Journal of Applied Analysis and Computation
Volume 14, Number 1, February 2024, 133–145
Website:http://www.jaac-online.com
DOI:10.11948/20230094

FRACTIONAL ADAMS-MOSER-TRUDINGER TYPE INEQUALITY WITH SINGULAR TERM IN LORENTZ SPACE AND L^P SPACE*

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Abstract For fractional derivatives $(-\Delta)^{\frac{s}{2}}$, we establish Adams-Moser-Trudinger type inequalities with singular term $\frac{1}{|x|^\alpha}$ under Lorentz norm and L^p norm on bounded open domains, and get the sharpness of all inequalities. Furthermore, we obtain the sharpness with a more general method.

Keywords Adams-Moser-Trudinger type inequality, fractional derivative, Lorentz space.

MSC(2010) 35B33, 35J60.

1. Introduction

Let Ω be an open domain with finite measure in $\mathbb{R}^n$. The Sobolev embedding theorem shows that $W_0^{k,p}(\Omega) \subset L^q(\Omega)$ for $1 \leq q \leq \frac{np}{n-kp}$ and $kp < n$. However, there is no $W_0^{k,p}(\Omega) \subset L^\infty(\Omega)$ in the borderline case $kp = n$. Yudovich [32], Pohozave [23] and Trudinger [29] proved that $W_0^{1,n}$ can be embedded in Orlicz space. In fact

$$\sup_{u\in W_0^{1,n}(\Omega),\ \|\nabla u\|_{L^n(\Omega)}\leq 1} \int_\Omega \exp(\beta\,|u|^{\frac{n}{n-1}})dx < +\infty \tag{1.1}$$

for some $\beta > 0$. Moser [21] gave the best constant $\beta > 0$ in the inequality (1.1), that is

$$\sup_{u\in W_0^{1,n}(\Omega),\ \|\nabla u\|_{L^n(\Omega)}\leq 1} \int_\Omega \exp(\beta_n\,|u|^{\frac{n}{n-1}})dx < +\infty, \quad \beta_n = n\omega_{n-1}^{\frac{1}{n-1}}. \tag{1.2}$$

The constant in (1.2) is sharp in the sense that for any $\beta > \beta_n$, the supermum in (1.1) is infinite. The inequality (1.2) is known as Moser-Trudinger inequality.

The Hardy inequality is

$$\left(\frac{n-1}{n}\right)^n \int_\Omega \frac{|u|}{|x|^n\left(\log\frac{R}{|x|}\right)^n}dx \leq \int_\Omega |\nabla u|^n,$$

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where $R \geq essup_{\Omega}|x|$. Thus it is very natural to establish in relationship between Hardy inequality and Moser-Trudinger inequality. Inspired by the Hardy inequality, Adimurthi and Sandeep proved a singular Moser-Trudinger inequality with the sharp constant in [2].

Theorem A ([2]). *Let Ω be an open and bounded set in $\mathbb{R}^n$. There exists a constant $C_0 = C_0(n, |\Omega|) > 0$ such that*

$$\int_{\Omega} \frac{\exp(\beta |u|^{\frac{n}{n-1}})}{|x|^{\alpha}} dx \leq C_0$$

for any $\alpha \in [0, n)$, $0 \leq \beta \leq (1 - \frac{\alpha}{n})\beta_n$, any $u \in W_0^{1,n}(\Omega)$ with $\int_{\Omega} |\nabla u|^n dx \leq 1$. Moreover, this constant $(1 - \frac{\alpha}{n})\beta_n$ is sharp in the sense that if $\beta > (1 - \frac{\alpha}{n})\beta_n$, then the above inequality can no longer hold with some C_0 independent of u.

There is another improved Moser-Trudinger inequality on the disk in $\mathbb{R}^2$, which was recently proved and studied in [3, 19]:

$$\sup_{u \in W_0^{1,2}(B),\ \|\nabla u\|_2 \leq 1} \int_B \frac{\exp\left(4\pi |u|^2 - 1\right)}{(1 - |x|^2)} dx < \infty.$$

Wang and Ye [30] proved an interesting Hardy-Moser-Trudinger inequality on the unit disk in $\mathbb{R}^2$, which improves the classical Moser-Trudinger inequality and the classical Hardy inequality at the same time. Namely, there exists a constant $C_0 > 0$ such that

$$\int_B \exp\left(\frac{4\pi u^2}{H(u)}\right) dx \leq C_0 < \infty,\ \forall u \in C_0^{\infty}(B) \setminus \{0\},$$

where

$$H(u) = \int_B |\nabla u|^2 dx - \int_B \frac{\exp\left(4\pi |u|^2 - 1\right)}{\left(1 - |x|^2\right)^2} dx.$$

Recently, there are many excellent results for the Moser-Trudinger inequality [6–8, 24].

In 1988, Adams [1] generalized the Moser-Trudinger inequality to high order Sobolev spaces. He gave the following theorem in [1].

