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EFFICIENT ALGORITHMS FOR REAL SYMMETRIC TOEPLITZ LINEAR SYSTEM WITH LOW-RANK PERTURBATIONS AND ITS APPLICATIONS*

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Abstract This study investigates efficient algorithms for solving the real symmetric Toeplitz linear system with low-rank perturbations. Based on a new factorization of the real symmetric Toeplitz matrix inversion, we propose a novel and effective method to solve a sequence of linear equations with a same symmetric Toeplitz matrix. Together with the application of the Sherman-Morrison-Woodbury formula, the symmetric Toeplitz linear system with low-rank perturbations can be solved by two proposed algorithms. Moreover, the (structured) perturbation analysis and the applications in image encryption and decryption are given. The numerical results are presented to verify the theoretical analysis.

Keywords Symmetric Toeplitz matrix, perturbations, order-reduction, encryption, decryption.

MSC(2010) 15A06, 15A09, 15B05, 65F05, 65T50.

1. Introduction

Toeplitz and quasi-Toeplitz matrices have been widely studied since they can be applied in mathematical and scientific fields, such as solving differential equations [1, 2, 18, 24, 35], scattering/radiation simulation on thin wires [9], the distribution of neurons across the mouse brain [25], the quantum random number generation [19], high-resolution optical imaging [31], the bandgap engineering of semiconductors [5] and the calculation of the hemodynamic response function [28, 32].

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In our study, we focus on a class of real symmetric Toeplitz matrices with low-rank perturbations, which are represented as $U = (u_{j,k})_{j,k=1}^{n}$:

$$u_{j,k} = \begin{cases} \tau_1 + t_1, & j = 1 \text{ and } k = 2, \\ \tau_2 + t_1, & j = n \text{ and } k = n-1, \\ t_{|j-k|}, & \text{otherwise.} \end{cases} \tag{1.1}$$

So U can be written as

$$U = A + \alpha e_2^T + \beta e_{n-1}^T, \tag{1.2}$$

where $A = (t_{|j-k|})_{j,k=1}^{n}$ is a nonsingular $n \times n$ real symmetrical Toeplitz matrix, $\alpha = [\tau_1, 0, \ldots, 0]^T$, $\beta = [0, \ldots, 0, \tau_2]^T$, $e_2 = [0, 1, 0, \ldots, 0]^T$ and $e_{n-1} = [0, \ldots, 0, 1, 0]^T$.

A large number of literatures exist on solving the Toeplitz and quasi-Toeplitz linear systems. Liu et al. presented fast iterative solvers with complexity $O(n \log_2 n)$ for Toeplitz linear systems [20–23]. In [12], a class of bordered tridiagonal systems of linear equations were solved. Fu et al. [7] developed two new methods to solve the CUPL-Toeplitz linear system [13, 14, 36]. Order-reduction algorithms were applied to the CUPL-Toeplitz linear system [33] for further reducing the complexity. In [6], the authors gave a new decomposition of tridiagonal quasi-Toeplitz matrix by exploiting the matrix structure. We also mention that Krylov subspace methods (KSMs) [27] have been widely used in solving the Toeplitz linear systems. Some preconditioners [10,26] were proposed for accelerating the convergence rate of KSMs.

In mathematical or engineering problems, we may be faced with solving a sequence of Toeplitz linear systems with a same coefficient matrix. In [11, 17], the authors solved these Toeplitz linear systems utilizing the decomposition of the inverse of Toeplitz matrix. Compared with the traditional methods of solving linear equations one by one, the proposed algorithms first solve two Toeplitz linear equations and then obtain the solutions by multiplying the Toeplitz matrix inversion and the vectors.

The stability analysis has been concerned by some scholars [14, 29, 30]. In this paper, we discuss the (structured) perturbation analysis of the factorization of the symmetric Toeplitz matrix inversion.

This article is organized as follows. In Section 2, we propose an algorithm for solving the linear systems with a same real symmetric Toeplitz matrix according to the factorization of the Toeplitz matrix inversion. In Section 3, we extend to the case of solving the symmetric Toeplitz linear system with low-rank perturbations. The stability of the decomposition formula of real symmetric Toeplitz matrix inversion is analyzed in Section 4. Applications and numerical experiments are presented in Section 5 and Section 6, respectively. Some of the conclusions drawn from this work are contained in Section 7.

