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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF MILD SOLUTIONS FOR FRACTIONAL MEASURE DIFFERENTIAL EQUATIONS*

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Abstract This paper is concerned with a class of nonlocal problem of multi-term time-fractional measure differential equations involving delay and nonlocal conditions in Banach spaces. We first introduce the concept of S-asymptotically ω-periodic mild solution, on the premise of by utilizing (β, γ_k)-resolvent family and measure functional (Henstock-Lebesgue-Stieltjes integral) under regulated functions. And then we show by using Schauder fixed point theorem. that the existence of S -asymptotically ω periodic mild solutions for the mentioned system are obtained. Finally, an example to illustrate the obtained results is given.

Keywords Regulated functions, Henstock-Lebesgue-Stieltjes integral, fractional calculus, generalized semigroup theory, multi-term time-fractional, fixed point theory.

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1. Introduction

Fractional calculus has been gaining increased attention over the past several decades, driven by the suitability of the nonlocal fractional-order differential operators to describe memory and hereditary properties of various real world processes [1, 52, 53], in comparison with their classical integer-order counterparts. Due to this fact, fractional-order operators have been employed in the mathematical modelling of various phenomena deriving from engineering, control theory, viscoelasticity, rheology, electrochemistry, biophysics, mechanics and mechatronics, signal and image processing, etc. [1, 52, 53]. As in most cases, the fractional-order differential equations and systems used in the mathematical modelling of practical problems are not explicitly solvable, their qualitative theory, and markedly the stability and asymptotic properties of their solutions are of uttermost importance, as depicted in two recent comprehensive surveys [1, 19, 46, 52, 53, 55, 59].

In view of the close relation to fractional-order linear systems, qualitative properties of multi-term fractional-order differential equations [3] have also been explored

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in recent years. It has been shown [33] that a linear multi-term fractional-order differential equation with rational-order derivatives is equivalent to a system of fractional-order differential equations with derivatives of the same order. This common order is in fact the inverse of the least common denominator of the considered rational orders, and hence, Matignon's stability theorem proves its utility once again. As a special case, the stability of the well-known Bagley-Torvik equation [58] can be analyzed along the same lines. However, to the best of our knowledge, in the presence of at least one irrational-order derivative, stability properties have only been explored in the case of two-term fractional-order differential equations [29]. Moreover, it is also worth noting that boundary value problems for multi-term fractional differential equations have been considered in [16,37].

Meanwhile, measure differential equations (MDEs, for short) covers some well known cases. When give an absolutely continuous function, astep function, or the sum of an absolutely continuous function with a step function, this kind of system corresponds to usual ordinary differential equations, difference equations or impulsive differential equations respectively. Another advantage of considering MDEs is that we can possibly model Zeno trajectories because gas a function of bounded variation may exhibit infinitely many discontinuities in a finite interval. This type of system arises in many areas of applied mathematics such as nonsmooth mechanics, game theory etc. see [6,57,60]. MDEs were investigated early by [17,45, 51,54]. One can refer to the review paper [13] for a complete introduction of measure differential systems. Recently, the theory of MDEs for $\mathbb{R}^n$ space has developed to some extent [22,23,42,63]. However, to the best of our knowledge, few literatures concerned abstract semilinear measure driven system in Banach spaces. In this paper, by Schauder fixed point theorem, we obtain existence results of semilinear measure driven system (1.1) with the states taking values in Hilbert space. Ref. [50] investigated existence of this type of nonlinear measure driven system in Kurzweil-Stieltjes integral setting (a class of very generalized integral including Lebesgue-Stieltjes integral as a special case). However, the separability of Banach spaces is demanded there. This assumption is dropped here in Lebesgue–Stieltjes integral setting. Further, in the case of $E = \mathbb{R}^n$ and Kurzweil-Stieltjes integral setting, [22, 23] discussed the retarded version of this type of nonlinear measure driven system by transforming measure equations into generalized ordinary differential equations. Here, in Lebesgue–Stieltjes integral setting, similar analysis to system (1.1) can lead to the existence result of nonlinear measure retarded equations without any assumptions of Lipschitz-type as those in [22,23].

On the other hand, owing to its intrinsic mathematical interest as well as its relevance in numerous applications to various fields such as physics, biology and control theory, the existence of solutions with some periodicity property is one of the most attractive subjects of the qualitative theory of differential equations, it is well known that the periodic law of the development or movement of things is a common phenomenon in nature and human activities. However, in real life, many phenomena do not have strict periodicity. In order to better characterize these mathematical models, many scholars have introduced other definitions of generalized periodicity, such as almost periodicity, asymptotic periodicity, asymptotic almost periodicity, pseudo almost periodicity and S-asymptotic periodicity. Since S-asymptotically periodic functions were first studied in Banach space by Henríquez et al. [27], is a more general approximate period function between asymptotically periodic function and asymptotically almost periodic function. There are some papers about S-

asymptotically periodic solutions for fractional evolution equations, one can refer to [14, 15, 35, 36, 47, 49].

In addition, the theory of nonlocal evolution equations is an important research branch in the field of analysis, which can better describe some specific phenomena in physics, biology, aerospace, and medicine than the traditional Cauchy problems. In [18], Deng studied a kind of reaction-diffusion equations with nonlocal initial condition $u(0) + \sum_{k=1}^{n} c_k u(t_k) = \psi$, and used this problem to describe the gas diffusion phenomenon in transparent tubes. In [7], Byszewski studied the nonlocal problems of a class of abstract functional differential equations, and pointed out that the abstract conclusions obtained can be used to determine the position change of a physical object in kinematics and dynamics. Therefore, the nonlocal conditions are better than the classical initial conditions both in theory and practical applications. For more research on nonlocal evolution equations, see [5, 8, 38, 39, 61, 62].

Based on the ideas and methods of the previously works in [11, 12, 25, 26, 43], the aim of this paper is to analyze the existence of S-asymptotically ω-periodic mild solution of the following multi-term time-fractional measure differential equations with delay and nonlocal conditions of the form

$$\begin{cases} {}^cD_t^{1+\beta}u(t) + \sum\limits_{k=1}^{n} \alpha_k {}^cD_t^{\gamma_k}u(t) = Au(t) + F(t, u(t), u_t)dg(t), \quad t \geq 0, \\ u(t) = \varphi(t), \qquad t \in [-r, 0], \\ u'(0) = Q_1(u) + \psi, \end{cases} \tag{1.1}$$

where $v(\cdot)$ take values in a Banach space E; ${}^\eta D^\alpha$ stand for the Caputo fractional derivative of order η, α_k 0 and all γ_k, $k = 1, 2, \cdots, n, n \in \mathbb{N}$, are positive real numbers such that $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$. We assume that $A : \mathcal{D}(A) \subset E \to E$ is a closed linear operator, and A generates a strongly continuous family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$ of bounded and linear operators on E, $f : \mathbb{R}^+ \times E \times \mathcal{B} \to E$ is a suitable nonlinear function, $\mathcal{B} := G([-r, 0], E)$ is Banach space of all regulated functions from $[-r, 0]$ to E with the norm $\|\phi\|_{\mathcal{B}} = \sup_{s\in[-r,0]} \|\phi(s)\|$. For $t \geq 0, u_t \in \mathcal{B}$ is the history state defined by $u_t(s) = u(t + s)$ for $s \in [-r, 0]$, and $g : \mathbb{R}^+ \to \mathbb{R}$ is nondeacreasing and continuous from the left, dg denote the distributional derivative of g (see [8]), $\varphi \in \mathcal{B}$ and $\varphi(0) \in \mathcal{D}(A), \psi \in E, r > 0$ is a constant. Furthermore, the functions $Q_1 : G(\mathbb{R}^+, E) \to E)$ will be specified later where $G(\mathbb{R}^+, E)$ denotes the space of regulated functions on $\mathbb{R}^+$, in which we consider our investigation.

Our motivation to study equation (1.1) comes from recent investigations where a related class appears in connection with partial differential equations and Cauchy time processes, a type of iterated stochastic processes (see [4]). Note that when $A = 2\Delta - \varepsilon^2\Delta^2$ (where Δ is the Laplace operator on $\mathbb{R}^N$), $\alpha = \beta = 1$ and $a_k = \frac{1}{\varepsilon^2}, g(t) = t$ without delay, the above equation was recently considered by Nane [44]. In particular, in case $a_k = 0, \alpha = \beta = 1, A = -\Delta^2$ and $u'(0) = -\frac{2}{\pi}\Delta u_0$ with u_0 belonging to the domain $D(\Delta)$ of the operator Δ the equation

$$u''(t) = -\Delta^2 u(t) - \frac{2}{\pi t}\Delta u_0 + f(t), \quad t > 0$$

has been studied in [44] in connection with partial differential equations and iterated processes. Very recently, the article [56] studied the nonlinear two-term time fractional diffusion wave equation (1.1) with $0 < \alpha < \beta - 1, 0 < \beta < 1$ and $A = \frac{d^2}{dx^2}$ without delay.

The highlights and advantages of this paper are presented in two aspects: firstly, an integer-order differential equation with n derivatives terms may be turned into an abstract version of a first-order differential system, but a fractional differential equation cannot. So, the technique used in this paper provides the tools to study a measure differential equation having more than one fractional derivative. Secondly, to the best of our knowledge, there is no study regarding the existence of S-asymptotically ω-periodic mild solution for multi-term time-fractional measure driven equations. These facts serve as both the motivation and the uniqueness of this work. Contributions of this work are listed as follows:

(1) Many significant classes of equations, such as ordinary differential equations, integral differential equations, and difference equations, can be framed as measure differential equations by selecting the function as the absolutely continuous function and the step function or the sum of the two preceding functions, respectively.