Theorem B ([1]). *Let Ω be an open and bounded domain in $\mathbb{R}^n$. If m is a positive integer less than n, then there exists a constant $C_0 = C(n, m) > 0$ such that for any $u \in W_0^{m, \frac{n}{m}}(\Omega)$ and $\|\nabla^m u\|_{L^{\frac{n}{k}}(\Omega)} \leq 1$, the inequality*

$$\sup_{u \in C_c^k(\Omega),\ \|\nabla^k u\|_{L^{\frac{n}{k}}(\Omega)} \leq 1} \int_{\Omega} \exp(\beta |u|^{\frac{n}{n-k}}) dx < C_0 |\Omega|$$

holds for all $\beta \leq \beta(k, n)$, where

$$\beta(k,n) = \begin{cases} \frac{n}{\omega_{n-1}} \left[\frac{\pi^{\frac{n}{2}} 2^k \Gamma(\frac{k+1}{2})}{\Gamma(\frac{n-k+1}{2})}\right]^{\frac{n}{n-k}}, & \text{where } k \text{ is odd,} \\ \frac{n}{\omega_{n-1}} \left[\frac{\pi^{\frac{n}{2}} 2^k \Gamma(\frac{k}{2})}{\Gamma(\frac{n-k}{2})}\right]^{\frac{n}{n-k}}, & \text{where } k \text{ is even,} \end{cases}$$

and, $\nabla^k := \nabla\Delta^{\frac{k-1}{2}}$ *for* k *odd and* $\nabla^k = \Delta^{\frac{k}{2}}$ *for* k *even. Moreover the constant* β *is sharp in the sense that*

$$\sup_{u\in C_c^k(\Omega),\ \|\nabla^k u\|_{L^{\frac{n}{k}}(\Omega)}\le 1}\int_\Omega \exp(\beta|u|^{\frac{n}{n-k}})dx=\infty$$

for $\beta > \beta(k,n)$*.*

Adams [1] gave the the result of fractional Laplace operator $f = (-\Delta)^{\frac{n}{2p}}u$, for more fractional operators we can see [28].

Theorem C ([1]). *Let* Ω *be an open and bounded domain in* $\mathbb{R}^n$*. Fix* $p \in (1,\infty)$*. For* $\alpha \in (0,n)$ *and* $f \in L^p(\Omega)$*, we consider the Riesz potential* $I_\alpha f$ *defined as*

$$I_\alpha f(x)=\int_\Omega \frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

Then

$$\sup_{f\in L^p(\Omega),\ \|f\|_{L^p(\Omega)}\le 1}\int_\Omega e^{\frac{n}{\omega_{n-1}}|I_{\frac{n}{p}}f|^{p'}}dx\le c_{n,p}|\Omega|,\quad p'=\frac{p}{p-1}.$$

The constant $\frac{n}{\omega_{n-1}}$ *is sharp in the sense that*

$$\sup_{f\in C_0^\infty(B_\delta),\ \|f\|_{L^p(B_\delta)}\le 1}\int_{B_\delta} e^{\gamma|I_{\frac{n}{p}}f|^{p'}}dx=\infty\ \text{ for every } \delta>0,\ \gamma>\frac{n}{\omega_{n-1}}.$$

In 1996, Alvino, Ferone and Trombetti established the Moser-Truding inequality on bounded domain in Lorentz-Sobolev space in [5].

Theorem D ([5]). *Let* Ω *be an open and bounded domain in* $\mathbb{R}^n$*,* $n \ge 2$*, and let* $f \in C_0^\infty(\Omega)$ *be a function compactly supported in* Ω *such that:* $\|\nabla f\|_{n,p} \le 1$*,* $1 \le q \le \infty$*. We have:*
(i) *if* $1 \le q < \infty$ *then:*

$$\frac{1}{|\Omega|}\int_\Omega \exp(\beta|f|^{\frac{q}{q-1}})dx\le c_0<\infty,\ \forall\beta\le\beta_{n,p}=(n\nu_n^{\frac{1}{n}})^{\frac{q}{q-1}};$$

(ii) *if* $q = \infty$ *then:*

$$\frac{1}{|\Omega|}\int_\Omega \exp(\beta|f|^{\frac{q}{q-1}})dx\le c_0<\infty,\ \forall\beta<\beta_{n,\infty}=n\nu_n^{\frac{1}{n}}.$$

This constant $\beta_{n,p}$ *is sharp in the sense that if* $\beta > \beta_{n,p}$*, then the above inequality can no longer hold with some* c_0 *independent of* f*.*

Lu and Tang gave the Moser-Trudinger inequalities on any bounded domain in $\mathbb{R}^n$ with Lorentz-Sobolev norms in [17].

For the fractional Adams-Moser-Trudinger type inequalities, Luca gave the following result.

Theorem E ([20]). *For any* $p \in (1,\infty)$ *and positive integer* n*, set* $K_{n,s} := \frac{\Gamma(\frac{n-s}{2})}{\Gamma(\frac{n}{2})2^s\pi^{\frac{n}{2}}}$*,* $\beta_{n,p} := \frac{n}{\omega_{n-1}}K_{n,\frac{n}{p}}^{-p'}$*. Then for any open set* $\Omega \in \mathbb{R}^n$ *with finite measure, we have*

$$\sup_{u\in W_0^{\frac{n}{2p},p}(\Omega),\ \left\|(-\Delta)^{\frac{n}{2p}}u\right\|_{L^p(\Omega)}\le 1}\int_\Omega \exp(\beta_{n,p}|u|^{p'})dx\le C_{n,p}|\Omega|. \tag{1.3}$$

Moreover the constant $\beta_{n,p}$ is sharp in the sense that we cannot replace it with any larger one without making the supremum in (1.3) infinite.

