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THE BEST MATCHING PARAMETERS AND NORM CALCULATION OF BOUNDED OPERATORS WITH SUPER-HOMOGENEOUS KERNEL*

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Abstract The concept of super-homogeneous function is introduced, sufficient and necessary condition for best matching parameters of bounded operator with super-homogeneous kernel is discussed, the norm formula for mutual mapping operators between weighted Lebesgue function space and weighted normed sequence space is obtained, and some special cases are given.

Keywords Super-homogeneous kernel, bounded operator, operator norm, weighted Lebesgue space, weighted normed sequence space, best matching parameters, Hilbert-type semi-discrete inequality.

MSC(2010) 26D15, 26D10.

1. Introduction and super-homogeneous function

Let $p > 1$, $\alpha \in \mathbb{R}$. Define the weighted normed sequence space l_p^α and the weighted Lebesgue function space L_p^α by respectively

$$l_p^\alpha = \left\{ \tilde{a} = \{a_n\} : \|\tilde{a}\|_{p,\alpha} = \left(\sum_{n=1}^{\infty} n^\alpha |a_n|^p \right)^{\frac{1}{p}} < +\infty \right\},$$

$$L_p^\alpha(0, +\infty) = \left\{ f(x) : \|f\|_{p,\alpha} = \left(\int_0^{+\infty} x^\alpha |f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}} < +\infty \right\}.$$

Let $K(n, x) \geq 0$. The discrete operator

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty} K(n, x) a_n, \quad \tilde{a} = \{a_n\} \in l_r^\alpha$$

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and integral operator

$$T_2(f)_n = \int_0^{+\infty} K(n,x)f(x)\mathrm{d}x, \quad f(x) \in L_r^\alpha(0,+\infty)$$

with $K(n,x)$ as the kernel can realize mappings from sequence space to function space and from function space to sequence space.

It is a basic problem in operator theory to discuss the boundedness of operators and the calculation of operator norm. Whether an operator is bounded or not is related not only to the kernel of the operator, but also to the various parameters of the space. Whether the operator norm can be computed is a much deeper question when this operator is bounded. If the norm expression of the operator can be obtained when the operator is known to be bounded, then the relevant parameters are said to be the best matching parameters.

In 1925, [1] obtained the famous semi-discrete Hilbert inequality

$$\int_0^{+\infty} \sum_{n=1}^{\infty} \frac{a_n}{n+x} f(x)\mathrm{d}x \le \frac{\pi}{\sin(\frac{\pi}{p})} \|\tilde{a}\|_p \|f\|_q, \tag{1.1}$$

where $\frac{1}{p}+\frac{1}{q}=1\,(p>1, q>1)$, $\tilde{a}=\{a_n\}\in l_p$, $f(x)\in L_q(0,+\infty)$, and the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best value. For operators

$$T'(\tilde{a})(x) = \sum_{n=1}^{\infty} \frac{a_n}{n+x}, \quad T''(f)_n = \int_0^{+\infty} \frac{f(x)}{n+x}\mathrm{d}x,$$

since (1.1) is equivalent to the operator inequalities $\|T'(\tilde{a})\|_p \le \frac{\pi}{\sin(\frac{\pi}{p})}\|\tilde{a}\|_p$ and $\|T''(f)\|_q \le \frac{\pi}{\sin(\frac{\pi}{p})}\|f\|_q$, it follows that T' is a bounded operator from l_p to $L_p(0,+\infty)$, T'' is a bounded operator from $L_q(0,+\infty)$ to l_q, and $\|T'\| = \|T''\| = \frac{\pi}{\sin(\frac{\pi}{p})}$.

Later, the above results were generalized to solve the problems of boundedness and operator norm calculation formula of many discrete and integral operators with homogeneous kernels, generalized homogeneous kernels and several non-homogeneous kernels in weighted normed sequence space and weighted Lebesgue space (see [3,4,9,11–16]).

In 2015, [5] abstractly discussed for the first time the problem of best matching parameters of discrete operators with quasi-homogeneous kernel in weighted Lebesgue space, and obtained sufficient condition for the best matching parameters and formula for calculating the operator norm. In 2016, [7] further solved the sufficient and necessary condition for best matching parameters of discrete operator with generalized homogeneous kernels and the norm calculation formula in weighted normed sequence space, which opened a new era of research on best matching parameters of operators, followed by a large number of research results (see [2,6,8,10]).

In order to take a broader perspective to explore the best matching parameters of operators and operator norm, we introduce the concept of super-homogeneous function, which is used to unify homogeneous functions, generalized homogeneous functions and several non-homogeneous functions.

Definition 1.1. Let $\sigma_1, \sigma_2, \tau_1, \tau_2 \in \mathbb{R}$. If $K(u,v)$ satisfies

$$K(tu,v) = t^{\sigma_1} K(u, t^{\tau_1} v), \quad K(u,tv) = t^{\sigma_2} K(t^{\tau_2} u, v)$$

for all $t > 0$, then we say $K(u, v)$ is a super-homogeneous function with parameters $\{\sigma_1, \sigma_2, \tau_1, \tau_2\}$.

Obviously, if $K_1(u, v)$ is a homogeneous function of order λ, then $K_1(u, v)$ is a super-homogeneous function with parameters $\{\lambda, \lambda, -1, -1\}$, and it can be seen that the super-homogeneous function is a generalization of the homogeneous function. If $G(x, y)$ is a homogeneous function of order λ, then $K_2(u, v) = G(u^{\lambda_1}, v^{\lambda_2})\ (\lambda_1\lambda_2 \neq 0)$ is a super-homogeneous function with parameters $\{\lambda\lambda_1, \lambda\lambda_2, -\frac{\lambda_1}{\lambda_2}, -\frac{\lambda_2}{\lambda_1}\}$. If $H(x)$ is a real function, then $K_3(u, v) = H(u^{\lambda_1} v^{\lambda_2})\ (\lambda_1\lambda_2 \neq 0)$ is a super-homogeneous function with parameters $\{0, 0, \frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}\}$.

Suppose that $K(u, v)$ is a super-homogeneous function with parameters $\{\sigma_1, \sigma_2, \tau_1, \tau_2\}$. Then

$$K(tu, v) = t^{\sigma_1} K(u, t^{\tau_1} v) = t^{\sigma_1 + \tau_1\sigma_2} K(t^{\tau_1\tau_2} u, v),$$

it follows that $\tau_1\tau_2 = 1$ and $\sigma_1 + \tau_1\sigma_2 = 0$ in the general case. Therefore, our discussions are all under the conditions that $\tau_1\tau_2 = 1$ and $\sigma_1 + \tau_1\sigma_2 = 0$.

