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INVARIANT ANALYSIS AND CONSERVATION LAWS FOR THE SPACE-TIME FRACTIONAL KDV-LIKE EQUATION

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Abstract Fractional calculus plays an essential role in describing nonlinear phenomena appears in applied sciences. In this article, we handle mainly the Korteweg-de Vries (KdV)-like equation which can be used to depicted the shallow water waves evolution mechanism in the sense of the space-time fractional derivative of the Riemann-Liouville. Firstly, on the basis of the Lie symmetry analysis technology, the symmetry of this considered model was constructed. Then, this equation can be changed into a fractional ordinary differential equation with the help of the Erdélyi-Kober fractional operators. Subsequently, the one-parameter group of Lie point transformation and a special type exact solution of this researched model were also obtained. Lastly, based on the nonlinear self-adjointness, conservation laws of the space-time fractional KdV-like equation can be found. These results can provide us with a new scheme for studying space-time fractional differential equations.

Keywords Invariant analysis, similarity reduction, one-parameter transformation lie group, nonlinear self-adjointness, conservation laws.

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1. Introduction

Due to fractional calculus in describing nonlinear phenomena has unique interpretation than integer calculus, there are a lot of papers in this areas [28, 36, 42]. Fractional differential equations often appear in various directions, such as applied mathematics, physics, optimized control and signal processing [10, 34, 40, 41], etc. It is known that there are many approaches to deal with the mathematical physics [1, 11, 25]. However, fractional differential equations were handled more difficult than integer differential equations. Regarding this situation, scientific workers advanced many powerful mathematical tools to deal with them, such as the finite

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difference method [27], the spectral methods [6] and the fractional Lie symmetry method [3, 9, 14, 23, 24, 35, 38, 43] and others.

In this article, we apply the fractional Lie symmetry method to solve the space-time fractional KdV-like equation [7] in the sense of the factional derivative of the Riemann-Liouville, which reads

$$D_t^{\alpha}u + D_x^{\beta}u + auu_{xxxxx} + bu_x u_{xxxx} + cu_{xx}u_{xxx} = 0,\ 0 < \alpha,\ \beta \leqslant 1, \tag{1.1}$$

where a, b and c are free parameter.

The Lie symmetry method plays an essential role in solving differential equations including partial differential equations and ordinary differential equations of fractional order or integer order [20, 44]. In recent years, the Lie symmetry method was used to handle the time fractional differential equations [5, 15, 21, 26, 37]. However, there are a few papers to solve the space-time fractional differential equations [2, 13, 18, 22]. The aims of this paper are utilized the fractional Lie symmetry approach to obtain the symmetry, similarity reduction, one-parameter group of Lie point transformation, exact solution and conservation laws of this considered equation.

2. Preliminaries

In this section, we show the definition of fractional derivative of Riemann-Liouville, which can be used to solve this problem in this article [16, 33].

Definition 2.1. [16, 33] The Riemann-Liouville fractional derivative of a function $u(t)$ with independent variable t, for $\alpha > 0$, is given by

$$D_t^{\alpha}u(t) = \begin{cases} \frac{d^m u(t)}{dt^m},\ \alpha = m \in N, \\ \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t \frac{u(s)}{(t-s)^{\alpha+1-m}}ds,\ m-1 < \alpha < m,\ m \in N, \end{cases} \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler's Gamma function.

Definition 2.2. [16, 33] The Riemann-Liouville fractional derivative of a function $u(x,t)$ with independent variables x and t, for $\alpha > 0$, is given by

$$D_t^{\alpha}u(x,t) = \begin{cases} \frac{\partial^m u(x,t)}{\partial t^m},\ \alpha = m \in N, \\ \frac{1}{\Gamma(m-\alpha)}\frac{\partial^m}{\partial t^m}\int_0^t \frac{u(x,s)}{(t-s)^{\alpha+1-m}}ds,\ m-1 < \alpha < m,\ m \in N. \end{cases} \tag{2.2}$$

The Riemann-Liouville fractional derivative has the following simple properties.

$$D_t^{\alpha}C = \frac{C}{\Gamma(1-\alpha)}t^{-\alpha} \quad \text{and} \quad D_t^{\alpha}t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}t^{\gamma-\alpha},$$

where C is an arbitrary constant.

3. Symmetry analysis of Eq.(1.1)

3.1. Description of the symmetry analysis for the space-time fractional PDEs

Here we give a general idea to deal with the space-time fractional partial differential equations (PDEs) with two independent variables x and t by using the fractional

Lie symmetry method [2,13,18,22]. That is to say

$$F(x,t,u,D_t^{\alpha}u,D_x^{\beta}u,u_{xx},u_{xt},...)=0,\quad \alpha>0,\quad \beta>0,\tag{3.1}$$

where D_t^{α} and D_x^{β} were defined in the sense of the Riemann-Liouville fractional derivative.