In 2008, Angel [4] gave the Adams type inequalities in bounded domain of Lorentz space:

Theorem F ([4]). *Let m be a positive integer satisfying $1 \leq m < n$ and let $q \in (1, +\infty)$,*

$$\beta_{n,m} = \begin{cases} \dfrac{\pi^{\frac{n}{2}} 2^m \Gamma(\frac{m}{2})}{\omega_n^{\frac{n-m}{n}} \Gamma(\frac{n-m}{2})}, & \text{where } m \text{ is even,} \\ \dfrac{\pi^{\frac{n}{2}} 2^m \Gamma(\frac{m+1}{2})}{\omega_n^{\frac{n-m}{n}} \Gamma(\frac{n-m+1}{2})}, & \text{where } m \text{ is odd.} \end{cases}$$

Then there exists a constant $C = C(n, m, |\Omega|, q)$ such that

$$\int_\Omega \exp(\beta_{n,m} |u|^{q'}) dx \leq C, \tag{1.4}$$

for every $u \in W_0^m L^{\frac{n}{m},q}(\Omega)$ fulfilling $\|\nabla^m u\|_{L^{\frac{n}{m},q}(\Omega)} \leq 1$. The result is sharp in the sense that the left-hand of (1.4) with $\beta_{n,m}$ replaced by any larger constant cannot be uniformly bounded as u ranges among all functions from $W_0^m L^{\frac{n}{m},q}(\Omega)$ satisfying $\|\nabla^m u\|_{L^{\frac{n}{m},q}(\Omega)} \leq 1$.

For the fractional Adams inequalities in Lorentz space, Xiao and Zhai [31] proved the following results in 2010.

Theorem G ([31]). *Suppose that $0 < s < n, q \in (1, \infty)$ and Ω is a bounded open domain. Then there exists a positive constant $C_{s,q,n}$ depending on s, q, and n such that*

$$\int_\Omega \exp\left(\beta \left|\frac{f(x)}{\|(\Delta)^{\frac{s}{2}} f\|_{L^{\frac{n}{s},q}(\Omega)}}\right|^{q'}\right) dx \leq C_{s,q,n} |\Omega| \tag{1.5}$$

holds for all

$$\beta \leq \left(\frac{n}{\omega_{n-1}}\right)^{q' \frac{n-s}{n}} K_{s,n}^{-q'}$$

and

$$supp((-\Delta)^{\frac{s}{2}} f) \subseteq \Omega, \quad \|(-\Delta)^{\frac{s}{2}} f\|_{L^{\frac{n}{s},q}(\Omega)} < \infty.$$

Furthermore, no number greater than $\left(\frac{n}{\omega_{n-1}}\right)^{q' \frac{n-s}{n}} K_{s,n}^{-q'}$ can replace β in (1.5), where $K_{s,n} = \frac{\Gamma\left(\frac{n-s}{2}\right)}{\pi^{\frac{n}{2}} 2^s \Gamma\left(\frac{s}{2}\right)}$.

For more fractional operator [11, 25–27].

In this paper, we will first give singular sharp Adams inequalities with fractional Laplace operators in Lorentz space and L^p space. We will establish the sharpness with a more general method.

Theorem 1.1. *If $q \in (1, \infty), 0 < \alpha < n$, then there exists a positive constant $C_{s,q,n}$ depending on s, q and n such that*

$$\int_\Omega \frac{\exp\left[(1 - \frac{\alpha}{n})\beta \left|\frac{f(x)}{\left\|(-\Delta)^{\frac{s}{2}} f\right\|_{L^{\frac{n}{s},\ q}(\Omega)}}\right|^{q'}\right]}{|x|^\alpha} dx \leq C_{s,q,n,\alpha} |\Omega|^{1-\frac{\alpha}{n}}, \tag{1.6}$$

for all

$$\beta \leq \left(\frac{n}{\omega_{n-1}}\right)^{q'\frac{n-s}{n}} K_{s,n}^{-q'}$$

and $supp((-\Delta)^{\frac{s}{2}} f) \subseteq \Omega$, $\left\|(-\Delta)^{\frac{s}{2}} f\right\|_{L^{\frac{n}{s},q}}(\Omega) < \infty$. *Furthermore, no number greater than* $\left(\frac{n}{\omega_{n-1}}\right)^{q'\frac{n-s}{n}} K_{s,n}^{-q'}$ *can replace* β *in (1.6).*

Now we consider the Adams-Moser-Trudinger type inequalities with singular term $\frac{1}{|x|^\alpha}$ under L^p norms. We observe that the requirement $supp((-\Delta)^{\frac{s}{2}} f) \subseteq \Omega$ is necessary in Theorem 1.1, and Adams applies this result to the function $f = (-\Delta)^{\frac{n}{2p}} u$ where u is smooth and supported in $\bar{\Omega}$, and $p = \frac{n}{k}$. Here it is crucial that when $\frac{n}{p} \in \mathbb{N}$, then the support of f does not exceed the the support of u, so that Adams' result in [1] can be applied. This is not the case when $\frac{n}{p} \notin \mathbb{N}$. In fact, the support of $(-\Delta)^{\frac{s}{2}} u$ can be the whole $\mathbb{R}^n$ for general $s > 0$ even if u is compactly supported.