2. An efficient method for solving linear systems with a same real symmetric Toeplitz matrix

In mathematical or engineering problems, several symmetric Toeplitz linear systems need to be solved. We could get into a situation where these linear systems have a same coefficient matrix. Therefore, we propose an efficient method for solving these linear systems by using the factorization of the Toeplitz matrix inversion.

The factorization of the Toeplitz matrix inversion is proposed by Gohberg et al. in [8, p738]. Supposing that the parameters $\varphi = 1$ and $\psi = i$ involved in Theorem 3.2, we have, if the vectors $p = [p_1, p_2, \ldots, p_n]^T$, $p_1 \neq 0$ and $q = [q_1, q_2, \ldots, q_n]^T$ satisfy

$$Ap = e_1 \quad \text{and} \quad Aq = e_n \tag{2.1}$$

with $e_1 = [1, 0, \ldots, 0]^T$, $e_n = [0, \ldots, 0, 1]^T$, then the Toeplitz matrix A is invertible and

$$A^{-1} = \frac{1}{p_1(1-i)}(S_{I1}C_1 - S_{I2}C_2), \tag{2.2}$$

where $S_{I1} = \text{circ}_{-i}(p_1, ip_n, ip_{n-1}, \ldots, ip_2)^T$, $S_{I2} = \text{circ}_{-i}(iq_n, iq_{n-1}, iq_{n-2}, \ldots, iq_1)^T$ are skew-imaginary circulant matrices [15]; $C_1 = \text{circ}(q_n, q_{n-1}, q_{n-2}, \ldots, q_1)^T$, $C_2 = \text{circ}(p_1, p_n, p_{n-1}, \ldots, p_2)^T$ are circulant matrices [16]. According to the structural characteristics of skew-imaginary circulant matrices and circulant matrices, the first columns of S_{I1}, S_{I2}, C_1 and C_2 are described as $p = [p_1, p_2, \ldots, p_n]^T$, $\hat{q} = [iq_n, q_1, q_2, \ldots, q_{n-1}]^T$, $\tilde{q} = [q_n, q_1, q_2, \ldots, q_{n-1}]^T$ and $p = [p_1, p_2, \ldots, p_n]^T$, respectively.

The vectors p and q satisfy $q = Jp$ when A is a real symmetric Toeplitz matrix, where J is a an anti-identity matrix. Instead of solving two equations $Ap = e_1$ and $Aq = e_n$, if $Ap = e_1$ has the solution $p = [p_1, p_2, \ldots, p_n]^T$ and $p_1 \neq 0$, then the real symmetric Toeplitz matrix A is invertible. Actually, we can get $S_{I2} = iS_{I1}^*$ and $C_1 = C_2^T$, the symbol S_{I1}^* means the conjugate transpose of S_{I1}. The equation (2.2) can be rewritten as

$$A^{-1} = \frac{1}{p_1(1-i)}(S_I C^T - iS_I^* C), \tag{2.3}$$

where the vector p is the first column of S_I and C simultaneously.

Performing the spectral decomposition on S_I and C, that is, $S_I = \Omega_n^* F_n^* \Lambda_{S_I} F_n \Omega_n$ and $C = F_n^* \Lambda_C F_n$, where $\Omega_n = \text{diag}(1, e^{\frac{-3i\pi}{2n}}, \ldots, e^{\frac{-3i(n-1)\pi}{2n}})$, $F_n = (F_{j,k})_{j,k=1}^n$, $F_{j,k} = \frac{1}{\sqrt{n}} e^{\frac{2\pi i(j-1)(k-1)}{n}}$, $1 \leq j, k \leq n$, Λ_{S_I} and Λ_C are diagonal matrices comprised of the eigenvalues of S_I and C, respectively. Then we can obtain

$$A^{-1} = \frac{1}{p_1(1-i)} \Omega_n^* F_n^* (\Lambda_{S_I} F_n \Omega_n F_n \Lambda_C F_n^{*2} - i\Lambda_{S_I}^* F_n \Omega_n F_n^* \Lambda_C) F_n, \tag{2.4}$$

where $F_n^{*2} = \text{lcirc}(1, 0, \ldots, 0)$ is a left circulant permutation matrix [16, p45]. The product of F_n^{*2} and the vector can be done by a simple permutation operation.

