Journal of Applied Analysis and Computation
Volume 14, Number 1, February 2024, 119–132
Website:http://www.jaac-online.com
DOI:10.11948/20230082

GLOBAL STRUCTURE OF POSITIVE SOLUTIONS FOR FIRST-ORDER DISCRETE PERIODIC BOUNDARY VALUE PROBLEMS WITH INDEFINITE WEIGHT*

Xiaoxiao Su[1,†], Meng Yan[1] and Ruyun Ma[1]

Abstract We are concerned with the first-order discrete periodic boundary value problem

$$\begin{cases} -Du(t) = \lambda a(t) f(u(t)), \quad t \in \{1, 2, \cdots, T\}, \\ u(0) = u(T), \end{cases} \tag{P}$$

where $\lambda > 0$ is a parameter, $T > 2$ is an integer, $Du(t) = u(t+1) - u(t)$, $a : \{1, 2, \cdots, T\} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(0) = 0$. Depending on the behavior of f near 0 and ∞, we obtain that there exist $0 < \lambda_* \leq \lambda^*$ such that for any $\lambda > \lambda^*$, problem (P) possesses at least two positive solutions, while it has no solution for $\lambda \in (0, \lambda_*)$. The proof of our main results are based upon bifurcation technique.

Keywords Periodic boundary value problem, difference equation, positive solution, indefinite weight, bifurcation.

MSC(2010) 39A27, 39A28, 39A70.

1. Introduction

Let $T > 2$ be an integer, $\mathbb{T} = \{1, \cdots, T\}, \hat{\mathbb{T}} = \{0, \cdots, T+1\}, Du(t) = u(t+1) - u(t)$. In this paper, we are concerned with the first-order discrete periodic boundary value problem

$$\begin{cases} -Du(t) = \lambda a(t) f(u(t)), \quad t \in \mathbb{T}, \\ u(0) = u(T), \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter and a, f obey the conditions specified later.

In recent years, a great deal of research has been devoted to the study of existence of positive periodic solutions for the nonlinear discrete equation

$$-Du(t) + \alpha(t) h(u(t)) u(t) = \lambda \beta(t) g(u(t)), \quad t \in \mathbb{Z}, \tag{1.2}$$

†The corresponding author.
1School of Mathematics and Statistics, Xidian University, Xi'an, Shannxi 710071, China
*The authors were supported by National Natural Science Foundation of China (12061064) and Shaanxi Fundamental Science Research Project for Mathematics and Physics (No. 22JSY018).
Email: suxiaoxiao2856@163.com(X. Su), yanmeng161999@163.com(M. Yan), ryma@xidian.edu.cn(R. Ma)

where $\alpha, \beta : \mathbb{Z} \to [0, \infty)$ are T-periodic, $h, g \in C([0, \infty), [0, \infty))$, $\lambda > 0$ is a parameter, see [5,10,11,13,15,20,22] and the references therein. For instance, by applying a cone theoretic fixed point theorem, Y. Raffoul [20] obtained that (1.2) with $h \equiv 1$ has at least one positive periodic solution for any $\lambda > 0$ under the assumption

(G1) $g_0 = \infty, g_\infty = 0$ or $g_0 = 0, g_\infty = \infty$, where

$$g_0 := \lim_{s \to 0} \frac{g(s)}{s}, \qquad g_\infty := \lim_{s \to \infty} \frac{g(s)}{s}.$$

R. Ma et.al [15] generalized the main results of Raffoul and assumed that $g_0 = 0$ and $g_\infty = 0$, then there exists $\lambda_0 > 0$, such that problem (1.2) has at least two positive T-periodic solutions for $\lambda > \lambda_0$. Later, Y. Lu [11] generalized the main results of [15] and provided more desirable intervals of λ.

However, fewer results seem to be available when $\alpha(t) \equiv 0$ in (1.2). One of the peculiar aspects of the periodic BVP associated with $\alpha(t) \equiv 0$ is the fact that the difference operator has a non-trivial kernel (which is made by the constant functions). And it is customary when working with such boundary value problems, whether in differential or difference equations, to display the desired solution in terms of a suitable Green function and then apply the methods based on the theory of positive operators for cones in Banach spaces [8,18,23,24].

The natural question is that what would happen if we do not need a positive operator. Fortunately, there have been some studies focused on the study of boundary value problems of differential equations [2–4,16]. For example, using critical point theory, A. Boscaggin etc [4] studied the periodic problem associated with the second order nonlinear differential equation

$$u''(t) + \lambda b(t) g(u) = 0, \tag{1.3}$$

where $\lambda > 0$ is sufficiently large, $b : \mathbb{R} \to \mathbb{R}$ is a locally integrable ω-periodic function and a sign-changing function, $g : [0, \infty) \to [0, \infty)$ is continuous and $g(0) = 0$. They obtained that

Theorem A. *Assume that $\int_0^\omega a(t)dt < 0$ and*

(H1) $g_0 = 0$ *and* $g_\infty = 0$;

(H2) g *is regularly oscillating at zero and infinity, that is*

$$\lim_{\omega \to 1, s \to 0} \frac{g(\omega s)}{g(s)} = 1, \qquad \lim_{\omega \to 1, s \to \infty} \frac{g(\omega s)}{g(s)} = 1.$$

Then there exists $\lambda^0 > 0$, such that problem (1.3) has at least two positive ω-periodic solutions for $\lambda > \lambda^0$.

