D}_2 = 2\left[1 + 2|g(0)| + 4\|g\|^2_{L^2(0,T^*)} + 2\sqrt{T^*}\|g'\|_{L^2(0,T^*)}\right].$$

Then, by (3.28) and (3.29), we obtain

$$S_m^{(k)}(t) \leq M^2\exp\left(-TD_2(M)\right) + D_2(M)\int_0^t S_m^{(k)}(s)ds. \tag{3.32}$$

By using Gronwall's lemma, we deduce from (3.32) that

$$S_m^{(k)}(t) \leq M^2\exp\left(-TD_2(M)\exp\left(tD_2(M) \leq M^2,\right.\right. \tag{3.33}$$

for all $t \in [0, T]$, for all m and k. This leads to the fact that

$$u_m^{(k)} \in W(M, T), \text{ for all } m \text{ and } k. \tag{3.34}$$

Step 3. *Limiting process.* From (3.34), there exists a subsequence of $\{u_m^{(k)}\}$ denoted by the same symbol such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty\left(0,T;H_\#^4\right) \text{ weak*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty\left(0,T;H^2\cap H_0^1\right) \text{ weak*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^2\left(0,T;H^3\cap H_0^1\right) \text{ weak}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in} \quad L^2(Q_T) \text{ weak}, \\ u_m \in W\left(M,T\right). \end{cases} \tag{3.35}$$

Taking the limitations in (3.14) as $k\to\infty$, we get that u_m satisfies (3.11), (3.12) in $L^2(0,T)$. On the other hand, by $(3.11)_1$ and $(3.35)_4$, we have that

$$u_m'' = \Delta u_m' - \Delta^2 u_m - \int_0^t g(t-s)\Delta u_m(s)ds + \mu_{mx} + F_m \in L^\infty\left(0,T;L^2\right),$$

$u_m \in W_1(M,T)$. Hence, Theorem 3.1 is proved completely. □

By Theorem 3.1, we prove the existence and uniqueness of weak solution of (1.1) which is given by the following theorem.

Theorem 3.2. *Let $(A_1)-(A_4)$ hold. There are two postive constants M and T such that $\{u_m\}$ converges strongly in $W_1(T)$ to $u\in W_1(M,T)$ being the unique weak solution of (1.1). Moreover, the following estimate is claimed*

$$\|u_m - u\|_{W_1(T)} \le C_T k_T^m, \text{ for all } m\in\mathbb{N}, \tag{3.36}$$

where $k_T\in(0,1)$ is defined by (3.30), (3.31) and C_T is a constant independent of m.

Proof. First, we shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1}-u_m$. Then $\bar{u}_m$ satisfies

$$\begin{aligned} &\langle w_m''(t),w\rangle + \langle w_{mx}'(t),w_x\rangle + \langle \Delta w_m(t),\Delta w\rangle \qquad (3.37)\\ &= \int_0^t g(t-s)\langle w_{mx}(s),w_x\rangle ds + \left\langle \bar{F}_m(t), w\right\rangle - \langle \bar{\mu}_m(t),w_x\rangle,\ \forall w\in H^2\cap H_0^1, \\ &w_m(0)=w_m'(0)=0, \end{aligned}$$

where

$$\begin{aligned} \bar{F}_m(t) &= F_{m+1}(t) - F_m(t) = f[u_m](t) - f[u_{m-1}](t), \\ \bar{\mu}_m(t) &= \mu_{m+1}(t) - \mu_m(t) = \mu[u_m](t) - \mu[u_{m-1}](t). \end{aligned} \tag{3.38}$$

Taking $w = w_m'(t)$ in (3.37) and after integrating in t, we get

$$\begin{aligned} \bar{S}_m(t) = &-2g(0)\int_0^t \|w_{mx}(s)\|^2 ds + 2\int_0^t g(t-s)\langle w_{mx}(s), w_{mx}(t)\rangle ds \\ &- 2\int_0^t dr \int_0^r g'(r-s)\langle w_{mx}(s), w_{mx}(r)\rangle ds \end{aligned} \tag{3.39}$$

$$
\begin{aligned}
&+2\int_0^t \left\langle \bar{F}_m(s), w_m'(s)\right\rangle ds - 2\int_0^t \langle \bar{\mu}_m(s), w_{mx}'(s)\rangle ds \\
&= J_1 + J_2 + J_3 + J_4 + J_5,
\end{aligned}
$$

where

$$
\bar{S}_m(t) = \|w_m'(t)\|^2 + \|\Delta w_m(t)\|^2 + 2\int_0^t \|w_{mx}'(s)\|^2 ds. \tag{3.40}
$$

It takes no difficulty to prove the lemma below.

Lemma 3.2. *If $(A_3)-(A_4)$ hold, then the following estimates are confirmed*

(i) $\left\|\bar{F}_m(t)\right\| \le 2K_M \|w_{m-1}\|_{W_1(T)}$,

(ii) $\|\bar{\mu}_m(t)\| \le 2(1+M)\bar{K}_M \|w_{m-1}\|_{W_1(T)}$.

Using Lemma 3.2, and evaluating in a similar way as above, the terms $J_1 - J_5$ are estimated as follows

$$
\begin{aligned}
J_1 &= -2g(0)\int_0^t \|w_{mx}(s)\|^2 ds \le 2|g(0)|\int_0^t \bar{S}_m(s)ds, \\
J_2 &= 2\int_0^t g(t-s)\langle w_{mx}(s), w_{mx}(t)\rangle ds \\
&\le \frac{1}{4}\|w_{mx}(t)\|^2 + 4\left(\int_0^t |g(t-s)|\,\|w_{mx}(s)\|\,ds\right)^2 \\
&\le \frac{1}{4}\bar{S}_m(t) + 4\|g\|_{L^2(0,T^*)}^2 \int_0^t \bar{S}_m(s)ds, \\
J_3 &= -2\int_0^t dr\int_0^r g'(r-s)\langle w_{mx}(s), w_{mx}(r)\rangle ds \\
&\le 2\sqrt{T^*}\,\|g'\|_{L^2(0,T^*)}\int_0^t \bar{S}_m(s)ds, \\
J_4 &= 2\int_0^t \left\langle \bar{F}_m(s), w_m'(s)\right\rangle ds \\
&\le 4TK_M^2\|w_{m-1}\|_{W_1(T)}^2 + \int_0^t \bar{S}_m(s)ds, \\
J_5 &= -2\int_0^t \langle \bar{\mu}_m(s), w_{mx}'(s)\rangle ds \\
&\le \frac{1}{2}\int_0^t \|w_{mx}'(s)\|^2 ds + 2\int_0^t \|\bar{\mu}_m(s)\|^2 ds \\
&\le \frac{1}{4}\bar{S}_m(t) + 8T(1+M)^2\bar{K}_M^2\|w_{m-1}\|_{W_1(T)}^2.
\end{aligned} \tag{3.41}
$$

It follows from (3.39), (3.41), that

$$
\bar{S}_m(t) \le T\tilde{D}_1(M)\|w_{m-1}\|_{W_1(T)}^2 + \tilde{D}_2\int_0^t \bar{S}_m(s)\,ds, \tag{3.42}
$$

where $\tilde{D}_1(M)$, $\tilde{D}_2$ are defined as in (3.31).

By using Gronwall's lemma, we derive from (3.42) that

$$\|w_m\|_{W_1(T)} \leq k_T \|w_{m-1}\|_{W_1(T)}, \ \forall m \in \mathbb{N}, \tag{3.43}$$

where $k_T < 1$ is defined as in (3.30), (3.31). This implies that

$$\|u_{m+p} - u_m\|_{W_1(T)} \leq \frac{M}{1-k_T} k_T^m, \ \forall m, \ p \in \mathbb{N}. \tag{3.44}$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \text{ strongly in } W_1(T). \tag{3.45}$$

On the other hand, $u_m \in W(M,T)$, there is a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in } L^\infty\left(0,T;H^4_\#\right) \text{ weak*}, \\ u'_{m_j} \to u' & \text{in } L^\infty\left(0,T;H^2\cap H^1_0\right) \text{ weak*}, \\ u'_{m_j} \to u' & \text{in } L^2\left(0,T;H^3\cap H^1_0\right) \text{ weak}, \\ u''_{m_j} \to u'' & \text{in } L^2(Q_T) \text{ weak}, \\ u \in W(M,T). \end{cases} \tag{3.46}$$

Note that

$$\begin{aligned} &\|F_m - f[u]\|_{L^\infty(0,T;L^2)} \leq 2K_M \|u_{m-1} - u\|_{W_1(T)}, \\ &\|\mu_m - \mu[u]\|_{L^\infty(0,T;L^2)} \leq 2(1+M)\bar{K}_M \|u_{m-1} - u\|_{W_1(T)}. \end{aligned} \tag{3.47}$$

Hence, we derive from (3.45) and (3.47) that

$$\begin{aligned} &F_m \to f[u] \text{ strongly in } L^\infty\left(0,T;L^2\right), \\ &\mu_m \to \mu[u] \text{ strongly in } L^\infty\left(0,T;L^2\right). \end{aligned} \tag{3.48}$$

Taking the limitations in (3.11) and(3.12) as $m = m_j \to \infty$, it implies from (3.45), (3.46) and (3.48) that $u \in W(M,T)$ and satisfies (3.1)-(3.3).