In terms of solving linear systems with a same symmetric Toeplitz matrix, we first give Algorithm 1 for computing p and the eigenvalues of S_I and C. Then based on the equation (2.4), Algorithm 2 is shown for the product of the real symmetric Toeplitz matirx inversion and the vector. All calculations are from right to left.

Algorithm 1: Computing p, Λ_{S_I} and Λ_C

Step 1: Solve $Ap = e_1$ by any symmetric Toeplitz linear solver
Step 2: S_I and C have the same first column $p = [p_1, p_2, \ldots, p_n]^T$
Step 3: The entries of Λ_{S_I} are obtained from $\sqrt{n} F_n \Omega_n p$ by fast Fourier transform (FFT)
Step 4: The entries of Λ_C are obtained from $\sqrt{n} F_n p$ by FFT

Algorithm 2: A general method for computing $x = A^{-1}y$
Step 1: Compute $\tilde{y} = F_n y$ by FFT
Step 2: Compute $\check{y} = \Lambda_{S_I} F_n \Omega_n F_n \Lambda_C F_n^{*2} \tilde{y}$ by FFT
Step 3: Compute $\hat{y} = i\Lambda_{S_I}^* F_n \Omega_n F_n^* \Lambda_C \tilde{y}$ by FFT and inverse FFT (IFFT)
Step 4: Compute $x = \frac{1}{p_1(1-i)} \Omega_n^* F_n^* (\check{y} - \hat{y})$ by the equation (2.4) and IFFT

KSMs are considered suitable for solving large Toeplitz linear systems. We apply conjugate gradient method with Strang's circulant preconditioner [11, 17] (PCG) to solve $Ap = e_1$ because A is set as a symmetric positive definite matrix in our simulations. It is well known that one FFT or IFFT needs $5n \log_2 n + O(n)$ real arithmetic operations [4, p75]. Algorithm 1 can be realized with $O(n \log_2 n)$ complexity. Algorithm 2 needs $30n \log_2 n + O(n)$ real arithmetic operations consist of four FFTs and two IFFTs with length n.

Instead of solving symmetric Toeplitz linear systems one by one, we first execute Algorithm 1 once, and then utilize the factorization of symmetric Toeplitz matrix inversion to solve the linear systems. The solutions can be obtained by the multiplications of the symmetric Toeplitz matrix inversion and the vectors, which can save a lot of time.

3. New algorithms for real symmetric Toeplitz linear system with low-rank perturbations

In this section, we consider solving the symmetric Toeplitz linear system with low-rank perturbations, that is,

$$Uz = b, \tag{3.1}$$

where U has a decomposition as in the equation (1.2), $b = [b_1, b_2, \ldots, b_n]^T$ and $z = [z_1, z_2, \ldots, z_n]^T$ is an unknown vector.

Substituting the equation (1.2) into the equation (3.1), then

$$(A + \alpha e_2^T + \beta e_{n-1}^T)z = b. \tag{3.2}$$

Extracting A from the left or the right of the decomposition of U, we can get

$$A(I_n + A^{-1}\alpha e_2^T + A^{-1}\beta e_{n-1}^T)z = b, \tag{3.3}$$

and

$$(I_n + \alpha e_2^T A^{-1} + \beta e_{n-1}^T A^{-1})Az = b, \tag{3.4}$$

where I_n is an n-by-n identity matrix, α, β, e_2 and e_{n-1} are the same as that in the equation (1.2).

Based on the equation (3.3), let $A^{-1}\alpha = \mu$, $A^{-1}\beta = \nu$ and $A^{-1}b = \eta$, where $\mu = [\mu_1, \mu_2, \ldots, \mu_n]^T$, $\nu = [\nu_1, \nu_2, \ldots, \nu_n]^T$ and $\eta = [\eta_1, \eta_2, \ldots, \eta_n]^T$. Then,

$$z = (I_n + \mu e_2^T + \nu e_{n-1}^T)^{-1}\eta. \tag{3.5}$$

We use Sherman-Morrison-Woodbury formula [3] to solve $(I_n + \mu e_2^T + \nu e_{n-1}^T)^{-1}$, and

$$(I_n + \mu e_2^T + \nu e_{n-1}^T)^{-1} = I_n - F, \tag{3.6}$$

where $F = (f_{j,k})_{j,k=1}^{n}$:

$$f_{j,k} = \begin{cases} \frac{(1+\nu_{n-1})\mu_j - \mu_{n-1}\nu_j}{(1+\mu_2)(1+\nu_{n-1})-\mu_{n-1}\nu_2}, & k = 2, \\ \frac{(1+\mu_2)\nu_j - \nu_2\mu_j}{(1+\mu_2)(1+\nu_{n-1})-\mu_{n-1}\nu_2}, & k = n-1, \\ 0, \quad \text{otherwise}. \end{cases} \tag{3.7}$$