In order to circumvent this issue, instead of using the Riesz potential we will write u in terms of a Green representation formular

$$u(x) = \int_\Omega G_{\frac{n}{p}}(x,y)(-\Delta)^{\frac{n}{2p}} u(y)dy,$$

which holds for a suitable Green function which we construct using variational methods, and which we can sharply bound in terms of the fundamental solution of $(-\Delta)^{\frac{n}{2p}}$ in $\mathbb{R}^n$.

Next, we will first give singular sharp inequality with fractional Laplace operator and establish the sharpness with a more general method.

Theorem 1.2. *For any* $p \in (1,\infty)$ *and positive integer* n, *set* $K_{n,s} = \frac{\Gamma\left(\frac{(n-s)}{2}\right)}{\Gamma(\frac{s}{2})2^s \pi^{\frac{n}{2}}}$, $\beta_{n,p} = \frac{n}{\omega_n} K_{n,\frac{n}{p}}^{-p'}$. *Then, for any open set* $\Omega \subset \mathbb{R}^n$ *with finite measure, we have*

$$\sup_{u \in W_0^{\frac{n}{p},p}(\Omega),\ \left\|(-\Delta)^{\frac{n}{2p}} u\right\|_{L^p(\Omega)} \leq 1} \frac{1}{|\Omega|^{1-\frac{\alpha}{n}}} \int_\Omega \frac{e^{\left(1-\frac{\alpha}{n}\right)\beta_{n,p}|u|^{\frac{p}{p-1}}}}{|x|^\alpha} dx \leq C(n,q,\alpha).$$

And the constant $\left(1-\frac{\alpha}{n}\right)\beta_{n,p}$ *is sharp in the sense that the above supremum is infinite for any* $\beta > \left(1-\frac{\alpha}{n}\right)\beta_{n,p}$.

2. Proof of Theorem 1.1

We know that from [31]

$$f(x) = K_{s,n} \int_{\mathbb{R}^n} \frac{(-\Delta)^{\frac{s}{2}} f(y)}{|y-x|^{n-s}} dy, \qquad \forall x \in \mathbb{R}^n \tag{2.1}$$

with

$$K_{s,n} = \frac{\Gamma(\frac{n-s}{2})}{\pi^{\frac{n}{2}} 2^s \Gamma(\frac{s}{2})}.$$

Theorem 1.1 is immediately obtained from (2.1) and Lemma 2.1 below-a sharp result about the Riesz potential.

Lemma 2.1. *If $q \in (1,\infty), 0<\alpha<n$, then there exists a positive constant $C_{s,q,n}$ depending on s, q and n such that*

$$\int_\Omega \frac{\exp\left[(1-\frac{\alpha}{n})\beta\left|\frac{I_s * f(x)}{\|f\|_{L^{\frac{n}{s},\ q}(\Omega)}}\right|^{q'}\right]}{|x|^\alpha}dx \le C_{n,p}|\Omega|^{1-\frac{\alpha}{n}} \tag{2.2}$$

holds, where

$$\beta \le \left(\frac{n}{\omega_{n-1}}\right)^{q'\frac{n-s}{n}},$$

and

$$supp(f) \subseteq \Omega, \quad \|f\|_{L^{\frac{n}{s},\ q}(\Omega)} < \infty.$$

Furthermore, no number greater than $\left(\frac{n}{\omega_{n-1}}\right)^{q'\frac{n-s}{n}}$ can replace β in (2.2).

Proof of Lemma 2.1. By O′Neil′s [22] result,

$$\begin{aligned}
u^*(t) &\le (I_s * f)^*(t)\\
&\le \frac{1}{t}\int_0^t I_s^*(\tau)\tau d\tau \int_0^t f^*(\tau)d\tau + \int_t^\infty I_s^*(\tau)f^*(\tau)d\tau\\
&\le (\frac{\omega_{n-1}}{n})^{\frac{n-s}{n}}\left[\frac{n}{s}t^{-\frac{n-s}{n}}\int_0^t f^*(\tau)d\tau + \int_t^{|\Omega|} f^*(\tau)\tau^{-\frac{n-s}{n}}d\tau\right].
\end{aligned}$$

Let $t=|\Omega|e^{-\tau}$. We get

$$\begin{aligned}
&\left(1-\frac{\alpha}{n}\right)^{\frac{1}{q'}}\left(\frac{\omega_{n-1}}{n}\right)^{\frac{n-s}{n}}u^*(|\Omega|e^{-\tau})\\
\le&\left(1-\frac{\alpha}{n}\right)^{\frac{1}{q'}}\frac{n}{s}(|\Omega|e^{-\tau})^{-\frac{n-s}{n}}\int_\tau^\infty f^*(|\Omega|e^{-\sigma})(|\Omega|e^{-\sigma})d\sigma\\
&+\int_0^\tau\left(1-\frac{\alpha}{n}\right)^{\frac{1}{q'}}f^*(|\Omega|e^{-\sigma})(|\Omega|e^{-\sigma})^{-\frac{n-s}{n}}(|\Omega|e^{-\sigma})d\sigma\\
:=&\int_0^\infty a(\sigma,\tau)\varphi(\sigma)d\sigma.
\end{aligned}$$