(2) In literature, first time (β, γ_k)-resolvent family is used to investigate the existence and approximate controllability results for measure driven equations.

(3) Schauder fixed point theorem is utilized to derive the existence results of S asymptotically ω periodic mild solutions, which will fill the research gap in this area by using regulated functions, Henstock-Lebesgue-Stieltjes integral settings for measure driven equation involving multi-term time fractional derivatives.

(4) The topological method that some authors have chosen to study existence of S-asymptotically ω-periodic solutions is the theory of fixed points, which has been a very powerful and important tool to the study of nonlinear phenomena. Specifically, authors have used contraction mapping principle, Leray-schauder alternative and Krasnoselkii's theorem. However, Schauder's fixed point theorem is the first time that it has been used to study our concerned problem. Therefore, our results are novel and meaningful.

The paper is structured as follows. Section 1 is devoted to a detailed introduction about the area. Some fundamental results are given in the Section 2. Existence theory for S-asymptotically ω-periodic mild solutions of the proposed problem is given in the Section 3 as main results. Section 4 is devoted to an illustrative example to demonstrate our analysis. In the last Section 5, we give a brief conclusion and some future direction for further research in this area.

2. Preliminaries

In this section, we briefly recall some basic known results which will be used in the sequel. Let $(E, \|\cdot\|_E)$ be a Banach space.

We denote by $C([a,b], E)$ the Banach space of continuous functions from $[a,b]$ with the supremum norms $\|u\|_C = \sup_{t\in[a,b]} \|u(t)\|$. $\mathtt{L}(E)$ denote the Banach space of all bounded and linear operators from E to E with the operator norm $\|\cdot\|_{\mathtt{L}(E)}$. Let $C_b(\mathbb{R}^+, E)$ denote the Banach space of all bounded and continuous functions from $\mathbb{R}^+$ to E equipped with the norm $\|u\|_C = \sup\limits_{t\in\mathbb{R}^+} \|u(t)\|$. Let $SAP_\omega(E)$ represent the subspace of $C_b(\mathbb{R}^+, E)$ consisting of all the E-value S-asymptotically ω-periodic functions endowed with the uniform convergence norm denoted by $\|\cdot\|_C$. Then $SAP_\omega(E)$ is a Banach space (see [27], Proposition 3.5]).

Throughout this work, $G(\mathbb{R}^+, E)$ denotes the space of regulated functions from $\mathbb{R}^+$ to E with the norm

$$\|u\|_\infty = \sup_{t\geq 0}\|u(t)\|, \quad u \in G(\mathbb{R}^+, E).$$

A partition of $[a, b]$ is a finite collection of pairs $\{([t_{i-1}, t_i], e_i), i = 1, 2, \cdots, n\}$, where $[t_{i-1}, t_i]$ are nonoverlapping subintervals of $[a, b]$, $e_i \in [t_{i-1}, t_i], i = 1, \cdots, n$ and $\bigcup_{i=1}^n [t_{i-1}, t_i] = [a, b]$. A guage δ on $[a, b]$ is a positive function on $[a, b]$. For a given guage δ we say that a partition is δ-fine if $[t_{i-1}, t_i] \subset (e_i - \delta(e_i), e_i + \delta(e_i))$, for any $i \in \{1, \cdots, n\}$. We denote by $v(t^-)$ and $v(t^+)$ denote the left limit and right limit of the function v at the point t, respectively.

We recall some basic definitions and properties of regulated function. For more details, we refer to [50].

Definition 2.1. [50] A function $v : [a, b] \to E$ is said to be regulated on $[a, b]$, if the limits

$$\lim_{s\to t^-} v(s) = v(t^-), \ t \in (a, b] \ \text{ and } \ \lim_{s\to t^+} v(s) = v(t^+), \ t \in [a, b)$$

exist and finite.

We denote by $G([a, b], E)$ the space of all regulated function from $[a, b]$ into E. The space $G([a, b], E)$ is a Banach space endowed with the supremum norm.

Definition 2.2. [50] A set $B \subset G([a, b], E)$ is called equiregulated, if for every $\epsilon > 0$ and $\tau \in [a, b]$, there exist $\delta > 0$ such that

(i) If $v \in B, t \in [a, b]$ and $t \in (\tau - \delta, \tau)$, then $\|v(\tau^-) - v(t)\|_E < \epsilon$.

(ii) If $v \in B, t \in [a, b]$ and $t \in (\tau, \tau + \delta)$, then $\|v(t) - v(\tau^+)\|_E < \epsilon$.

Lemma 2.1. [50] *Let $\{v_n\}_{n=1}^\infty$ be a sequence of functions from $[a, b]$ to E. If v_n converge pointwisely to v_0 as $n \to \infty$ and the sequence $\{v_n\}_{n=1}^\infty$ is equiregulated, then v_n converges uniformely to v_0.*

Lemma 2.2. [50] *Let $B \subset G([a, b], E)$. If B is bounded and bounded and equiregulated, then the set $\overline{co}(B)$ is also bounded and equiregulated, where $\overline{co}(B)$ define the convex hull of B.*

The following result is a variant of the Arzela-Ascoli theorem for regulated functions with Banach space values.

Lemma 2.3. [41] *Assume that $B \subset G([a, b], E)$ is equiregulated and, for every $t \in [a, b]$ the set $\{v(t) : v \in B\}$ is relatively compact in E. Then the set B is relatively compact in $G([a, b], E)$.*

Next, we recall the definition of Henstock-Lebesgue-Stieljes integral.

Definition 2.3. [50] A function $\psi : [a, b] \to E$ is said to be Henstock-Lebesgue-Stieltjes integrable over $[a, b)$, if there exists a function denoted by

$$(HLS)\int_a^{(\cdot)} : [a, b] \to E$$

such that, for every $\epsilon > 0$, there is a gauge δ_ϵ satisfying

$$\Big\| \sum_{i=1}^{n} \psi(e_i)(g(t_i)-g(t_{i-1})) - \Big((HLS)\int_a^{t_i} \psi(s)dg(s) - (HLS)\int_a^{t_{i-1}} \psi(s)dg(s)\Big)\Big\| < \epsilon,$$

for every δ_ϵ-fine partition $\{(e_i, [t_{i-1}, t_i]) : i = 1, 2, \ldots, n\}$ of $[a, b)$.

In this work, we denote by $\mathbb{HLS}_g^p([a, b], \mathbb{R})(p > 1)$ the space of all p-ordered Henstock-Lebesgue- Stieltjes integral regulated from $[a, b]$ to $\mathbb{R}$ with respect to g, with norm $\| \cdot \|_{\mathbb{HLS}_g^p}$ defined by

$$\|\psi\|_{\mathbb{HLS}_g^p} = \Big((HLS)\int_a^b \|\psi(s)\|^p dg(s)\Big)^{\frac{1}{p}}.$$

Lemma 2.4. [26] *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $\Psi \in \mathbb{HLS}_g^p([a, b], \mathbb{R}^+)$ and $g : [a, b] \to \mathbb{R}$ be regulated. Then the function $H(t) = \int_0^t (t-s)^\beta \Psi(s)dg(s)$ is regulated and*

$$H(t) - H(t^-) \le \Big(\int_{t^-}^{t} (t-s)^{q\beta} dg(s)\Big)^{\frac{1}{q}} \Psi(t)(\Delta^- g(t))^{\frac{1}{p}}, \quad t \in (a, b],$$
$$H(t^+) - H(t) \le \Big(\int_{t^+}^{t} (t^+ - s)^{q\beta} dg(s)\Big)^{\frac{1}{q}} \Psi(t)(\Delta^+ g(t))^{\frac{1}{p}}, \quad t \in [a, b),$$

where $\Delta^+ g(t) = g(t^+) - g(t)$ and $\Delta^- g(t) = g(t) - g(t^-)$.

Lemma 2.5. [19] *Let for $t \in [a, b]$ be weakly relatively compact in E. Suppose that $B \subset L^1_\mu([a, b], E)$ is a bounded set and there is a function $N(\cdot) \in L^1_\mu([a, b], \mathbb{R}+)$ such that $\|b(t)\|_E \le N(t)$ μ-a.e. $t \in [a, b]$ for all $b \in B$. If for every $b \in B, b(t) \in Z(t)$ for μ-a.e. $t \in [a, b]$ then B is weakly relatively compact in $L^1_\mu([a, b], E)$, where $L^1_\mu([a, b], E)$ is the set of all μ-integrable functions, μ is a measure.*

We recall the following Schauder's fixed point theorem

Lemma 2.6. [21] *Let $\mathcal{O} \subseteq E$ be a bounded, closed and convex set. If the operator $Q : \mathcal{O} \to \mathcal{O}$ is completely continuous, then Q has a fixed point in $\mathcal{O}$.*

The most frequently encountered tools in the theory of fractional calculus are provided by the Riemann-Liouville and Caputo fractional differential operators.

Definition 2.4. The Riemann-Liouville fractional integral of a function $f \in L^1_{loc}([0, \infty), E)$ of order $\eta > 0$ with lower limit zero is defined as follows

$$\mathbb{I}^\eta f(t) = \int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} f(s)ds, \quad t > 0$$

and $\mathbb{I}^0(t) = f(t)$, provided that side integral is point-wise defined in $[0, \infty)$.