An one-parameter (ε) group of Lie point transformation has the forms [2,4,13,18,22,30,32]

$$\begin{aligned}
&\widetilde{t}=t+\varepsilon\tau(x,t,u)+O(\varepsilon^2),\\
&\widetilde{x}=x+\varepsilon\xi(x,t,u)+O(\varepsilon^2),\\
&\widetilde{u}=u+\varepsilon\eta(x,t,u)+O(\varepsilon^2),\\
&\frac{\partial^{\alpha}\widetilde{u}}{\partial\widetilde{t^{\alpha}}}=\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+\varepsilon\eta^{\alpha,t}(x,t,u)+O(\varepsilon^2),\\
&\frac{\partial^{\beta}\widetilde{u}}{\partial\widetilde{x^{\beta}}}=\frac{\partial^{\beta}u}{\partial x^{\beta}}+\varepsilon\eta^{\beta,x}(x,t,u)+O(\varepsilon^2),\\
&\frac{\partial^{2}\widetilde{u}}{\partial\widetilde{x}^{2}}=\frac{\partial^{2}u}{\partial x^{2}}+\varepsilon\eta^{2x}(x,t,u)+O(\varepsilon^2),\\
&...,\\
&\frac{\partial^{k}\widetilde{u}}{\partial\widetilde{t}^{k}}=\frac{\partial^{k}u}{\partial x^{k}}+\varepsilon\eta^{kx}(x,t,u)+O(\varepsilon^2),
\end{aligned}\tag{3.2}$$

where ξ,τ,η are infinitesimals, and $\eta^{kx},\eta^{\alpha,t},\eta^{\beta,x}$ are the extended infinitesimals.

Eq.(3.1) with the one-parameter Lie group (3.2) has the following vector field

$$X=\tau(x,t,u)\frac{\partial}{\partial t}+\xi(x,t,u)\frac{\partial}{\partial x}+\eta(x,t,u)\frac{\partial}{\partial u},\tag{3.3}$$

such that the space-time fractional prolonged generator can be seen that [2,13,18,22]

$$pr^{(\alpha,\beta,n)}X=X+\eta^{\alpha,t}\frac{\partial}{\partial D_t^{\alpha}u}+\eta^{\beta,x}\frac{\partial}{\partial D_x^{\beta}u}+\eta^{2x}\frac{\partial}{\partial u_{xx}}+....+\eta^{nx}\frac{\partial}{\partial u_{nx}},\tag{3.4}$$

where n is the order of Eq.(3.1), and $u_{nx}=\frac{\partial^n u}{\partial x^n}$.

Meanwhile, the operators $\eta^{nx}(n=2,3,...)$ and $\eta^{\alpha,t},\eta^{\beta,x}$ can be shown as [2,4,13,18,22,30,32]

$$\begin{aligned}
&\eta^{\alpha,t}=D_t^{\alpha}(\eta)+\xi D_t^{\alpha}(u_x)-D_t^{\alpha}(\xi u_x)+D_t^{\alpha}(D_t(\tau)u)-D_t^{\alpha+1}(\tau u)+\tau D_t^{\alpha+1}(u),\\
&\eta^{\beta,x}=D_x^{\beta}(\eta)+D_x^{\beta}(u(D_x(\xi)))-D_x^{\beta+1}(\xi u)+\xi D_x^{\beta+1}(u)+\tau D_x^{\beta}(u_t)-D_x^{\beta}(\tau u_t),\\
&\eta^{2x}=D_x(\eta^x)-u_{2x}D_x(\xi)-u_{xt}D_x(\tau),\\
&...,\\
&\eta^{kx}=D_x(\eta^{(k-1)x})-u_{kx}D_x(\xi)-u_{(k-1)xt}D_x(\tau),
\end{aligned}\tag{3.5}$$

where D_x and D_t are the total derivative operators [4,30,32]

$$\begin{aligned}
D_t&=\frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{tt}\frac{\partial}{\partial u_t}+u_{xt}\frac{\partial}{\partial u_x}+...\ ,\\
D_x&=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xt}\frac{\partial}{\partial u_t}+...\ .
\end{aligned}$$

Here applying the generalized Leibnitz's rule [31]

$$D_t^\alpha(f(t)g(t)) = \sum_{n=0}^{\infty} \begin{pmatrix} \alpha \\ n \end{pmatrix} D_t^{\alpha-n} f(t) D_t^n g(t), \alpha > 0, \tag{3.6}$$

with

$$\begin{pmatrix} \alpha \\ n \end{pmatrix} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}$$

and the chain rule for composite function [29]

$$\frac{d^m g(y(t))}{dt^m} = \sum_{k=0}^{m}\sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} \frac{1}{k!}[-y(t)]^r \frac{d^m}{dt^m}[y(t)^{k-r}] \frac{d^k g(y(t))}{dy^k},$$

the fractional extended operators $\eta^{\alpha,t}$ and $\eta^{\beta,x}$ through a series of rigorous calculations can be given by the following forms [2, 13, 18, 22]