Motivated by the results mentioned above, it is of interest to know whether Theorem A still hold true for the first-order discrete periodic boundary value problem (1.1). We note that while these earlier considerations furnished the motivation for our work, it is clear that they cannot be easily applied to problems (1.1). Indeed, if $u(t) > 0$ is a solution of (1.1), then, after summing the equation from 0 to $T-1$, one has that

$$\sum_{t=0}^{T-1} a(t) f(u(t)) = 0$$

with $f(u(t)) > 0$ for every t, this implies that $a(t)$ cannot be of constant sign. Thus, the first difficulty we have to face is that the spectrum structure of corresponding linear eigenvalue problem with indefinite weight is unknown. Moreover, to the best of our knowledge, the study of global structure of positive solutions for problem (1.1) is completely new and has not been described before in related problems.

On the other hand, there are many essential differences between the discrete equation and the differential equation. Such as mean value theorem, Poincaré inequality and the existence, uniqueness, and multiplicity of solutions may not be shared between the continuous differential equation and related discrete equation, see [1, p.520]. In addition, The continuous logistic model find that population growth will reach a certain limit due to resource constraints, whereas its discrete

$$v(t+1) = kv(t)(1 - v(k)), \quad t \in \mathbb{N}$$

is sensitive to changes in initial values and parameter values, and often small changes can cause chaos, where $k > 0$ is a parameter. Thus, new challenges are faced.

For the convenience of the reader, we shall recall the following assumptions:

(A0) $a : \mathbb{T} \to \mathbb{R}$ is a sign-changing function and satisfies $\sum\limits_{t=0}^{T} a(t) > 0$;

(F0) $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $f(0) = 0$ and $f(s) > 0$ for $s > 0$;

(F1) $f_0 := \lim\limits_{s\to 0} \frac{f(s)}{s} = 0$, $f_\infty := \lim\limits_{s\to\infty} \frac{f(s)}{s} = 0$;

(F2) $\lim\limits_{\omega\to 1, s\to 0} \frac{f(\omega s)}{f(s)} = 1$, $\lim\limits_{\omega\to 1, s\to\infty} \frac{f(\omega s)}{f(s)} = 1$.

Let $Y := \{u \mid u : \mathbb{T} \to \mathbb{R}\}$ be a Banach space with the norm $\|u\|_Y = \max\limits_{t\in\mathbb{T}} |u|$. Let $E := \{u \mid u : \hat{\mathbb{T}} \to \mathbb{R}, u(0) = u(T)\}$ be a Banach space with the norm $\|u\|_0 = \max\limits_{t\in\hat{\mathbb{T}}} |u|$. Let Σ be the closure of the set of nontrivial solutions of (1.1) in $[0, \infty) \times E$.

The main results of this paper are the following

Theorem 1.1. *Assume (A0) and (F0)-(F2) hold. Then there exists a connected component $\mathcal{C}_+ \subset \Sigma$ such that*
(i) $\mathcal{C}_+$ joins $(\infty, 0)$ with infinity in λ direction;
(ii) there exist two constants $\Lambda > 0$ and $\rho > 0$ such that

$$\mathcal{C}_+ \cap \{(\lambda, u) \in \Sigma \mid \lambda \geq \Lambda, ||u||_0 = \rho\} = \emptyset;$$

(iii) there exists $\lambda_* > 0$ such that

$$\mathcal{C}_+ \cap ((0, \lambda^*) \times E) = \emptyset.$$

Corollary 1.1. *Assume (A0) and (F0)-(F2) hold. Then there exist $0 < \lambda_* \leq \lambda^*$ such that problem (1.1) has at least two positive solutions for $\lambda > \lambda^*$, while it has no solution for $\lambda \in (0, \lambda_*)$.*

Remark 1.1. A function satisfying condition (F2) is called a "regularly oscillating" function. It was done first by R. Schmidt [21] in 1925s in a sequential form. For the specific definition considered in our paper, as well as for some historical remarks, see [7] and the references therein. Observe that any function $f(s)$ such that $f(s) = Ks^q, K, q > 0$ is regularly oscillating both at zero and at infinity.

Remark 1.2. Note that, for problem (1.1), constants are always solutions as $\lambda \to 0$, thanks to regularly oscillating conditions, we proved that the connected branch and the vertical axis are disjoint, i.e., there exists $\lambda_* > 0$ such that (1.1) has no solution for $\lambda \in (0, \lambda_*)$.

Remark 1.3. For other related results on the existence of positive solutions of first-order ordinary differential equations, see [9, 12, 17] and references therein.

The rest of this work is organized as follows. In Section 2, we state some properties of the superior limit of a certain infinity collection of connected sets and prove some preliminary results. Section 3 is devoted to showing the proof of main results.

2. Preliminaries

Firstly, we state some properties of the superior limit of a certain infinity collection of connected sets. Let N be a metric space and $\{C_n | \ n = 1, 2, \cdots\}$ be a family of subsets of N. Then the superior limit $\mathbb{D}$ of C_n is defined by

$$\mathbb{D} := \limsup_{n\to\infty} \ C_n = \{x \in N | \ \exists \ n_k \subset \mathbb{N}, x_{n_k} \in C_{n_k} \text{ such that } x_{n_k} \to x\}.$$

A *component* of a set M means a maximal connected subset of M, see [24] for the detail.