Definition 2.5. Let $\eta > 0$ be given and denote $m = [\eta]$. The Caputo fractional derivative of order $\eta > 0$ of a function $f \in C^m([0, \infty), E)$ with lower limit zero is given by

$${}^c D^\eta f(t) = \mathbb{I}^{m-\eta} D^m f(t) = \int_0^t \frac{(t-s)^{m-\eta-1}}{\Gamma(m-\eta)} D^m f(s)ds,$$

and ${}^c D^0 f(t) = f(t)$, where $D^m = dm/dtm$ and $[\cdot]$ is ceiling function.

For more progress and important properties about fractional calculus and its applications, we refer the reader to [31, 57] and references therein.

Let A be a closed linear operator on the Banach space E with domain $\mathcal{D}(A)$ and denote by $\rho(A)$ the resolvent set of A.

Definition 2.6. [30] Let E be a Banach space and let $\beta > 0$, $\gamma_k, \alpha_k, k = 1, 2, \dots n$ be real positive numbers. Then A is called the generator of (β, γ_k)-resolvent family if there exists $\kappa \geq 0$ and a strongly continuous function $S_{\beta,\gamma_k} : \mathbb{R}^+ \to \mathtt{L}(E)$ such that

$$\Big\{\lambda^{\beta+1} + \sum_{k=1}^{n} \alpha_k \lambda^{\gamma_k} : Re(\lambda) > \kappa\Big\} \subset \rho(A)$$

and

$$\lambda^{\beta}\Big(\lambda^{\beta+1} + \sum_{k=1}^{n} \alpha_k \lambda^{\gamma_k} - A\Big)^{-1} u = \int_0^{\infty} S_{\beta,\gamma_k}(t) u dt,$$

where $Re(\lambda) > \kappa$ and $u \in E$.

A operator A is said to be κ-sectorial of angle θ if there exist $\theta \in [0, \frac{\pi}{2})$ and $\kappa \in \mathbb{R}$ such that its resolvent is in the sector

$$\kappa + S_\theta := \Big\{\kappa + \lambda : \lambda \in \mathbb{C}, |arg(\lambda)| < \frac{\pi}{2} + \theta\Big\} \backslash \{\omega\},$$

and

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in \omega + S_\theta.$$

Lemma 2.7. [30] *Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_k \geq 0, k = 1, 2, \dots n$ be given, $\mu > 0$ and $\kappa < 0$. Assume that A is a κ-sectorial operator of angle $\frac{\gamma_k \pi}{2}$. Then A generates a (β, γ_k)-resolvent family $S_{\beta,\gamma_k}(t)$ satisfying the estimate*

$$\|S_{\beta,\gamma_k}(t)\| \leq \frac{C}{1 + |\kappa|(t^{\beta+1} + \sum_{k=1}^{n} \alpha_k t^{\gamma_k})}, \quad t \geq 0, \tag{2.1}$$

for some constant $C > 0$ depending only on β, γ_k.

Definition 2.7. [27] A function $u \in C_b(\mathbb{R}^+, E)$ is called S-asymptotically ω-periodic if there exists ω such that

$$\lim_{t \to \infty} \|u(t + \omega) - u(t)\| = 0, \quad \forall t \geq 0.$$

In this case, we say that ω is an asymptotic of u. It is clear that if ω is an asymptotic period for u, then every $k\omega, k = 1, 2, \cdots$, is also an asymptotic period of u.

We now look for suitable concept of mild to problem (1.1).

Lemma 2.8. *Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_k \geq 0, k = 1, 2, \dots n$ be given and A be a generator of a bounded (β, γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t \geq 0}$. Then mild solution of the problem (1.1) is given by*

$$\begin{aligned} u(t) = & S_{\beta,\gamma_k}(t)\varphi(0) + (\varphi_1 * S_{\beta,\gamma_k})(t)[Q_1(u) + \psi] \\ & + \sum_{k=1}^{n} \alpha_k (\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)\varphi(0) \end{aligned}$$

$$+\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s)$$

for $t \geq 0$, *where* $T_{\beta,\gamma_k}(t) = (\varphi_\beta * S_{\beta,\gamma_k})(t)$.

Proof. Putting $h(t) = F(t,u(t),u_t)$, let $\mathcal{L}(\cdot)$ and $(\cdot * \cdot)(\cdot)$ denote the Laplace transformation and convolution, respectively. With the goal of constructing a representation of the solution of (1.1) in terms of the family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$, and applying the Laplace transform to (1.1), we have

$$\begin{aligned}
&\lambda^{\beta+1}\mathcal{L}[u](\lambda) - \sum_{j=0}^{[\beta+1]-1} u^{(j)}(0)\lambda^{\beta-j}\\
&+\sum_{k=1}^{n}\alpha_k\Big[\lambda^{\gamma_k}\mathcal{L}[u](\lambda) - \sum_{j=0}^{[\gamma_k]-1} u^{(j)}(0)\lambda^{\gamma_k-1-j}\Big]\\
=&A\mathcal{L}[u](\lambda) + \mathcal{L}[h](\lambda).
\end{aligned}$$

Applying the given nonlocal conditions, we have

$$\begin{aligned}
&\lambda^{\beta+1}\mathcal{L}[u](\lambda) - \lambda^{\beta}\varphi(0) - \lambda^{\beta-1}[Q_1(u)+\psi]\\
&+\sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k}\mathcal{L}[u](\lambda) - \sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k-1}\varphi(0)\\
=&A\mathcal{L}[u](\lambda) + \mathcal{L}[h](\lambda).
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\Big(\lambda^{\beta+1} + \sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k} - A\Big)\mathcal{L}[u](\lambda)\\
=&\lambda^{\beta}\varphi(0) + \lambda^{\beta-1}[Q_1(u)+\psi]\\
&+\sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k-1}\varphi(0) + \mathcal{L}[h](\lambda), \quad Re(\lambda) > \omega.
\end{aligned}$$

Hence, assuming the existence of the family $S_{\beta,\gamma_k}(t)$ we obtain

$$\begin{aligned}
\mathcal{L}[u](\lambda) =&\lambda^{\beta}\Big(\lambda^{\beta+1} + \sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k} - A\Big)^{-1}\varphi(0)\\
&+\lambda^{\beta-1}\Big(\lambda^{\beta+1} + \sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k} - A\Big)^{-1}[Q_1(u)+\psi]\\
&+\sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k-1}\Big(\lambda^{\beta+1} + \sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k} - A\Big)^{-1}\varphi(0)\\
&+\Big(\lambda^{\beta+1} + \sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k} - A\Big)^{-1}\mathcal{L}[h](\lambda)
\end{aligned}$$

for all λ such that $Re(\lambda) > \omega$, $\lambda^{\beta+1} + \sum\limits_{k=1}^{n}\alpha_k\lambda^{\gamma_k} \in \rho(A)$, then

$$\mathcal{L}[u](\lambda) =\mathcal{L}[S_{\beta,\gamma_k}](\lambda)\varphi(0) + \mathcal{L}[\varphi_1]\mathcal{L}[S_{\beta,\gamma_k}](\lambda)[Q_1(u)+\psi]$$

$$+\sum_{k=1}^{n}\alpha_k\mathcal{L}[\varphi_{1+\beta-\gamma_k}]\mathcal{L}[S_{\beta,\gamma_k}](\lambda)\varphi(0)$$

$$+\mathcal{L}[S_{\beta,\gamma_k}](\lambda)\mathcal{L}[\varphi_\beta](\lambda)\mathcal{L}[h](\lambda),\quad Re(\lambda)>\omega,$$

where

$$\varphi_\beta(t)=\frac{t^{\beta-1}}{\Gamma(\beta)},\quad t>0,\ \beta>0.$$

Inversion of the Laplace transform shows that

$$\begin{aligned}u(t)=&S_{\beta,\gamma_k}(t)\varphi(0)+(\varphi_1*S_{\beta,\gamma_k})(t)[Q_1(u)+\psi]\\&+\sum_{k=1}^{n}\alpha_k(\varphi_{1+\beta-\gamma_k}*S_{\beta,\gamma_k})(t)\varphi(0)\\&+\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s).\end{aligned}$$

This completed the proof. □

The above representation formula allows us to give the following definition.

Definition 2.8. Let $0<\beta\le\gamma_n\le\cdots\le\gamma_1\le 1$ and $\alpha_k\ge 0, k=1,2,\dots n$ be given and A be a generator of a bounded (β,γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\ge 0}$. Then a regulated function $u(\cdot):\mathbb{R}^+\to E$ is said to be mild solution of probelm (1.1) if $u(t)=\varphi(t)$ for $t\in[-r,0]$, $u'(0)=Q_1(u)+\psi$ and satifies the following integral equation

$$\begin{aligned}u(t)=&S_{\beta,\gamma_k}(t)\varphi(0)+(\varphi_1*S_{\beta,\gamma_k})(t)[Q_1(u)+\psi]\\&+\sum_{k=1}^{n}\alpha_k(\varphi_{1+\beta-\gamma_k}*S_{\beta,\gamma_k})(t)\varphi(0)\\&+\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s),\quad t\ge 0.\end{aligned}$$

Moreover, if u is S-asymptotically ω-periodic, then it is called S-asymptotically ω-periodic mild solution of probelm (1.1).

The following result is a variant of the Arzelà-Ascoli theorem for regulated functions with Banach space-values.