$$\begin{aligned} \eta^{\alpha,t} = {} & \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau)) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} \\ & + \sum_{n=1}^{\infty} [\begin{pmatrix} \alpha \\ n \end{pmatrix} \frac{\partial^n \eta_u}{\partial t^n} - \begin{pmatrix} \alpha \\ n+1 \end{pmatrix} D_t^{n+1}(\tau)] D_t^{\alpha-n}(u) \\ & - \sum_{n=1}^{\infty} \begin{pmatrix} \alpha \\ n \end{pmatrix} D_t^n(\xi) D_t^{\alpha-n}(u_x) + \mu_\alpha, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \eta^{\beta,x} = {} & \frac{\partial^\beta \eta}{\partial x^\beta} + (\eta_u - \beta D_\xi(\xi)) \frac{\partial^\beta u}{\partial x^\beta} - u \frac{\partial^\beta \eta_u}{\partial x^\beta} \\ & + \sum_{n=1}^{\infty} [\begin{pmatrix} \beta \\ n \end{pmatrix} \frac{\partial^n \eta_u}{\partial x^n} - \begin{pmatrix} \beta \\ n+1 \end{pmatrix} D_x^{n+1}(\xi)] D_x^{\beta-n}(u) \\ & - \sum_{n=1}^{\infty} \begin{pmatrix} \beta \\ n \end{pmatrix} D_x^n(\tau) D_x^{\beta-n}(u_t) + \mu_\beta, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \mu_\alpha = {} & \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1} \begin{pmatrix} \alpha \\ n \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} k \\ r \end{pmatrix} \frac{1}{k!} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \\ & \times (-u)^r \cdot D_t^m(u^{k-r}) \frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k} \end{aligned}$$

and

$$\mu_\beta = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1} \begin{pmatrix} \beta \\ n \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} k \\ r \end{pmatrix} \frac{1}{k!} \frac{x^{n-\beta}}{\Gamma(n+1-\alpha)}$$

$$\times (-u)^r \cdot D_x^m (u^{k-r}) \frac{\partial^{n-m+k}\eta}{\partial x^{n-m}\partial u^k}.$$

The invariance criterion for the space-time fractional PDEs (3.1) can be expressed as [2,13,18,22]

$$pr^{(\alpha,\beta,n)} X(F)|_{F=0} = 0, \tag{3.9}$$

where $pr^{(\alpha,\beta,n)}$ is the space-time fractional prolonged generator (3.4).

In what follows, we apply the above processes to yield the vector field (3.3) to the space-time fractional KdV-like equation (1.1).

Remark 3.1. In order to maintain the structure of the Riemann-Liouville fractional derivative, the following conditions need to satisfy

$$\tau(x,t,u)|_{t=0} = 0 \ \text{ and } \ \xi(x,t,u)|_{x=0} = 0. \tag{3.10}$$

3.2. Symmetry of Eq.(1.1)

Here we apply this processes of the above stated to get the vector field (3.3) of Eq.(1.1). At the beginning of, we assume that Eq.(1.1) is invariant with the aid of the one-parameter Lie group (3.2), then the transformed equation writes

$$D_t^\alpha \tilde{u} + D_x^\beta \tilde{u} + a\tilde{u}\widetilde{u_{xxxxxx}} + b\widetilde{u_x}\widetilde{u_{xxxx}} + c\widetilde{u_{xx}}\widetilde{u_{xxx}} = 0, \ 0 < \alpha, \beta \le 1. \tag{3.11}$$

By means of one-parameter Lie group (3.2) and invariant equation (3.11), the symmetry equation of this researched model can be obtained

$$\eta^{\alpha,t} + \eta^{\beta,x} + a\eta u_{xxxxxx} + au\eta^{5x} + b\eta^x u_{xxxx} + bu_x\eta^{4x} + c\eta^{2x}u_{xxx} + cu_{xx}\eta^{3x} = 0. \tag{3.12}$$

Substituting (3.5) together along with equations (3.7) and (3.8) into (3.12), equating different powers of function u and its various derivatives to zero, then after a series of treatments, the over-determined system of linear fractional differential equations can be yielded by

$$\begin{aligned}
&\eta_{uu} = \tau_u = \tau_x = \xi_u = \xi_t = 0,\\
&\alpha\tau_t - \beta\xi_x = 0, \alpha u\tau_t - 5u\xi_x + \eta = 0,\\
&\beta\xi_{xx} - \xi_{xx} - 2\eta_{xu} = 0, -\alpha\tau_{tt} + \tau_{tt} + 2\eta_{tu} = 0,\\
&au\eta_{xxxxxx} + D_x^\beta(\eta) - uD_x^\beta(\eta_u) + D_t^\alpha(\eta) - uD_t^\alpha(\eta_u) = 0,\\
&\begin{pmatrix}\alpha\\ n\end{pmatrix} D_t^n(\eta_u) - \begin{pmatrix}\alpha\\ n+1\end{pmatrix} D_t^{n+1}(\tau) = 0, \ n = 1,2,\dots,\\
&\begin{pmatrix}\beta\\ n\end{pmatrix} D_x^n(\eta_u) - \begin{pmatrix}\beta\\ n+1\end{pmatrix} D_x^{n+1}(\xi) = 0, \ n = 1,2,\dots .
\end{aligned} \tag{3.13}$$