According to the equations (3.5), (3.6) and (3.7), the solution is

$$z = (I_n - F)\eta = (\eta_j - f_{j,2}\eta_2 - f_{j,n-1}\eta_{n-1})_{j=1}^{n}. \tag{3.8}$$

Based on the equations (3.7) and (3.8), Algorithm 1 and Algorithm 2, we give a new algorithm to solve $Uz = b$.

Algorithm 3: An algorithm for solving real symmetric Toeplitz linear system with low-rank perturbations $Uz = b$

Step 1: Implement Algorithm 1
Step 2: Get $\mu = A^{-1}\alpha$, $\nu = A^{-1}\beta$ and $\eta = A^{-1}b$ by Algorithm 2
Step 3: Compute $f_{j,k}$ by the equation (3.7)
Step 4: Compute the solution z by the equation (3.8)

Similarly, for the equation (3.4), if we set $e_2^T A^{-1} = \rho^T$, $e_{n-1}^T A^{-1} = \sigma^T$ and $Az = \varrho$, where $\rho = [\rho_1, \rho_2, \ldots, \rho_n]^T$, $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]^T$ and $\varrho = [\varrho_1, \varrho_2, \ldots, \varrho_n]^T$, then

$$\varrho = (I_n + \alpha\rho^T + \beta\sigma^T)^{-1} b. \tag{3.9}$$

Based on the Sherman-Morrison-Woodbury formula, $(I_n + \alpha\rho^T + \beta\sigma^T)^{-1}$ can be represented as

$$(I_n + \alpha\rho^T + \beta\sigma^T)^{-1} = I_n - G, \tag{3.10}$$

where $G = (g_{j,k})_{j,k=1}^{n}$:

$$g_{j,k} = \begin{cases} \frac{(1+\sigma_n\tau_2)\tau_1\rho_k - \rho_n\tau_1\tau_2\sigma_k}{(1+\rho_1\tau_1)(1+\sigma_n\tau_2)-\sigma_1\rho_n\tau_1\tau_2}, & j = 1, \\ \frac{(1+\rho_1\tau_1)\tau_2\sigma_k - \sigma_1\tau_1\tau_2\rho_k}{(1+\rho_1\tau_1)(1+\sigma_n\tau_2)-\sigma_1\rho_n\tau_1\tau_2}, & j = n, \\ 0, \quad \text{otherwise}. \end{cases} \tag{3.11}$$

According to the equations (3.9), (3.10) and (3.11), we can get

$$\varrho = (I_n - G)b = b - [\sum_{k=1}^{n} g_{1,k}b_k, 0, \ldots, 0, \sum_{k=1}^{n} g_{n,k}b_k]^T. \tag{3.12}$$

Finally, the solution z can be obtained by solving the linear system $Az = \varrho$.

Based on Algorithm 1, Algorithm 2, the equations (3.11) and (3.12), another algorithm for solving $Uz = b$ is shown as follows

Algorithm 4: An algorithm for solving real symmetric Toeplitz linear system with low-rank perturbations $Uz = b$

Step 1: Implement Algorithm 1
Step 2: Get $\rho = A^{-1}e_2$ and $\sigma = A^{-1}e_{n-1}$ by Algorithm 2
Step 3: Compute ϱ by the equations (3.11) and (3.12)
Step 4: Get the solution $z = A^{-1}\varrho$ by Algorithm 2

It is shown that solving the real symmetric Toeplitz linear system with low-rank perturbations needs to solve three symmetric Toeplitz linear systems. The complexity of Algorithm 3 or Algorithm 4 is $90n\log_2 n + O(n)$ besides Algorithm 1. If three are more perturbations, we must solve more symmetric Toeplitz linear systems, then it is more efficient to utilize the factorization of matrix inversion for computation. The proposed algorithms are also fit for solving many linear systems with a constant symmetric Toeplitz matrix with low-rank perturbations.