Lemma 2.9. [41] *Assume that $B\subset G([a,b],E)$ is equiregulated and, for every $t\in[a,b]$ the set $\{v(t):v\in B\}$ is relatively compact in E. Then the set B is relatively compact in $G([a,b],E)$.*

Lemma 2.10. *Let $u:[-r,+\infty)\to E$ be a function with $\phi(0)\in G([-r,0],E)$ and $u_{[0,+\infty)}\in SAP_\omega(E)$. Then the function $t\to u_t$ belongs to $SAP_\omega(G([-r,0],E))$.*

Proof. Since u_t is continuous on $[-r,0]$, we see that there exists $\overline{\theta}\in[-r,0]$ such that

$$\|u_{t+\omega}-u_t\|_{[-r,0]}=\sup_{-r\le\theta\le 0}\|u(t+\omega+\theta)-u(t+\theta)\|=\|u(t+\omega+\overline{\theta})-u(t+\overline{\theta})\|.$$

Setting $\tau = t + \overline{\theta}$, we have

$$\lim_{t\to+\infty} \|u(t+\omega+\overline{\theta}) - u(t+\overline{\theta})\| = \lim_{\tau\to+\infty} \|u(\tau+\omega) - u(\tau)\| = 0.$$

□

Lemma 2.11. [34] (Inversion formula) *Suppose that the resolvent family $S_{\beta,\gamma_k}(t)$ satisfying $\|S_{\beta,\gamma_k}(t)\| \le C_0 e^{\kappa t}$. Then for $x \in \mathcal{D}(A)$, when $1+\beta-\gamma_1 \ge 1$,*

$$S_{\beta,\gamma_k}(t)x = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\lambda t}\lambda^{\beta}\Big(\lambda^{\beta+1} + \sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k} - A\Big)^{-1} u d\lambda$$

where $a > \kappa$.

Moreover, we noted that by the estimate (2.1) we have

$$\begin{aligned}
\|T_{\beta,\gamma_k}(t)\|_{L(\mathbb{E})} &= \|(\varphi_\beta * S_{\beta,\gamma_k})(t)\|_{L(\mathbb{E})}\\
&\le \int_0^t \varphi_\beta(t-\tau)\|T_{\beta,\gamma_k}(\tau)\|d\tau\\
&\le \Gamma(1-\gamma_k)\int_0^t \varphi_\beta(t-\tau)\varphi_{1-\gamma_k}(\tau)\tau^{\gamma_k}\|T_{\beta,\gamma_k}(\tau)\|d\tau\\
&\le \frac{\Gamma(1-\gamma_k)}{|\kappa|\sum_{k=1}^n \alpha_k}\int_0^t \varphi_\beta(t-\tau)\varphi_{1-\gamma_k}(\tau)d\tau\\
&= \frac{\Gamma(1-\gamma_k)}{|\kappa|\sum_{k=1}^n \alpha_k}\varphi_{\beta-\gamma_k+1}(t),
\end{aligned}$$

for all $t > 0$. Hence there exists a constant $C > 0$ such that

$$\|T_{\beta,\gamma_k}(t)\|_{L(\mathbb{E})} \le Ct^{\beta-\gamma_k}. \tag{2.2}$$

We estimate $\|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)\|$ as follows. Let $0 < \varepsilon < \gamma_k - \beta$ be given, then

$$\begin{aligned}
&\|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\|\\
=&\|\Gamma(\gamma_k-\beta-\varepsilon)\int_0^t \varphi_{1+\beta-\gamma_k}(t-\tau)\varphi_{\gamma_k-\beta-\varepsilon}(\tau)\tau^{\beta-\gamma_k+\varepsilon+1}S_{\beta,\gamma_k}(\tau)d\tau\|\\
\le&\Gamma(\gamma_k-\beta-\varepsilon)\int_0^t \varphi_{1+\beta-\gamma_k}(t-\tau)\varphi_{\gamma_k-\beta-\varepsilon}(\tau)\tau^{\beta-\gamma_k+\varepsilon+1}\|S_{\beta,\gamma_k}(\tau)d\tau\|,
\end{aligned}$$

where, thanks to (2.1), we have that

$$\begin{aligned}
\Gamma(\gamma_k-\beta-\varepsilon)\tau^{\beta-\gamma_k+\varepsilon+1}\|S_{\beta,\gamma_k}(\tau)\| &\le \frac{M\tau^{\beta-\gamma_k+\varepsilon-1}}{1+|\kappa|\tau^{\beta+1}}\\
&= \frac{M\tau^{-\gamma_k+\varepsilon}}{\frac{1}{\tau^{\beta+1}}+|\kappa|}, \qquad \varepsilon > 0.
\end{aligned}$$

Since $\varepsilon < \gamma_k$, there exists a constant $C > 0$ such that

$$\tau^{\beta-\gamma_k+\varepsilon+1}\|S_{\beta,\gamma_k}(\tau)\| \le C.$$

Therefore,

$$\begin{aligned}\|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\| &\leq C\int_0^t \varphi_{1+\beta-\gamma_k}(t-\tau)\varphi_{\gamma_k-\beta-\varepsilon}(\tau)d\tau \\ &= C\varphi_{1-\varepsilon}(t) \\ &= Ct^{-\varepsilon}. \end{aligned} \tag{2.3}$$

First of all, in view of (2.1), we denote $M := \sup_{t\geq 0}\|S_{\beta,\gamma_k}(t)\| < +\infty, M > 0$. By Lemma 2.15 and (2.1), we note that

$$\int_0^t S_{\beta,\gamma_k}(s)xds = \lim_{r\to\infty}\frac{1}{2\pi}\int_{-r}^{r} e^{(a_i\lambda)t}\widehat{S}_{\beta,\gamma_k}(a+i\lambda)x\frac{d\lambda}{a+i\lambda}$$

for each $x \in E$ and the limit exists uniformly for $t > 0$, where $\widehat{S}_{\beta,\gamma_k}(\lambda) = \int_0^\infty e^{-\lambda t}S_{\beta,\gamma_k}(t)ds$ is the Laplace transform of $S_{\beta,\gamma_k}(t)$. Hence, we

$$\sup_{t>\tau}\|(\varphi_1 * S_{\beta,\gamma_k})(t)\| = \sup_{t>\tau}\left\|\int_0^t S_{\beta,\gamma_k}(s)ds\right\| < +\infty$$

for each $\tau > 0$. Moreover, we denote

$$\widetilde{M} := \sup_{t\geq\tau>0}\||(\varphi_1 * S_{\beta,\gamma_k})(t)\| > 0. \tag{2.4}$$

In view of (2.3), we denote

$$M_1 := \sup_{t\geq 0}\|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\| < +\infty. \tag{2.5}$$

3. Main result

Moreover, for a given $\varphi \in \mathcal{B}$ and $u \in G([-r,+\infty),E)$, we define $u_\varphi : [-r,+\infty) \to E$ by

$$u_\varphi = \begin{cases} u(t), & t \geq 0, \\ \varphi(t), & t \in [-r,0]. \end{cases}$$

And define a closed subspace of $G([-r,+\infty),E)$ by

$$G_\varphi(E) = \{u \in G([-r,+\infty),E) |\ u(0) = \varphi(0)\}$$

with the norm $\|u\| = \max\{\|u\|_\infty, \|\varphi\|_{\mathcal{B}}\}$.

Now, we establish the existence result of mild solutions for equation (1.1).

To obtain the existence result for the problem (1.1), we introduce the following hypotheses:

(A1) Assume that A is an κ-sectorial operator of angle $\frac{\gamma_k\pi}{2}, k = 1,2,\cdots,n$ with $\kappa < 0$, and A generates a strongly continuous resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$, and $\{S_{\beta,\gamma_k}(t) : t \geq 0\}$ and $\{T_{\beta,\gamma_k}(t) : t \geq 0\}$ are compact.

(A2) The function $F : \mathbb{R}^+ \times E \times \mathcal{B} \to E$ is continuous and satisfies the following

1. For any bounded sets $D \subset E$, the set $\{F(t,u,\phi)|t \geq 0, u \in D, \phi \in \mathcal{B}\}$ is bounded, and

$$\lim_{t\to\infty} \|F(t+\omega,u,\phi) - F(t,u,\phi)\| = 0$$

for all $u \in E$.

2. $F(\cdot,u,\phi)$ is measurable for all $u \in G(\mathbb{R}^+,E)$.

(A3) For each constant $r > 0$, there exists $P(\cdot) \in \mathbb{HLS}_g^p(\mathbb{R}^+,\mathbb{R}^+)$ for some $p > 1$ such that

$$\sup_{\|u\|\leq r} \|F(t,u,\phi)\| \leq P(t)W(r),\ t \in [a,b],$$

where $W : [0,+\infty) \to \mathbb{R}^+$ is a continuous nondecreasing function and

$$\lim_{r\to\infty} \inf \frac{W(r)}{r} = w_0 < \infty.$$

(A4) $Q_1 : G(\mathbb{R}^+,E) \to E$ is continuous and compact mapping, and there exists a nondecreasing function $\mathcal{Q} \in G(\mathbb{R}^+,\mathbb{R}^+)$ with

$$\liminf_{t\to+\infty} \frac{\mathcal{Q}(t)}{t} := \eta < +\infty,$$

such that

$$\|Q_1(u)\| \leq \mathcal{G}(\|u\|_\infty), \quad u \in G(\mathbb{R}^+,E).$$

Theorem 3.1. *Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_k \geq 0, k = 1,2,\dots n$ be given. If the assumptions (A1)-(A4) hold, then problem (1.1) has at least one S-asymptotically ω-periodic mild solution $u \in G([-r,+\infty),E)$ provided that*

$$\Theta = \Big[\widetilde{M}\eta + Cw_0 \sup_{t\geq 0}\Big(\int_0^t (t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\|P\|_{\mathbb{HLS}_g^p}\Big] < 1,$$

where $\frac{1}{q}+\frac{1}{p} = 1$.