Solving the over-determined system (3.13), we can obtain the values of τ, ξ and η of the forms

$$\tau = c_3\beta t + c_2, \quad \xi = c_3\alpha x + c_1, \quad \eta = -c_3\alpha\beta u + 5c_3\alpha u, \tag{3.14}$$

where c_1, c_2 and c_3 are arbitrary free constants.

Considering the conditions (3.10), we get

$$c_1 = c_2 = 0. \tag{3.15}$$

Hence, the vector field (3.3) becomes

$$X = \frac{t}{\alpha}\frac{\partial}{\partial t} + \frac{x}{\beta}\frac{\partial}{\partial x} + u(\frac{5-\beta}{\beta})\frac{\partial}{\partial u}. \tag{3.16}$$

In what follows, according to the above general vector field (3.16), this original equation (1.1) can be further reduced.

4. Similarity reduction of Eq.(1.1)

In this section, on the basis of the known vector field (3.16), the space-time fractional KdV-like can be changed into lower dimensional differential equation of fractional order through the Erdélyi-Kober fractional differential operators.

For vector field $X = \frac{t}{\alpha}\frac{\partial}{\partial t} + \frac{x}{\beta}\frac{\partial}{\partial x} + u(\frac{5-\beta}{\beta})\frac{\partial}{\partial u}$, it has the following characteristic equation

$$\frac{\beta dx}{x} = \frac{\alpha dt}{t} = \frac{\beta du}{u(5-\beta)}. \tag{4.1}$$

As a result, the similarity transformation and similarity variable

$$u = t^{\frac{\alpha(5-\beta)}{\beta}} f(\xi),\ \xi = xt^{-\frac{\alpha}{\beta}}, \tag{4.2}$$

can be yielded, respectively.

We can see that the following result.

Theorem 4.1. *Upon the similarity transformation $u = t^{\frac{\alpha(5-\beta)}{\beta}} f(\xi)$ with similarity variable $\xi = xt^{-\frac{\alpha}{\beta}}$, the space-time fractional KdV-like equation (1.1) can be reduced into a nonlinear ordinary differential equation of fractional order of the form*

$$(P_{\frac{\beta}{\alpha}}^{1-\alpha+\frac{\alpha(5-\beta)}{\beta},\alpha} f)(\xi) + \xi^{-\beta}(D_1^{-\beta,\beta} f)(\xi) + aff_{\xi\xi\xi\xi\xi} + bf_\xi f_{\xi\xi\xi\xi} + cf_{\xi\xi}f_{\xi\xi\xi} = 0 \tag{4.3}$$

with the left Erdélyi-Kober fractional operator and the right Erdélyi-Kober fractional operator [17]

$$(P_{\varpi}^{\sigma,\alpha} f)(\zeta) := \sum_{j=0}^{m-1}(\sigma + j - \frac{1}{\varpi}\zeta\frac{d}{d\zeta})(K_{\varpi}^{\sigma+\alpha,m-\alpha} f)(\zeta), \zeta > 0, \varpi > 0, \alpha > 0,$$

$$m = \begin{cases} [\alpha] + 1, & \text{if } \alpha \notin N, \\ \alpha, & \text{if } \alpha \in N, \end{cases}$$

and

$$(D_{\varpi}^{\sigma,\beta} f)(\zeta) := \sum_{j=0}^{m}(\sigma + j + \frac{1}{\varpi}\zeta\frac{d}{d\zeta})(I_{\varpi}^{\sigma+\beta,m-\beta} f)(\zeta), \zeta > 0, \varpi > 0, \beta > 0,$$

$$m = \begin{cases} [\beta] + 1, & \text{if } \beta \notin N, \\ \beta, & \text{if } \beta \in N, \end{cases}$$

where

$$(K_{\varpi}^{\sigma,\alpha}f)(\zeta) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^{\infty}(\chi-1)^{\alpha-1}\chi^{-(\sigma+\alpha)}f(\zeta\chi^{\frac{1}{\varpi}})d\chi, \ \textit{if } \alpha>0, \\ f(\zeta), \ \textit{if } \alpha=0, \end{cases}$$

and

$$(I_{\varpi}^{\sigma,\beta}f)(\zeta) := \begin{cases} \frac{1}{\Gamma(\beta)}\int_0^{1}(1-\chi)^{\beta-1}\chi^{\sigma}f(\zeta\chi^{\frac{1}{\varpi}})d\chi, \ \textit{if } \beta>0, \\ f(\zeta), \ \textit{if } \beta=0, \end{cases}$$

are the left Erdélyi-Kober integral operator and the right Erdélyi-Kober fractional integral operator [17].