To avoid repetition, in this paper we always write

$$\tilde{A}(K, \tilde{a}, f) = \int_0^{+\infty} \sum_{n=1}^{\infty} K(n, x) a_n f(x) \mathrm{d}x = \sum_{n=1}^{\infty} \int_0^{+\infty} K(n, x) a_n f(x) \mathrm{d}x,$$

$$W_1(s) = \int_0^{+\infty} K(1, t) t^s \mathrm{d}t, \quad W_2(s) = \int_0^{+\infty} K(t, 1) t^s \mathrm{d}t,$$

where $\tilde{a} = \{a_n\}$.

2. Some lemmas

Lemma 2.1. *$\tau_1 b - a = \tau_1 - \sigma_1 - 1$ and $\tau_2 a - b = \tau_2 - \sigma_2 - 1$ are equivalent when and only when $\tau_1\tau_2 = 1$ and $\sigma_1 + \tau_1\sigma_2 = 0$.*

Proof. A sufficient and necessary condition for equivalence of $\tau_1 b - a = \tau_1 - \sigma_1 - 1$ and $\tau_2 a - b = \tau_2 - \sigma_2 - 1$ is that the augmented matrix of the system of linear equations

$$\begin{cases} x_1 - \tau_1 x_2 = -\tau_1 + \sigma_1 + 1, \\ \tau_2 x_1 - x_2 = \tau_2 - \sigma_2 - 1 \end{cases}$$

has rank 1, i.e.

$$\begin{aligned} 1 &= \mathrm{Rank} \begin{pmatrix} 1 & -\tau_1 & -\tau_1 + \sigma_1 + 1 \\ \tau_2 & -1 & \tau_2 - \sigma_2 - 1 \end{pmatrix} \\ &= \mathrm{Rank} \begin{pmatrix} 1 & -\tau_1 & \sigma_1 \\ \tau_2 & -1 & -\sigma_2 \end{pmatrix} \\ &= \mathrm{Rank} \begin{pmatrix} 1 - \tau_1\tau_2 & 0 & \sigma_1 + \tau_1\sigma_2 \\ \tau_2 & -1 & \sigma_2 \end{pmatrix}, \end{aligned}$$

this is equivalent to $\tau_1\tau_2 = 1$ and $\sigma_1 + \tau_1\sigma_2 = 0$, so Lemma 2.1 holds. □

Lemma 2.2. *Let $K(u,v)$ be a super-homogeneous function with parameters $\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$, and $\tau_1\tau_2\neq 0$.*

(i) If $\tau_1 b-a=\tau_1-\sigma_1-1$, then $W_2(-a)=\frac{1}{|\tau_1|}W_1(-b)$;

(ii) If $\tau_2 a-b=\tau_2-\sigma_2-1$, then $W_1(-b)=\frac{1}{|\tau_2|}W_2(-a)$;

(iii) If $K(t,1)t^{-a}$ decreases on $(0,+\infty)$, then

$$\begin{aligned}\tilde{\omega}_1(n,b)&=\int_0^{+\infty}K(n,x)x^{-b}\mathrm{d}x=n^{\sigma_1+\tau_1(b-1)}W_1(-b),\\ \tilde{\omega}_2(x,a)&=\sum_{n=1}^{\infty}K(n,x)n^{-a}\leq x^{\sigma_2+\tau_2(a-1)}W_2(-a).\end{aligned}$$

Proof. (i) From $\tau_1 b-a=\tau_1-\sigma_1-1$, we have $\frac{1}{\tau_1}(\sigma_1-a+1)-1=-b$, so

$$\begin{aligned}W_2(-a)&=\int_0^{+\infty}K(1,t^{\tau_1})t^{\sigma_1-a}\mathrm{d}t\\ &=\frac{1}{|\tau_1|}\int_0^{+\infty}K(1,u)u^{\frac{1}{\tau_1}(\sigma_1-a+1)-1}\mathrm{d}u\\ &=\frac{1}{|\tau_1|}\int_0^{+\infty}K(1,u)u^{-b}\mathrm{d}u\\ &=\frac{1}{|\tau_1|}W_1(-b).\end{aligned}$$

(ii) Similarly, it can be proved that $W_1(-b)=\frac{1}{|\tau_2|}W_2(-a)$.

(iii) Since $K(u,v)$ is a super-homogeneous function, we have

$$\begin{aligned}\tilde{\omega}_1(n,b)&=n^{\sigma_1}\int_0^{+\infty}K(1,n^{\tau_1}x)x^{-b}\mathrm{d}x\\ &=n^{\sigma_1+\tau_1 b-\tau_1}\int_0^{+\infty}K(1,t)t^{-b}\mathrm{d}t\\ &=n^{\sigma_1+\tau_1(b-1)}W_1(-b),\end{aligned}$$

and notice that $K(t,1)t^{-a}$ decreases on $(0,+\infty)$, it follows that

$$\begin{aligned}&\tilde{\omega}_2(x,a)\\ &=x^{\sigma_2}\sum_{n=1}^{\infty}K(x^{\tau_2}n,1)n^{-a}\\ &=x^{\sigma_2+\tau_2 a}\sum_{n=1}^{\infty}K(x^{\tau_2}n,1)(x^{\tau_2}n)^{-a}\\ &\leq x^{\sigma_2+\tau_2 a}\int_0^{+\infty}K(x^{\tau_2}u,1)(x^{\tau_2}u)^{-a}\mathrm{d}u\\ &=x^{\sigma_2+\tau_2(a-1)}\int_0^{+\infty}K(t,1)t^{-a}\mathrm{d}t\\ &=x^{\sigma_2+\tau_2(a-1)}W_2(-a).\end{aligned}$$

□

Lemma 2.3. *Let $\tau_1\tau_2 = 1$, $\sigma_1 + \tau_1\sigma_2 = 0$. If $K(u, v)$ is a super-homogeneous function with parameters $\{\sigma_1, \sigma_2, \tau_1, \tau_2\}$, and $\tau_1 b - a = \tau_1 - \sigma_1 - 1$, then*

$$W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a) = \left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}} W_1(-b) = \left(\frac{1}{|\tau_2|}\right)^{\frac{1}{p}} W_2(-a). \tag{2.1}$$