Lemma 2.1 ([14]). *Let X be a Banach space and let C_n be a family of closed connected subsets of X. Assume that*

(a) there exist $z_n \in C_n, n = 1, 2, \cdots$ and $z_* \in X$ such that $z_n \to z_*$;

(b) $\lim\limits_{n\to\infty} r_n = \infty$, where $r_n = \sup\{\|x\| \mid x \in C_n\}$;

(c) for every $R > 0, (\cup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact set of X, where

$$B_R = \{x \in X \mid \|x\| \le R\}.$$

Then $\mathcal{D} := \limsup\limits_{n\to\infty} \ C_n$ contains an unbounded component $\mathbb{C}$ *with $z_* \in \mathbb{C}$.*

Define $T : E \to Y$ by

$$Tu := -Du(t).$$

As underlined in the introduction, due to the difference operator T has a non-trivial kernel (which is made by the constant functions), in order to using bifurcation theory, we consider the following auxiliary problem

$$\begin{cases} -Du(t) + \frac{1}{m}u(t) = \lambda a(t)f(u(t)), \quad t \in \mathbb{T}, \\ u(0) = u(T), \end{cases} \tag{2.1}$$

where $m \in \mathbb{N}$ is a constant.

It is easy to see that u is a solution of (2.1) if and only if

$$u(t) = \lambda \sum_{s=t}^{t+T-1} G(t,s)a(s)f(u(s)),$$

where

$$G(t,s)=\frac{\prod\limits_{i=s+1}^{t+T-1}(1+\frac{1}{m})}{\prod\limits_{i=t}^{t+T-1}(1+\frac{1}{m})-1}=\frac{(1+\frac{1}{m})^{t-s+T-1}}{(1+\frac{1}{m})^{T}-1},\quad t\le s\le t+T-1.$$

Notice that

$$\frac{1}{(1+\frac{1}{m})^{T}-1}\le G(t,s)\le\frac{(1+\frac{1}{m})^{T-1}}{(1+\frac{1}{m})^{T}-1}.$$

Next, we will consider the linear eigenvalue problem

$$\begin{cases}-Du(t)+\frac{1}{m}u(t)=\lambda a(t)u(t), & t\in\mathbb{T},\\ u(0)=u(T).\end{cases}\tag{2.2}$$

Let

$$J=\begin{pmatrix}1+\frac{1}{m} & -1 & 0 & \cdots & 0 & 0\\ 0 & 1+\frac{1}{m} & -1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1+\frac{1}{m} & -1\\ 0 & 0 & 0 & \cdots & 0 & 1+\frac{1}{m}\end{pmatrix}$$

and $F=\mathrm{diag}(a(1),a(2),\cdots,a(T))$. Then (2.2) can be written as a linear pencil problem

$$Ju=\lambda Fu.$$

Let J_j denote the j-th principal submatrix of J. Then J and J_i are positive definite. For $j=1,2,\cdots,T$, let $Q_j(\lambda)$ denote the j-th principal subdeterminant of $J-\lambda F$ and suppose that $Q_0(\lambda)=1$. Then $Q_T(\lambda)=\det(J-\lambda F)$ and

$$\begin{aligned}&Q_0(\lambda)=1;\\ &Q_1(\lambda)=1+\frac{1}{m}-\lambda a(1);\\ &Q_2(\lambda)=[1+\frac{1}{m}-\lambda a(1)][1+\frac{1}{m}-\lambda a(2)];\\ &Q_j(\lambda)=\prod_{i=1}^{j}[1+\frac{1}{m}-\lambda a(i)],\quad j=3,\cdots,T.\end{aligned}$$

As we know, finding the eigenvalues of (2.2) is equivalent to finding the zeros of $Q_T(\lambda)$.

For $j\in\{1,\cdots,T\}$, let j^+ be the number of the elements in $\{a(i)\mid a(i)>0\ \text{for some}\ i\in\{1,\cdots,j\}\}$, and j^- the number of the elements in $\{a(i)\mid a(i)<0\ \text{for some}\ i\in\{1,\cdots,j\}\}$.

Obviously, $Q_1(\lambda)=1+\frac{1}{m}-\lambda a(1)$. If $a(1)>0$, then $j=1, j^+=1, j^-=0$ and $\lambda_{1,1}^+=\frac{1+\frac{1}{m}}{a(1)}>0$. If $a(1)<0$, then $j=1, j^+=0, j^-=1$ and $\lambda_{1,1}^-=\frac{1+\frac{1}{m}}{a(1)}<0$.

Recall $Q_2(\lambda) = [1 + \frac{1}{m} - \lambda a(1)][1 + \frac{1}{m} - \lambda a(2)]$. Then $Q_2(\lambda) = 0$ has two roots as follows:

$$\lambda_1 = \frac{1 + \frac{1}{m}}{a(1)}, \quad \lambda_2 = \frac{1 + \frac{1}{m}}{a(2)}.$$

Case 1. If $a(1) > 0, a(2) > 0$, then $j = 2, j^+ = 2, j^- = 0$. Let $\lambda^+_{2,1} = \lambda_1, \lambda^+_{2,2} = \lambda_2$. Then $\lambda^+_{2,1}, \lambda^+_{2,2} > 0$.

Case 2. If $a(1) < 0, a(2) < 0$, then $j = 2, j^+ = 0, j^- = 2$. Let $\lambda^-_{2,1} = \lambda_1, \lambda^-_{2,2} = \lambda_2$. Then $\lambda^-_{2,1}, \lambda^-_{2,2} < 0$.

Case 3. If $a(1) > 0, a(2) < 0$, then $j = 2, j^+ = 1, j^- = 1$. Let $\lambda^+_{2,1} = \lambda_1, \lambda^-_{2,1} = \lambda_2$. Then $\lambda^+_{2,1} > 0, \lambda^-_{2,1} < 0$.