Proof. Taking $n-1<\alpha<n, n=1,2,3,....$ According to the definition of fractional derivative of Riemann-Liouville, we have

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\partial^n}{\partial t^n}[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}s^{\frac{\alpha(5-\beta)}{\beta}}f(xs^{-\frac{\alpha}{\beta}})ds]. \tag{4.4}$$

Considering $v=\frac{t}{s}$, the above equation becomes

$$\begin{aligned}\frac{\partial^{\alpha}u}{\partial t^{\alpha}} &= \frac{\partial^n}{\partial t^n}[\frac{1}{\Gamma(n-\alpha)}t^{n-\alpha+\frac{\alpha(5-\beta)}{\beta}}\int_1^{\infty}(v-1)^{n-\alpha-1}v^{-(n-\alpha+1+\frac{\alpha(5-\beta)}{\beta})}f(\xi v^{\frac{\alpha}{\beta}})dv] \\ &= \frac{\partial^n}{\partial t^n}[t^{n-\alpha+\frac{\alpha(5-\beta)}{\beta}}(K_{\frac{\beta}{\alpha}}^{1+\frac{\alpha(5-\beta)}{\beta},n-\alpha}f)(\xi)].\end{aligned} \tag{4.5}$$

We note that the $\xi=xt^{-\frac{\alpha}{\beta}}$ and $\phi(\xi)\in C^1(0,\infty)$ have the relations of the form

$$t\frac{d}{dt}\phi(\xi) = -\frac{\alpha}{\beta}\xi\frac{d}{d\xi}\phi(\xi). \tag{4.6}$$

Further, equation (4.5) becomes

$$\begin{aligned}\frac{\partial^{\alpha}u}{\partial t^{\alpha}} &= \frac{\partial^{n-1}}{\partial t^{n-1}}[t^{n-\alpha-1+\frac{\alpha(5-\beta)}{\beta}}\times(n-\alpha+\frac{\alpha(5-\beta)}{\beta}-\frac{\alpha}{\beta}\xi\frac{d}{d\xi})(K_{\frac{\beta}{\alpha}}^{1+\frac{\alpha(5-\beta)}{\beta},n-\alpha}f)(\xi))] \\ &= ... \ \ (n-1 \text{ times}) \\ &= t^{-\alpha+\frac{\alpha(5-\beta)}{\beta}}\prod_{j=0}^{n-1}(1-\alpha+\frac{\alpha(5-\beta)}{\beta}+j-\frac{\alpha}{\beta}\xi\frac{d}{d\xi})(K_{\frac{\beta}{\alpha}}^{1+\frac{\alpha(5-\beta)}{\beta},n-\alpha}f)(\xi)).\end{aligned} \tag{4.7}$$

As a result, we have

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = t^{-\alpha+\frac{\alpha(5-\beta)}{\beta}}(P_{\frac{\beta}{\alpha}}^{1-\alpha+\frac{\alpha(5-\beta)}{\beta},\alpha}f)(\xi). \tag{4.8}$$

Similarity, the space fractional derivative $\frac{\partial^{\beta}u}{\partial x^{\beta}}$ has

$$\frac{\partial^{\beta}u}{\partial x^{\beta}} = \frac{\partial^n}{\partial x^n}(\frac{1}{\Gamma(n-\beta)}\int_0^x(x-\varphi)^{n-\beta-1}t^{\frac{\alpha(5-\beta)}{\beta}}f(\varphi t^{-\frac{\alpha}{\beta}})d\varphi). \tag{4.9}$$

Let $s = \frac{\varphi}{x}$, then Eq.(4.9) becomes

$$\begin{aligned}\frac{\partial^\beta u}{\partial x^\beta} &= \frac{\partial^n}{\partial x^n}(\frac{1}{\Gamma(n-\beta)}x^{n-\beta}t^{\frac{\alpha(5-\beta)}{\beta}}\int_0^1(1-s)^{n-\beta-1}f(s\xi)ds) \\ &= t^{\frac{\alpha(5-\beta)}{\beta}}\frac{\partial^n}{\partial x^n}[x^{n-\beta}\frac{1}{\Gamma(n-\beta)}\int_0^1(1-s)^{n-\beta-1}f(s\xi)ds] \\ &= t^{\frac{\alpha(5-\beta)}{\beta}}\frac{\partial^n}{\partial x^n}[x^{n-\beta}I_1^{0,n-\beta}f](\xi).\end{aligned} \tag{4.10}$$