Proof. Define an operator $\aleph : G([-r,+\infty),E) \to G([-r,+\infty),E)$ as follows:

$$(\aleph u)(t) = \begin{cases} S_{\beta,\gamma_k}(t)\varphi(0) + (\varphi_1 * S_{\beta,\gamma_k})(t)[Q_1(u)+\psi] \\ +\sum_{k=1}^n \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)\varphi(0) \\ +\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s), \quad t \geq 0, \\ \varphi(t), \quad t \in [-r,0]. \end{cases} \tag{3.1}$$

From (A2), the integral $\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s)$ is well defined. Clearly, if $\aleph$ admit a fixed point in $G([-r,+\infty),E)$, then the system (1.1) admits a mild solution. For every $r > 0$, define $\Omega := \{u \in G_\varphi(E) \cap C_b([-r,+\infty),E) :\ \|u\| \leq r,\ u|_{[-r,0]} = 0, u|_{[0,+\infty)} \in SPA_\omega(E))\}$. It is clear that Ω is a closed, bounded and convex subset of $G([-r,+\infty),E)$.

In the following, we will prove that $\aleph$ has one fixed point by using Schauder's fixed point. The proof is given in the following four steps.

Step.1. There exists $r > 0$ such that $\aleph(\Omega) \subset \Omega$.

To do this, suppose that our claim does not hold, then for each $r > 0$, there exist a $u^r \in \Omega$ and $t^* \geq 0$ such that $\|(\aleph u^r)(t^*)\| > r$. Then, by (2.2), (2.4), (2.5), we have

$$
\begin{aligned}
r <&\|(\aleph u^r)(t^*)\| \\
\leq&\|S_{\beta,\gamma_k}(t^*)\varphi(0) + (\varphi_1 * S_{\beta,\gamma_k})(t^*)[Q_1(u^r) + \psi] \\
&+ \sum_{k=1}^{n} \alpha_k \int_0^{t^*} \frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)} S_{\beta,\gamma_k}(s)\varphi(0)ds \\
&+ \int_0^{t^*} T_{\beta,\gamma_k}(t^*-s)F(s, u^r(s), u^r_s)dg(s)\| \\
\leq&\|S_{\beta,\gamma_k}(t^*)\varphi(0)\| + \|(\varphi_1 * S_{\beta,\gamma_k})(t^*)[Q_1(u_r) + \psi]\| \\
&+ \|\sum_{k=1}^{n} \alpha_k (\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t^*)\varphi(0)\| \\
&+ \|\int_0^{t^*} T_{\beta,\gamma_k}(t^*-s)F(s, u^r(s), u^r_s)dg(s)\| \\
\leq&\|S_{\beta,\gamma_k}(t^*)\varphi(0)\| + \|(\varphi_1 * S_{\beta,\gamma_k})(t^*)[Q_1(u_r) + \varphi]\| \\
&+ \|\sum_{k=1}^{n} \alpha_k (\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t^*)\varphi(0)\| \\
&+ \|\int_0^{t^*} T_{\beta,\gamma_k}(t^*-s)\| \cdot \|F(s, u^r(s), u^r_s)\|dg(s) \\
\leq&M(\|\varphi(0)\|) + \widetilde{M}(\mathcal{Q}(\|u_r\|) + \|\psi\|) + \Big(\sum_{k=1}^{n} \alpha_k M_1\Big)\|\varphi(0)\| \\
&+ CW(r)\Big(\int_0^{t^*} (t^*-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\Big(\int_0^{t^*} [P(s)]^p dg(s)\Big)^{\frac{1}{p}} \\
\leq&M(\|\varphi(0)\|) + \widetilde{M}(\mathcal{Q}(\|u_r\|) + \|\psi\|) + \Big(\sum_{k=1}^{n} \alpha_k M_1\Big)\|\varphi(0)\| \\
&+ CW(r)\Big(\int_0^{t^*} (t^*-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\|P\|_{\mathbb{HLS}^p_g}. \qquad (3.2)
\end{aligned}
$$

Dividing both sides of (3.2) by r and taking $r \to \infty$, we have

$$
\widetilde{M}\eta + Cw_0 \sup_{t^* \geq 0} \Big(\int_0^{t^*} (t^*-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\|P\|_{\mathbb{HLS}^p_g} \geq 1.
$$

This contradicts $\Theta < 1$.

Next we show that $\aleph(SPA_\omega(E)) \subset SPA_\omega(E)$. For any $u \in \Omega$, it is clear that $\aleph u$ is defined on $[-r, +\infty)$, and because $\varphi \in \mathcal{B}$, we have $\aleph u|_{[-r,0]} \in \mathcal{B}$.

Let

$$
\begin{aligned}
f : t \to& S_{\beta,\gamma_k}(t)\varphi(0) + (\varphi_1 * S_{\beta,\gamma_k})(t)[Q_1(u) + \psi] \\
&+ \sum_{k=1}^{n} \alpha_k (\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)\varphi(0)
\end{aligned}
$$

$$+\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s)\in SAP_\omega(E),\quad t\geq 0.$$

In view of Lemma 2.10, since $u|_{\mathbb{R}^+}\in SPA_\omega(E)$ and $u_t\in SPA_\omega(\mathcal{B})$ for all $t\geq 0$, hence, for any $\epsilon>0$, there exists a constant $t_{\epsilon,1}$ such that $\|u(t+\omega)-u(t)\|\leq\epsilon$ and $\|u_{t+\omega}-u_t\|_{\mathcal{B}}\leq\epsilon$ for every $t\geq t_{\epsilon,1}$. Thus, by the condition (A2), for $t\geq t_{\epsilon,1}$, we have

$$\|F(t,u(t+\omega),u_{t+\omega})-F(t,u(t),u_t)\|\leq\frac{\epsilon}{2},\tag{3.3}$$

and we can find a positive constant $t_{\epsilon,2}$ sufficiently large such that for $t\geq t_{\epsilon,2}$,

$$\|F(t+\omega,u(t+\omega),u_{t+\omega})-F(t,u(t+\omega),u_{t+\omega})\|\leq\frac{\epsilon}{2}.\tag{3.4}$$

Then for $t>t_\epsilon:=\max\{t_{\epsilon,1},t_{\epsilon,2}\}$, from (3.1), it follows that

$$\begin{aligned}
&f(t+\omega)-f(t)\\
=&S_{\beta,\gamma_k}(t+\omega)\varphi(0)+(\varphi_1*S_{\beta,\gamma_k})(t+\omega)[Q_1(u)+\psi]\\
&+\sum_{k=1}^n\alpha_k(\varphi_{1+\beta-\gamma_k}*S_{\beta,\gamma_k})(t+\omega)\varphi(0)\\
&+\int_0^t T_{\beta,\gamma_k}(t+\omega-s)F(s,u(s),u_s)dg(s)\\
&-S_{\beta,\gamma_k}(t)\varphi(0)-(\varphi_1*S_{\beta,\gamma_k})(t)[Q_1(u)+\psi]\\
&-\sum_{k=1}^n\alpha_k(\varphi_{1+\beta-\gamma_k}*S_{\beta,\gamma_k})(t)\varphi(0)-\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s)\\
=&S_{\beta,\gamma_k}(t+\omega)\varphi(0)-S_{\beta,\gamma_k}(t)\varphi(0)+(\varphi_1*S_{\beta,\gamma_k})(t+\omega)[Q_1(u)+\psi]\\
&-(\varphi_1*S_{\beta,\gamma_k})(t)\varphi(0)\\
&+\sum_{k=1}^n\alpha_k(\varphi_{1+\beta-\gamma_k}*S_{\beta,\gamma_k})(t+\omega)\varphi(0)-\sum_{k=1}^n\alpha_k(\varphi_{1+\beta-\gamma_k}*S_{\beta,\gamma_k})(t)\varphi(0)\\
&+\int_0^\omega T_{\beta,\gamma_k}(t+\omega-s)F(s,u(s),u_s)dg(s)\\
&+\int_0^t T_{\beta,\gamma_k}(t-s)(F(s+\omega,u(s+\omega))-F(s,u(s),u_s)dg(s)\\
:=&J_1(t)+J_2(t)+J_3(t)+J_4(t)+J_5(t).
\end{aligned}$$

Then

$$\|f(t+\omega)-f(t)\|\leq\|J_1(t)\|+\|J_2(t)\|+\|J_3(t)\|+\|J_4(t)\|+\|J_5(t)\|.$$

