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EXISTENCE OF WEAK SOLUTIONS TO THE BGK EQUATION AND AN APPROXIMATE CONSERVATION LAWS WITH LARGE INITIAL DATA*

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Abstract This paper studies the Cauchy problem of a BGK model and the corresponding nonlinear hyperbolic conservation laws. Given bounded initial data for the kinetic equation, the existence of weak solutions to the BGK model is obtained by the time-splitting method. Moreover, weak solutions to the limiting hyperbolic system are obtained by passing the relaxation parameter to zero in a modified BGK model.

Keywords BGK equation, conservation laws, existence, time splitting method.

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1. Introduction

In this paper, we consider the following BGK kinetic system:

$$\begin{cases} f_t^\epsilon + v\partial_x f^\epsilon = \dfrac{1}{\epsilon}\big(M_{f^\epsilon} - f^\epsilon\big), \\ f^\epsilon(0,x,v) = f_0(x,v), \end{cases} \tag{1.1}$$

where $f^\epsilon(t,x,v) \geq 0$ represents the density distribution function of particles at time $t \in \mathbb{R}^+$ around position $x \in \mathbb{R}$ with velocity $v \in (-1,1)$, ϵ is the relaxation parameter, and M is defined later in (1.4).

Let $d\mu := 1/2dv$ and

$$\langle g \rangle := \int_{-1}^{1} g(v)d\mu(v) = \frac{1}{2}\int_{-1}^{1} g(v)dv$$

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be the mean of a function g with respect to v in distribution sense. The macroscopic quantities of f include density ρ and momentum J, which are defined by

$$\begin{pmatrix}\rho_f\\ J_f\end{pmatrix} := \begin{pmatrix}\langle f\rangle\\ \langle vf\rangle\end{pmatrix} = \int_{-1}^{1}\begin{pmatrix}1\\ v\end{pmatrix} f(t,x,v)d\mu(v). \tag{1.2}$$

For simplicity, we use ρ and J to denote ρ_f and J_f respectively when there is no ambiguities. The entropy of f is given by

$$H(f) := \langle f\ln f\rangle = \int_{-1}^{1} f(t,x,v)\ln f(t,x,v)d\mu(v). \tag{1.3}$$

Define the equilibrium state of f as

$$M_f = M_{(\rho_f,J_f)} := \begin{cases} 0, & \text{if } \rho_f = 0,\\ \dfrac{\rho_f}{\mathbb{F}\circ\mathbb{G}^{-1}(\frac{J_f}{\rho_f})}\exp\big(v\mathbb{G}^{-1}(\dfrac{J_f}{\rho_f})\big), & \text{otherwise,}\end{cases} \tag{1.4}$$

where

$$\mathbb{F}(z) := \begin{cases}\dfrac{\sinh z}{z}, & z\neq 0,\\ 1, & z=0,\end{cases} \qquad \mathbb{G}(z) := \begin{cases}\dfrac{\mathbb{F}'(z)}{\mathbb{F}(z)} = \coth z - \dfrac{1}{z}, & z\neq 0,\\ 0, & z=0.\end{cases}$$

Easy to see that function $\mathbb{G}$ is a C^∞ diffeomorphism from $\mathbb{R}$ onto $(-1,1)$.

Given any $\rho(t,x), J(t,x)$, one can check that $M_{(\rho,J)}$ is a minimizer of $H(f)$ among all f satisfying $\langle f\rangle=\rho$, $\langle vf\rangle = J$. Namely,

$$\langle M_{(\rho,J)}\rangle = \rho, \qquad \langle vM_{(\rho,J)}\rangle = J, \qquad H(M_{(\rho,J)}) = \min_{\langle f\rangle=\rho,\langle vf\rangle=J} H(f). \tag{1.5}$$

By using the nonlinear entropy-based moment closure method in [15], we can derive the following hyperbolic conservation laws:

$$\begin{cases}\rho_t + \partial_x J = 0, & t>0,\ x\in\mathbb{R},\\ J_t + \partial_x\big(\rho\psi(\dfrac{J}{\rho})\big) = 0, & t>0,\ x\in\mathbb{R},\\ \rho(x,0)=\rho_0(x), \qquad J(x,0)=J_0(x), & \end{cases} \tag{1.6}$$

with $|J|<\rho$ and $\psi : (-1,1)\to(0,+\infty)$ defined by

$$\psi(s) := s^2 + \mathbb{G}'\big(\mathbb{G}^{-1}(s)\big) = \frac{\mathbb{F}''}{\mathbb{F}}\big(\mathbb{G}^{-1}(s)\big), \qquad s\in(-1,1).$$