Case 4. If $a(1) < 0, a(2) > 0$, then $j = 2, j^+ = 1, j^- = 1$. Let $\lambda^+_{2,1} = \lambda_2, \lambda^-_{2,1} = \lambda_1$. Then $\lambda^+_{2,1} > 0, \lambda^-_{2,1} < 0$.

Since a is sign-changing, Case 1 and Case 2 are impossible. Now by the same argument for $j = T$, we may deduce that $Q_T(\lambda) = 0$ has T roots as follows:

$$\lambda_1 = \frac{1 + \frac{1}{m}}{a(1)}, \quad \lambda_2 = \frac{1 + \frac{1}{m}}{a(2)}, \quad \cdots, \quad \lambda_T = \frac{1 + \frac{1}{m}}{a(T)},$$

where $j^+ > 0$ positive roots and $j^- > 0$ negative roots. Let $\lambda^+_{T,1}$ is the first positive eigenvalue.

Lemma 2.2. *$\lambda^+_{T,1}$ is simple, and the corresponding eigenfunction does not change sign. Thus we can choose $\phi^+_{T,1}$ such that $\phi^+_{T,1} > 0$.*

Proof. Let w be any eigenfunction corresponding to $\lambda^+_{T,1}$. We write $w = w^+ - w^-$. Then we have

$$\sum_{t=0}^{T-1} w^+(t+1)Dw^+(t) + \frac{1}{m}\sum_{t=0}^{T-1}(w^+)^2 = \lambda^+_{T,1}\sum_{t=0}^{T-1} a(t)(w^+)^2$$

and

$$\sum_{t=0}^{T-1} w^-(t+1)Dw^-(t) + \frac{1}{m}\sum_{t=0}^{T-1}(w^-)^2 = \lambda^+_{T,1}\sum_{t=0}^{T-1} a(t)(w^-)^2.$$

If w does not have constant sign in $\mathbb{T}$, then both w^+ and w^- are not identically zero. Hence both w^+ and w^- are eigenfunctions for $\lambda^+_{T,1}$, i.e.,

$$-Dw^+(t) + \frac{1}{m}w^+(t) + \lambda^+_{T,1}a^-(t)w^+(t) = \lambda^+_{T,1}a^+(t)w^+(t)$$

and

$$-Dw^-(t) + \frac{1}{m}w^-(t) + \lambda^+_{T,1}a^-(t)w^-(t) = \lambda^+_{T,1}a^+(t)w^-(t).$$

By the strong maximum principle, $w^+ > 0$ and $w^- > 0$. But it is impossible since w^+ must vanish when $w^- \neq 0$ and vice versa. Thus, w does not change sign in $\mathbb{T}$. Hence we can choose $\phi^+_{T,1} > 0$ which is an eigenfunction corresponding to $\lambda^+_{T,1}$.

If $\lambda^+_{T,1}$ is not simple, then there exists another eigenfunction orthogonal to $\phi^+_{T,1} > 0$, say, $\hat{\phi}^+_{T,1} > 0$ which also does not change sign in $\mathbb{T}$. It is impossible to $\sum_{t=0}^{T-1} \phi^+_{T,1}\hat{\phi}^+_{T,1} = 0$. Thus $\lambda^+_{T,1}$ is simple. $\square$

3. The proof of main results

For any $n \in \mathbb{N}$, define $f^{[n]} : \mathbb{R} \to \mathbb{R}$ by

$$f^{[n]}(s) = \begin{cases} nf(\frac{1}{n})s, & s \in [0, \frac{1}{n}], \\ f(s), & s \in (\frac{1}{n}, +\infty), \\ -f^{[n]}(-s), & s < 0. \end{cases}$$

Then $f^{[n]}$ is an odd function in $\mathbb{R}$ and

$$(f^{[n]})_0 := \lim_{s\to 0} \frac{f^{[n]}(s)}{s} = nf(\frac{1}{n}), \quad (f^{[n]})_\infty := \lim_{s\to\infty} \frac{f^{[n]}(s)}{s} = \lim_{s\to\infty} \frac{f(s)}{s}.$$

By the first condition (F1), it follows that

$$\lim_{n\to\infty} (f^{[n]})_0 = 0.$$

Now let us consider the auxiliary family of the problem

$$\begin{cases} -Du(t) + \frac{1}{m}u(t) = \lambda a(t) f^{[n]}(u(t)), \quad t \in \mathbb{T}, \\ u(0) = u(T). \end{cases} \tag{3.1}$$

From the definition of $f^{[n]}$, it follows that for every $s \in \mathbb{R}$,

$$f^{[n]}(s) = (f^{[n]})_0 s + \zeta^{[n]}(s) = nf(\frac{1}{n})s + \zeta^{[n]}(s),$$

where $\zeta^{[n]} : \mathbb{R} \to \mathbb{R}$ is continuous and

$$\lim_{s\to 0} \frac{\zeta^{[n]}(s)}{s} = 0.$$

Define linear operator $L : E \to Y$ by

$$Lu := -Du(t) + \frac{1}{m}u(t).$$

Then $L : E \to E$ is compact and continuous since E is finite dimensional.