4. Stability analysis

Algorithms for solving real symmetric Toeplitz linear system with low-rank perturbations are based on the factorization of symmetric Toeplitz inversion. Therefore, the numerical stability of the symmetric Toeplitz inversion factorization is an important problem. In this section, we show the error bounds of the equation (2.3) and the error estimate between the numerical solution and the exact solution. Assume that $\hat{p} = [\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n]^T$ and $\hat{p}_1 \neq 0$ is the numerical solution of $Ap = e_1$, we denote

$$\hat{A}^{-1} = \frac{1}{\hat{p}_1(1-i)}(\hat{S}_I\hat{C}^T - i\hat{S}_I^{\,*}\hat{C}) \tag{4.1}$$

as a perturbation to A^{-1}.

Theorem 4.1. *Let $\epsilon > 0$, if $p_1 \neq 0$, $\hat{p}_1 \neq 0$, suppose $\hat{\epsilon} = \frac{|1/p_1 - 1/\hat{p}_1|}{|1/p_1|}$ as the relative error of $1/p_1$, and*

$$\frac{\|\hat{p} - p\|_1}{\|p\|_1} \leq \epsilon, \tag{4.2}$$

then the absolute error

$$\|A^{-1} - \hat{A}^{-1}\|_c \leq |\frac{\sqrt{2}}{p_1}|\{\epsilon + [\epsilon + (1+\epsilon)\hat{\epsilon}](1+\epsilon)\}\|p\|_1^2, \tag{4.3}$$

where $c = 1, 2$ or ∞. The relative errors

$$\frac{\|A^{-1} - \hat{A}^{-1}\|_1}{\|A^{-1}\|_1} \leq |\frac{\sqrt{2}}{p_1}|\{\epsilon + [\epsilon + (1+\epsilon)\hat{\epsilon}](1+\epsilon)\}\|p\|_1, \tag{4.4}$$

$$\frac{\|A^{-1} - \hat{A}^{-1}\|_\infty}{\|A^{-1}\|_\infty} \leq |\frac{\sqrt{2}}{p_1}|\{\epsilon + [\epsilon + (1+\epsilon)\hat{\epsilon}](1+\epsilon)\}\|p\|_1, \tag{4.5}$$

and

$$\frac{\|A^{-1} - \hat{A}^{-1}\|_2}{\|A^{-1}\|_2} \leq |\frac{\sqrt{2n}}{p_1}|\{\epsilon + [\epsilon + (1+\epsilon)\hat{\epsilon}](1+\epsilon)\}\|p\|_1. \tag{4.6}$$

Proof. The proof is similar to the proof in [34]. □

Theorem 4.2. *Under the same assumptions of Theorem 4.1, if $x = A^{-1}y$ and $\hat{x} = \hat{A}^{-1}y$ are the computed solution and the exact solution of the real symmetric Toeplitz linear system, respectively. We have*

$$\frac{\|x - \hat{x}\|_c}{\|x\|_c} \leq |\frac{\sqrt{2}}{p_1}|\{\epsilon + [\epsilon + (1+\epsilon)\hat{\epsilon}](1+\epsilon)\} \cdot \|p\|_1^2 \cdot \|A\|_c, \tag{4.7}$$

where $c = 1, 2$ or ∞.

Proof. It is indicate that

$$\begin{aligned}\|x-\hat{x}\|_c &= \|A^{-1}y-\hat{A}^{-1}y\|_c \\ &= \|A^{-1}-\hat{A}^{-1}\|_c \cdot \|Ax\|_c \\ &\le \|A^{-1}-\hat{A}^{-1}\|_c \cdot \|A\|_c \cdot \|x\|_c,\end{aligned} \tag{4.8}$$

where $c = 1, 2$ or ∞. By the equation (4.3), we can write

$$\frac{\|x-\hat{x}\|_c}{\|x\|_c} \le |\frac{\sqrt{2}}{p_1}|\{\epsilon + [\epsilon + (1+\epsilon)\hat{\epsilon}](1+\epsilon)\} \cdot \|p\|_1^2 \cdot \|A\|_c. \tag{4.9}$$

□

5. Applications

Image encryption and decryption have received a lot of attention recently. The real symmetric Toeplitz matrix with low-rank perturbations can be used effectively in the process of image encryption and decryption. Now, we consider a real symmetric Toeplitz matrix with perturbations in the first and last rows.