Further, by the assumptions $(A_2) - (A_4)$, we obtain from (3.1) and $(3.46)_5$ that

$$u'' = \Delta u' - \Delta^2 u - \int_0^t g(t-s)\Delta u(s)ds + \frac{\partial}{\partial x}\mu[u] + f[u] \in L^\infty\left(0,T;L^2\right).$$

Thus, $u \in W_1(M,T)$. This confirms that u is a weak solution of (1.1).

Next, we prove the uniqueness of weak solutions of (1.1) as follows.

Let u_1, $u_2 \in W_1(M,T)$ be two weak solutions of (1.1). Then $v = u_1 - u_2$ satisfies

$$\begin{cases} \quad \langle v''(t), w\rangle + \langle v'_x(t), w_x\rangle + \langle \Delta v(t), \Delta w\rangle + \langle \mu[u_1](t) - \mu[u_2](t), w_x\rangle \\ = \displaystyle\int_0^t g(t-s)\langle v_x(s), w_x\rangle ds + \langle f[u_1](t) - f[u_2](t), w\rangle, \quad \forall w \in H^2\cap H^1_0, \\ v(0) = v'(0) = 0. \end{cases} \tag{3.49}$$

Taking $w = v'(t)$ in (3.49) and integrating in t , we get

$$\bar{Z}(t) = 2\int_0^t dr \int_0^r g(r-s)\langle v_x(s), v_x'(r)\rangle ds + 2\int_0^t \langle f[u_1](s) - f[u_2](s), v'(s)\rangle\, ds$$
$$- 2\int_0^t \langle \mu[u_1](s) - \mu[u_2](s), v_x'(s)\rangle ds, \tag{3.50}$$

where

$$\bar{Z}(t) = \|v'(t)\|^2 + \|\Delta v(t)\|^2 + 2\int_0^t \|v_x'(s)\|^2\, ds. \tag{3.51}$$

Then it follows from (3.50)-(3.51) that

$$\bar{Z}(t) \le \rho_M \int_0^t \bar{Z}(s)\, ds, \tag{3.52}$$

where $\rho_M = 4T^* \|g\|^2_{L^2(0,T^*)} + 8\sqrt{2}K_M + 16(1+M)^2\bar{K}_M^2$.

By using Gronwall's lemma, we obtain that $\bar{Z}(t) \equiv 0$, i.e., $u = u_1 - u_2 = 0$. This claims the uniqueness of solutions of (1.1). Thus, Theorem 3.2 is proved completely. $\square$

4. Blow-up of solutions

This section is devoted to studying of the finite-time blow up of the solution of the problem (1.1) in the case

$$f = -\lambda u_t + f(u, u_x),$$
$$\mu = \mu\left(\|u_x(t)\|^2\right) u_x + \mu_1(u, u_x).$$

Precisely, we consider the initial-boundary value problem as follows

$$\begin{cases} \quad u_{tt} + \lambda u_t + \Delta^2 u - \Delta u_t - \dfrac{\partial}{\partial x}\left[\mu\left(\|u_x(t)\|^2\right) u_x + \mu_1(u, u_x)\right] \\ \quad - \displaystyle\int_0^t g(t-s) u_{xx}(x,s) ds \\ = f(u, u_x),\ 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ \ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{4.1}$$

where $\lambda > 0$ is a given constant and $\tilde{u}_0$, $\tilde{u}_1$, μ, μ_1, g, f are given functions satisfiying the following assumptions

(A_1) $\quad (\tilde{u}_0, \tilde{u}_1) \in H^4_{\#} \times (H^2 \cap H^1_0)$;

$(\hat{A}_2)$ $\quad \mu \in C^2(\mathbb{R}_+)$ and there is a positive constant μ_* such that

$\quad \mu(z) \ge \mu_*,\ \forall z \in \mathbb{R}_+$;

$(\hat{A}_3)$ $\quad g \in C^1(\mathbb{R}_+)$;

$(\hat{A}_4)$ $\quad f \in C^2\left(\mathbb{R}^2\right),\ \mu_1 \in C^2\left(\mathbb{R}^2\right)$.

By the same method used for the proof of Theorem 3.2, the problem (4.1) has a unique local solution provided by the following theorem.

Theorem 4.1. *Suppose that* (A_1), $(\hat{A}_2)-$ $(\hat{A}_4)$ *hold then the problem (4.1) admits a unique local solution satisfiying*

$$\begin{aligned}
&u \in L^{\infty}(0,T;H^4_{\#}) \cap C^0([0,T];H^3 \cap H^1_0) \cap C^1([0,T];L^2), \\
&u' \in L^{\infty}(0,T;H^2 \cap H^1_0) \cap L^2\left(0,T;H^3 \cap H^1_0\right) \cap C^0([0,T];L^2), \\
&u'' \in L^{\infty}(0,T;L^2),
\end{aligned} \tag{4.2}$$

for a sufficiently small $T > 0$.

In what follows, we will consider the existence of local solution of (4.1) in the case that the initial datum is less regularized.

$(\bar{A}_1)$ $(\tilde{u}_0, \tilde{u}_1) \in \left(H^2 \cap H^1_0\right) \times L^2$;

$(\hat{A}_2)$ $\mu \in C^2(\mathbb{R}_+)$ and there is a positive constant μ_* such that

$\mu(z) \geq \mu_*,\ \forall z \in \mathbb{R}_+$;

$(\hat{A}_3)$ $g \in C^1(\mathbb{R}_+)$;

$(\tilde{A}_4)$ There is a function $\mathcal{F} \in C^3(\mathbb{R}^2;\mathbb{R})$ and positive constants α, β, $\hat{d}_2$, with α, $\beta > 2$, such that

(i) $\dfrac{\partial \mathcal{F}}{\partial u}(u,v) = f(u,v)\,,\ \dfrac{\partial \mathcal{F}}{\partial v}(u,v) = -\mu_1(u,v)\,,$ for all $(u,v) \in \mathbb{R}^2$,

(ii) $\mathcal{F}(u,v) \leq \hat{d}_2\left(1 + |u|^{\alpha} + |v|^{\beta}\right)$, for all $(u,v) \in \mathbb{R}^2$.

Then, we also get the local solution existence given by the theorem below.

Theorem 4.2. *Assume that* $(\bar{A}_1)$, $(\hat{A}_2)$, $(\hat{A}_3)$, $(\tilde{A}_4)$ *hold, then the problem (4.1) has a unique local solution*

$$u \in W_1(T) = \{v \in C^0([0,T];H^2 \cap H^1_0) \cap C^1([0,T];L^2) : v' \in L^2(0,T;H^1_0)\}, \tag{4.3}$$

for a sufficiently small $T > 0$.

Proof. In order to obtain the existence of a weak solution, we use standard arguments of density.

With $T^* > 0$, let us consider $(\tilde{u}_0, \tilde{u}_1) \in \left(H^2 \cap H^1_0\right) \times L^2$.