Let us consider

$$Lu(t) = \lambda a(t)(f^{[n]})_0 u(t) + \lambda a(t)\zeta^{[n]}(u(t)) \tag{3.2}$$

as a bifurcation problem from the trivial solution $u \equiv 0$. Then problem (3.2) can be equivalently written as

$$u = \lambda L^{-1}[a(t)(f^{[n]})_0 u](t) + \lambda L^{-1}[a(t)\zeta^{[n]}u](t).$$

Further

$$\begin{aligned} ||L^{-1}[a(\cdot)\zeta^{[n]}u](\cdot)||_0 &= \max_{t\in\mathbb{T}} \Big| \sum_{s=t}^{t+T-1} G(t,s)a(s)\zeta^{[n]}(u(s)) \Big| \\ &\le \frac{(1+\frac{1}{m})^{T-1}}{(1+\frac{1}{m})^T - 1} \max_{t\in\mathbb{T}} |a(t)| \cdot ||\zeta^{[n]}(u(\cdot))||_0, \end{aligned}$$

therefore we have that $||L^{-1}[a(\cdot)\zeta^{[n]}(u(\cdot))]||_0 = \circ(||u||_0)$ for u near 0 in E.

Let S denote the set of functions in E which have no generalized zeros in $\mathbb{T}$, $S^- = -S^+$ and $S = S^+ \cup S^-$. Let $\Sigma_m^{[n]}$ be the closure of the set of nontrivial solutions of (3.2) in $[0,\infty) \times E$.

By the Rabinowitz global bifurcation theorem [19], there exists a continuum $\mathcal{C}_m^{[n]} \subset \Sigma_m^{[n]}$ of solutions of (3.2) bifurcating from $(\frac{\lambda_{T,1}^+}{(f^{[n]})_0}, 0)$ which is either unbounded or contains a pair $(\frac{\lambda_{T,j}^+}{(f^{[n]})_0}, 0)$ for some $j > 1$. Note that the fact, if there exists t_0 such $u(t_0) = 0$, then $u \equiv 0$, prevents the second alternative occurring.

Furthermore, by [6, Theorem 2], there are two continua $\mathcal{C}_{m,+}^{[n]}$ and $\mathcal{C}_{m,-}^{[n]}$, consisting of the bifurcation branch $\mathcal{C}_m^{[n]}$, which satisfy either $\mathcal{C}_{m,+}^{[n]}$ and $\mathcal{C}_{m,-}^{[n]}$ are both unbounded or $\mathcal{C}_{m,+}^{[n]} \cap \mathcal{C}_{m,-}^{[n]} \neq \{(\frac{\lambda_{T,1}^+}{(f^{[n]})_0}, 0)\}$.

Indeed, since $\alpha\phi_{T,1}^+ \in S^\pm$ if $0 \neq \alpha \in \mathbb{R}^\pm$ or $\mathbb{R}^\mp$, we have that

$$\Big(\mathcal{C}_{m,\pm}^{[n]} \setminus \{(\frac{\lambda_{T,1}^+}{(f^{[n]})_0}, 0)\} \cap \mathbb{B}_\epsilon(\frac{\lambda_{T,1}^+}{(f^{[n]})_0}, 0)\Big) \subset \mathbb{R} \times S^\pm$$

for all positive ϵ small enough. Similar to the above argument, we can show that $\mathcal{C}_{m,\pm}^{[n]} \setminus \{(\frac{\lambda_{T,1}^+}{(f^{[n]})_0}, 0)\}$ cannot leave $\mathbb{R} \times S^\pm$ outside of a neighborhood of $(\frac{\lambda_{T,1}^+}{(f^{[n]})_0}, 0)$. Therefore, we have that

$$\mathcal{C}_{m,\pm}^{[n]} \subset \Big(\mathbb{R} \times S^\pm \cup \{(\frac{\lambda_{T,1}^+}{(f^{[n]})_0}, 0)\}\Big).$$

It follows that both $\mathcal{C}_+^{[n]}$ and $\mathcal{C}_-^{[n]}$ are unbounded. Otherwise, with loss of generality, we may suppose that $\mathcal{C}_{m,-}^{[n]}$ is bounded. Then there exists $(\lambda_*, u_*) \in \mathcal{C}_{m,+}^{[n]} \cap \mathcal{C}_{m,-}^{[n]}$ such that $(\lambda_*, u_*) \neq (\frac{\lambda_{T,1}^+}{(f^{[n]})_0}, 0)$ and $u_* \in S^+ \cap S^-$. This contradicts the definition of S^+ and S^-.

Since this paper is only concerned with the existence of positive solutions, we will only consider the properties of $\mathcal{C}_{m,+}^{[n]}$ in the following.

Lemma 3.1. *Assume that (A0) and (F0)-(F2) hold and let ρ be a fixed constant. Then there exists a positive constant Λ such that*

$$\mathcal{C}_{m,+}^{[n]} \cap \{(\lambda, u) \in S \mid \lambda \geq \Lambda, \rho - \frac{\rho}{8} \leq \|u\|_0 \leq \rho + \frac{\rho}{8}\} = \emptyset.$$

Proof. In fact, if $(\lambda, u) \in \mathcal{C}_{m,+}^{[n]}$ is a solution with

$$\rho - \frac{\rho}{8} \leq \|u\|_0 \leq \rho + \frac{\rho}{8}.$$

Let $N_* \in \mathbb{N}$ be an integer such that

$$\frac{1}{N_*} < \frac{1}{2}\rho.$$

Then, for $n \geq N_*$, we have

$$f^{[n]}(s) = f(s), \quad s \in [\frac{1}{2}\rho, \infty).$$

Denote

$$\mathbb{I}^+ = \{t \mid a(t) > 0, t \in \mathbb{T}\}, \quad \mathbb{I}^- = \{t \mid a(t) < 0, t \in \mathbb{T}\}.$$