Hence, we have

$$\begin{aligned}
\|J_1(t)\|&\leq\|S_{\beta,\gamma_k}(t+\omega)\varphi(0)\|+\|S_{\beta,\gamma_k}(t)\varphi(0)\|\\
&\leq(\|S_{\beta,\gamma_k}(t+\omega)\|+\|S_{\beta,\gamma_k}(t)\|)\cdot\|\varphi(0)\|\\
&\leq\frac{2C\|\varphi\|}{1+|\kappa|(t^{\beta+1}+\sum_{k=1}^n\alpha_k t^{\gamma_k})},
\end{aligned}$$

it is implies that $\|J_1(t)\|$ tend to 0 as $t \to \infty$.
Notice that by the estimate (2.1), we have

$$\begin{aligned}\|\varphi_\beta * S_{\beta,\gamma_k}(t)\| &\le \int_0^t \varphi_\beta(t-\tau)\|S_{\beta,\gamma_k}(\tau)\|d\tau \\ &\le \Gamma(1-\gamma_k)\int_0^t \varphi_\beta(t-\tau)\varphi_{1-\gamma_k}(\tau)\tau^{\gamma_k}\|S_{\beta,\gamma_k}(\tau)\|d\tau \\ &\le \frac{\Gamma(1-\gamma_k)}{|\kappa|\sum_{k=1}^n \alpha_k}\int_0^t \varphi_\beta(t-\tau)\varphi_{1-\gamma_k}(\tau)d\tau \\ &= \frac{\Gamma(1-\gamma_k)}{|\kappa|\sum_{k=1}^n \alpha_k}\varphi_{\beta-\gamma_k+1},\end{aligned}$$

for all $t > 0$. Hence there exists a constant $C > 0$ such that

$$\|\varphi_\beta * S_{\beta,\gamma_k}(t)\| \le Ct^{\beta-\gamma_k}. \tag{3.5}$$

Thus,

$$\|T_{\beta,\gamma_k}(t)\| = \|\varphi_\beta * S_{\beta,\gamma_k}(t)\| \le Ct^{\beta-\gamma_k}. \tag{3.6}$$

On the other hand, note that by (2.1) we have $\sup_{t>\tau}\|tS_{\beta,\gamma_k}(t)\| < \infty$, for each $\tau > 0$. Since A is an ω-sectorial of angle $\gamma_k\frac{\pi}{2}$ then $\|\mathcal{L}[S_{\beta,\gamma_k}](\lambda)\| \to 0$ as $\lambda \to 0$. Thus, by the vector-valued Hardy-Littlewood theorem (see [2], Theorem 4.2.9) we conclude that

$$\|(\varphi_1 * S_{\beta,\gamma_k})(t)\| \to 0 \quad \text{as} \quad t \to \infty. \tag{3.7}$$

We estimate $\|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)\|$ as follows. Let $0 < \varepsilon < \gamma_k - \beta$ be given, then

$$\begin{aligned}&\|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\| \\ =&\|\Gamma(\gamma_k-\beta-\varepsilon)\int_0^t \varphi_{1+\beta-\gamma_k}(t-\tau)\varphi_{\gamma_k-\beta-\varepsilon}(\tau)\tau^{\beta-\gamma_k+\varepsilon+1}S_{\beta,\gamma_k}(\tau)d\tau\| \\ \le&\Gamma(\gamma_k-\beta-\varepsilon)\int_0^t \varphi_{1+\beta-\gamma_k}(t-\tau)\varphi_{\gamma_k-\beta-\varepsilon}(\tau)\tau^{\beta-\gamma_k+\varepsilon+1}\|S_{\beta,\gamma_k}(\tau)d\tau\|,\end{aligned}$$

where, thanks to (2.1), we have that

$$\begin{aligned}\Gamma(\gamma_k-\beta-\varepsilon)\tau^{\beta-\gamma_k+\varepsilon+1}\|S_{\beta,\gamma_k}(\tau)\| &\le \frac{M\tau^{\beta-\gamma_k+\varepsilon-1}}{1+|\kappa|\tau^{\beta+1}} \\ &= \frac{M\tau^{-\gamma_k+\varepsilon}}{\frac{1}{\tau^{\beta+1}}+|\kappa|}, \qquad \varepsilon > 0.\end{aligned}$$

Since $\varepsilon < \gamma_k$, there exists a constant $C > 0$ such that

$$\tau^{\beta-\gamma_k+\varepsilon+1}\|S_{\beta,\gamma_k}(\tau)\| \le C.$$

Therefore,

$$\|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\| \le C\int_0^t \varphi_{1+\beta-\gamma_k}(t-\tau)\varphi_{\gamma_k-\beta-\varepsilon}(\tau)d\tau$$

$$= C\varphi_{1-\varepsilon}(t)$$
$$= Ct^{-\varepsilon},$$

which shows that

$$\|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\| \to 0 \text{ as } t \to \infty. \tag{3.8}$$

$$\begin{aligned}\|J_2(t)\| &\leq \|(\varphi_1 * S_{\beta,\gamma_k})(t+\omega)[Q_1(u)+\psi] - (\varphi_1 * S_{\beta,\gamma_k})(t)[Q_1(u)+\psi]\| \\ &\leq \|(\varphi_1 * S_{\beta,\gamma_k})(t+\omega) - (\varphi_1 * S_{\beta,\gamma_k})(t)\| \cdot (\mathcal{Q}(r) + \|\psi\|).\end{aligned}$$

By (3.7), we deduce that $\|J_2(t)\|$ tend to 0 as $t \to \infty$.

$$\begin{aligned}\|J_3(t)\| \leq & \|\sum_{k=1}^{n} \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t+\omega)\varphi(0) \\ & - \sum_{k=1}^{n} \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)\varphi\| \\ \leq & \sum_{k=1}^{n} \alpha_k \|[(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t+\omega) \\ & - (\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)]\| \cdot \|\varphi(0)\|.\end{aligned}$$

By (3.8), we deduce that $\|J_3(t)\|$ tend to 0 as $t \to \infty$.
In view of (A2), we find that for any $u \in E$, there exists a constant M_0 such that

$$\sup_{t\geq 0}\{\|F(t, u(t), u_t)\|\} \leq M_0. \tag{3.9}$$

By (3.6) and (3.9), we have

$$\begin{aligned}\|J_4\| &\leq \int_0^{\omega} \|T_{\beta,\gamma_k}(t+\omega-s)\| \cdot \|F(s, u(s), u_s)\| dg(s) \\ &\leq CM_0 \int_0^{\omega} (t+\omega-s)^{\beta-\gamma_k} dg(s),\end{aligned}$$

we deduce that $\|J_4(t)\|$ tend to 0 as $t \to \infty$. By (3.3), (3.4) and (3.6), we have

$$\begin{aligned}\|J_5\| \leq & \int_0^t \|T_{\beta,\gamma_k}(t-s)\| \cdot \|F(s+\omega, u(s+\omega), u_{s+\omega}) \\ & - F(s, u(s+\omega), u_{s+\omega}\| dg(s) \\ & + \int_0^t \|T_{\beta,\gamma_k}(t-s)\| \cdot \|F(s, u(s+\omega), u_{s+\omega}) - F(s, u(s), u_s)\| dg(s) \\ \leq & 4M_0C \int_0^{t_\epsilon} (t-s)^{\beta-\gamma_k} dg(s) + \epsilon \int_{t_\epsilon}^{t} \|T_{\beta,\gamma_k}(t-s)\| dg(s) \\ \leq & 4M_0C \int_0^{t_\epsilon} (t-s)^{\beta-\gamma_k} dg(s) + \epsilon \int_0^{t} \|T_{\beta,\gamma_k}(t-s)\| dg(s),\end{aligned}$$

which implies that $\|J_5(t)\|$ tends to 0 ad $t \to \infty$.
Thus, from the above results, we can deduce that

$$\lim_{t\to\infty} \|f(t+\omega) - f(t)\| = 0.$$

Combining this with the definition $\aleph$, we can conclude that $\aleph(SPA_\omega(E) \subset SPA_\omega(E)$.

Step.2. The set $\{\aleph u : u(\cdot) \in \Omega\}$ is equiregulated.

For any $b \in (0,\infty)$, restrict $u(t)$ to interval $[0,b)$. For any $t_0 \in [0,b)$, we have

$$\begin{aligned}
&\|(\aleph u)(t) - (\aleph u)(t_0^+)\| \\
\leq &\|(S_{\beta,\gamma_k}(t) - S_{\beta,\gamma_k}(t_0^+))\varphi(0)\| \\
&+ \|[(\varphi_1 * S_{\beta,\gamma_k})(t) - (\varphi_1 * S_{\beta,\gamma_k})(t_0^+)]\varphi(0)\| \\
&+ \sum_{k=1}^{n} \frac{\alpha_k M}{\Gamma(1+\beta-\gamma_k)} \Big| \int_0^t (t-s)^{\beta-\gamma_k} ds \\
&- \int_0^{t_0^+} (t_0^+ - s)^{\beta-\gamma_k} ds \Big| \|\varphi(0)\| + \int_0^{t_0^+} \|[T_{\beta,\gamma_k}(t-s) \\
&- T_{\beta,\gamma_k}(t_0^+ - s)]F(s,u(s),u_s)\| dg(s) \\
&+ \int_{t_0^+}^{t} \|T_{\beta,\gamma_k(t-s)}(t-s)F(s,u(s),u_s)\| dg(s) \\
\leq &\|S_{\beta,\gamma_k}(t) - S_{\beta,\gamma_k}(t_0^+)\|_{\mathrm{L}(E)}(\|\varphi(0)\|) + M|t-t_0^+|(\mathcal{Q}(\|u\|)+\psi) \\
&+ \sum_{k=1}^{n} \alpha_k M \Big| \frac{t^{1+\beta-\gamma_k} - (t_0^+)^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)} \Big| \|\varphi(0)\| \\
&+ W(r) \int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{\mathrm{L}(E)} P(s) dg(s) \\
&+ CW(r) \int_{t_0^+}^{t} (t-s)^{\beta-\gamma_k} P(s) dg(s) \\
= &I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t),
\end{aligned}$$

where

$$\begin{aligned}
I_1(t) &= \|S_{\beta,\gamma_k}(t) - S_{\beta,\gamma_k}(t_0^+)\|_{\mathrm{L}(E)} \|\varphi\|, \\
I_2(t) &= M|t-t_0^+|(\mathcal{Q}(\|u\|)+\psi), \\
I_3(t) &= \sum_{k=1}^{n} \alpha_k M \Big| \frac{t^{1+\beta-\gamma_k} - (t_0^+)^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)} \Big| \|\varphi\|, \\
I_4(t) &= W(r) \int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{\mathrm{L}(E)} P(s) dg(s), \\
I_5(t) &= CW(r) \int_{t_0^+}^{t} (t-s)^{\beta-\gamma_k} P(s) dg(s).
\end{aligned}$$