Proof. It follows from Lemma 2.1 that $\tau_1 b - a = \tau_1 - \sigma_1 - 1$ and $\tau_2 a - b = \tau_2 - \sigma_2 - 1$ are equivalent, then $\tau_1 b - a = \tau_1 - \sigma_1 - 1$ and $\tau_2 a - b = \tau_2 - \sigma_2 - 1$ are true at the same time. Hence, from Lemma 2.2, we can obtain (2.1). □

3. Sufficient and necessary condition for the best matching parameters of the semi-discrete Hilbert-type inequality with super-homogeneous kernel

Theorem 3.1. *Suppose that $\frac{1}{p} + \frac{1}{q} = 1\,(p > 1, q > 1)$, $a, b \in \mathbb{R}$, $\tau_1\tau_2 \neq 0$, $K(u, v) \geq 0$ is a super-homogeneous function with parameters $\{\sigma_1, \sigma_2, \tau_1, \tau_2\}$, $0 < W_1(-b) < +\infty$, $0 < W_2(-a) < +\infty$, $\tau_1 b - a - (\tau_1 - \sigma_1 - 1) = c$, both $K(t, 1)t^{-a}$ and $K(t, 1)t^{-a+\frac{\tau_1 c}{p}}$ are decreasing on $(0, +\infty)$.*

(i) Denote $\alpha = a(p-1) + \tau_1(b-1) + \sigma_1$ and $\beta = b(q-1) + \tau_2(a-1) + \sigma_2$. Then we have the following Hilbert-type semi-discrete inequality

$$\tilde{A}(K, \tilde{a}, f) = \int_0^{+\infty} \sum_{n=1}^{\infty} K(n, x) a_n f(x) \mathrm{d}x \leq W_1^{\frac{1}{p}}(-b) W_2^{\frac{1}{q}}(-a) \|\tilde{a}\|_{p,\alpha} \|f\|_{q,\beta}, \tag{3.1}$$

where $\tilde{a} = \{a_n\} \in l_p^{\alpha}$, $f(x) \in L_q^{\beta}(0, +\infty)$. When $\tau_1\tau_2 = 1$, $\sigma_1 + \tau_1\sigma_2 = 0$, and $\tau_1 b - a = \tau_1 - \sigma_1 - 1$, (3.1) is reduced to

$$\begin{aligned} \tilde{A}(K, \tilde{a}, f) &\leq \left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}} W_1(-b) \|\tilde{a}\|_{p,ap-1} \|f\|_{q,bq-1} \\ &= \left(\frac{1}{|\tau_2|}\right)^{\frac{1}{p}} W_2(-a) \|\tilde{a}\|_{p,ap-1} \|f\|_{q,bq-1}. \end{aligned} \tag{3.2}$$

(ii) If $\tau_1\tau_2 = 1$ and $\sigma_1 + \tau_1\sigma_2 = 0$, the constant factor of (3.1) is optimal when and only when $\tau_1 b - a = \tau_1 - \sigma_1 - 1$, i.e. a and b are the best matching parameters.

Proof. (i) Introducing the matching parameters a and b, according to the mixed Hölder inequality, we have

$$\begin{aligned} &\tilde{A}(K, \tilde{a}, f) \\ &= \int_0^{+\infty} \sum_{n=1}^{\infty} \left(\frac{n^{\frac{a}{q}}}{x^{\frac{b}{p}}} a_n\right) \left(\frac{x^{\frac{b}{p}}}{n^{\frac{a}{q}}} f(x)\right) K(n, x) \mathrm{d}x \\ &\leq \left(\int_0^{+\infty} \sum_{n=1}^{\infty} n^{\frac{ap}{q}} x^{-b} |a_n|^p K(n, x) \mathrm{d}x\right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^{+\infty} \sum_{n=1}^{\infty} x^{\frac{bq}{p}} n^{-a} |f(x)|^q K(n, x) \mathrm{d}x\right)^{\frac{1}{q}} \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{n=1}^{\infty} n^{a(p-1)}|a_n|^p \tilde{\omega}_1(n,b)\right)^{\frac{1}{p}} \left(\int_0^{+\infty} x^{b(q-1)}|f(x)|^q \tilde{\omega}_2(x,a)\mathrm{d}x\right)^{\frac{1}{q}} \\
&\leq W_1^{\frac{1}{p}}(-b) W_2^{\frac{1}{q}}(-a) \left(\sum_{n=1}^{\infty} n^{a(p-1)+\tau_1(b-1)+\sigma_1}|a_n|^p\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^{\infty} x^{b(q-1)+\tau_2(a-1)+\sigma_2}|f(x)|^q \mathrm{d}x\right)^{\frac{1}{q}} \\
&= W_1^{\frac{1}{p}}(-b) W_2^{\frac{1}{q}}(-a) \|\tilde{a}\|_{p,\alpha} \|f\|_{q,\beta},
\end{aligned}
$$

so (3.1) is proved. □

If $\tau_1\tau_2 = 1$, $\sigma_1 + \tau_1\sigma_2 = 0$, and $\tau_1 b - a = \tau_1 - \sigma_1 - 1$, according to Lemma 2.1, we find $\tau_2 a - b = \tau_2 - \sigma_2 - 1$. According to $\tau_1 b - a = \tau_1 - \sigma_1 - 1$ and $\tau_2 a - b = \tau_2 - \sigma_2 - 1$, the calculation gives $\alpha = ap - 1$ and $\beta = bq - 1$, then, according to Lemma 2.3, we know that (3.1) is reduced to (3.2).