Thus

$$\begin{aligned}
\frac{9}{8}\rho &= ||u||_0 \\
&= \lambda \max_{t\in\mathbb{T}} \Big| \sum_{s=t}^{t+T-1} G(t,s)a(s)f(u(s))\Big| \\
&\geq \lambda \max_{t\in\mathbb{I}^+} \Big| \sum_{s=t}^{t+T-1} G(t,s)a(s)f(u(s))\Big| \\
&\geq \frac{\lambda}{(1+\frac{1}{m})^T - 1} \max_{t\in\mathbb{I}^+} \Big| \sum_{s=t}^{t+T-1} a(s)f(u(s))\Big| \\
&\geq \frac{\lambda f_{\min} \max\limits_{t\in\mathbb{I}^+} \sum\limits_{s=t}^{t+T-1} a(s)}{(1+\frac{1}{m})^T - 1} \\
&\geq \frac{\lambda f_{\min} \max\limits_{t\in\mathbb{I}^+} \sum\limits_{s=t}^{t+T-1} a(s)}{2^T - 1},
\end{aligned}$$

where

$$f_{\min} := \min\{f(u) : \frac{1}{2}\rho \leq u \leq \rho\}.$$

Choose

$$\Lambda := \frac{9}{8}\rho(2^T - 1)\Big(f_{\min} \max_{t\in\mathbb{I}^+} \sum_{s=t}^{t+T-1} a(s)\Big)^{-1} + \frac{1}{8}\rho.$$

Obviously, Λ is independent of n and m. □

Lemma 3.2. *Assume that (A0) and (F0)-(F2) hold and let $I \in (0, \infty)$ be a closed and bounded interval. Then there exists a positive constant c such that*

$$\sup\{||u||_0 : (\lambda, u) \in \mathcal{C}_{m,+}^{[n]}, \lambda \in I\} \leq c.$$

Proof. On the contrary, we suppose that there exists $(\lambda_k, u_k) \in \mathcal{C}_{m,+}^{[n]}$, $\lambda_k \in I$ such that $||u_k||_0 \to \infty$. Then we have

$$-Du_k(t) + \frac{1}{m}u_k(t) = \lambda_k a(t) f^{[n]}(u_k).$$

Set $v_k(t) := \frac{u_k(t)}{||u_k||_0}$, then $||v_k||_0 = 1$. Now, choosing a subsequence and relabelling if necessary, it follows that there exists $(\lambda_*, v_*) \in I \times E$ with $||v_*||_0 = 1$, such that

$$\lim_{k\to\infty} (\lambda_k, u_k) = (\lambda_*, v_*) \text{ in } \mathbb{R} \times E.$$

On the other hand, we have

$$-Dv_k(t) + \frac{1}{m}v_k(t) = \lambda_k a(t)\frac{f^{[n]}(u_k)}{u_k}v_k(t),$$

and

$$v_k(t) = \lambda_k \sum_{s=t}^{t+T-1} G(t,s)a(s)\frac{f^{[n]}(u_k)}{u_k}v_k(s).$$

Combining this with $(f^{[n]})_\infty = 0$ and using the Lebesgue dominated convergence theorem, it follows that

$$v_*(t) = \lambda_* \sum_{s=t}^{t+T-1} G(t,s)a(s)\cdot 0 \cdot v_*(s),$$

and consequently, $v_* \equiv 0$. This contradicts $||v_*||_0 = 1$. $\square$

Lemma 3.3. *Assume that (A0) and (F0)-(F2) hold. Then there exists* $\rho^* > 0$ *such that*

$$(\bigcup_{n=1}^{\infty} \mathcal{C}_{m,+}^{[n]}) \cap ((0,\rho^*)\times E) = \emptyset.$$

Proof. Assume on the contrary that there exists $\{(\lambda_k,u_k)\} \subset (\bigcup_{n=1}^{\infty} \mathcal{C}_{m,+}^{[n]}) \cap ((0,\infty)\times E)$ such that $\lambda_k \to 0$. Set $v_k(t) = \frac{u_k(t)}{||u_k||_0}$, then $||v_k||_0 = 1$. And we have

$$\begin{aligned} v_k(t) &= \lambda_k \sum_{s=t}^{t+T-1} G(t,s)a(s)\frac{f^{[n]}(u_k)}{u_k}v_k(s) \\ &\leq \lambda_k \sum_{s=t}^{t+T-1} G(t,s)a(s)\frac{f^{[n]}(u_k)}{u_k}||v_k(s)||_0 \to 0, \end{aligned}$$

which contradicts the fact $||v_k||_0 = 1$. $\square$

Lemma 3.4. *Assume that (A0) and (F0)-(F2) hold. Then*

$$(\mathcal{C}_{m,+}^{[n]}) \cap ((0,+\infty)\times\{0\}) = \emptyset.$$

Proof. Suppose, on the contrary, that there exists $(\lambda_k,u_k) \in \Big((\mathcal{C}_{m,+}^{[n]}) \cap ((0,+\infty)\times \{0\})\Big)$ such that $\lambda_k \to \mu \in (0,\infty)$ and $u_k \to 0$ as $k\to\infty$. Hence, for any $N_0 \in \mathbb{N}$, there exists $n_0 \geq N_0$ such that $(\lambda_k,u_k) \in \mathcal{C}_{m,+}^{[n_0]}$. It follows that $\mu = \frac{\lambda_{T,1}^+}{n_0}$. The arbitrary of N_0 implies that $\mu = 0$, which contradicts the assumption of $\mu \in (0,\infty)$. $\square$