From the expression of $I_2(t)$ and $I_3(t)$, we derive that $I_2(t) \to 0$ and $I_3(t) \to 0$ as $t \to t_0^+$ independently of $u \in \Omega$. From (A1), the compactness of $S_{\beta,\gamma_k}(t)$ and $T_{\beta,\gamma_k}(t)$ for $t > 0$ yields the continuity in the sense of uniform operator topology. We dedude that $I_1(t) \to 0$ and applying dominated convergence theorem on $I_4(t)$ and , we can derive that $I_4(t) \to 0$ as $t \to t_0^+$ independently of $u \in \Omega$. Let $H(t) = \int_0^t (t-s)^\beta P(s) dg(s)$. Thanks to Lemma 2.4, we known that $H(t)$ is a regulated function on $\mathbb{R}^+$. Therefore, we have

$$\begin{aligned}I_5(t) &= CW(r)\int_{t_0^+}^{t}(t-s)^{\beta-\gamma_k}P(s)dg(s)\\&\leq CW(r)\Big(\|H(t)-H(t_0^+)\|\\&\quad+\int_0^{t_0^+}\|((t-s)^{\beta-\gamma_k}-(t_0^+-s)^{\beta-\gamma_k})P(s)\|dg(s)\Big)\\&\to 0 \ \text{ as } \ t\to t_0^+ \ \text{ independently of } \ u.\end{aligned}$$

Therefore, $\|(\aleph u)(t)-(\aleph u)(t_0^+)\|\to 0$ as $t\to t_0^+$. independently of $u\in\Omega$. Similarly, one can demonstrate that for any $t_0\in(0,b]$, $\|(\aleph u)(t)-(\aleph u)(t_0^+)\|\to 0$ as $t\to t_0^+$.
According to the arbitrariness of b, one can find that $u(t)$ is defined on $[0,\infty)$. On the other hand, it is easy to see $\lim_{t\to\infty}\|u(t+\omega)-u(t)\|=0$. Hence, assert that $\{\aleph u : u(\cdot)\in\Omega\}$ is equiregulated.

Step.3. $\aleph:\Omega\to\Omega$ is continuous operator. Let $\{u^n\}\subset\Omega$ be a sequence such that u^n converges to u as $n\to\infty$. By assumption conditions (A2) and (A3) it conclude that

$$F(s,u^n(s),u_s^n)\to F(s,u(s),u_s)$$

and

$$\|F(s,u^n(s),u_s^n)-F(s,u(s),u_s)\|\leq 2P(s)W(r) \tag{3.10}$$

and moreover for each $t\geq 0$, we have

$$\begin{aligned}&\|\aleph(u_n)(t)-\aleph(u)(t)\|\\\leq&\|(\varphi_1 * S_{\beta,\gamma_k}(t)\|\cdot\|Q_1(u^n)-Q_1(u)\|\\&+C\int_0^t(t-s)^{\beta-\gamma_k}\|F(s,u^n(s),u_s^n)-F(s,u(s),u_s)\|dg(s).\end{aligned} \tag{3.11}$$

Using the strong continuity of S_{β,γ_k}, assumptions (A2)-(A4), inequalities (3.10)-(3.11) and the dominated convergence theorem incorporated for the Henstock-Lebesgue-Stieltjes integral $\|\aleph(u_n)(t)-\aleph(u)(t)\|\to 0$ as $n\to\infty$. Moreover, by Step 2, it can shown that $\{\aleph(u_n)\}_{n=1}^{\infty}$ is equiregulated. Therefore, taking account to Lemma 2.1, we derive that $\aleph(u_n)$ converge uniformly to $\aleph(u)$. Hence, $\aleph$ is a continuous operator.

Step.4. For each $t\geq$, $\mathcal{C}(t):=\{\aleph(u)(t):u\in\Omega\}$ is a relatively compact subset of E. Clearly, $\mathcal{C}(0)$ is compact. Let $t\geq 0$ be fixed and $\epsilon\in(0,t)$. by the compactness of $S_{\beta,\gamma_k}(t)$ and $T_{\beta,\gamma_k}(t)$, observe that is $\mathcal{C}_\epsilon(t):=\{\aleph^\epsilon(u)(t):u\in\Omega\}$ relatively compact in E.

Then for any $u\in\Omega$, we have that

$$\begin{aligned}&\|\aleph(u)(t)-\aleph^\epsilon(u)(t)\|\\\leq&\|((\varphi_1 * S_{\beta,\gamma_k}(t)-\varphi_1 * S_{\beta,\gamma_k}(t-\epsilon))[Q_1(u)+\psi]\|\\&+\Big\|\sum_{k=1}^{n}\alpha_k\int_{t-\epsilon}^{t}\frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)}S_{\beta,\gamma_k}(s)\varphi(0)ds\Big\|\end{aligned}$$

$$
\begin{aligned}
&+\Big\|\int_{t-\epsilon}^{t}T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s)\Big\|\\
\le& M\epsilon(\mathcal{Q}(\|u\|)+\psi)+\sum_{k=1}^{n}\frac{\alpha_k M\epsilon^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)}\|\varphi\|+CW(r)\int_{t-\epsilon}^{t}(t-s)^{\beta-\gamma_k}P(s)dg(s)\\
\le& M\epsilon(\mathcal{Q}(\|u\|))+\sum_{k=1}^{n}\frac{\alpha_k M\epsilon^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)}\|\varphi(0)\|\\
&+CW(r)\Big(\|H(t)-H(t-\epsilon)\|+\int_{0}^{t-\epsilon}|(t-s)^{\beta-\gamma_k}-(t-\epsilon-s)^{\beta-\gamma_k}|P(s)dg(s)\Big).
\end{aligned}
$$

Now, letting $\epsilon\to 0$, we observe that relatively compact set arbitrary close to $\mathcal{C}(t)$ for each $t\ge 0$. Thus for all $t\ge 0$, $\mathcal{C}(t)$ is relatively compact in E. Therefore, the set $\mathcal{C}(t)$ is relatively compact for all $t\ge 0$. From Step 3, Step 4 and Lemma 2.9, we deduce that $\aleph$ is a completely continuous operator. According to Schauder fixed point theorem (Lemma 2.6), we deduce that $\aleph$ has at least a fixed point in Ω, which is a mild solution for problem (1.1). The proof is completed. $\square$

Our purpose in this section is to study the uniqueness of S-asymptotically ω-periodic mild solutions for the equation (1.1). In order to obtain the S-asymptotically ω-periodic mild solutions, we need to impose an additional condition on the nonlinear term.

(A5) The function $F:\mathbb{R}^+\times E\times\mathcal{B}\to E$ is continuous and there exists $L_2>0$ such that
$$\|F(t,u_1,\phi_1)-F(t,u_2,\phi_2)\|\le L_1\|u_1-u_2\|+L_2\|\phi_1-\phi_2\|$$
for any $t\in[0,\infty)$ and $u_1,u_2\in E$ $\phi_1,\phi_2\in\mathcal{B}$.

(A6) $Q_1:G(\mathbb{R}^+,E)\to E$ are continuous and there is positive constants L_3 such that
$$\|Q_1(u_2)-Q_2(u_1)\|\le L_3\|u_1-u_2\|,$$
for any $u_1,u_2\in E$.

We now present the main results of this section.

Theorem 3.2. *Let $0<\beta\le\gamma_n\le\cdots\le\gamma_1\le 1$ and $\alpha_k\ge 0, k=1,2,\dots n$ be given. If the assumptions (A1)-(A6) hold, then problem (1.1) has a unique S-asymptotically ω-periodic mild solution on $\mathbb{R}^+$ provided that*

$$L_3\widetilde{M}+C(L_1+L_2)\sup_{t\ge 0}\Big(\int_0^t(t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}<1, \tag{3.12}$$

where

$$
\begin{aligned}
R=&\Big(M+\sum_{k=1}^{n}\alpha_k M_1\Big)\|\varphi(0)\|+\widetilde{M}(\mathcal{Q}(\|u\|)+\|\psi\|)\\
&+CW(R)\sup_{t\ge 0}\Big(\int_0^t(t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\|P\|_{\mathbb{HLS}_g^p}.
\end{aligned}
$$

Proof. Let $\Omega^+:=\{u\in G(\mathbb{R}^+,E)\cap C_b(\mathbb{R}^+,E):u(0)=\varphi(0),\|u\|\le R,u|_{[\mathbb{R}^+}\in SPA_\omega(E))\}$ and an operator $Q:\Omega^+\to G(\mathbb{R}^+,E)$ be defined by

$$(Qu)(t)=S_{\beta,\gamma_k}(t)\varphi(0)+(\varphi_1*S_{\beta,\gamma_k})(t)[Q_1(u)+\psi]$$

$$+\sum_{k=1}^{n}\alpha_k(\varphi_{1+\beta-\gamma_k}*S_{\beta,\gamma_k})(t)\varphi(0)$$
$$+\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s),\quad t\geq 0. \tag{3.13}$$