(ii) Sufficiency: Suppose that $\tau_1 b - a = \tau_1 - \sigma_1 - 1$. According to (i), it is known that (3.1) becomes (3.2). If the constant factor $(\frac{1}{|\tau_1|})^{\frac{1}{q}} W_1(-b)$ in (3.2) is not the best possible, then there exists a positive constant $M_0 < (\frac{1}{|\tau_1|})^{\frac{1}{q}} W_1(-b)$ satisfying

$$
\tilde{A}(K, \tilde{a}, f) \leq M_0 \|\tilde{a}\|_{p,ap-1} \|f\|_{q,bq-1}. \tag{3.3}
$$

If $\tau_1 < 0$, for sufficiently small $\varepsilon > 0$ and sufficiently large $N > 0$, taking

$$
a_n = \begin{cases} 0, & n = 1, 2, \cdots, N-1, \\ n^{-\frac{ap - \tau_1\varepsilon}{p}}, & n = N, N+1, \cdots \end{cases},
$$

$$
f(x) = \begin{cases} 0, & 0 < x < 1, \\ x^{-\frac{bq+\varepsilon}{q}}, & x \geq 1, \end{cases}
$$

then

$$
\begin{aligned}
&M_0 \|\tilde{a}\|_{p,ap-1} \|f\|_{q,bq-1} \\
&= M_0 \left(\sum_{n=N}^{\infty} n^{-1+\tau_1\varepsilon}\right)^{\frac{1}{p}} \left(\int_1^{+\infty} x^{-1-\varepsilon}\mathrm{d}x\right)^{\frac{1}{q}} \\
&\leq M_0 \left(\int_1^{+\infty} t^{-1+\tau_1\varepsilon}\mathrm{d}t\right)^{\frac{1}{p}} \left(\int_1^{+\infty} x^{-1-\varepsilon}\mathrm{d}x\right)^{\frac{1}{q}} \\
&= \frac{M_0}{\varepsilon} \left(\frac{1}{|\tau_1|}\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.4}
$$

Since $\tau_1 < 0$, we have $n^{\tau_1} \leq N^{\tau_1}$ when $n \geq N$, thus

$$
\begin{aligned}
&\tilde{A}(K, \tilde{a}, f) \\
&= \sum_{n=N}^{\infty} n^{-a+\frac{\tau_1\varepsilon}{p}} \left(\int_1^{+\infty} K(n,x) x^{-b-\frac{\varepsilon}{q}}\mathrm{d}x\right) \\
&= \sum_{n=N}^{\infty} n^{\sigma_1 - a + \frac{\tau_1\varepsilon}{p}} \left(\int_1^{+\infty} K(1, n^{\tau_1}x) x^{-b-\frac{\varepsilon}{q}}\mathrm{d}x\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=N}^{\infty} n^{\sigma_1 - a + \frac{\tau_1 \varepsilon}{p} + \tau_1(b + \frac{\varepsilon}{q}) - \tau_1} \left(\int_{n^{\tau_1}}^{+\infty} K(1,t) t^{-b-\frac{\varepsilon}{q}} \mathrm{d}t \right) \\
&= \sum_{n=N}^{\infty} n^{\sigma_1 - a + \tau_1(b-1) + \tau_1 \varepsilon} \left(\int_{n^{\tau_1}}^{+\infty} K(1,t) t^{-b-\frac{\varepsilon}{q}} \mathrm{d}t \right) \\
&= \sum_{n=N}^{\infty} n^{-1+\tau_1 \varepsilon} \left(\int_{n^{\tau_1}}^{+\infty} K(1,t) t^{-b-\frac{\varepsilon}{q}} \mathrm{d}t \right) \\
&\geq \sum_{n=N}^{\infty} n^{-1+\tau_1 \varepsilon} \int_{N^{\tau_1}}^{+\infty} K(1,t) t^{-b-\frac{\varepsilon}{q}} \mathrm{d}t \\
&\geq \int_{N}^{+\infty} t^{-1+\tau_1 \varepsilon} \mathrm{d}t \int_{N^{\tau_1}}^{+\infty} K(1,t) t^{-b-\frac{\varepsilon}{q}} \mathrm{d}t \\
&= \frac{1}{|\tau_1| \varepsilon} N^{\tau_1 \varepsilon} \int_{N^{\tau_1}}^{+\infty} K(1,t) t^{-b-\frac{\varepsilon}{q}} \mathrm{d}t. \qquad (3.5)
\end{aligned}
$$

By (3.3), (3.4) and (3.5), we have

$$
\frac{1}{|\tau_1|} N^{\tau_1 \varepsilon} \int_{N^{\tau_1}}^{+\infty} K(1,t) t^{-b-\frac{\varepsilon}{q}} \mathrm{d}t \leq M_0 \left(\frac{1}{|\tau_1|} \right)^{\frac{1}{p}},
$$

thus

$$
\left(\frac{1}{|\tau_1|} \right)^{\frac{1}{q}} N^{\tau_1 \varepsilon} \int_{N^{\tau_1}}^{+\infty} K(1,t) t^{-b-\frac{\varepsilon}{q}} \mathrm{d}t \leq M_0. \qquad (3.6)
$$

Considering ε as a sequence of positive terms tending to 0, by the well-known Fatou Lemma, we have

$$
\begin{aligned}
\int_{N^{\tau_1}}^{+\infty} K(1,t) t^{-b} \mathrm{d}t &= \int_{N^{\tau_1}}^{+\infty} \liminf_{\varepsilon \to 0^+} K(1,t) t^{-b-\frac{\varepsilon}{q}} \mathrm{d}t \\
&\leq \liminf_{\varepsilon \to 0^+} \int_{N^{\tau_1}}^{+\infty} K(1,t) t^{-b-\frac{\varepsilon}{q}} \mathrm{d}t,
\end{aligned}
$$

so by setting $\varepsilon \to 0^+$ in (3.6), we get

$$
\left(\frac{1}{|\tau_1|} \right)^{\frac{1}{q}} \int_{N^{\tau_1}}^{+\infty} K(1,t) t^{-b} \mathrm{d}t \leq M_0,
$$

then letting $N \to +\infty$, and noting $\tau_1 < 0$, we have

$$
\left(\frac{1}{|\tau_1|} \right)^{\frac{1}{q}} W_1(-b) = \left(\frac{1}{|\tau_1|} \right)^{\frac{1}{q}} \int_{0}^{+\infty} K(1,t) t^{-b} \mathrm{d}t \leq M_0,
$$

this contradicts $M_0 < (\frac{1}{|\tau_1|})^{\frac{1}{q}} W_1(-b)$. Therefor, the constant factor in (3.2) is the best possible.