Let $\{(\tilde{u}_{0m}, \tilde{u}_{1m})\} \subset C_0^{\infty}([0,1]) \times C_0^{\infty}([0,1])$ be a sequence such that

$$\begin{cases} \tilde{u}_{0m} \to \tilde{u}_0 \quad \text{strongly in} \quad H^2 \cap H^1_0, \\ \tilde{u}_{1m} \to \tilde{u}_1 \quad \text{strongly in} \quad L^2. \end{cases} \tag{4.4}$$

For all $m \in \mathbb{N}$, suppose that $\{\tilde{u}_{0m}\}$ satisfies compatibility conditions

$$\Delta\tilde{u}_{0m}(0) = \Delta\tilde{u}_{0m}(1) = 0. \tag{4.5}$$

Then, for each $m \in \mathbb{N}$ and the conditions of Theorem 4.1, there exists a unique function u_m such that

$$\begin{cases} \quad \langle u_m''(t), v\rangle + \lambda \langle u_m'(t), v\rangle + \langle u_{mx}'(t), v_x\rangle + \langle \Delta u_m(t), \Delta v\rangle \\ \quad +\mu\left(\|u_{mx}(t)\|^2\right)\langle u_{mx}(t), v_x\rangle \\ = \int_0^t g(t-s)\langle u_{mx}(s), v_x\rangle\, ds, \forall v \in H^2 \cap H_0^1 \\ \quad + \langle f\left(u_m(t), u_{mx}(t)\right), v\rangle - \langle \mu_1\left(u_m(t), u_{mx}(t)\right), v_x\rangle, \\ u_m(0) = \tilde{u}_{0m}, u_m'(0) = \tilde{u}_{1m}, \end{cases} \tag{4.6}$$

and

$$\begin{cases} u_m \in L^\infty(0, T_m; H_\#^4) \cap C^0([0, T_m]; H^3 \cap H_0^1) \cap C^1([0, T_m]; L^2), \\ u_m' \in L^\infty(0, T_m; H^2 \cap H_0^1) \cap L^2\left(0, T_m; H^3 \cap H_0^1\right) \cap C^0([0, T_m]; L^2), \\ u_m'' \in L^\infty(0, T_m; L^2). \end{cases} \tag{4.7}$$

Priori estimates. Replacing v with $u_m'(t)$ and then integrating, we obtain

$$\begin{aligned} \bar{\mu}_* \bar{S}_m(t) &\le S_m(t) \\ &= S_m(0) - 2\int_0^1 \mathcal{F}(\tilde{u}_{0m}(x), \tilde{u}_{0mx}(x))dx - 2g(0)\int_0^t \|u_{mx}(s)\|^2\, ds \\ &\quad + 2\int_0^t g(t-s)\langle u_{mx}(s), u_{mx}(t)\rangle\, ds \\ &\quad - 2\int_0^t dr \int_0^r g'(r-s)\langle u_{mx}(s), u_{mx}(r)\rangle\, ds \\ &\quad + 2\int_0^1 \mathcal{F}(u_m(x,t), u_{mx}(x,t))dx \\ &= S_m(0) - 2\int_0^1 \mathcal{F}(\tilde{u}_{0m}(x), \tilde{u}_{0mx}(x))dx + I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{4.8}$$

where $\bar{\mu}_* = \min\{1, \mu_*, 2\lambda\}$ and

$$\begin{aligned} \bar{S}_m(t) &= \|u_m'(t)\|^2 + \|\Delta u_m(t)\|^2 + \|u_{mx}(t)\|^2 + \int_0^t \|u_{mx}'(s)\|^2\, ds + \int_0^t \|u_m'(s)\|^2\, ds, \\ S_m(t) &= \|u_m'(t)\|^2 + \|\Delta u_m(t)\|^2 + \int_0^{\|u_{mx}(t)\|^2} \mu(z)dz + 2\int_0^t \|u_{mx}'(s)\|^2\, ds \\ &\quad + 2\lambda\int_0^t \|u_m'(s)\|^2\, ds, \\ S_m(0) &= \|\tilde{u}_{1m}\|^2 + \|\Delta\tilde{u}_{0m}\|^2 + \int_0^{\|\tilde{u}_{0mx}\|^2} \mu(z)dz. \end{aligned} \tag{4.9}$$

Similarly to the above estimates, we evaluate the terms $I_1 - I_3$ as follows

$$I_1 = -2g(0)\int_0^t \|u_{mx}(s)\|^2\, ds \le 2\,|g(0)|\int_0^t \bar{S}_m(s)ds, \tag{4.10}$$

$$I_2 = 2\int_0^t g(t-s)\left\langle u_{mx}(s), u_{mx}(t)\right\rangle ds \le \gamma \bar{S}_m(t) + \frac{1}{\gamma}\left\| g\right\|^2_{L^2(0,T^*)}\int_0^t \bar{S}_m(s)ds,$$
$$I_3 = -2\int_0^t dr\int_0^r g'(r-s)\left\langle u_{mx}(s), u_{mx}(r)\right\rangle ds \le 2\sqrt{T^*}\left\| g'\right\|_{L^2(0,T^*)}\int_0^t \bar{S}_m(s)ds.$$

For the term I_4, using the hypothesis $(\tilde{A}_4), (ii)$, we get

$$I_4 = 2\int_0^1 \mathcal{F}(u_m(x,t), u_{mx}(x,t))dx \le 2\hat{d}_2 + 2\hat{d}_2\left[\left\| u_m(t)\right\|^\alpha_{L^\alpha} + \left\| u_{mx}(t)\right\|^\beta_{L^\beta}\right]. \tag{4.11}$$

In order to evaluate $\left\| u_m(t)\right\|^\alpha_{L^\alpha}$. We use the estimate below

$$\begin{aligned}\left| u_m(x,t)\right|^\alpha &= \left|\tilde{u}_{0m}(x)\right|^\alpha + \alpha\int_0^t \left| u_m(x,s)\right|^{\alpha-2} u_m(x,s)u'_m(x,s)ds\\ &\le \left|\tilde{u}_{0m}(x)\right|^\alpha + \alpha\int_0^t \left\| u_{mx}(s)\right\|^{\alpha-1}\left| u'_m(x,s)\right| ds,\end{aligned}$$

then

$$\begin{aligned}\left\| u_m(t)\right\|^\alpha_{L^\alpha} &\le \left\|\tilde{u}_{0m}\right\|^\alpha_{L^\alpha} + \alpha\int_0^1\left(\int_0^t \left\| u_{mx}(s)\right\|^{\alpha-1} ds\right)\left| u'_m(x,s)\right| dx\\ &= \left\|\tilde{u}_{0m}\right\|^\alpha_{L^\alpha} + \alpha\int_0^t \left\| u_{mx}(s)\right\|^{\alpha-1} ds\int_0^1 \left| u'_m(x,s)\right| dx\\ &\le \left\|\tilde{u}_{0m}\right\|^\alpha_{L^\alpha} + \alpha\int_0^t \left\| u_{mx}(s)\right\|^{\alpha-1}\left\| u'_m(s)\right\| ds \qquad (4.12)\\ &\le \left\|\tilde{u}_{0m}\right\|^\alpha_{L^\alpha} + \alpha\int_0^t \left(\sqrt{\bar{S}_m(s)}\right)^{\alpha-1}\sqrt{\bar{S}_m(s)}ds\\ &= \left\|\tilde{u}_{0m}\right\|^\alpha_{L^\alpha} + \alpha\int_0^t \left(\sqrt{\bar{S}_m(s)}\right)^{\alpha} ds.\end{aligned}$$

In order to evaluate $\left\| u_{mx}(t)\right\|^\beta_{L^\beta}$. We use the estimates below

$$\begin{aligned}\left| u_{mx}(x,t)\right|^\beta &= \left|\tilde{u}_{0mx}(x)\right|^\beta + \beta\int_0^t \left| u_{mx}(x,s)\right|^{\beta-2} u_{mx}(x,s)u'_{mx}(x,s)ds\\ &\le \left|\tilde{u}_{0mx}(x)\right|^\beta + \beta\int_0^t \left| u_{mx}(x,s)\right|^{\beta-1}\left| u'_{mx}(x,s)\right| ds,\end{aligned}$$

and

$$\left| u_{mx}(x,s)\right| \le \sqrt{2}\sqrt{\left\| u_{mx}(s)\right\|^2 + \left\|\Delta u_m(s)\right\|^2} \le \sqrt{2}\sqrt{\bar{S}_m(s)}.$$

Then

$$\begin{aligned}\left| u_{mx}(x,t)\right|^\beta &\le \left|\tilde{u}_{0mx}(x)\right|^\beta + \beta\int_0^t \left| u_{mx}(x,s)\right|^{\beta-1}\left| u'_{mx}(x,s)\right| ds\\ &\le \left|\tilde{u}_{0mx}(x)\right|^\beta + \beta\left(\sqrt{2}\right)^{\beta-1}\int_0^t \left(\sqrt{\bar{S}_m(s)}\right)^{\beta-1}\left| u'_{mx}(x,s)\right| ds.\end{aligned}$$