Suppose the matrix U has the structure of the equation (1.2), where the first column of the Toeplitz matrix A and two perturbations are randomly taken from $(0, 1)$. To keep U as a diagonally dominant matrix, a parameter is added to the diagonal elements of U.

We take three pictures with different pixels, then encrypt and decrypt them using the matrix U. Let the original image matrix be $\Phi = [\phi_1, \phi_2, \ldots, \phi_n]$, the encrypted image matrix is $\Psi = [\psi_1, \psi_2, \ldots, \psi_n] = U(U[\phi_1, \phi_2, \ldots, \phi_n])$ and the decrypted image matrix is obtained by $\tilde{\Phi} = [\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_n] = U^{-1}(U^{-1}[\psi_1, \psi_2, \ldots, \psi_n])$. Fig. 1 – Fig. 3 show the results of image encryption and decryption of different pixels.

Figure 1. Image encryption and decryption of 256 × 256 pixels

The matrix-vector multiplication needs to be calculated in image encryption, and multiple linear systems with a same coefficient matrix U need to be solved in image decryption. Here, we utilize matrix-vector multiplication in MATLAB for encryption and Algorithm 3 for decryption. When Algorithm 3 is performed in the process of image decryption, Algorithm 1 and $\mu = A^{-1}\alpha$, $\nu = A^{-1}\beta$ can be done only once. The results show that effective image encryption and decryption are realized.

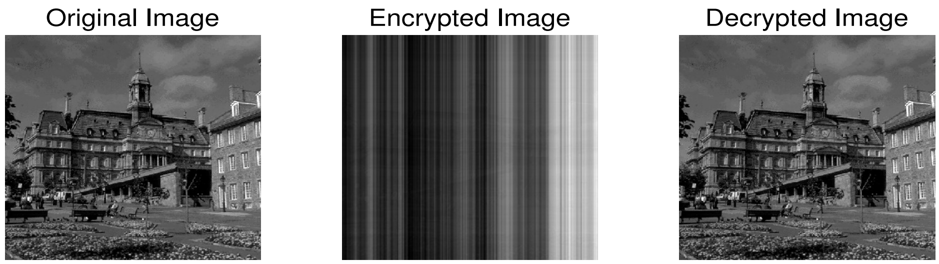

Figure 2. Image encryption and decryption of 512×512 pixels

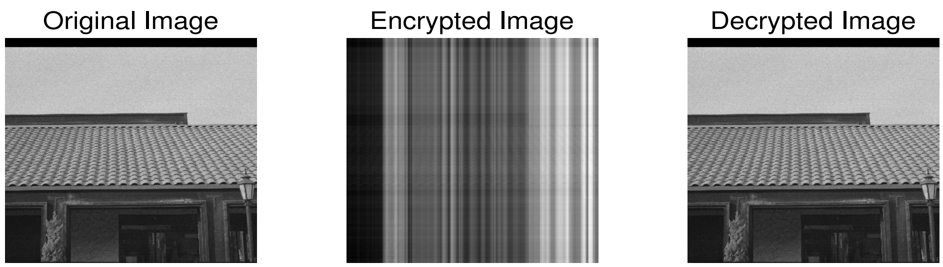

Figure 3. Image encryption and decryption of 1024×1024 pixels

6. Numerical examples

In this section, we give some numerical experiments to illustrate the theoretical results. All experiments are run using MATLAB R2022a on an AMD Ryzen 7 5800H processor with 3.20 GHz and 16GB RAM.

Example 6.1. Consider $Ax = y$ as a symmetric Toeplitz linear system. The first column of A is randomly taken from $(0, 1)$. Then we set the diagonal elements of A to be the sum of the first column. The exact solution is $x_{\text{exact}} = [1, 1, ..., 1]^T$, so the vector $y = Ax_{\text{exact}}$.

Table 1 shows the errors and CPU times of different methods for solving the symmetric Toeplitz linear system. "Error" $= \|\frac{x - x_{\text{exact}}}{x_{\text{exact}}}\|_\infty$, "$n$" denotes matrix order and "Time" is measured in seconds. Besides Back-slash, the stopping criterion for the other methods is set to 1×10^{-7}.

Back-slash method using backward slash operator in MATLAB, Alg 1 and Alg 2 denote the proposed algorithms. In addition to Back-slash method, the other methods can deal with high-order linear system. The latter two methods utilize the decomposition of matrix inversion to solve the linear system. Alg 1 + Alg 2 performs better than Alg Huang due to considering the special structure of symmetric Toeplitz matrix. However, compared with PCG method, the advantage of the decomposition is not realized when solving a linear system.