For $\xi = xt^{-\frac{\alpha}{\beta}}$, the following relation holds:

$$x\frac{d}{dx}\phi(\xi) = \xi\frac{d}{d\xi}\phi(\xi). \tag{4.11}$$

Hence, Eq.(4.9) as follows:

$$\begin{aligned}\frac{\partial^\beta u}{\partial x^\beta} &= t^{\frac{\alpha(5-\beta)}{\beta}}\frac{\partial^{n-1}}{\partial x^{n-1}}[\frac{\partial}{\partial x}(x^{n-\beta}I_1^{0,n-\beta}f)(\xi)] \\ &= t^{\frac{\alpha(5-\beta)}{\beta}}\frac{\partial^{n-1}}{\partial x^{n-1}}[x^{n-\beta-1}((n-\beta+\xi\frac{d}{d\xi})I_1^{0,n-\beta}f)(\xi)] \\ &= \ldots \quad (n-1 \text{ times}) \\ &= t^{\frac{\alpha(5-\beta)}{\beta}}x^{-\beta}\prod_{j=1}^{n}((-\beta+j+\xi\frac{d}{d\xi})I_1^{0,n-\beta}f)(\xi) \\ &= t^{\frac{\alpha(5-\beta)}{\beta}}x^{-\beta}(D_1^{-\beta,\beta}f)(\xi).\end{aligned} \tag{4.12}$$

As a result, we have

$$\frac{\partial^\beta u}{\partial x^\beta} = t^{\frac{\alpha(5-\beta)}{\beta}}x^{-\beta}(D_1^{-\beta,\beta}f)(\xi). \tag{4.13}$$

Substituting equations (4.8) and (4.13) with (4.2) into Eq.(1.1), it can be translated into the following form

$$(P_{\frac{\beta}{\alpha}}^{1-\alpha+\frac{\alpha(5-\beta)}{\beta},\alpha}f)(\xi) + \xi^{-\beta}(D_1^{-\beta,\beta}f)(\xi) + aff_{\xi\xi\xi\xi\xi} + bf_\xi f_{\xi\xi\xi\xi} + cf_{\xi\xi}f_{\xi\xi\xi} = 0.$$

This proof is completely finished. □

5. One-parameter Lie group and exact solutions of Eq.(1.1)

5.1. One-parameter Lie group of Eq.(1.1)

In this subsection, we want to find the one-parameter group of Lie point transformation of the space-time fractional KdV-like equation through the vector fields (3.16). Here we need to consider firstly the following initial problems [4,30,32]

$$\begin{aligned}\frac{d(x^*(\varepsilon))}{d\varepsilon} &= \xi(x^*(\varepsilon),t^*(\varepsilon),u^*(\varepsilon)), x^*(0) = x, \\ \frac{d(t^*(\varepsilon))}{d\varepsilon} &= \tau(x^*(\varepsilon),t^*(\varepsilon),u^*(\varepsilon)), t^*(0) = t, \\ \frac{d(u^*(\varepsilon))}{d\varepsilon} &= \eta(x^*(\varepsilon),t^*(\varepsilon),u^*(\varepsilon)), u^*(0) = u,\end{aligned} \tag{5.1}$$

where $\varepsilon \ll 1$ is a small parameter.

Solving the initial problems (5.1) with the vector fields (3.16), the one-parameter group of Lie point transformation $g(\varepsilon)$ of Eq.(1.1) has

$$g(\varepsilon):(x,t,u)\to(e^{\frac{1}{\beta}\cdot\varepsilon}x,e^{\frac{1}{\alpha}\cdot\varepsilon}t,e^{\frac{5-\beta}{\beta}\cdot\varepsilon}u). \tag{5.2}$$

We can obtain the following result of the one-parameter group of Lie point transformation of Eq.(1.1).

Theorem 5.1. *If $u=f(x,t)$ is a solution of the space-time fractional KdV-like equation, then*

$$u=e^{\frac{\beta-5}{\beta}\varepsilon}f(e^{-\frac{1}{\beta}\cdot\varepsilon}x,e^{-\frac{1}{\alpha}\cdot\varepsilon}t) \tag{5.3}$$

is also solution to Eq.(1.1).

5.2. Exact solutions of Eq.(1.1)

In this subsection, we consider two types special exact solutions for Eq.(1.1). First of all, we suppose that Eq.(1.1) has the solution of the form

$$u=u(t). \tag{5.4}$$

Inserting equation (5.4) into equation (1.1), we have

$$D_t^{\alpha}u(t)+\Lambda\cdot u(t)=0, \tag{5.5}$$

where $\Lambda=\frac{1}{\Gamma(1-\beta)}x^{-\beta}$.

Applying the Laplace transform method [19], a solution of Eq.(5.5) can be written as

$$u=\sum_{k=0}^{n-1}b_k t^{\alpha-k-1}E_{\alpha,\alpha}(-\Lambda\cdot t^{\alpha}), \tag{5.6}$$

where $E_{\alpha,\alpha}(\cdot)$ is two-parameter Mittag-Leffler function [39].