Thus, integrating in x from 0 to 1 for the above inequality, we have

$$\left\| u_{mx}(t)\right\|^\beta_{L^\beta} \le \left\|\tilde{u}_{0mx}\right\|^\beta_{L^\beta} + \beta\left(\sqrt{2}\right)^{\beta-1}\int_0^1 dx\int_0^t \left(\sqrt{\bar{S}_m(s)}\right)^{\beta-1}\left| u'_{mx}(x,s)\right| ds$$

$$= \|\tilde{u}_{0mx}\|_{L^\beta}^\beta + \beta\left(\sqrt{2}\right)^{\beta-1}\int_0^t\left(\sqrt{\bar{S}_m(s)}\right)^{\beta-1}ds\int_0^1|u'_{mx}(x,s)|\,dx$$
$$\leq \|\tilde{u}_{0mx}\|_{L^\beta}^\beta + \gamma\int_0^t\|u'_{mx}(s)\|^2\,ds + \frac{1}{4\gamma}\beta^2 2^{\beta-1}\int_0^t\left(\bar{S}_m(s)\right)^{\beta-1}ds$$
$$\leq \|\tilde{u}_{0mx}\|_{L^\beta}^\beta + \gamma\bar{S}_m(t) + \frac{1}{4\gamma}\beta^2 2^{\beta-1}\int_0^t\left(\bar{S}_m(s)\right)^{\beta-1}ds. \tag{4.13}$$

We deduce from (4.11), (4.12) and (4.13), the term I_4 is evaluated as follows

$$\begin{aligned}
I_4 &= 2\int_0^1\mathcal{F}(u_m(x,t),u_{mx}(x,t))dx \leq 2\hat{d}_2\left[1+\|u_m(t)\|_{L^\alpha}^\alpha + \|u_{mx}(t)\|_{L^\beta}^\beta\right]\\
&= 2\hat{d}_2 + 2\hat{d}_2\left[\|u_m(t)\|_{L^\alpha}^\alpha + \|u_{mx}(t)\|_{L^\beta}^\beta\right]\\
&\leq 2\hat{d}_2 + 2\hat{d}_2\left[\|\tilde{u}_{0m}\|_{L^\alpha}^\alpha + \alpha\int_0^t\left(\sqrt{\bar{S}_m(s)}\right)^\alpha ds\right] \qquad (4.14)\\
&\quad + 2\hat{d}_2\left[\|\tilde{u}_{0mx}\|_{L^\beta}^\beta + \gamma\bar{S}_m(t) + \frac{1}{4\gamma}\beta^2 2^{\beta-1}\int_0^t\left(\bar{S}_m(s)\right)^{\beta-1}ds\right]\\
&= 2\hat{d}_2 + 2\hat{d}_2\left(\|\tilde{u}_{0m}\|_{L^\alpha}^\alpha + \|\tilde{u}_{0mx}\|_{L^\beta}^\beta\right) + 2\hat{d}_2\gamma\bar{S}_m(t)\\
&\quad + 2\hat{d}_2\alpha\int_0^t\left(\sqrt{\bar{S}_m(s)}\right)^\alpha ds + \frac{\hat{d}_2}{4\gamma}\beta^2 2^\beta\int_0^t\left(\bar{S}_m(s)\right)^{\beta-1}ds.
\end{aligned}$$

By combining (4.8), (4.10) and (4.14), and choosing $\gamma = \dfrac{\bar{\mu}_*}{2\left(1+2\hat{d}_2\right)}$, we get

$$\bar{S}_m(t) \leq \bar{S}_{0m} + \int_0^t\Psi\left(\bar{S}_m(s)\right)ds,\ 0\leq t\leq T_m, \tag{4.15}$$

where

$$\begin{aligned}
\bar{S}_{0m} &= \frac{2}{\bar{\mu}_*}\left[S_m(0) - 2\int_0^1\mathcal{F}(\tilde{u}_{0m}(x),\tilde{u}_{0mx}(x))dx\right.\\
&\qquad\left. +2\hat{d}_2\left(1+\|\tilde{u}_{0m}\|_{L^\alpha}^\alpha + \|\tilde{u}_{0mx}\|_{L^\beta}^\beta\right)\right],\\
\Psi(S) &= \frac{4}{\bar{\mu}_*}\left(|g(0)| + \sqrt{T^*}\,\|g'\|_{L^2(0,T^*)} + \frac{\left(1+2\hat{d}_2\right)}{\bar{\mu}_*}\|g\|_{L^2(0,T^*)}^2\right)S \qquad (4.16)\\
&\quad + \frac{2\hat{d}_2}{\bar{\mu}_*}\left[2\alpha S^{\frac{\alpha}{2}} + \frac{\left(1+2\hat{d}_2\right)\beta^2 2^{\beta-1}}{\bar{\mu}_*}S^{\beta-1}\right].
\end{aligned}$$

By the convergences given in (4.4), there exists a positive constant M_1 independent of m, such that

$$\bar{S}_{0m} \leq M_1,\ \text{for all } m\in\mathbb{N}. \tag{4.17}$$

It follows from (4.15) and (4.17) that

$$\bar{S}_m(t) \leq M_1 + \int_0^t\Psi\left(\bar{S}_m(s)\right)ds,\ 0\leq t\leq T_m. \tag{4.18}$$

Then, by solving the nonlinear Volterra integral equation (4.18) (based on the methods in [10]), we get the following lemma.

Lemma 4.1. *There exists a positive constant T depending on T^* (independent of m) such that*

$$\bar{S}_m(t) \leq C_T,\ \forall m \in \mathbb{N},\ \forall t \in [0,T], \tag{4.19}$$

where C_T is a constant depending on T only.

Lemma 4.1 allows us to take $T_m = T$ for all m.

Next, we shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Put $w_{m,\,l} = u_m - u_l$, $\tilde{u}_{0m,l} = \tilde{u}_{0m} - \tilde{u}_{0l}$, $\tilde{u}_{1m,l} = \tilde{u}_{1m} - \tilde{u}_{1l}$. It follows from (4.6) that

$$\begin{cases} \quad \left\langle w''_{m,l}(t), v\right\rangle + \lambda \left\langle w'_{m,l}(t), v\right\rangle + \left\langle w'_{m,l\;x}(t), v_x\right\rangle + \langle \Delta w_{m,l}(t), \Delta v\rangle \\ \quad +\mu\left(\|u_{mx}(t)\|^2\right)\langle w_{m,l\;x}(t), v_x\rangle \\ = \left[\mu\left(\|u_{mx}(t)\|^2\right) - \mu\left(\|u_{lx}(t)\|^2\right)\right]\langle \Delta u_l(t), v\rangle \\ \quad + \displaystyle\int_0^t g(t-s)\langle w_{m,l\;x}(s), v_x\rangle\, ds + \langle f(u_m(t), u_{mx}(t)) - f(u_l(t), u_{lx}(t)), v\rangle \\ \quad - \langle \mu_1(u_m(t), u_{mx}(t)) - \mu_1(u_l(t), u_{lx}(t)), v_x\rangle\,, \text{ for all } v \in H^2 \cap H_0^1, \\ w_{m,\,l}(0) = \tilde{u}_{0m,l}, \ \ w'_{m,l}(0) = \tilde{u}_{1m,l}. \end{cases} \tag{4.20}$$

Taking $v = w'_{m,l} = u'_m - u'_l$ in (4.20) and then integrating with respect to t, we obtain

$$\begin{aligned} \lambda_* \bar{S}_{m,l}(t) &\leq S_{m,l}(t) \\ &= S_{m,l}(0) + 2\int_0^t dr \int_0^r g(r-s)\left\langle w_{m,l\;x}(s), w'_{m,l\;x}(r)\right\rangle ds \\ &\quad - 2\int_0^t \mu\left(\|u_{mx}(s)\|^2\right)\left\langle w_{m,l\;x}(s), w'_{m,l\;x}(s)\right\rangle ds \\ &\quad + 2\int_0^t \left[\mu\left(\|u_{mx}(s)\|^2\right) - \mu\left(\|u_{lx}(s)\|^2\right)\right]\left\langle \Delta u_l(s), w'_{m,l}(s)\right\rangle ds \\ &\quad + 2\int_0^t \left\langle f(u_m(s), u_{mx}(s)) - f(u_l(s), u_{lx}(s)), w'_{m,l}(s)\right\rangle ds \\ &\quad - 2\int_0^t \left\langle \mu_1(u_m(s), u_{mx}(s)) - \mu_1(u_l(s), u_{lx}(s)), w'_{m,l\;x}(s)\right\rangle ds \\ &= S_{m,l}(0) + \sum_{j=1}^{5} \bar{I}_j, \end{aligned} \tag{4.21}$$

where $\lambda_* = \min\{1, 2\lambda\}$, and

$$S_{m,l}(t) = \left\|w'_{m,l}(t)\right\|^2 + \|\Delta w_{m,l}(t)\|^2 + 2\lambda\int_0^t \left\|w'_{m,l}(s)\right\|^2 ds + 2\int_0^t \left\|w'_{m,l\;x}(s)\right\|^2 ds, \tag{4.22}$$

$$\bar{S}_{m,l}(t) = \left\|w'_{m,l}(t)\right\|^2 + \|\Delta w_{m,l}(t)\|^2 + \int_0^t \left\|w'_{m,l}(s)\right\|^2 ds + \int_0^t \left\|w'_{m,l\;x}(s)\right\|^2 ds,$$