If $\tau_1 > 0$, for sufficiently small $\varepsilon > 0$ and sufficiently large $N > 0$, taking

$$
a_n = \begin{cases} 0, & n = 1, \\ n^{-\frac{ap+\tau_1 \varepsilon}{p}}, & = 2, 3, \cdots \end{cases},
$$

$$f(x)=\begin{cases}x^{-\frac{bq-\varepsilon}{q}}, & 0<x\le N,\\ 0, & x>N,\end{cases}$$

then

$$\begin{aligned}&M_0\|\tilde{a}\|_{p,ap-1}\|f\|_{q,bq-1}\\ &=M_0\left(\sum_{n=2}^{\infty}n^{-1-\tau_1\varepsilon}\right)^{\frac{1}{p}}\left(\int_0^N x^{-1+\varepsilon}\mathrm{d}x\right)^{\frac{1}{q}}\\ &\le M_0\left(\int_1^{+\infty}t^{-1-\tau_1\varepsilon}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^N x^{-1+\varepsilon}\mathrm{d}x\right)^{\frac{1}{q}}\\ &=\frac{M_0}{\varepsilon}\left(\frac{1}{\tau_1}\right)^{\frac{1}{p}}N^{\frac{\varepsilon}{q}},\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\tilde{A}(K,\tilde{a},f)\\ &=\sum_{n=2}^{\infty}n^{-a-\frac{\tau_1\varepsilon}{p}}\left(\int_0^N K(n,x)x^{-b+\frac{\varepsilon}{q}}\mathrm{d}x\right)\\ &=\sum_{n=2}^{\infty}n^{\sigma_1-a-\frac{\tau_1\varepsilon}{p}}\left(\int_0^N K(1,n^{\tau_1}x)x^{-b+\frac{\varepsilon}{q}}\mathrm{d}x\right)\\ &=\sum_{n=2}^{\infty}n^{\sigma_1-a-\frac{\tau_1\varepsilon}{p}+\tau_1(b-\frac{\varepsilon}{q})-\tau_1}\left(\int_0^{Nn^{\tau_1}}K(1,t)t^{-b+\frac{\varepsilon}{q}}\mathrm{d}t\right)\\ &=\sum_{n=2}^{\infty}n^{-1-\tau_1\varepsilon}\left(\int_0^{Nn^{\tau_1}}K(1,t)t^{-b+\frac{\varepsilon}{q}}\mathrm{d}t\right)\\ &\ge\sum_{n=2}^{\infty}n^{-1-\tau_1\varepsilon}\left(\int_0^{2^{\tau_1}N}K(1,t)t^{-b+\frac{\varepsilon}{q}}\mathrm{d}t\right)\\ &\ge\int_2^{+\infty}t^{-1-\tau_1\varepsilon}\mathrm{d}t\int_0^{2^{\tau_1}N}K(1,t)t^{-b+\frac{\varepsilon}{q}}\mathrm{d}t\\ &=\frac{1}{\tau_1\varepsilon}2^{-\tau_1\varepsilon}\int_0^{2^{\tau_1}N}K(1,t)t^{-b+\frac{\varepsilon}{q}}\mathrm{d}t.\end{aligned} \tag{3.8}$$

It follows from (3.3), (3.7) and (3.8) that

$$\left(\frac{1}{\tau_1}\right)^{\frac{1}{q}}2^{-\tau_1\varepsilon}N^{-\frac{\varepsilon}{q}}\int_0^{2^{\tau_1}N}K(1,t)t^{-b+\frac{\varepsilon}{q}}\mathrm{d}t\le M_0. \tag{3.9}$$

Similarly, using the Fatou Lemma yields

$$\int_0^{2^{\tau_2}N}K(1,t)t^{-b}\mathrm{d}t\le\liminf_{\varepsilon\to0^+}\int_0^{2^{\tau_2}N}K(1,t)t^{-b+\frac{\varepsilon}{q}}\mathrm{d}t.$$

Thus, by setting $\varepsilon\to0^+$ in (3.9), we have

$$\left(\frac{1}{\tau_1}\right)^{\frac{1}{q}}\int_0^{2^{\tau_2}N}K(1,t)t^{-b}\mathrm{d}t\le M_0.$$

Then, letting $N \to +\infty$ yields

$$\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}} W_1(-b) = \left(\frac{1}{\tau_1}\right)^{\frac{1}{q}} \int_0^{+\infty} K(1,t)t^{-b}\mathrm{d}t \le M_0.$$

This still contradicts $M_0 < (\frac{1}{|\tau_1|})^{\frac{1}{q}} W_1(-b)$, so the constant factor in (3.2) is also the best possible.

Necessary: Suppose that the constant factor $W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a)$ in (3.1) is the best value. Since $\tau_1\tau_2 = 1$ and $\sigma_1 + \tau_1\sigma_2 = 0$, it follows that $\sigma_1\sigma_2 \neq 0$ or $\sigma_1 = \sigma_2 = 0$.

If $\sigma_1\sigma_2 \neq 0$, then $\tau_1 = -\frac{\sigma_1}{\sigma_2}$ and $\tau_2 = -\frac{\sigma_2}{\sigma_1}$, thus $\tau_1 b - a = \tau_1 - \sigma_1 - 1$ is transformed into $\sigma_1 b + \sigma_2 a = \sigma_1 + \sigma_2 + \sigma_1\sigma_2$, and from $\tau_1 b - a - (\tau_1 - \sigma_1 - 1) = c$ we get $\sigma_1 b + \sigma_2 a - (\sigma_1 + \sigma_2 + \sigma_1\sigma_2) = -c\sigma_2$. Let

$$\sigma_1 b + \sigma_2 a - (\sigma_1 + \sigma_2 + \sigma_1\sigma_2) = c',\ a' = a - \frac{c'}{\sigma_2 p},\ b' = b - \frac{c'}{\sigma_1 q}.$$

It is easy to see that $\sigma_1 b' + \sigma_2 a' = \sigma_1 + \sigma_2 + \sigma_1\sigma_2$, $\alpha = a'p - 1$ and $\beta = b'q - 1$. And since

$$\begin{aligned}
&W_2(-a)\\
&= \int_0^{+\infty} K(t,1)t^{-a}\mathrm{d}t = \int_0^{+\infty} K(1,t^{-\frac{\sigma_1}{\sigma_2}})t^{\sigma_1 - a}\mathrm{d}t\\
&= \left|\frac{\sigma_2}{\sigma_1}\right| \int_0^{+\infty} K(1,u)u^{-b+\frac{c'}{\sigma_1}}\mathrm{d}u\\
&= \left|\frac{\sigma_2}{\sigma_1}\right| W_1(-b + \frac{c'}{\sigma_1}),
\end{aligned}$$