Here $\varphi(\sigma)=f^*(|\Omega|e^{-\sigma})(|\Omega|e^{-\sigma})^{\frac{s}{n}}$, if $\sigma>0$, and

$$a(\sigma,\tau)=\begin{cases}1-\frac{\alpha}{n}, & 0\le\sigma<\tau<\infty,\\ (1-\frac{\alpha}{n})\frac{n}{s}(|\Omega|e^{-\tau})^{-\frac{n-s}{n}}(|\Omega|e^{-\sigma})^{-\frac{n-s}{n}}, & 0\le\tau<\sigma<\infty.\end{cases}$$

Now, we need a result similar to the Lemma 1 in Adams [1]: Let $a(s,t)$ be a nonnegative measure function in $[0,\infty]\times[0,\infty]$ such that $a(s,t)\le 1$, for *a.e.* $0\le s<t$. Assume that

$$\sup_{t>0}\left(\int_t^\infty a(s,t)^{q'}ds\right)^{\frac{1}{q'}} = b < \infty.$$

Then there exists a positive constant $C=C(q,b,\alpha)$ such that

$$\int_0^\infty e^{-F_\alpha(t)}dt \le C,$$

for every nonnegative measure function φ in $(0,\infty)$ satisfying $\int_0^\infty \varphi(s)^q ds \le 1$, where

$$F_\alpha(t) = (1-\frac{\alpha}{n})t - (1-\frac{\alpha}{n})\left(\int_0^\infty a(s,t)\varphi(s)ds\right)^{q'}.$$

It is easy to know that $a(\sigma,\tau) \le 1$, for $a.\ e.$ $0 \le \sigma < \tau$, when $0 \le \tau < \sigma < \infty$,

$$\begin{aligned}
\int_\tau^\infty a(\sigma,\tau)^{q'} d\sigma &= (1-\frac{\alpha}{n})\int_\tau^\infty \left[\frac{n}{s}(|\Omega|e^{-\tau})^{-\frac{n-s}{n}}(|\Omega|e^{-\sigma})^{-\frac{n-s}{n}}\right]^{q'} d\sigma \\
&= \left(1-\frac{\alpha}{n}\right)(\frac{n}{s})^{q'}(|\Omega|e^{-\tau})^{-q'\frac{n-s}{n}}\int_\tau^\infty (|\Omega|e^{-\sigma})^{q'\frac{n-s}{n}} d\sigma \\
&= \left(1-\frac{\alpha}{n}\right)(\frac{n}{s})^{q'}\frac{n}{q'(n-s)} < \infty
\end{aligned}$$

for $\tau > 0$, which can imply

$$\sup_{\tau>0}\int_\tau^\infty a(\sigma,\tau)^{q'} d\sigma < \infty,$$

and

$$\begin{aligned}
\|\varphi\|^q_{L^q_{(0,\infty]}} &= \int_0^\infty [f^*(|\Omega|e^{-\sigma})(|\Omega|e^{-\sigma})^{q'\frac{s}{n}}]^q d\sigma \\
&= \int_0^{|\Omega|} (f^*(t)t^{\frac{s}{n}})^q \frac{dt}{t} \\
&= \|f\|^q_{L^{\frac{n}{s},q}(\Omega)} = 1.
\end{aligned}$$

Using above result, we obtain

$$\begin{aligned}
&\int_\Omega \frac{\exp\left((1-\frac{\alpha}{n})(\frac{n}{\omega_{n-1}})^{q'\frac{n-s}{n}}|u(x)|^{q'}\right)}{|x|^\alpha}dx \\
\le& \int_0^{|\Omega|} \exp\left((1-\frac{\alpha}{n})(\frac{n}{\omega_{n-1}})^{q'\frac{n-s}{n}}|u^*(t)|^{q'}\right)\left(\frac{\omega_{n-1}}{n}t^{-1}\right)^{\frac{\alpha}{n}} dt \\
=& |\Omega|^{1-\frac{\alpha}{n}}\int_0^\infty \exp\left((1-\frac{\alpha}{n})(\frac{n}{\omega_{n-1}})^{q'\frac{n-s}{n}}|u^*(|\Omega|e^{-\tau})|^{q'} - (1-\frac{\alpha}{n})\tau\right) d\tau \\
\le& |\Omega|^{1-\frac{\alpha}{n}}\int_0^\infty \exp\left(\left(\int_0^\infty a(\sigma,\tau)\varphi(\sigma)d\sigma\right)^{q'} - q'(1-\frac{\alpha}{n})\tau\right) d\tau \\
=& |\Omega|^{1-\frac{\alpha}{n}}\int_0^\infty e^{-F_\alpha(\tau)} d\tau \le C|\Omega|^{1-\frac{\alpha}{n}},
\end{aligned}$$

where $F_\alpha(\tau) = (1-\frac{\alpha}{n})\tau - (1-\frac{\alpha}{n})\left(\int_0^\infty a(\sigma,\tau)\varphi(\sigma)d\sigma\right)^{q'}$.