Example 6.2. Consider a series of symmetric Toeplitz linear systems $Ax_m = y_m,\ m = 1, 2, \ldots, M$ in the real number field, the first column of the symmetric Toeplitz matrix A is randomly taken from $(0, 1)$. The sum of the first column is set to the diagonal elements of A. The vectors y_m are all randomly taken from $(0, 1)$.

Table 1. Comparison of different methods for solving $Ax = y$ for Example 6.1

n	Back-slash		PCG		Alg Huang [11,17]		Alg 1 + Alg 2	
	Error	Time	Error	Time	Error	Time	Error	Time
2^{12}	1.6653e-14	0.2426	2.2595e-07	0.0045	7.2351e-09	0.0083	5.9447e-09	0.0058
2^{13}	2.5313e-14	1.4247	7.9559e-08	0.0055	1.4935e-09	0.0122	9.9938e-10	0.0070
2^{14}	3.4195e-14	9.2321	3.8915e-08	0.0070	1.0192e-07	0.0181	7.1900e-08	0.0089
2^{15}	—	—	1.7127e-08	0.0109	1.3075e-08	0.0275	2.2860e-07	0.0152
⋮	⋮	⋮	⋮	⋮	⋮	⋮		
2^{23}	—	—	3.3373e-08	2.4233	1.9537e-08	6.6573	4.8824e-08	3.5626
2^{24}	—	—	9.9464e-09	5.2435	2.5576e-08	14.0127	2.1068e-08	7.2237

"—" denotes exceeding the memory of MATLAB.

Table 2. CPU time in seconds for solving $Ax_m = y_m$ for Example 6.2

n	M	Back-slash	PCG	Alg Huang [11,17]	Alg 1 + Alg 2
2^{12}	2	0.5119	0.0087	0.0174	0.0106
	5	1.3229	0.0230	0.0273	0.0170
	10	2.5481	0.0468	0.0457	0.0338
2^{13}	2	3.1010	0.0106	0.0206	0.0129
	5	7.3105	0.0293	0.0334	0.0228
	10	15.2951	0.0517	0.0514	0.0350
2^{14}	2	18.5335	0.0139	0.0250	0.0146
	5	45.8370	0.0373	0.0425	0.0241
	10	90.2456	0.0705	0.0589	0.0408
2^{15}	2	—	0.0237	0.0389	0.0207
	5	—	0.0564	0.0578	0.0374
	10	—	0.1117	0.0845	0.0569
⋮	⋮	⋮	⋮	⋮	⋮
2^{23}	2	—	4.8812	8.2343	4.8370
	5	—	11.8070	12.5119	8.3027
	10	—	24.2191	19.7335	14.1546
2^{24}	2	—	10.8995	16.5022	9.3189
	5	—	26.3403	23.0020	14.6806
	10	—	52.8359	34.6852	23.8413

"—" denotes exceeding the memory of MATLAB.

A series of symmetric Toeplitz linear systems are solved by different methods in Example 2. "M" denotes the number of linear equations. As can be seen from Table 2, when the number of equations is greater than 5, Alg 1 + Alg 2 provides a dramatic reduction in running time over PCG method. Alg Huang method also has improved performance compared with PCG method when solving 10 symmetric Toeplitz linear systems.

It is proved that using the decomposition of matrix inversion to solve multiple linear equations with a same coefficient matrix is meaningful and has remarkable performance.

Example 6.3. Consider $Uz = b$ as a symmetric Toeplitz linear system with low-rank perturbations. According to the equation (1.2), U can be seen as the sum of a symmetric Toeplitz matrix A and two rank-one matrices, where the elements of A are $t_s = \frac{1}{s}, s = 1, 2, \ldots, n$, the perturbations τ_1 and τ_2 are randomly taken in $(0, 1)$. The vector b is chosen so that the exact solution is $z_{\text{exact}} = [1, 1, ..., 1]^T$.