(3.1) is reduced to the equivalence inequality

$$\tilde{A}(K,\tilde{a},f) \le \left|\frac{\sigma_2}{\sigma_1}\right|^{\frac{1}{q}} W_1^{\frac{1}{p}}(-b)W_1^{\frac{1}{q}}(-b + \frac{c'}{\sigma_1})\|\tilde{a}\|_{p,a'p-1}\|f\|_{q,b'q-1}. \tag{3.10}$$

Note that the constant factor of (3.1) is the best possible, and thus the best constant factor of (3.10) equivalent to it is

$$\left|\frac{\sigma_2}{\sigma_1}\right|^{\frac{1}{q}} W_1^{\frac{1}{p}}(-b)W_1^{\frac{1}{q}}(-b + \frac{c'}{\sigma_1}).$$

In view of $\sigma_1 b' + \sigma_2 a' = \sigma_1 + \sigma_2 + \sigma_1\sigma_2$, we have $\tau_1 b' - a' = \tau_1 - \sigma_1 - 1$. And since

$$K(t,1)t^{-a'} = K(t,1)t^{-a+\frac{c'}{\sigma_2 p}} = K(t,1)t^{-b-\frac{\sigma_1 c}{\sigma_2 p}} = K(t,1)t^{-a+\frac{\tau_1 c}{p}}$$

decreases on $(0,+\infty)$, it follows from the previous proof of sufficiency that the best constant factor for (3.10) should be

$$\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}} W_1(-b') = \left|\frac{\sigma_2}{\sigma_1}\right|^{\frac{1}{q}} W_1(-b + \frac{c'}{\sigma_1 q}),$$

so we get

$$W_1(-b + \frac{c'}{\sigma_1 q}) = W_1^{\frac{1}{p}}(-b)W_1^{\frac{1}{q}}(-b + \frac{c'}{\sigma_1}). \tag{3.11}$$

According to Hölder integral inequality, we have

$$
\begin{aligned}
& W_1(-b+\frac{c'}{\sigma_1 q}) \\
={}& \int_0^{+\infty} t^{\frac{c'}{\sigma_1 q}} K(1,t)t^{-b}\mathrm{d}t \\
\leq{}& \left(\int_0^{+\infty} K(1,t)t^{-b}\mathrm{d}t\right)^{\frac{1}{p}} \left(\int_0^{+\infty} t^{\frac{c'}{\sigma_1}} K(1,t)t^{-b}\mathrm{d}t\right)^{\frac{1}{q}} \\
={}& W_1^{\frac{1}{p}}(-b)W_1^{\frac{1}{q}}(-b+\frac{c'}{\sigma_1}).
\end{aligned} \tag{3.12}
$$

From (3.11) we known that (3.12) should take the equal sign, and according to the condition that Hölder integral inequality takes the equal sign, we have $t^{\frac{c'}{\sigma_1}} =$ constant, so $c' = 0$, which gives $\tau_1 b - a = \tau_1 - \sigma_1 - 1$.

If $\sigma_1 = \sigma_2 = 0$, then $\tau_1 b - a = \tau_1 - \sigma_1 - 1$ is reduced to $\tau_1 b - a = \tau_1 - 1$. Since $\tau_1\tau_2 = 1$, we can set $\tau_1 = \frac{\lambda_1}{\lambda_2}$ and $\tau_2 = \frac{\lambda_2}{\lambda_1}$, so $\tau_1 b - a = \tau_1 - 1$ is further reduced to $\lambda_1(b-1) = \lambda_2(a-1)$. And letting

$$
\lambda_1(b-1) - \lambda_2(a-1) = c'',\ a'' = a + \frac{c''}{\lambda_2 p},\ b'' = b - \frac{c''}{\lambda_1 q},
$$

then by calculation we can get $\lambda_1(b''-1) = \lambda_2(a''-1)$, $\alpha = a''p-1$, $\beta = b''q-1$, and

$$
\begin{aligned}
& W_2(-a) \\
={}& \int_0^{+\infty} K(t,1)t^{-a}\mathrm{d}t \\
={}& \int_0^{+\infty} K(1,t^{\frac{\lambda_1}{\lambda_2}})t^{-a}\mathrm{d}t \\
={}& \left|\frac{\lambda_2}{\lambda_1}\right| \int_0^{+\infty} K(1,u)u^{-b+\frac{c''}{\lambda_1}}\mathrm{d}u \\
={}& \left|\frac{\lambda_2}{\lambda_1}\right| W_1(-b+\frac{c''}{\lambda_1}),
\end{aligned}
$$

thus (3.1) is reduced to the equivalence inequality

$$
\tilde{A}(K,\tilde{a},f) \leq \left|\frac{\lambda_2}{\lambda_1}\right|^{\frac{1}{q}} W_1^{\frac{1}{p}}(-b)W_1^{\frac{1}{q}}(-b+\frac{c''}{\lambda_1})\|\tilde{a}\|_{p,a''p-1}\|f\|_{q,b''q-1}. \tag{3.13}
$$

Since the constant factor of (3.1) is the best possible, and thus the best constant factor of (3.13) equivalent to it is

$$
\left|\frac{\lambda_2}{\lambda_1}\right|^{\frac{1}{q}} W_1^{\frac{1}{p}}(-b)W_1^{\frac{1}{q}}(-b+\frac{c''}{\lambda_1}).
$$

Since $\lambda_1(b''-1) = \lambda_2(a''-1)$, $\tau_1 b'' - a'' = \tau_1 - 1$. Similarly, from the previous proof of sufficiency, it follows that the best constant factor in (3.13) should be

$$
\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}} W_1(-b'') = \left|\frac{\lambda_2}{\lambda_1}\right|^{\frac{1}{q}} W_1(-b+\frac{c''}{\lambda_1 q}),
$$

thereby having

$$W_1(-b+\frac{c''}{\lambda_1 q})=W_1^{\frac{1}{p}}(-b)W_1^{\frac{1}{q}}(-b+\frac{c''}{\lambda_1}).$$

Similarly, using the condition that Hölder integral inequality takes an equal sign, we can also obtain $t^{\frac{c''}{\lambda_1}}$ =canstant, so $c''=0$, which gives $\tau_1 b-a=\tau_1-1$.