Now we will prove the $(1-\frac{\alpha}{n})(\frac{n}{\omega_{n-1}})^{q'\frac{n-s}{n}}$ is the best one for $\Omega = B$: the unite ball centered at the origin, let $f \ge 0$ such that $I_s * f \ge 1$, for $x \in B_r := \{x \in \mathbb{R}^n : |x| \le r\}$ with $0 < r < 1$, from (1.6) it follows that

$$\left|\frac{B_r}{B}\right|^{1-\frac{\alpha}{n}} |B_r|^{\frac{\alpha}{n}}\frac{1}{r^\alpha} e^{\frac{(1-\frac{\alpha}{n})\beta}{\|f\|^{q'}_{L^{\frac{n}{s},q}}}}$$

$$
\begin{aligned}
&\le \left|\frac{B_r}{B}\right|^{1-\frac{\alpha}{n}} \frac{1}{|B_r|^{1-\frac{\alpha}{n}}} \int_{B_r} \frac{e^{\frac{(1-\frac{\alpha}{n})\beta}{\|f\|^{q'}_{L^{\frac{n}{s},q}}}}}{|x|^\alpha} dx \\
&\le \left|\frac{B_r}{B}\right|^{1-\frac{\alpha}{n}} \frac{1}{|B_r|^{1-\frac{\alpha}{n}}} \int_{B_r} \frac{e^{\frac{(1-\frac{\alpha}{n})\beta I_s * f(x)}{\|f\|^{q'}_{L^{\frac{n}{s},q}}}}}{|x|^\alpha} dx \\
&\le \frac{1}{|B_r|^{1-\frac{\alpha}{n}}} \int_{B} \frac{e^{\frac{(1-\frac{\alpha}{n})\beta I_s * f(x)}{\|f\|^{q'}_{L^{\frac{n}{s},q}}}}}{|x|^\alpha} dx \\
&\le C,
\end{aligned}
$$

and

$$
\begin{aligned}
\Big(1-\frac{\alpha}{n}\Big)\beta &\le \|f\|^{q'}_{L^{\frac{n}{s},q}(B)} \left((1-\frac{\alpha}{n}) \log\left|\frac{B}{B_r}\right| + \log(r^\alpha |B_r|^{-\frac{\alpha}{n}}) + \log C\right) \\
&\le \|f\|^{q'}_{L^{\frac{n}{s},q}(B)} \left((1-\frac{\alpha}{n}) \log\left|\frac{B}{B_r}\right| + \log|B|^{\frac{\alpha}{n}} + \log C\right).
\end{aligned}
$$

So $\beta \le n \lim_{r\to 0}(\log \frac{1}{r})[Cap_{WL^{\frac{n}{s},q}}(B_r;B)]^{q'}$, where

$$
Cap_{WL^{\frac{n}{s},q}}(E;B) = \inf \|f\|_{L^{\frac{n}{s},q}(B)},
$$

and E is a compact subset of B. Where the infimum is taken over all $f \ge 0$ vanishing on the complement of B, and

$$
I_s * f(x) \ge 1, \quad \text{on} \quad E.
$$

By the proof of Theorem 1.2 in [1], for any $\varepsilon > 0$, we can choose $0 < r < 1$ small enough such that

$$
I_s * f_r(y) \ge 1, \quad \text{on} \quad B_r,
$$

with

$$
f_r(y) = \begin{cases} \frac{1}{\omega_{n-1}(1-\varepsilon)}(\log\frac{1}{r})^{-1}|y|^{-s}, & r < |y| < 1, \\ 0, & \text{otherwise}, \end{cases}
$$

and

$$
h(y) = \begin{cases} |y|^{-s}, & r < |y| < 1, \\ 0, & \text{otherwise}. \end{cases}
$$

Then the domain of $h^*(t)$ is $(r^n \frac{\omega_{n-1}}{n},\ \infty)$, where

$$
h^*(t) = \begin{cases} (\frac{tn}{\omega_{n-1}})^{-\frac{s}{n}}, & r^n\frac{\omega_{n-1}}{n} < t < \frac{\omega_{n-1}}{n}, \\ 0, & \text{otherwise}. \end{cases}
$$

Consequently,

$$
\begin{aligned}
\|f_r\|_{L^{\frac{n}{s},q}(B)} &= \|t^{\frac{s}{n}-\frac{1}{q}} f_r^*(t)\|_{L^q(0,|B|)} \\
&\le \frac{1}{\omega_{n-1}(1-\varepsilon)}(\log\frac{1}{r})^{-1} \left(\int_{r^n\frac{\omega_{n-1}}{n}}^{\frac{\omega_{n-1}}{n}} \left[(\frac{tn}{\omega_{n-1}})^{-\frac{s}{n}} t^{\frac{s}{n}-\frac{1}{q}}\right]^q dt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$= \frac{n^{\frac{1}{q}}}{\omega_{n-1}(1-\varepsilon)}(\frac{\omega_{n-1}}{n})^{\frac{s}{n}}(\log\frac{1}{r})^{\frac{1-q}{q}}.$$

This gives

$$Cap_{\dot{w}L^{\frac{n}{s},q}}(B_r;B) \le \|f_r\|_{L^{\frac{n}{s},q}(B)}$$
$$= \frac{n^{\frac{1}{q}}}{\omega_{n-1}(1-\varepsilon)}(\frac{\omega_{n-1}}{n})^{\frac{s}{n}}(\log\frac{1}{r})^{\frac{1-q}{q}}.$$

Finally, a simple computation yields

$$\beta \le n\lim_{n\to 0}\log\frac{1}{r}\left(\frac{n^{\frac{1}{q}}}{\omega_{n-1}(1-\varepsilon)}(\frac{\omega_{n-1}}{n})^{\frac{s}{n}}(\log\frac{1}{r})^{\frac{1-q}{q}}\right)^{q'}$$
$$= \left(\frac{n}{\omega_{n-1}}\right)^{q'\frac{n-s}{n}}.$$

This completes the proof of Lemma 2.1. □

3. The proof of Theorem 1.2

In order to prove the Theorem 1.2, we need some preparatory work.