$S_{m,l}(0) = \|u_{1m} - u_{1l}\|^2 + \|\Delta u_{0m} - \Delta u_{0l}\|^2 .$

We will evaluate the terms on the right-hand side of (4.21) as follows.
Put

$$\begin{aligned} \mu_{\max} &= \sup_{0\le z\le C_T} \mu(z), \\ L_T(\mu) &= \sup_{0\le z\le C_T} \mu'(z), \\ L_T(f) &= \sup_{0\le y\le\sqrt{C_T},\ 0\le z\le 2\sqrt{C_T}} \left(|D_1 f(y,z)| + |D_2 f(y,z)|\right), \\ L_T(\mu_1) &= \sup_{0\le y\le\sqrt{C_T},\ 0\le z\le 2\sqrt{C_T}} \left(|D_1 \mu_1(y,z)| + |D_2 \mu_1(y,z)|\right). \end{aligned} \tag{4.23}$$

Using the following inequalities $\|w_{m,l\ x}(t)\|^2 \le \|\Delta w_{m,l}(t)\|^2$, and

$$2ab \le \gamma a^2 + \frac{1}{\gamma} b^2,\ \forall a, b \in \mathbb{R},\ \forall \gamma > 0, \tag{4.24}$$

we get estimates of $\bar{I}_1 - \bar{I}_5$, as follows.

The term $\bar{I}_1$.

$$\begin{aligned} \bar{I}_1 &= 2\int_0^t dr \int_0^r g(r-s)\left\langle w_{m,l\ x}(s), w'_{m,l\ x}(r)\right\rangle ds \\ &\le 2\int_0^t \left\|w'_{m,l\ x}(r)\right\| dr \int_0^r |g(r-s)|\, \|w_{m,l\ x}(s)\|\, ds \\ &\le 2\int_0^t \left\|w'_{m,l\ x}(r)\right\| dr \int_0^r |g(r-s)| \sqrt{\bar{S}_{m,l}(s)} ds \\ &\le 2\left[\int_0^t \left\|w'_{m,l\ x}(r)\right\|^2 dr\right]^{1/2} \left[\int_0^t \left(\int_0^r |g(r-s)|\sqrt{\bar{S}_{m,l}(s)} ds\right)^2 dr\right]^{1/2} \\ &\le 2\sqrt{\bar{S}_{m,l}(t)}\sqrt{T^*}\, \|g\|_{L^2(0,T^*)} \left[\int_0^t \bar{S}_{m,l}(s) ds\right]^{1/2} \\ &\le \gamma \bar{S}_{m,l}(t) + \frac{1}{\gamma} T^* \|g\|^2_{L^2(0,T^*)} \int_0^t \bar{S}_{m,l}(s) ds. \end{aligned} \tag{4.25}$$

The term $\bar{I}_2$.

$$\begin{aligned} \bar{I}_2 &= -2\int_0^t \mu\left(\|u_{mx}(s)\|^2\right)\left\langle w_{m,l\ x}(s), w'_{m,l\ x}(s)\right\rangle ds \\ &\le 2\mu_{\max}\int_0^t \|\Delta w_{m,l}(s)\| \left\|w'_{m,l\ x}(s)\right\| ds \\ &\le \gamma \int_0^t \left\|w'_{m,l\ x}(s)\right\|^2 ds + \frac{1}{\gamma}\mu^2_{\max}\int_0^t \|\Delta w_{m,l}(s)\|^2 ds \\ &\le \gamma \bar{S}_{m,l}(t) + \frac{1}{\gamma}\mu^2_{\max}\int_0^t \bar{S}_{m,l}(s) ds. \end{aligned} \tag{4.26}$$

The term $\bar{I}_3$. From the below inequality

$$\left|\mu\left(\|u_{mx}(s)\|^2\right) - \mu\left(\|u_{lx}(s)\|^2\right)\right|$$

$$
\begin{aligned}
&\leq L_T(\mu)\left|\|u_{mx}(s)\|^2 - \|u_{lx}(s)\|^2\right| \\
&\leq 2\sqrt{C_T}L_T(\mu)\,\|w_{m,l\;x}(s)\| \\
&\leq 2\sqrt{C_T}L_T(\mu)\,\|\Delta w_{m,l}(s)\|\,,
\end{aligned}
$$

we infer that

$$
\begin{aligned}
\bar{I}_3 &= 2\int_0^t \left[\mu\left(\|u_{mx}(s)\|^2\right) - \mu\left(\|u_{lx}(s)\|^2\right)\right]\left\langle \Delta u_l(s), w'_{m,l}(s)\right\rangle ds \\
&\leq 4C_T L_T(\mu)\int_0^t \|\Delta w_{m,l}(s)\|\left\|w'_{m,l}(s)\right\| ds \\
&\leq 2C_T L_T(\mu)\int_0^t \bar{S}_{m,l}(s)ds.
\end{aligned} \tag{4.27}
$$

The term $\bar{I}_4$. From the below inequality

$$
\begin{aligned}
&\|f(u_m(s), u_{mx}(s)) - f(u_l(s), u_{lx}(s))\| \\
\leq& L_T(f)\left[\|w_{m,l}(s)\| + \|w_{m,l\;x}(s)\|\right] \\
\leq& 2L_T(f)\,\|w_{m,l\;x}(s)\| \\
\leq& 2L_T(f)\,\|\Delta w_{m,l}(s)\|\,;
\end{aligned}
$$

we have

$$
\begin{aligned}
\bar{I}_4 &= 2\int_0^t \left\langle f(u_m(s), u_{mx}(s)) - f(u_l(s), u_{lx}(s)), w'_{m,l}(s)\right\rangle ds \\
&\leq 4L_T(f)\int_0^t \|\Delta w_{m,l}(s)\|\left\|w'_{m,l}(s)\right\| ds \\
&\leq 2L_T(f)\int_0^t \bar{S}_{m,l}(s)ds.
\end{aligned} \tag{4.28}
$$

The term $\bar{I}_5$. Similarly, from the following inequality

$$
\begin{aligned}
\|\mu_1(u_m(s), u_{mx}(s)) - \mu_1(u_l(s), u_{lx}(s))\| &\leq 2L_T(\mu_1)\,\|w_{m,l\;x}(s)\| \\
&\leq 2L_T(\mu_1)\,\|\Delta w_{m,l}(s)\|\,,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\bar{I}_5 &= -2\int_0^t \left\langle \mu_1(u_m(s), u_{mx}(s)) - \mu_1(u_l(s), u_{lx}(s)), w'_{m,l\;x}(s)\right\rangle ds \\
&\leq \gamma\int_0^t \left\|w'_{m,l\;x}(s)\right\|^2 ds + \frac{1}{\gamma}\int_0^t \|\mu_1(u_m(s), u_{mx}(s)) - \mu_1(u_l(s), u_{lx}(s))\|^2 ds \\
&\leq \gamma\int_0^t \left\|w'_{m,l\;x}(s)\right\|^2 ds + \frac{4}{\gamma}L_T^2(\mu_1)\int_0^t \|\Delta w_{m,l}(s)\|^2 ds \\
&\leq \gamma\bar{S}_{m,l}(t) + \frac{4}{\gamma}L_T^2(\mu_1)\int_0^t \bar{S}_{m,l}(s)ds.
\end{aligned} \tag{4.29}
$$