Table 3. Comparison of different methods for solving $Uz = b$ for Example 6.3

n	Back-slash		PGMRES		Algorithm 3		Algorithm 4	
	Error	Time	Error	Time	Error	Time	Error	Time
2^{12}	1.8985e-13	0.3832	1.4858e-06	0.0080	4.9529e-07	0.0105	3.5194e-07	0.0101
2^{13}	3.6282e-13	2.3162	5.0524e-06	0.0098	1.6417e-06	0.0149	1.8373e-06	0.0141
2^{14}	7.8981e-13	16.1937	8.4249e-07	0.0147	4.2389e-06	0.0248	3.4561e-06	0.0228
2^{15}	—	—	5.1378e-06	0.0241	7.2713e-06	0.0412	7.1734e-06	0.0436
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
2^{23}	—	—	6.4451e-06	8.7136	1.1978e-05	13.2083	1.1524e-05	12.8630
2^{24}	—	—	7.4835e-05	16.8070	2.1252e-05	43.7741	2.4763e-05	34.4123

"—" denotes exceeding the memory of MATLAB.

For solving a symmetric Toeplitz linear system with low-rank perturbations, Table 3 presents the comparison of different methods. "Error" is given as the difference between the exact solution and the numerical solution under infinity norm. PGMRES method means the generalized minimal residual method with Strang's circulant preconditioner. Since U is a non-symmetric matrix, we choose PGMRES method in KSMs to solve the linear system as a comparison. The proposed Algorithm 3 and Algorithm 4 can not only solve higher-order linear systems, but also take much less time than Back-slash solver in MATLAB. Similar to Example 1, PGMRES method has the best performance in solving symmetric Toeplitz linear systems with low-rank perturbations and outperforms our proposed algorithm.

Example 6.4. Consider a series of linear systems $Uz_m = b_m,\ m = 1, 2, \ldots, M$ in the real number field, where U is a symmetric Toeplitz matrix with two perturbations. Based on the equation (1.2), the elements of A are $t_s = \frac{1}{s}, s = 1, 2, \ldots, n$, and the perturbations τ_1 and τ_2 are randomly taken in $(0, 1)$. The vectors b_m are all randomly taken from $(0, 1)$.

In this experiment, we apply different methods to solve the multiple linear systems with a same perturbed symmetric Toeplitz matrix. Table 4 displays the computational times in different cases. "n" and "M" represent the matrix order and the number of linear systems, respectively.

The times taken by the Back-slash and PGMRES methods multiplied as the number of linear equations solved increases. However, with the increase of "M" in the same matrix order, the entire times spent by Algorithm 3 and Algorithm 4 tend to decrease, compared with the PGMRES method. Basically, when the number of equations reaches 5, the proposed algorithms are already superior to the other algorithms.

Table 4. CPU time in seconds for solving $Uz_m = b_m$ for Example 6.4

n	M	Back-slash	PGMRES	Algorithm 3	Algorithm 4
	2	0.7941	0.0167	0.0169	0.0166
2^{12}	5	2.0613	0.0384	0.0324	0.0338
	10	3.9121	0.0690	0.0594	0.0584
	2	4.6807	0.0200	0.0249	0.0236
2^{13}	5	11.8098	0.0520	0.0428	0.0457
	10	23.6419	0.1060	0.0834	0.0861
	2	30.2341	0.0273	0.0369	0.0362
2^{14}	5	76.8728	0.0744	0.0708	0.0704
	10	152.2609	0.1476	0.1311	0.1237
	2	—	0.0461	0.0616	0.0669
2^{15}	5	—	0.1206	0.1133	0.1187
	10	—	0.2394	0.2145	0.2205
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	2	—	15.5762	17.2683	17.3519
2^{23}	5	—	39.0341	31.5532	29.7404
	10	—	72.2756	51.4408	49.4211
	2	—	32.9732	33.1619	35.7834
2^{24}	5	—	80.5099	57.5120	64.4113
	10	—	160.4509	99.2317	110.5142

"—" denotes exceeding the memory of MATLAB.

7. Conclusions

In this paper, we apply a new factorization of the inverse of real symmetric Toeplitz matrix to solve multiple linear equations with a constant coefficient matrix. For the solution of the quasi-Toeplitz linear system which has two different perturbations, two efficient algorithms are proposed in this paper. Numerical experiments show that the proposed algorithms can save the computational time significantly when solving a series of (quasi-)symmetric Toeplitz linear systems with a same coefficient matrix. The stability analysis of the decomposition of the inverse of the real symmetric Toeplitz matrix has been shown, and the applications of the proposed algorithms in image encryption and decryption has been expressed.

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