Then, we assume that Eq.(1.1) has the solution of the form

$$u=u(x). \tag{5.7}$$

Plugging Eq.(5.7) into Eq.(1.1) has

$$D_x^{\beta}u+\Pi\cdot u_x+auu_{xxxxx}+bu_xu_{xxxx}+cu_{xx}u_{xxx}=0, \tag{5.8}$$

where $\Pi=\frac{1}{\Gamma(1-\alpha)}t^{-\alpha}$.

We once again apply the fractional Lie symmetry method to deal with equation (5.8). As a result, the vector field of Eq.(5.8) can be shown as

$$X=-\frac{1}{\beta-5}x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}. \tag{5.9}$$

On the basis of the vector field (5.9), we can do similar discussions as the above given. Here we omit them.

Remark 5.1. According to Theorem 5.1, we know that

$$u=e^{\frac{\beta-5}{\beta}\varepsilon}\cdot\sum_{k=0}^{n-1}b_k(e^{-\frac{1}{\alpha}\cdot\varepsilon}t)^{\alpha-k-1}E_{\alpha,\alpha}(-\frac{e^{-\varepsilon}}{\Gamma(1-\beta)}x^{-\beta}\cdot(e^{-\frac{1}{\alpha}\cdot\varepsilon}t)^{\alpha}),$$

is also solution of Eq.(1.1).

6. Nonlinear self-adjointness and conservation laws of Eq.(1.1)

6.1. Nonlinear self-adjointness of Eq.(1.1)

In this subsection, we apply the concept of the nonlinear self-adjoint to construct conservation laws of the space-time fractional KdV-like equation [2,8,12,13,18,22].

A formal Lagrangian function [2,13] for Eq.(1.1) can be written as

$$L{=}p(x,t)(D_t^{\alpha}u+D_x^{\beta}u+auu_{xxxxx}+bu_xu_{xxxx}+cu_{xx}u_{xxx}), \tag{6.1}$$

where $p(x,t)$ is a new function.

The adjoint equation of the space-time fractional KdV-like equation (1.1) is given by

$$A^*=\frac{\delta L}{\delta u}, \tag{6.2}$$

where $\frac{\delta}{\delta u}$ is the Euler-Lagrange operator [2,13]

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}+(D_t^{\alpha})^*\frac{\partial}{\partial(D_t^{\alpha}u)}+...+(D_x^{\beta})^*\frac{\partial}{\partial(D_x^{\beta}u)}+\sum_{k=1}^{\infty}(-1)^kD_{i_1i_2...i_k}\frac{\partial}{\partial(u_j)_{i_1i_2...i_k}}. \tag{6.3}$$

Here $(D_t^{\alpha})^*$ and $(D_x^{\beta})^*$ are adjoint operators of D_t^{α} and D_x^{β}.

Therefore, the adjoint equation (6.2) of Eq.(1.1) has the form

$$\begin{aligned}A^*=&(D_t^{\alpha})^*p+(D_x^{\beta})^*p-c(p_{xxx}u_{xx}+2p_{xx}u_{xxx})+b(p_{xxxx}u_x+4p_{xxx}u_{xx}\\&+6p_{xx}u_{xxx}+3p_xu_{xxxx})-a(p_{xxxxx}u+5p_{xxxx}u_x+10p_{xxx}u_{xx}+5p_xu_{xxxx})\\=&0.\end{aligned} \tag{6.4}$$

Taking $p=\psi(x,t,u)(\psi(x,t,u)\neq 0)$, its different partial derivatives are shown by

$$\begin{aligned}p_x=&\psi_x+\psi_uu_x,\\p_{xx}=&\psi_{xx}+2\psi_{xu}u_x+\psi_uu_{xx}+\psi_{uu}u_x^2,\\p_{xxx}=&\psi_{xxx}+3\psi_{xxu}u_x+3\psi_{xuu}u_x^2+3\psi_{xu}u_{xx}+3\psi_{uu}u_xu_{xx}+\psi_uu_{xxx}+\psi_{uuu}u_x^3,\\p_{xxxx}=&\psi_{xxxx}+4\psi_{xxxu}u_x+6\psi_{xxuu}u_x^2+6\psi_{xxu}u_{xx}+4\psi_{xuuu}u_x^3+12\psi_{xuu}u_xu_{xx}\\&+4\psi_{xu}u_{xxx}+4\psi_{uu}u_xu_{xxx}+6\psi_{uuu}u_x^2u_{xx}+3\psi_{uu}u_{xx}^2+\psi_uu_{xxxx}\\&+\psi_{uuuu}u_x^4,\\p_{xxxxx}=&\psi_{xxxxx}+\psi_uu_{xxxxx}+5\psi_{xuuuu}u_x^4+10\psi_{xxuuu}u_x^3+10\psi_{xxxuu}u_x^2\\&+\psi_{uuuuu}u_x^5+15\psi_{xuu}u_{xx}^2+5\psi_{xxxxu}u_x+10\psi_{xxxu}u_{xx}+10\psi_{xxu}u_{xxx}\\&+5\psi_{xu}u_{xxxx}+10\psi_{uuuu}u_{xx}u_x^3+10\psi_{uuu}u_{xxx}u_x^2+10\psi_{uu}u_{xx}u_{xxx}\\&+30\psi_{xuuu}u_{xx}u_x^2+30\psi_{xxuu}u_xu_{xx}+20\psi_{xuu}u_xu_{xxx}+15\psi_{uuu}u_xu_{xx}^2\\&+5\psi_{uu}u_xu_{xxxx}.\end{aligned} \tag{6.5}$$