By the conditions (A1)-(A4), we have

$$\begin{aligned}
&\|(Qu)(t)\|\\
\leq&\|S_{\beta,\gamma_k}(t)\varphi(0)+(\varphi_1*S_{\beta,\gamma_k})(t)[Q_1(u)+\psi]\\
&+\sum_{k=1}^{n}\alpha_k\int_0^t\frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)}S_{\beta,\gamma_k}(s)\varphi(0)ds+\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s)\|\\
\leq&\|S_{\beta,\gamma_k}(t)\varphi(0)\|+\|(\varphi_1*S_{\beta,\gamma_k})(t)[Q_1(u_r)+\psi]\|\\
&+\|\sum_{k=1}^{n}\alpha_k(\varphi_{1+\beta-\gamma_k}*S_{\beta,\gamma_k})(t)\varphi(0)\|+\|\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s)\|\\
\leq&\|S_{\beta,\gamma_k}(t)\varphi(0)\|+\|(\varphi_1*S_{\beta,\gamma_k})(t)[Q_1(u_r)+\varphi]\|\\
&+\|\sum_{k=1}^{n}\alpha_k(\varphi_{1+\beta-\gamma_k}*S_{\beta,\gamma_k})(t)\varphi(0)\|+\|\int_0^t T_{\beta,\gamma_k}(t-s)\|\cdot\|F(s,u(s),u_s)\|dg(s)\\
\leq&M(\|\varphi(0)\|)+\widetilde{M}(\mathcal{Q}(\|u\|)+\|\psi\|)+\Big(\sum_{k=1}^{n}\alpha_kM_1\Big)\|\varphi(0)\|\\
&+CW(R)\Big(\int_0^t(t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\Big(\int_0^t[P(s)]^pdg(s)\Big)^{\frac{1}{p}}\\
\leq&M\|\varphi(0)\|+\widetilde{M}(\mathcal{Q}(\|u\|)+\|\psi\|)+\Big(\sum_{k=1}^{n}\alpha_kM_1\Big)\|\varphi(0)\|\\
&+CW(R)\sup_{t\geq0}\Big(\int_0^t(t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\|P\|_{\mathbb{HLS}_g^p}.
\end{aligned}$$

Thus, we have
$$\|Qu\|\leq R.$$
Moreover, we can similarly prove the operator $Qu\in SAP_\omega(H)$ as Step 1 in Theorem 3.1.

Next, we then show that Q is a contractive map. In fact, for $x,y\in SAP_\omega(H)$, due to the conditions (A5)-(A6) we find that

$$\begin{aligned}
&\|(Qx)(t)-(Qy)(t)\|\\
\leq&\|(\varphi_1*S_{\beta,\gamma_k})(t)[Q_1(x)-Q_1(y)]\|\\
&+\Big\|\int_0^t T_{\beta,\gamma_k}(t-s)\|\cdot\|F(s,x(s),x_s)-F(s,y(s),y_s)\Big\|dg(s)\\
\leq&L_3\|(\varphi_1*S_{\beta,\gamma_k})(t)\|\cdot\|x-y\|\\
&+C\Big(\int_0^t(t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}(L_1\|x-y\|+L_2\|x_s-y_s\|)\\
\leq&L_3\widetilde{M}\cdot\|x-y\|+C(L_1+L_2)\sup_{t\geq0}\Big(\int_0^t(t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\cdot\|x-y\|
\end{aligned}$$

$$\leq\Big(L_3\widetilde{M}+C(L_1+L_2)\sup_{t\geq 0}\Big(\int_0^t(t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\Big)\|x-y\|,$$

for all $t\in[0,\infty)$, from which it follows that

$$\|Qx-Qy\|\leq\Big(L_3\widetilde{M}+C(L_1+L_2)\sup_{t\geq 0}\Big(\int_0^t(t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\Big)\|x-y\|.$$

So, (3.12) implies Q is contractive on Ω^+ and hence we infer that there exists a unique fixed point $u^*(\cdot)$ for the operator Q on Ω^+ by the Banach fixed point theorem again, then $u^*\in\Omega^+$ defined by (3.13) is the unique mild solution of nonlocal problem (1.1).

The proof is thus completed. □

4. Applications

In this section, we give an example to illustrate our main results. Let $\beta,\gamma_k>0(k=1,2,\ldots,n)$ be such that $0<\beta\leq\gamma_n\leq\cdots\leq\gamma_1\leq 1$. Let $E=L^2([0,\pi],\mathbb{R})$ and consider the following measure driven differential equation:

$$\begin{cases} \quad {}^cD_{0+}^{1+\beta}u(t,x)+\sum\limits_{k=1}^{n}\alpha_k{}^cD^{\gamma_k}u(t,x)\\ =\Delta u(t,x)+\tau u(t,x)+(\xi_0\frac{\sin(u(t,x))}{1+e^{2t}}+\frac{b\cos t}{e^{2t}}u(t+s,x)+1)dg(t),\\ \quad x\in[0,\pi],\ t\in\mathbb{R}^+,\\ u(t,0)=u(t,\pi)=0,\quad t\in\mathbb{R}^+,\\ u(t,x)=\varphi(t,x),\quad (t,x)\in[-r,0]\times[0,\pi],\\ \frac{\partial u(t,x)}{\partial t}|_{t=0}=\frac{\xi_3|u|}{7+|u|}+\psi(x),\quad x\in[0,\pi], \end{cases}\tag{4.1}$$

where $\varphi\in G([-r,0]\times[0,\pi],\mathbb{R}^+),\psi\in E,\ \xi_0,\xi_1,\xi_2\in\mathbb{R}^+-\{0\}$ and $g:[0,\pi]\to\mathbb{R}$ is a nondeacresing, left continuous function. Furthermore, define the operator $A:D(A)\subset E\to E$ by $Au=\Delta u+\tau u$, where $\tau<0$ and

$$D(A)=\{u\in E:u,u'\quad\text{are absolutely continuous}, u''\in E, u(0)=u(\pi)=0\}.$$

Then it is wellknown that the operator A is κ-sectorial with $\kappa=\tau<0$ and angle $\frac{\pi}{2}$ (and hence of angle $\frac{\gamma_k\pi}{2}$) for all $\gamma_k\leq 1,k=1,2,\cdots,n$).

Since $\beta,\gamma_k>0,k=1,2,\cdots,m$ be such that $0<\beta\leq\gamma_m\leq\cdots\leq\gamma_1\leq 1$, by Theorem 2.7 , we deduce that A generates a bounded (β,γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$. We define

$$\varphi(t)(x)=\varphi(t,x),v(t)(x)=u(t,x),v_t(s)(x)=u(t+s,x),\quad\text{for}\ \ t\in E,\ \ x\in[0,\pi].$$

For every $t\in E,x\in[0,\pi]$, define

$$F(t,v,v_t)=\frac{\xi_0\sin(v(t))}{1+e^{2t}}+\frac{b\cos t}{e^{2t}}u(t+s,x)+1,$$

$$Q_1:G([0,\pi],E)\to E,\quad Q_1(v(t,x))=\frac{\xi_3|v(t,x)|}{7+|v(t,x)|}.$$

Then problem (4.1) can be rewritten in the abtract form of problem (1.1). Further, from the definition of functions F and Q_1 it turns out that Q_1 are Lipschitz continuous with Lipschitz constant $\frac{\xi_0}{2}$ and $\frac{\xi_3}{7}$ respectively, and

$$\|F(t,v,v_t)\| \leq \frac{\xi_0}{2}\|v\| + \frac{b}{2}\|v_t\| \leq (\frac{\xi_0}{2}+\frac{b}{2})\|v\|, \qquad \|Q_1(v(t,x))\| \leq \frac{\xi_3}{7}.$$

Additionnaly, (A2) is satisfied with $P(t) = \frac{\xi_0}{2} + \frac{b}{2}$ and $W(r) = r$. Let

$$\Theta = \Big[\frac{\widetilde{M}\xi_3}{7} + C(\frac{\xi_0}{2}+\frac{b}{2})\sup_{t\geq 0}\Big(\int_0^t (t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\|P\|_{\mathbb{HLS}_g^p}\Big] < 1.$$

Choosing ξ_1, ξ_0 small enought, we can obtian $\Theta < 1$, then the conditions in Theorem 3.1 are satisfied, the problem (4.1) has a S-asymptotically ω-periodic mild solutions.

5. Conclusions

Multi-term linear fractional-order differential equations including three fractional-order derivatives have been analyzed, in extension to some recent results that have been obtained for linear incommensurate bidimensional fractional-order systems. This paper has established some of results concerning the existence of the S-asymptotically ω-periodic mild solutions for a class of multi-term time-fractional measure differential equations involving nonlocal conditions in Hilbert spaces. It has been shown that the obtained theoretical results can be successfully applied for the particular cases of measure driven differential equations. In the furture work, we study the existence of maximal and minimal S-asymptotically ω-periodic mild solutions for this class of multi-term time-fractional measure differential equations in order Banach space, by means of the method of lower and upper solution. Moreover, we study a class of multi-term fractional differential equation describing an inextensible pendulum with fractional damping terms as well as for a fractional harmonic oscillator. Possible generalizations to the case of general multi-term fractional-order differential equations will be investigated in a future work.

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