By combining the estimates (4.21), (4.25)-(4.29) and choosing $\gamma = \dfrac{\lambda_*}{6}$, we obtain

$$
\bar{S}_{m,l}(t) \leq R_{m,l} + D_T\int_0^t \bar{S}_{m,l}(s)ds, \tag{4.30}
$$

where

$$R_{m,l} = \frac{2}{\lambda_*} S_{m,l}(0) = \frac{2}{\lambda_*} \left(\|u_{1m} - u_{1l}\|^2 + \|\Delta u_{0m} - \Delta u_{0l}\|^2 \right), \tag{4.31}$$

$$D_T = \frac{4}{\lambda_*} \left[\frac{3}{\lambda_*} \left(T^* \|g\|^2_{L^2(0,T^*)} + \mu^2_{\max} + 4L^2_T(\mu_1) \right) + C_T L_T(\mu) + L_T(f) \right].$$

By Gronwall's lemma, we deduce from (4.30), that

$$\bar{S}_{m,l}(t) \le R_{m,l} \exp(TD_T) = \bar{R}^2_{m,l}, \ \forall t \in [0,T], \ \forall m, \ l \in \mathbb{N}. \tag{4.32}$$

Hence

$$\begin{aligned} \|w_{m,l}\|_{W_1(T)} &= \|\Delta w_{m,l}\|_{C^0([0,T];L^2)} + \left\|w'_{m,l}\right\|_{C^0([0,T];L^2)} + \left\|w'_{m,l}\right\|_{L^2(0,T;H^1_0)} \\ &\le 3\bar{R}_{m,l} \to 0, \ \ as\ m, \ l \to \infty. \end{aligned} \tag{4.33}$$

This confirms that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \text{ strongly in } W_1(T). \tag{4.34}$$

On the other hand, by (4.19), there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty\left(0,T; H^2 \cap H^1_0\right) \text{ weak*}, \\ u'_{m_j} \to u' & \text{in} \quad L^\infty\left(0,T; L^2\right) \text{ weak*}, \\ u'_{m_j} \to u' & \text{in} \quad L^2\left(0,T; H^1_0\right) \text{ weak}. \end{cases} \tag{4.35}$$

Moreover, by (4.23), we deduce that

$$\begin{aligned} &\|f(u_m, u_{mx}) - f(u,u_x)\|_{C([0,T];L^2)} \le 2L_f(C_T)\|u_m - u\|_{W_1(T)}, \\ &\|\mu_1(u_m, u_{mx}) - \mu_1(u,u_x)\|_{C([0,T];L^2)} \le 2L_{\mu_1}(C_T)\|u_m - u\|_{W_1(T)}, \\ &\left\|\mu\left(\|u_{mx}\|^2\right) - \mu\left(\|u_x\|^2\right)\right\|_{C([0,T])} \le 2\sqrt{C_T} L_\mu(C_T)\|u_m - u\|_{W_1(T)}. \end{aligned} \tag{4.36}$$

It follows from that (4.34), (4.36) that

$$\begin{cases} f(u_m, u_{mx}) - f(u,u_x) & \text{in} \quad C([0,T];L^2) \text{ strongly}, \\ \mu_1(u_m, u_{mx}) - \mu_1(u,u_x) & \text{in} \quad C([0,T];L^2) \text{ strongly}, \\ \mu\left(\|u_{mx}\|^2\right) - \mu\left(\|u_x\|^2\right) & \text{in} \quad C([0,T]) \text{ strongly}. \end{cases} \tag{4.37}$$

By using the convergences (4.34), (4.35) and (4.37) to pass the limitations in (4.6), we have $u \in W_1(T)$ satisfying the problem

$$\begin{cases} \quad \dfrac{d}{dt}\langle u'(t), v\rangle + \lambda\langle u'(t), v\rangle + \langle u'_x(t), v_x\rangle + \langle \Delta u(t), \Delta v\rangle \\ \quad +\mu\left(\|u_x(t)\|^2\right)\langle u_x(t), v_x\rangle \\ = \displaystyle\int_0^t g(t-s)\langle u_x(s), v_x\rangle\, ds + \langle f(u(t), u_x(t)), v\rangle \\ \quad -\langle \mu_1(u(t), u_x(t)), v_x\rangle, \text{ for all } v \in H^2 \cap H^1_0, \\ u(0) = \tilde{u}_0, \ \ u'(0) = \tilde{u}_1. \end{cases} \tag{4.38}$$

Finally, the uniqueness of a weak solution is obtained by using the well-known regularization procedure due to Lions, see Ngoc et. al. [20] as an example.

Theorem 4.2 is proved completely. □

In order to study of finite-time blow up of the solution of the problem (4.1), we make the following assumptions

(B_2) $\mu \in C^1(\mathbb{R}_+)$ and there are two positive constants μ_*, $\bar{\chi}_*$ such that

(i) $\mu(y) \geq \mu_* > 0$, for all $y \geq 0$,

(ii) $y\mu(y) \leq \bar{\chi}_* \int_0^y \mu(z)dz$, for all $y \geq 0$;

(B_3) $g \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ such that

(i) $0 < \bar{g}(\infty) \equiv \int_0^\infty g(s)\,ds < 1$,

(ii) $g'(t) \leq 0 < g(t)$, for all $t \geq 0$;

(B_4) There is a function $\mathcal{F} \in C^3(\mathbb{R}^2;\mathbb{R})$ and positive constants d_1, $\bar{d}_1$, $\bar{d}_2$, α, $\bar{\alpha}$, $\bar{\beta}$, with α, $\bar{\alpha}$, $\bar{\beta} > 2$, $i = 1, \cdots, N$, such that

(i) $\frac{\partial \mathcal{F}}{\partial u}(u,v) = f(u,v)$, $\frac{\partial \mathcal{F}}{\partial v}(u,v) = -\mu_1(u,v)$, for all $(u,v) \in \mathbb{R}^2$,

(ii) $uf(u,v) - v\mu_1(u,v) \geq d_1\mathcal{F}(u,v)$, for all $(u,v) \in \mathbb{R}^2$;

(iii) $\bar{d}_1 |v|^\alpha \leq \mathcal{F}(u,v) \leq \bar{d}_2\left(1 + |u|^{\bar{\alpha}} + |v|^{\bar{\beta}}\right)$, for all $(u,v) \in \mathbb{R}^2$;

(B_5) $2\bar{\chi}_* < p < d_1$; $0 < \bar{g}(\infty) < \frac{p(p-2\bar{\chi}_*)\mu_*}{(p-1)^2}$.

Remark 4.1. The example below shows that there are two functions $f(u,v)$, $\mu_1(u,v)$ satisfying (B_4). Indeed, we consider

$$f(u,v) = \beta k_2 |u|^{\beta-2} u |v|^\gamma,$$
$$\mu_1(u,v) = -\alpha k_1 |v|^{\alpha-2} v - \gamma k_2 |u|^\beta |v|^{\gamma-2} v,$$

where k_1, $k_2 > 0$, α, β, $\gamma > 3$ are constants and $\min\{\alpha, \beta+\gamma\} > p$.

It is clear that

$$\mathcal{F}(u,v) = k_1 |v|^\alpha + k_2 |u|^\beta |v|^\gamma,$$

is a $C^3(\mathbb{R}^2;\mathbb{R})$ function and satisfies

(i) $\frac{\partial \mathcal{F}}{\partial u}(u,v) = f(u,v)$, $\frac{\partial \mathcal{F}}{\partial v}(u,v) = -\mu_1(u,v)$, $\forall (u,v) \in \mathbb{R}^2$,

(ii) $uf_1(u,v) - v\mu_1(u,v) \geq d_1\mathcal{F}(u,v)$, $\forall (u,v) \in \mathbb{R}^2$,

(iii) $\bar{d}_1 |v|^\alpha \leq \mathcal{F}(u,v) \leq \bar{d}_2\left(1 + |u|^{\bar{\alpha}} + |v|^{\bar{\beta}}\right)$,

where $d_1 = \min\{\alpha, \beta+\gamma\} > p$, $\bar{d}_1 = k_1$, $\bar{d}_2 = k_1 + \frac{k_2}{2}$, $\bar{\alpha} = 2\beta$, $\bar{\beta} = \max\{\alpha, 2\gamma\}$. So (B_4) is satisfied.