4. The best matching parameters and norm formulas for the operators with super-homogeneous kernel

Let $K(n,x)\geq 0$. For the operators

$$T_1(\tilde{a})(x)=\sum_{n=1}^{\infty}K(n,x)a_n,\quad T_2(f)_n=\int_0^{+\infty}K(n,x)f(x)\mathrm{d}x \tag{4.1}$$

with $K(n,x)$ as the kernel, according to the basic theory of Hilbert-type inequalities (see [4]), the semi-discrete Hilbert-type inequality (3.1) is equivalent to the following operator inequalities

$$\|T_1(\tilde{a})\|_{p,\beta(1-p)}\leq W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a)\|\tilde{a}\|_{p,\alpha},$$

$$\|T_2(f)\|_{q,\alpha(1-q)}\leq W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a)\|f\|_{q,\beta},$$

thus, the equivalence theorem of Theorem 3.1 can be obtained.

Theorem 4.1. *Suppose that* $\frac{1}{p}+\frac{1}{q}=1\,(p>1,q>1)$, $a,b\in\mathbb{R}$, $\tau_1\tau_2\neq 0$, $K(u,v)\geq 0$ *is super-homogeneous function with parameters* $\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$, $0<W_1(-b)<+\infty$, $0<W_2(-a)<+\infty$, $\tau_1 b-a-(\tau_1-\sigma_1-1)=c$, $K(t,1)t^{-a}$ *and* $K(t,1)t^{-a+\frac{\tau_1 c}{p}}$ *are decreasing on* $(0,+\infty)$, *and the discrete operator* T_1 *and the integral operator* T_2 *are defined by (4.1).*

(i) Denote $\alpha=a(p-1)+\tau_1(b-1)+\sigma_1$ *and* $\beta=b(q-1)+\tau_2(a-1)+\sigma_2$. *Then* T_1 *is a bounded operator from* l_p^{α} *to* $L_p^{\beta(1-p)}(0,+\infty)$, T_2 *is a bounded operator from* $L_q^{\beta}(0,+\infty)$ *to* $l_q^{\alpha(1-q)}$, *and*

$$\|T_1\|\leq W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a),\quad \|T_2\|\leq W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a).$$

(ii) If $\tau_1\tau_2=1$ *and* $\sigma_1+\tau_1\sigma_2=0$, *then when and only when* $\tau_1 b-a=\tau_1-\sigma_1-1$, a *and* b *are the best matching parameters, i.e.*

$$\|T_1\|=\|T_2\|=W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a).$$

When $\tau_1 b-a=\tau_1-\sigma_1-1$, *the operator norms of* $T_1: l_p^{ap-1}\to L_p^{(bq-1)(1-p)}(0,+\infty)$ *and* $T_2: L_q^{bq-1}(0,+\infty)\to l_p^{(ap-1)(1-q)}$ *are*

$$\|T_1\|=\|T_2\|=\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}W_1(-b)=\left(\frac{1}{|\tau_2|}\right)^{\frac{1}{p}}W_2(-a).$$

Taking $a = \frac{1}{p}$ and $b = \frac{1}{q}$ in Theorem 4.1, then $\tau_1 b - a = \tau_1 - \sigma_1 - 1$ reduces to $\frac{\tau_1}{p} = \frac{1}{q} + \sigma_1$, and when $\frac{\tau_1}{p} = \frac{1}{q} + \sigma_1$, there holds $\alpha = \beta = 0$, so from Theorem 4.1 we have:

Corollary 4.1. *Suppose that* $\frac{1}{p} + \frac{1}{q} = 1\,(p > 1, q > 1)$, $\tau_1\tau_2 \neq 0$, $K(u,v) \geq 0$ *is a super-homogeneous function with parameters* $\{\sigma_1, \sigma_2, \tau_1, \tau_2\}$, $0 < W_1(-\frac{1}{q}) < +\infty$, $0 < W_2(-\frac{1}{p}) < +\infty$, $\frac{1}{q} - \frac{\tau_1}{p} + \sigma_1 = c$, $K(t,1)t^{-\frac{1}{p}}$ *and* $K(t,1)t^{-\frac{1}{p}+\frac{\tau_1 c}{p}}$ *are decreasing on* $(0,+\infty)$, *and the operators* T_1 *and* T_2 *are defined by (4.1).*

(i) Denoting $\alpha = \frac{1}{q} - \frac{\tau_1}{p} + \sigma_1$ *and* $\beta = \frac{1}{p} - \frac{\tau_2}{q} + \sigma_2$, *then* T_1 *is a bounded operator from* l_p *to* $L_p(0,+\infty)$, T_2 *is a bounded operator from* $L_q(0,+\infty)$ *to* l_q, *and*

$$\|T_1\| \leq W_1^{\frac{1}{p}}(-\frac{1}{q})W_2^{\frac{1}{q}}(-\frac{1}{p}), \quad \|T_2\| \leq W_1^{\frac{1}{p}}(-\frac{1}{q})W_2^{\frac{1}{q}}(-\frac{1}{p}).$$

(ii) If $\tau_1\tau_2 = 1$ *and* $\sigma_1 + \tau_1\sigma_2 = 0$, *then*

$$\|T_1\| = \|T_2\| = W_1^{\frac{1}{p}}(-\frac{1}{q})W_2^{\frac{1}{q}}(-\frac{1}{p})$$

when and only when $\frac{\tau_1}{p} = \frac{1}{q} + \sigma_1$. *When* $\frac{\tau_1}{p} = \frac{1}{q} + \sigma_1$, *the operator norms of* $T_1 : l_p \to L_p(0,+\infty)$ *and* $T_2 : L_q(0,+\infty) \to l_q$ *are*

$$\|T_1\| = \|T_2\| = \left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}} W_1(-\frac{1}{q}) = \left(\frac{1}{|\tau_2|}\right)^{\frac{1}{p}} W_2(-\frac{1}{p}).$$

Corollary 4.2. *Suppose that* $\frac{1}{p} + \frac{1}{q} = 1\,(p > 1, q > 1)$, $\lambda_1 > 0$, $\lambda_2 > 0$, *and* $0 \leq c_1 < c_2$. *Then the discrete operator* T_1 *defined by*