Lemma 3.1. ([16]) *The fundamental solution of* $(-\Delta)^{-\frac{s}{2}}$ *on* $\mathbb{R}^n$ *is* $F_s(x) = K_{n,s}|x|^{s-n}$*, in the sense that* $F_s \in L_s(\mathbb{R}^n)$ *and* $(-\Delta)^{-\frac{s}{2}}F_s = \delta_0$ *in the sense of tempered distributions. Moreover* $(-\Delta)^{-\frac{s}{2}}(F_s * f) = f$ *for every* $f \in \mathcal{S}(\mathbb{R}^n)$.

Lemma 3.2. ([14]) *Let* $0 < p < \infty,\ 0 \le \alpha < n$*, and* $\Omega \subset \mathbb{R}^n$ *be an open set with bounded measure. Then there is a constant* $C_0 = C_0(p,q)$ *such that for* $f \in L^p(\mathbb{R}^n)$ *with support contained in* Ω*, then*

$$\int_\Omega \frac{\exp\left((1-\frac{\alpha}{n})\frac{n}{\omega_{n-1}}\left|\frac{I_\gamma * f(x)}{\|f\|_p}\right|^{p'}\right)}{|x|^\alpha}dx \le C_0, \tag{3.1}$$

where $I_\gamma * f(x) = \int |x-y|^{\gamma-n}f(y)dy$ *is the Riesz potential of order* γ.

For the proof of sharpness of Theorem 1.1, we will give a more general conclusion. We hope obtain the following result:

$$\sup_{u\in\tilde{W}_0^{\frac{n}{p},p}(\Omega),\ \left\|(-\Delta)^{\frac{n}{2p}}u\right\|^p_{L^p(\Omega)}\le 1}\int_\Omega \frac{f(|u|)e^{(1-\frac{\alpha}{n})\beta_{n,p}|u|^{p'}}}{|x|^\alpha}dx = \infty,$$

for any $f : [0,\infty) \to [0,\infty)$ with

$$\lim_{t\to\infty} f(t) = \infty, \quad f \text{ is Borel measurable.} \tag{3.2}$$

Obviously, the above $f(t)$ is not necessarily increasing with exponent.

Similar to standard Moser function, Adams' test function is smooth and compact, he obtains the $W_0^{k,\frac{n}{k}}$-norms with a precisely method. However, when k is no longer an integer even number, the Adams way is much more complex. So we have to prove the fractional Laplacians norms $\left\|(-\Delta)^{\frac{n}{2p}}u\right\|_{L^p(\Omega)}$.

In fact, we can prove the following result in a slightly stronger form.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain with finite measure and let $f : [0,\infty) \to [0,\infty)$ be a Borel measurable satisfying $\lim_{t\to\infty} f(t) = \infty$. Then*

$$\sup_{u\in \tilde{W}_0^{\frac{n}{p},p}(\Omega),\ \|u\|^p_{L^p(\Omega)}+\left\|(-\Delta)^{\frac{n}{2p}}u\right\|^p_{L^p(\mathbb{R}^n)}\le 1} \int_\Omega \frac{f(|u|)e^{(1-\frac{\alpha}{n})\beta_{n,p}|u|^{p'}}}{|x|^\alpha}dx = \infty,$$

where $\beta_{n,p} = \frac{n}{\omega_{n-1}}(K_{n,\frac{n}{p}})^{p'}$.

In order to prove Lemma 3.3, we need the following Moser type functions. Set two smooth functions $\eta,\ \varphi$ such that $0 \le \eta,\ \varphi \le 1$,

$$\eta \in C_c^\infty(-1,1), \quad \eta = 1 \text{ on } (-\frac{3}{4}, \frac{3}{4}),$$

and

$$\varphi \in C_c^\infty(-2,2), \quad \eta = 1 \text{ on } (-1,1).$$

For $\varepsilon > 0$, we set

$$\psi_\varepsilon(t) = \begin{cases} 1-\varphi_\varepsilon(t), & \text{if } 0 \le t \le \frac{1}{2}, \\ \eta(t), & \text{if } t \ge \frac{1}{2}, \end{cases}$$

and

$$v_\varepsilon\varepsilon(x) = \left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{p}} \left(\log\left(\frac{1}{\varepsilon}\right)\varphi_\varepsilon(|x|) + \log\left(\frac{1}{|x|}\right)\varphi_\varepsilon(|x|)\right), \quad x \in \mathbb{R}^n,$$

where

$$\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(\frac{t}{\varepsilon}).$$

Lemma 3.4. ([12]) *Let*

$$u_\varepsilon(x) = |\omega^{n-1}|^{-\frac{1}{p}} 2^{\frac{n}{p'}} \pi^{\frac{n}{2}} \Gamma(\frac{n}{2p'}) \frac{1}{\Gamma(\frac{n}{2p})\gamma_n} v_\varepsilon(x).$$

For $1 < p < \infty$ there exists a constant $C > 0$ such that

$$\left\|(-\Delta)^{\frac{n}{2p}}u_\varepsilon\right\|_{L^p(\mathbb{R}^n)} \le \left(1 + C\left(\log\frac{1}{\varepsilon}\right)^{-1}\right)^{\frac{1}{p}},$$

where Γ is gamma function and $\gamma_n = \frac{(n-1)!}{2}|\omega^n|$.