Then, substituting equation (6.5) into equation (6.4), we have the self-adjoint condition

$$\frac{\delta L}{\delta u}|_{p=\psi(x,t,u)}=\mu E, \tag{6.6}$$

where

$$E = D_t^\alpha u + D_x^\beta u + auu_{xxxxx} + bu_x u_{xxxx} + cu_{xx}u_{xxx}$$

and μ is an undermined coefficient.

Expanding (6.6) and comparing the coefficients ψ and its various partial derivatives, we obtain

$$v = \widehat{a} = constant. \tag{6.7}$$

In what follows, we will use it to construct conservation laws of Eq.(1.1).

Remark 6.1. For adjoint operators $(D_t^\alpha)^*$ and $(D_x^\beta)^*$, they have the following relations with the Caputo factional derivative as follows:

$$(D_t^\alpha)^* = (-1)^n I_v^{n-\alpha}(D_t^n) = {}_t^C D_v^\alpha,$$
$$(D_x^\beta)^* = (-1)^m I_z^{m-\beta}(D_x^m) = {}_t^C D_z^\beta,$$

with

$$I_v^{n-\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_t^v (s-t)^{n-\alpha-1} f(x,s)ds, \ n = [\alpha]+1,$$
$$I_z^{m-\beta} f(x,t) = \frac{1}{\Gamma(m-\beta)} \int_t^z (s-x)^{m-\beta-1} f(s,t)ds, \ m = [\beta]+1.$$

6.2. Conservation laws of Eq.(1.1)

On the basis of the vector field (3.16) and nonlinear self-adjointness, conservation laws of the space-time fractional KdV-like equation were obtained. We have been known that a vector (C^t, C^x) needs to meet the following equation

$$D_t(C^t) + D_x(C^x)|_{(1)} = 0 \tag{6.8}$$

that contains for all solutions of Eq.(1.1).

For fractional derivatives cases, we require the existence of the fractional generalization of the Noether operators. Fortunately, this thing have been done in papers [2, 8, 12, 13, 18, 22]. Here we directly give the t-component of conserved vector of the form

$$C^t = \sum_{k=0}^{n-1} (-1)^k D_t^{\alpha-1-k}(W) D_t^k \frac{\partial L}{\partial(D_t^\alpha u)} - (-1)^n J_1(W, D_t^n(\frac{\partial L}{\partial(D_t^\alpha u)})), \ n = [\alpha]+1, \tag{6.9}$$

where

$$W = \eta - \tau u_t - \xi u_x$$

is the Lie characteristic equation of vector field (3.3) and

$$J_1(f,g) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^T \frac{f(s,x)g(\upsilon,x)}{(\upsilon-s)^{\alpha+1-n}} d\upsilon ds,$$

is an integral transformation.

In the same idea, the x-component of conserved vector can be similarly given by [2, 8, 12, 13, 18, 22]

$$C^x = \sum_{k=0}^{m-1} (-1)^k D_x^{\beta-1-k}(W) D_x^k (\frac{\partial L}{\partial(D_x^\beta u)}) - (-1)^m J_2(W, D_x^m(\frac{\partial L}{\partial(D_x^\beta u)})), \tag{6.10}$$

$m = [\beta]+1,$

where

$$J_2(f,g) = \frac{1}{\Gamma(m-\beta)} \int_0^x \int_x^X \frac{f(s,t)g(r,t)}{(r-s)^{\alpha+1-n}} drds.$$

For vector field (3.16) of Eq.(1.1), the Lie characteristic equation becomes

$$W = -\frac{x}{\beta}u_x - \frac{t}{\alpha}u_t + \frac{5-\beta}{\beta}u. \tag{6.11}$$

Without loss of generality, we taking $v = 1$ to construct conservation laws of Eq.(1.1) with Eqs.(6.9) and (6.10), respectively.

The t-component of conserved vectors of Eq.(1.1) is found by

$$C^t = \frac{5-\beta}{\beta} I_t^{1-\alpha}(u) - \frac{x}{\beta} I_t^{1-\alpha}(u_x) - \frac{1}{\alpha} I_t^{1-\alpha}(tu_t).$$

The x-component of conserved vectors of Eq.(1.1) is given by

$$C^x = \frac{5-\beta}{\beta} I_x^{1-\beta}(u) - \frac{1}{\beta} I_x^{1-\beta}(xu_x) - \frac{t}{\alpha} I_x^{1-\beta}(u_t).$$

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