Putting

$$H(0) = \int_0^1 \mathcal{F}(\tilde{u}_0(x), \tilde{u}_{0x}(x))dx - \frac{1}{2}\|\tilde{u}_1\|^2 - \frac{1}{2}\|\Delta\tilde{u}_0\|^2 - \frac{1}{2}\int_0^{\|\tilde{u}_{0x}\|^2} \mu(z)dz. \quad (4.39)$$

Theorem 4.3. *Let $(B_2)-(B_5)$ hold, then for any $(\tilde{u}_0, \tilde{u}_1) \in \left(H^2 \cap H_0^1\right) \times L^2$ such that $H(0) > 0,$ the weak solution u of (4.1) blows up in finite time, i.e., there is a positive constant T_∞ such that*

$$\lim_{t \to T_\infty^-} \left(\|u(t)\|_{H^2 \cap H_0^1} + \|u'(t)\| \right) = +\infty.$$

Proof. First, we prove that

$$\text{the problem (4.1) does not admit any global weak solutions.} \tag{4.40}$$

Arguing by contradiction, we assume that the problem (4.1) admits a global weak solution

$$u \in C^0(\mathbb{R}_+; H^2 \cap H_0^1) \cap C^1(\mathbb{R}_+; L^2),\ u' \in L^2_{loc}(0, \infty; H_0^1). \tag{4.41}$$

We define the energy functional associated with (4.1) by

$$\begin{aligned} E(t) = {} & \frac{1}{2} \|u'(t)\|^2 + \frac{1}{2}(g * u)(t) \\ & + \frac{1}{2} \left[\|\Delta u(t)\|^2 + \int_0^{\|u_x(t)\|^2} \mu(z) dz - \bar{g}(t) \|u_x(t)\|^2 \right] \\ & - \int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right) dx. \end{aligned} \tag{4.42}$$

Put $H(t) = -E(t)$, $\forall t \geq 0$. Multiplying both sides of $(4.1)_1$ by $u'(x,t)$ and then integrating the obtained equation, we have

$$H'(t) = \|u_x'(t)\| + \lambda \|u'(t)\|^2 - \frac{1}{2}\left(g' * u\right)(t) + \frac{1}{2} g(t) \|u_x(t)\|^2 \geq 0. \tag{4.43}$$

This implies that

$$0 < H(0) \leq H(t),\ \forall t \geq 0, \tag{4.44}$$

so we deduce from (4.42) that

$$\begin{aligned} & \|u'(t)\|^2 + (g * u)(t) + \|\Delta u(t)\|^2 + \int_0^{\|u_x(t)\|^2} \mu(z) dz - \bar{g}(t)) \|u_x(t)\|^2 \\ & \leq 2 \int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right) dx,\ \forall t \geq 0. \end{aligned} \tag{4.45}$$

Now, we define

$$L(t) = H^{1-\eta}(t) + \varepsilon \psi(t), \tag{4.46}$$

where

$$\psi(t) = \langle u'(t), u(t) \rangle + \frac{\lambda}{2} \|u(t)\|^2 + \frac{1}{2} \|u_x(t)\|^2, \tag{4.47}$$

for ε small enough and

$$0 < \eta < \frac{1}{2},\ 2/(1-2\eta) \leq \alpha. \tag{4.48}$$

In what follows, we show that there is a constant $\zeta_1 > 0$ such that

$$L'(t) \geq \zeta_1 \left[H(t) + \|u'(t)\|^2 + \|u_x(t)\|^2 + \|\Delta u(t)\|^2 + \|u_x(t)\|^\alpha_{L^\alpha} \right]. \tag{4.49}$$

Multiplying both sides of $(4.1)_1$ by $u(x,t)$ and integrating over $[0,1]$, it leads to

$$\begin{aligned}\psi'(t) &= \|u'(t)\|^2 - \|\Delta u(t)\|^2 - \|u_x(t)\|^2 \mu\left(\|u_x(t)\|^2\right) \\ &\quad + \int_0^t g(t-s)\langle u_x(s), u_x(t)\rangle ds + \langle f(u(t), u_x(t)), u(t)\rangle \\ &\quad - \langle \mu_1(u(t), u_x(t)), u_x(t)\rangle .\end{aligned} \tag{4.50}$$

Therefore

$$L'(t) = (1-\eta)H^{-\eta}(t)H'(t) + \varepsilon\psi'(t) \geq \varepsilon\psi'(t). \tag{4.51}$$

By $\left(\bar{A}_{b4}\right)$, we obtain

$$\begin{cases} \langle f(u(t), u_x(t)), u(t)\rangle - \langle \mu_1(u(t), u_x(t)), u_x(t)\rangle \geq d_1 \displaystyle\int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right) dx, \\ \displaystyle\int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right) dx \geq \bar{d}_1 \|u_x(t)\|_{L^\alpha}^\alpha . \end{cases} \tag{4.52}$$

On the other hand, by (B_2), (B_3), we get

$$- \|u_x(t)\|^2 \mu\left(\|u_x(t)\|^2\right) \geq -\bar{\chi}_* \int_0^{\|u_x(t)\|^2} \mu(z)dz, \tag{4.53}$$

$$\left(\frac{p}{2} - \bar{\chi}_*\right) \int_0^{\|u_x(t)\|^2} \mu(z)dz \geq \left(\frac{p}{2} - \bar{\chi}_*\right) \mu_* \|u_x(t)\|^2 ,$$

$$\int_0^t g(t-s)\langle u_x(s), u_x(t)\rangle ds \geq -\frac{p}{2}(g*u)(t) + \left(1 - \frac{1}{2p}\right) \bar{g}(t) \|u_x(t)\|^2 . \tag{4.54}$$

It implies from (4.50), $(4.53)_1$ and (4.54) that

$$\begin{aligned}\psi'(t) &\geq \|u'(t)\|^2 - \|\Delta u(t)\|^2 - \bar{\chi}_* \int_0^{\|u_x(t)\|^2} \mu(z)dz - \frac{p}{2}(g*u)(t) \\ &\quad + \left(1 - \frac{1}{2p}\right) \bar{g}(t) \|u_x(t)\|^2 + \delta_1 d_1 \int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right) dx \\ &\quad + (1-\delta_1)d_1 \left[\int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right) dx\right] \qquad (4.55) \\ &= \|u'(t)\|^2 - \|\Delta u(t)\|^2 - \bar{\chi}_* \int_0^{\|u_x(t)\|^2} \mu(z)dz - \frac{p}{2}(g*u)(t) \\ &\quad + \left(1 - \frac{1}{2p}\right) \bar{g}(t) \|u_x(t)\|^2 + \delta_1 d_1 \int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right) dx \\ &\quad + (1-\delta_1)d_1 H(t) + \frac{1}{2}(1-\delta_1)d_1 \|u'(t)\|^2 + \frac{1}{2}(1-\delta_1)d_1(g*u)(t) \\ &\quad + \frac{1}{2}(1-\delta_1)d_1 \left[\|\Delta u(t)\|^2 + \int_0^{\|u_x(t)\|^2} \mu(z)dz - \bar{g}(t)) \|u_x(t)\|^2\right] \\ &= \left[1 + \frac{1}{2}(1-\delta_1)d_1\right] \|u'(t)\|^2 + \delta_1 d_1 \int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right) dx \\ &\quad + (1-\delta_1)d_1 H(t) + \frac{1}{2}\left[(1-\delta_1)d_1 - p\right](g*u)(t)\end{aligned}$$

$$+\left(\frac{1}{2}(1-\delta_1)d_1-\bar{\chi}_*\right)\int_0^{\|u_x(t)\|^2}\mu(z)dz+\left(\frac{1}{2}(1-\delta_1)d_1-1\right)\|\Delta u(t)\|^2$$
$$+\left[1-\frac{1}{2p}-\frac{1}{2}(1-\delta_1)d_1\right]\bar{g}(t))\|u_x(t)\|^2.$$