Let us verify that $\{\mathcal{C}_{m,+}^{[n]}\}$ satisfies all of the conditions of Lemma 2.1. Since

$$\lim_{n\to\infty}\frac{\lambda_{T,1}^+}{(f^{[n]})_0} = \lim_{n\to\infty}\frac{\lambda_{T,1}^+}{nf(\frac{1}{n})} = \infty.$$

Condition (a) in Lemma 2.1 is satisfied with $z^* = (\infty, 0)$. Obviously

$$r_n = \sup\{|\lambda| + ||u||_0 \mid (\lambda, u) \in \{\mathcal{C}_{m,+}^{[n]}\}\} = \infty,$$

and accordingly, (b) holds. (c) can be deduced directly from the Arzelà-Ascoli theorem and the definition of $f^{[n]}$. Therefore, the superior limit of $\{\mathcal{C}_{m,+}^{[n]}\}$, i.e. $\mathcal{D}$, contains an unbounded connected component $\mathcal{C}_{m,+}$ joins $(\infty, 0)$ with infinity in the direction of λ.

Now by the same argument in the proof of Lemma 3.2, with obvious changes, we may deduce the desired results for $\mathcal{C}_{m,+}$.

Lemma 3.5. *Assume that (A0) and (F0)-(F2) hold and let $I_1 \in (0, \infty)$ be a closed and bounded interval. Then there exists a positive constant c_1 such that*

$$\sup\{||u||_0 : (\lambda, u) \in \mathcal{C}_{m,+}, \lambda \in I_1\} \leq c_1.$$

Lemma 3.6. *Assume that (A0) and (F0)-(F2) hold. Then for fixed λ, there exists a positive constant c_2 such that*

$$\sup\{||u||_0 : (\lambda, u) \in \mathcal{C}_{m,+}\} \leq c_2 \quad \textit{uniformly for } m \in \mathbb{N}.$$

Proof of Theorem 1.1. By Lemma 3.5 and 3.6, for $(\lambda, u_m) \in \mathcal{C}_{m,+}$, choosing a subsequence and relabling if necessary, it follows from the Arzalà-Ascoli theorem that there exist an unbounded connected component $\mathcal{C}_+ \subseteq \mathcal{C}_{m,+}$, which joins $(\infty, 0)$ with infinity in the direction of λ, that is, (i) of Theorem 1.1 holds. Since Λ in Lemma 3.1 is independent of n, m, we also have

$$\mathcal{C}_+ \cap \{(\lambda, u) \in \Sigma \mid \lambda \geq \Lambda, ||u||_0 = \rho\} = \emptyset,$$

i.e. (ii) of Theorem 1.1 is satisfied.

Next we will show the proof of (iii) of Theorem 1.1. Let

$$\lambda_* := \inf\{\lambda : (\lambda, u) \in \mathcal{C}_+\}.$$

We claim that $\lambda_* \in (0, \infty)$.

Assume on the contrary that there exists $\{(\lambda_k, u_k)\} \subset \mathcal{C}_+ \cap ((0, \infty) \times E)$ such that $\lambda_k \to 0$, then $u_k \to c_3$, where c_3 is a constant.

We first claim that there exist b_*, B_* such that $b_* \leq c_3 \leq B^*$. Suppose on the contrary that there exists a sequence $\{u_k\}$ of non-negative solutions of (1.1) for $\lambda = \lambda_k$ satisfying $||u_k||_0 = b_k \to 0$. Define

$$v_k(t) = \frac{u_k(t)}{b_k}, \quad t \in \mathbb{T},$$

which solves

$$\begin{cases} -Dv_k(t) = \lambda_k a(t) \frac{f(u_k(t))}{u_k(t)} v_k(t), & t \in \mathbb{T}, \\ v_k(0) = v_k(T). \end{cases} \tag{3.3}$$

Sum both sides of (3.3) from 1 to $t-1$, using (F1), we have

$$-\sum_{s=1}^{t-1} Dv_k(s) = \lambda_k \sum_{s=1}^{t-1} a(s) \frac{f(u_k(s))}{u_k(s)} v_k(s) \leq \lambda_k T ||a||_0 \sup \Big| \frac{f(u_k(s))}{u_k(s)} \Big| ||v_k||_0 \to 0.$$

Thus, combining the fact $||v_k||_0 \leq 1$, we obtain that $Dv_k \to 0$ uniformly. As a consequence, $v_k \to 1$ uniformly in $\mathbb{T}$, since

$$|v_k(t) - 1| = |v_k(t) - v_k(t_0)| \leq \sum_{s=1}^{t-1} |Dv_k(s)|,$$

where $t_0 \in \mathbb{T}$ such that $v_k(t_0) = b_k$. On the other hand, summing equation (3.3) we obtain that

$$0 = -\sum_{s=0}^{T-1} a(s) f(v_k(s)) = -\sum_{s=0}^{T-1} a(s) f(b_k) + \sum_{s=0}^{T-1} a(s)(f(b_k) - f(b_k v_k(s)))$$

and dividing by $f(b_k) > 0$, we have

$$0 < \sum_{s=1}^{T-1} a(s) \leq (T-1)||a||_0 \sup \Big|1 - \frac{f(b_k v_k(s))}{f(b_k)}\Big|.$$

Using the first condition in (F2) and $v_k(t) \to 1$ uniformly as $k \to \infty$, we have $0 < \sum\limits_{s=1}^{T-1} a(s) \leq 0$. This is absurd. Therefore we can fix b_* such that $b_* \leq c_3$. Using the second condition in (F2), the argument would be similar for the case $c_3 \leq B^*$ and we omit its details here.