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty} \ln\left(\frac{n^{\lambda_1} + c_2 x^{\lambda_2}}{n^{\lambda_1} + c_1 x^{\lambda_2}}\right) a_n$$

is a bounded operator from $l_p^{(1+\frac{\lambda_1}{2})p-1}$ *to* $L_p^{\frac{\lambda_2}{2}p-1}(0,+\infty)$, *the integral operator* T_2 *defined by*

$$T_2(f)_n = \int_0^{+\infty} \ln\left(\frac{n^{\lambda_1} + c_2 x^{\lambda_2}}{n^{\lambda_1} + c_1 x^{\lambda_2}}\right) f(x)\mathrm{d}x$$

is a bounded operator from $L_q^{(1-\frac{\lambda_2}{2})q-1}(0,+\infty)$ *to* $l_q^{-\frac{\lambda_1}{2}q-1}$, *and*

$$\|T_1\| = \|T_2\| = \frac{2\pi}{\lambda_1^{1/q}\lambda_2^{1/p}}\left(\frac{1}{\sqrt{c_1}} - \frac{1}{\sqrt{c_2}}\right).$$

Proof. Let

$$K(n,x) = \ln\left(\frac{n^{\lambda_1} + c_2 x^{\lambda_2}}{n^{\lambda_1} + c_1 x^{\lambda_2}}\right).$$

Then $K(u,v) \geq 0$ is a super-homogeneous function with parameters $\{0, 0, -\frac{\lambda_1}{\lambda_2}, -\frac{\lambda_2}{\lambda_1}\}$. Take $a = 1 + \frac{\lambda_1}{2}$ and $b = 1 - \frac{\lambda_2}{2}$. Since $\sigma_1 = 0$, $\sigma_2 = 0$, $\tau_1 = -\frac{\lambda_1}{\lambda_2}$ and $\tau_2 = -\frac{\lambda_2}{\lambda_1}$, we have $\tau_1\tau_2 = 1$ and $\tau_1 b - a = \tau_1 - \sigma_1 - 1$. Letting $\varphi(t) = \ln(t^{\lambda_1} + c_2) - \ln(t^{\lambda_1} + c_1)$, then

$$\varphi'(x) = \frac{\lambda_1 t^{\lambda_1 - 1}}{t^{\lambda_1} + c_2} - \frac{\lambda_1 t^{\lambda_1 - 1}}{t^{\lambda_1} + c_1} = -\frac{(c_2 - c_1)\lambda_1 t^{\lambda_1 - 1}}{(t^{\lambda_1} + c_2)(t^{\lambda_1} + c_1)} < 0,$$

so $\varphi(x)$ is decreasing on $(0,+\infty)$. And in view of $\lambda_1>0$, we deduce that

$$K(t,1)t^{-a}=\ln\left(\frac{t^{\lambda_1}+c_2}{t^{\lambda_1}+c_1}\right)t^{-1-\frac{\lambda_1}{2}}=\varphi(t)t^{-1-\frac{\lambda_1}{2}}$$

is decreasing on $(0,+\infty)$. And since

$$\begin{aligned}
&W_1(-b)\\
&=\int_0^{+\infty}K(1,t)t^{-b}\mathrm{d}t\\
&=\int_0^{+\infty}\ln\left(\frac{1+c_2t^{\lambda_2}}{1+c_1t^{\lambda_2}}\right)t^{\frac{\lambda_2}{2}-1}\mathrm{d}t\\
&=\frac{2}{\lambda_2}\left[t^{\frac{\lambda_2}{2}}\ln\left(\frac{1+c_2t^{\lambda_2}}{1+c_1t^{\lambda_2}}\right)\Bigg|_0^{+\infty}+\int_0^{+\infty}\frac{\lambda_2(c_2-c_1)t^{\frac{3}{2}\lambda_2-1}}{(1+c_1t^{\lambda_2})(1+c_2t^{\lambda_2})}\mathrm{d}t\right]\\
&=2(c_2-c_1)\int_0^{+\infty}\frac{t^{\frac{3}{2}\lambda_2-1}}{(1+c_1t^{\lambda_2})(1+c_2t^{\lambda_2})}\mathrm{d}t\\
&=\frac{4(c_2-c_1)}{\lambda_2}\left(\frac{1}{c_2-c_1}\int_0^{+\infty}\frac{1}{1+c_1u^2}\mathrm{d}u-\frac{1}{c_2-c_1}\int_0^{+\infty}\frac{1}{1+c_2u^2}\mathrm{d}u\right)\\
&=\frac{4(c_2-c_1)}{\lambda_2}\left(\frac{\pi}{2\sqrt{c_1}(c_2-c_1)}-\frac{\pi}{2\sqrt{c_2}(c_2-c_1)}\right)\\
&=\frac{2\pi}{\lambda_2}\left(\frac{1}{\sqrt{c_1}}-\frac{1}{\sqrt{c_2}}\right),
\end{aligned}$$

setting $t=\frac{1}{u}$, we have

$$\begin{aligned}
&W_2(-a)\\
&=\int_0^{+\infty}K(t,1)t^{-a}\mathrm{d}t=\int_0^{+\infty}\ln\left(\frac{c_2+t^{\lambda_1}}{c_1+t^{\lambda_1}}\right)t^{-1-\frac{\lambda_1}{2}}\mathrm{d}t\\
&=\int_0^{+\infty}\ln\left(\frac{1+c_2u^{\lambda_1}}{1+c_1u^{\lambda_1}}\right)u^{\frac{\lambda_1}{2}-1}\mathrm{d}u\\
&=\frac{2\pi}{\lambda_1}\left(\frac{1}{\sqrt{c_1}}-\frac{1}{\sqrt{c_2}}\right).
\end{aligned}$$

After a simple calculation, one can also obtain

$$\begin{aligned}
\alpha&=a(p-1)+\tau_1(b-1)+\sigma_1=\left(1+\frac{\lambda_1}{2}\right)p-1,\\
\beta&=b(q-1)+\tau_2(a-1)+\sigma_2=\left(1-\frac{\lambda_2}{2}\right)q-1,\\
\alpha(1-q)&=\left[\left(1+\frac{\lambda_1}{2}\right)p-1\right](1-q)=-\frac{\lambda_1}{2}q-1,\\
\beta(1-p)&=\left[\left(1-\frac{\lambda_2}{2}\right)q-1\right](1-p)=\frac{\lambda_2}{2}p-1,
\end{aligned}$$

and

$$\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}W_1(-b)=\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{q}}\frac{2\pi}{\lambda_2}\left(\frac{1}{\sqrt{c_1}}-\frac{1}{\sqrt{c_2}}\right)=\frac{2\pi}{\lambda_1^{1/q}\lambda_2^{1/p}}\left(\frac{1}{\sqrt{c_1}}-\frac{1}{\sqrt{c_2}}\right).$$

In summary and according to Theorem 4.1, we know that Corollary 4.2 holds. □

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