Proof of Lemma 3.3. Without loss of generality, we can assume that $B_1 \subseteq \Omega$, let u_ε be defined as in Lemma 3.4, we claim that there exists a constant $\delta > 0$ such that

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} \frac{\exp\left(\frac{(1-\frac{\alpha}{n})\beta_{n,p}|u_\varepsilon|^{p'}}{\|u_\varepsilon\|^p_{L^p(\mathbb{R}^n)}+\left\|(-\Delta)^{\frac{n}{2p}}u_\varepsilon\right\|^p_{L^p(\mathbb{R}^n)}}\right)}{|x|^\alpha}dx = \lim_{\varepsilon\to 0} I_\varepsilon \ge \delta.$$

Then Lemma 3.3 follows at once, since $u_\varepsilon \to \infty$, on B_ε as $\varepsilon \to 0$, and

$$\sup_{u\in \tilde{W}_0^{\frac{n}{p},p}(\Omega),\ \|u\|^p_{L^p(\Omega)}+\left\|(-\Delta)^{\frac{n}{2p}}u\right\|^p_{L^p(\mathbb{R}^n)}\le 1} \int_\Omega \frac{f(|u|)e^{(1-\frac{\alpha}{n})\beta_{n,p}|u|^{p'}}}{|x|^\alpha}dx$$
$$\ge \frac{I_\varepsilon}{|x|^\alpha}\inf_{x\in B_\varepsilon} f(|u_\varepsilon(x)|).$$

Let $\varepsilon = e^{-k}$. Noting that

$$\lim_{k\to\infty}[-k+k(1+\frac{C}{k})^{-\frac{p'}{p}}] = -C\frac{p'}{p}$$

and using the results of Lemma 3.4,

$$\begin{aligned}
\frac{I_\varepsilon}{|x|^\alpha} &\ge \frac{|B_1|\varepsilon^n e^{n\log\frac{1}{\varepsilon}(1+C(\log\frac{1}{\varepsilon})^{-1})^{-\frac{p'}{p}}}}{e^{-k\alpha}}\\
&= |B_1|e^{-kn}e^{\frac{n-\alpha}{n}(n\log e^k(1+C\frac{1}{\log e^k})^{-\frac{p'}{p}})}\\
&= |B_1|e^{-kn+\frac{n-\alpha}{n}(kn+(1+C\frac{1}{k})^{-\frac{p'}{p}})+k\alpha}\\
&= |B_1|e^{-kn+k\alpha+kn(1+\frac{C}{k})^{-\frac{p'}{p}}}\\
&= |B_1|e^{(n-\alpha)(-k+k(1+\frac{C}{k})^{-\frac{p'}{p}})}.
\end{aligned}$$

So,

$$\frac{I_\varepsilon}{|x|^\alpha} \ge |B_1|e^{(n-\alpha)C\frac{p'}{p}}.$$

This completes the proof of Lemma 3.3. □

Now, we begin to prove Theorem 1.2.

Proof of Theorem 1.2. Set $f := |(-\Delta)^{\frac{n}{2p}}u| \in L^p(\Omega)$. Using Lemma 3.1, we get

$$|u(x)| \le \int_\Omega |G_{\frac{n}{p}}(x,y)|f(y)dy \le K_{n,\frac{n}{p}}I_{\frac{n}{p}}f(x).$$

Then, for $\|u\|^p_{L^p(\Omega)} \le 1$, by Lemma 3.2,

$$\int_\Omega \frac{\exp\left((1-\frac{\alpha}{n})\frac{n}{\omega_{n-1}}\left|\frac{I_{\frac{n}{p}}*f(x)}{\|f\|_p}\right|^{p'}\right)}{|x|^\alpha} \le C_0,$$

we obtain

$$\begin{aligned}
\frac{1}{|\Omega|^{1-\frac{\alpha}{n}}}\int_\Omega \frac{\exp{(1-\frac{\alpha}{n})\beta_{n,p}}\,||u|^{p'}}{|x|^\alpha}dx &\le \frac{1}{|\Omega|^{1-\frac{\alpha}{n}}}\int_\Omega \frac{\exp{(1-\frac{\alpha}{n})\frac{n}{\omega_{n-1}}}\left|\frac{I_{\frac{n}{p}}*f(x)}{\|f\|_p}\right|^{p'}}{|x|^\alpha}dx\\
&\le C_0|\Omega|^{1-\frac{\alpha}{n}}.
\end{aligned}$$

We can obtain the sharpness by Lemma 3.3.

This completes the proof. □

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