Furthermore, $\{(\lambda_k, u_k)\} \subset \mathcal{C}_+ \cap ((0, \infty) \times E)$ satisfy

$$\begin{cases} -\frac{1}{\lambda_k} Du_k(t) = a(t) f(u_k(t)), \quad t \in \mathbb{T}, \\ u_k(0) = u_k(T) \end{cases} \tag{3.4}$$

and $u_k \to c_3 \in [b_*, B_*]$. Summing (3.4) from 0 to T, then

$$0 = -\sum_{s=0}^{T-1} a(s) f(u_k(s)).$$

By dominated convergence theorem, it implies that $0 = -f(c_3) \sum\limits_{s=0}^{T-1} a(s) < 0$, this is a contradiction. Thus, there exists $\lambda_* > 0$ such that

$$\mathcal{C}_+ \cap ((0, \lambda^*) \times E) = \emptyset,$$

i.e., the property (iii) of the continuum $\mathcal{C}_+$ listed in Theorem 1.1 is satisfied. In conclusion, the proof of Theorem 1.1 is completed. □

The proof of Corollary 1.1. It is an immediate consequence of Theorem 1.1. □

Acknowledgements

The authors are grateful to the anonymous referees for their useful comments and suggestions.

References

[1] R. P. Agarwal, *On multipoint boundary value problems for discrete equations*, J. Math. Anal. Appl., 1983, 96(2), 520–534.

[2] A. Boscaggin and G. Feltrin, *Positive periodic solutions to an indefinite Minkowski-curvature equation*, J. Differential Equations, 2020, 269(7), 5595–5645.

[3] A. Boscaggin, G. Feltrin and F. Zanolin, *Pairs of positive periodic solutions of nonlinear ODEs with indefinite weight: a topological degree approach for the super-sublinear case*, Proc. Roy. Soc. Edinburgh Sect. A, 2016, 146(3), 449–474.

[4] A. Boscaggin and F. Zanolin, *Pairs of positive periodic solutions of second order nonlinear equations with indefinite weight*, J. Differential Equations, 2012, 252(3), 2900–2921.

[5] A. Cabada, V. Otero-Espinar and R. L. Pouso, *Existence and approximation of solutions for first-order discontinuous difference equations with nonlinear global conditions in the presence of lower and upper solutions*, Comput. Math. Appl., 2000, 39(1–2), 21–33.

[6] E. Dancer, *On the structure of solutions of non-linear eigenvalue problems*, Indiana Univ. Math. J., 1974, 23, 1069–1076.

[7] D. Djurčić and A. Torgašev, *Strong asymptotic equivalence and inversion of functions in the class K_c*, J. Math. Analysis Applic., 2001, 255(2), 383–390.

[8] W. Guan, S. Ma and D. Wang, *Periodic boundary value problems for first order difference equations*, Electron. J. Qual. Theory Differ. Equ., 2012, 52, 1–9.

[9] P. Korman and D. S. Schmidt, *Global solution curves for first order periodic problems with applications*, Nonlinear Anal. Real World Appl., 2020, 56, 1–18.

[10] Y. Li, L. Zhu and P. Liu, *Positive periodic solutions of nonlinear functional difference equations depending on a parameter*, Comput. Math. Appl., 2004, 48(10–11), 1453–1459.

[11] Y. Lu, *Positive periodic solutions of nonlinear first-order functional difference equations with a parameter*, Electron. J. Differential Equations, 2011, 96, 1–9.

[12] Y. Lu and J. Wang, *Continuum branch of one-signed periodic solutions of first-order functional equations involving the nonlinearity with zeros*, Monatsh. Math., 2019, 190(4), 769–788.

[13] M. Ma and J. Yu, *Existence of multiple positive periodic solutions for nonlinear functional difference equations*, J Math. Anal. Appl., 2005, 305(2), 483–490.

[14] R. Ma and Y. An, *Global structure of positive solutions for nonlocal boundary value problems involving integral conditions*, Nonlinear Anal., 2009, 71(10), 4364–4376.

[15] R. Ma, T. Chen and Y. Lu, *Positive periodic solutions of nonlinear first-order functional difference equations*, Discrete Dyn. Nat. Soc., 2010. DOI: 10.1155/2010/419536.

[16] R. Ma, X. Su and Z. Zhao, *Global structure of positive solutions for a Neumann problem with indefinite weight in Minkowski space*, J. Fixed Point Theory Appl., 2023, 25(2), 1–13.

[17] L. Maia, N. EI Khattabi and M. Frigon, *Existence and multiplicity results for first-order Stieltjes differential equations*, Adv. Nonlinear Stud., 2022, 22(1), 684–710.

[18] S. Peng, *Positive solutions for first order periodic boundary value problem*, Appl. Math. Comput., 2004, 158(2), 345–351.

[19] P. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal., 1971, 7, 487–513.

[20] Y. N. Raffoul, *Positive periodic solutions of nonlinear functional difference equations*, Electron. J. Differential Equations, 2002, 55, 1–8.

[21] R. Schmidt, *Über divergente Folgen und lineare Mittelbildungen*, Math. Z., 1925, 22(1), 89–152.

[22] J. Sugie, *Number of positive periodic solutions for first-order nonlinear difference equations with feedback*, Appl. Math. Comput., 2021, 391, 1–13.

[23] J. Sun, *Positive solution for first-order discrete periodic boundary value problem*, Appl. Math. Lett. 2006, 19(11), 1244–1248.

[24] D. Wang, Multiple solutions of periodic boundary value problems for first-order difference equations, Bull. Aust. Math. Soc., 2008, 78(1), 1–11.