Because of $d_1>p$, hence we can choose $\delta_1\in(0,1)$ such that $(1-\delta_1)d_1=p$, then it follows from $(4.52)_2$, $(4.53)_2$, (4.55) that

$$\begin{aligned}\psi'(t)\geq&\left(1+\frac{p}{2}\right)\|u'(t)\|^2+\delta_1d_1\bar{d}_1\|u_x(t)\|^\alpha_{L^\alpha}+pH(t)+\left(\frac{p}{2}-1\right)\|\Delta u(t)\|^2\\&+\left(\frac{p}{2}-\bar{\chi}_*\right)\int_0^{\|u_x(t)\|^2}\mu(z)dz+\left(1-\frac{1}{2p}-\frac{p}{2}\right)\bar{g}(t))\|u_x(t)\|^2\\\geq&\left(1+\frac{p}{2}\right)\|u'(t)\|^2+\delta_1d_1\bar{d}_1\|u_x(t)\|^\alpha_{L^\alpha}+pH(t)+\left(\frac{p}{2}-1\right)\|\Delta u(t)\|^2\\&+\left(\frac{p}{2}-\bar{\chi}_*\right)\mu_*\|u_x(t)\|^2-\frac{(p-1)^2}{2p}\bar{g}(t)\|u_x(t)\|^2\\\geq&\left(1+\frac{p}{2}\right)\|u'(t)\|^2+\delta_1d_1\bar{d}_1\|u_x(t)\|^\alpha_{L^\alpha}+pH(t)+\left(\frac{p}{2}-1\right)\|\Delta u(t)\|^2\\&+\frac{(p-1)^2}{2p}\left[\frac{p\,(p-2\bar{\chi}_*)\,\mu_*}{(p-1)^2}-\bar{g}(\infty)\right]\|u_x(t)\|^2.\end{aligned}\tag{4.56}$$

By $0<\bar{g}(\infty)<\dfrac{p\,(p-2\bar{\chi}_*)\,\mu_*}{(p-1)^2}$, we deduce from (4.51) and (4.56) that there is a constant $\zeta_1>0$ such that (4.49) holds.

From the formula of $L(t)$ and (4.49), we can choose $\varepsilon>0$ small enough such that

$$L(t)\geq L(0)>0,\ \forall t\geq 0.\tag{4.57}$$

Using the inequality

$$\left(\sum_{i=1}^4 x_i\right)^r\leq 4^{r-1}\sum_{i=1}^4 x_i^r,\ \text{for all } r>1,\ \text{and } x_1,\cdots,x_4\geq 0,\tag{4.58}$$

we deduce from (4.46) and (4.47) that

$$\begin{aligned}&L^{1/(1-\eta)}(t)\\\leq&Const\ \left[H(t)+|\langle u(t),u'(t)\rangle|^{1/(1-\eta)}+\|u(t)\|^{2/(1-\eta)}+\|u_x(t)\|^{2/(1-\eta)}\right].\end{aligned}\tag{4.59}$$

Using Young's inequality, we have

$$\begin{aligned}|\langle u(t),u'(t)\rangle|^{1/(1-\eta)}&\leq\|u(t)\|^{1/(1-\eta)}\|u'(t)\|^{1/(1-\eta)}\\&\leq\frac{1-2\eta}{2(1-\eta)}\|u(t)\|^s+\frac{1}{2(1-\eta)}\|u'(t)\|^2\\&\leq Const\left(\|u(t)\|^s+\|u'(t)\|^2\right),\end{aligned}\tag{4.60}$$

where $s=2/(1-2\eta)\leq\alpha$ as in (4.48).

Combining (4.59) and (4.60), it implies that

$$L^{1/(1-\eta)}(t)\leq Const\ \left[H(t)+\|u'(t)\|^2+\|u(t)\|^s+\|u(t)\|^{2/(1-\eta)}+\|u_x(t)\|^{2/(1-\eta)}\right].\tag{4.61}$$

We need the following lemma.

Lemma 4.2. *Let $2 \leq r_1$, r_2, $r_3 \leq \alpha$, then we have*

$$\|v\|^{r_1} + \|v\|^{r_2} + \|v_x\|^{r_3} \leq 3\left(\|v_x\|^2 + \|v_x\|_{L^\alpha}^\alpha\right), \ \forall v \in H_0^1.$$

The proof of Lemma 4.2 is not difficult, so we omit the details.

Using (4.61) and Lemma 4.2 with $r_2 = s = 2/(1-2\eta)$, $r_2 = r_3 = 2/(1-\eta)$, we obtain

$$\begin{aligned} &L^{1/(1-\eta)}(t) \\ &\leq Const\left[H(t) + \|u'(t)\|^2 + \|u_x(t)\|^2 + \|u_x(t)\|_{L^\alpha}^\alpha + \|\Delta u(t)\|^2\right], \quad \forall t \geq 0. \end{aligned} \tag{4.62}$$

It follows from (4.49) and (4.62) that

$$L'(t) \geq \zeta_2 L^{1/(1-\eta)}(t), \ \forall t \geq 0, \tag{4.63}$$

where ζ_2 is a positive constant.

Integrating (4.63) over $(0, t)$, it leads to

$$L^{\eta/(1-\eta)}(t) \geq \frac{1-\eta}{\zeta_2 \eta}\frac{1}{T_* - t}, \ 0 \leq t < T_* = \frac{1-\eta}{\zeta_2\eta}L^{-\eta/(1-\eta)}(0). \tag{4.64}$$

The estimate (4.64) shows that

$$\lim_{t \to T_*^-} L(t) = +\infty. \tag{4.65}$$

On the other hand, by (4.46) and (4.47), we have

$$0 < L(t) \leq 1 + H(t) + \varepsilon\left(\|u'(t)\|\,\|u(t)\| + \frac{\lambda}{2}\|u(t)\|^2 + \frac{1}{2}\|u_x(t)\|^2\right). \tag{4.66}$$

However, because of the fact that $u \in C^0([0, T_*]; H^2 \cap H_0^1) \cap C^1([0, T_*]; L^2)$, it implies that the function $t \longmapsto H(t) + \varepsilon\left(\|u'(t)\|\,\|u(t)\| + \frac{\lambda}{2}\|u(t)\|^2 + \frac{1}{2}\|u_x(t)\|^2\right)$ is continuous. Hence, the right-hand side of (4.66) is bounded. This is contrary to (4.65). Thus, (4.40) holds.

Next, we put

$$T_\infty = \sup\{T > 0 : (4.1) \text{ admits a unique solution } u \in W_1(T)\}. \tag{4.67}$$

By (4.40), we have $T_\infty < +\infty$. We now prove that

$$\lim_{t \to T_\infty^-}\left(\|u(t)\|_{H^2 \cap H_0^1} + \|u'(t)\|\right) = +\infty. \tag{4.68}$$

Indeed, assume that (4.68) is not true, there exist a constant $M > 0$ and an increasing sequence $\{T_m\} \subset (0, T_\infty)$, $T_m \to T_\infty$ such that

$$\|u(T_m)\|_{H^2 \cap H_0^1} + \|u'(T_m)\| \leq M, \ \forall m \in \mathbb{N}.$$

Then, for each $m \in \mathbb{N}$, the problem (4.1) has a unique weak solution

$$u_* \in W_1(T_m, T_m + \sigma) = \{v \in C^0([T_m, T_m + \sigma]; H^2 \cap H_0^1) \cap C^1([T_m, T_m + \sigma]; L^2) : \\ v' \in L^2(T_m, T_m + \sigma; H_0^1)\},$$

corresponding to the initial data

$$(u_*(T_m),\ u'_*(T_m)) = (u(T_m),\ u'(T_m)),$$

with $\sigma > 0$ independent of $m \in \mathbb{N}$.

By the fact $T_m \to T_\infty$ and the definition of T_∞, we obtain $T_m + \sigma > T_\infty$ for $m \in \mathbb{N}$ sufficiently large. Then the function $\tilde{u}$ defined by

$$\tilde{u}(t) = \begin{cases} u(t), & 0 \le t \le T_m, \\ u_*(t), & T_m \le t \le T_m + \sigma, \end{cases}$$

is a weak solution of (4.1) on $[0, T_m + \sigma]$. This is contrary to the fact that the interval $[0, T_\infty]$ is maximal interval on which the equation (4.1) admits a weak solution. Thus, (4.68) holds. Theorem 4.3 is proved. □

Remark 4.2. In some previous papers, the results of the finite time blowup for negative initial energy were derived in case of that the nonlinearity is less complex, for example such as in [4] and [30] with the nonlinearity $f(u_x)_x$. Recently, with the same nonlinearity, the authors in [9] and [27] have proved that the blow-up property occurs in finite time for arbitrary positive initial energy and suitable initial data. In our paper, the nonlinear quantities in the first equation of the problem (4.1) are considered with more complicated forms, precisely that are $-\frac{\partial}{\partial x}\left[\mu\left(\|u_x(t)\|^2\right)u_x + \mu_1(u, u_x)\right]$ and $f(u, u_x)$. Therefore, it is very difficult to construct sufficient conditions for which the finite-time blow up for positive initial energy of the solutions for the problem (4.1) is established, and hence this is still open.

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