Obviously, $\mathbb{G}$ is an odd function, $\mathbb{F}$, ψ are even, ψ is strictly convex, and

$$\mathbb{F}(0)=1, \quad \mathbb{G}(0)=0, \quad \psi(0)=\mathbb{G}'(0)=\frac{1}{3}, \quad \psi'(0)=0,$$
$$\lim_{s\to\pm1}\psi(s)=1, \qquad \lim_{s\to\pm1}\psi'(s)=\pm2.$$

Direct calculation implies that the eigenvalues and corresponding eigenvectors of (1.6) are given by:

$$\lambda_i(s) = \frac{\psi'(s) \pm \sqrt{\left(\psi'(s)\right)^2 - 4s\psi'(s) + 4\psi(s)}}{2}, \quad r_i(s) = \begin{pmatrix} 1 \\ \lambda_i(s) \end{pmatrix}, \quad i = 1, 2,$$

with $\lambda_1(s) < s < \lambda_2(s)$ and $\lambda_i'(s) > 0$. The system (1.6) is strictly hyperbolic and genuinely nonlinear. All the properties above were given in [8], and it was shown that the corresponding homogeneous Riemann problem can be solved without smallness assumption.

There have been extensive studies on the mathematical theory for the systems of hyperbolic conservation laws. One feature of the hyperbolic system is that discontinuities always happen within a finite time no matter how smooth the initial data is. In the framework of solutions with small total variation, there are two classic methods to obtain the global existence: In the fundamental work of Glimm [10], the weak solution is established by introducing the Glimm scheme which uses as building blocks the solutions to Riemann problems solved by Lax, refer to [5,12,14,16,17] and the references therein. On the other hand, Bianchini and Bressan [3] constructed solutions for hyperbolic systems by the method of vanishing viscosity. Bianchini provided in [2] a general framework to extend the relative entropy method to a class of diffusive relaxation systems with discrete velocities. The same approach has been successfully used to show the strong convergence of a vector-BGK model to the 2D incompressible Navier-Stokes equations. Buttá et al. [6] proved the convergence of a suitable particle system towards the BGK model. Hwang et al. [13] proved the unique existence and asymptotic behavior of classical solutions to the relativistic polyatomic BGK model when the initial data is sufficiently close to a global equilibrium.

For weak solutions with large initial data, the method of compensated compactness is used to establish weak solutions to the scalar or 2×2 systems of hyperbolic conservation laws with initial data in L^p, refer to [7,18,22]. The key point of this method is to establish a series of entropies for conservation laws, but the equation of entropy is always a second-order nonlinear PDE, so it is hard to obtain the entropy function by solving it. Yang, Zhu and Zhao in [25] studied the strong convergence of a sequence of uniform bounded approximate solutions $u^\epsilon(x,t)$ to a weak solution of scalar conservation laws in $L^p_{loc}(\mathbb{R} \times \mathbb{R}^+)$, $1 < p \leq +\infty$. Lu in [19] established an existence theorem for global entropy solutions of the nonstrictly hyperbolic system by use the theory of compensated compactness coupled with some basic ideas of the kinetic formulation. Wang, Liu and Zhao in [24] studied the Riemann problem of a one-dimensional nonlinear wave systems with different gamma laws. They constructed the Riemann solution and prove the stability of Riemannian solutions for some disturbance of the initial data by utilizing the interaction of the elementary waves. Under the assumptions of spherical symmetry and self-similarity, Zhang and Hu considered in [26] the self-similar flow of multidimensional isentropic compressible Euler equations caused by uniform expansion of a spherically-symmetric piston into the undisturbed fluid. Sun and Lu studied in [21] the global existence of weak solutions for the Cauchy problem of the nonlinear hyperbolic system with bounded initial data. They introduced a variant of the viscosity argument, added the artificial viscosity to the Riemann invariant system, and constructed the approximate solutions of the nonlinear conservation system. Then they proved the pointwise convergence of the viscosity solutions by the compensated compactness theory.

Formal moment closure of the kinetic equation can give rise to hyperbolic systems. This idea can also be utilized to construct schemes for hyperbolic systems by solving the underlying kinetic equation, which is the approach that we take in this paper, as well as in [8, 15, 20]. If the entropy of the kinetic equation is chosen to be linear, the minimizer will be a Dirac function, see [20, 23], and the moments of minimizer belong to a bounded subset of $BV \cap L^\infty$. However, if we use the physical entropy (1.3), the minimizer given by (1.4) is unbounded in L^∞. This fact leads to a lack of strong compactness. Our main goal is to prove that given rough initial data for the kinetic equation, the solutions converge to a weak solution of the limiting hyperbolic system.

Now we give definitions of the weak solutions to (1.1) and (1.6).

Definition 1.1. We say that f^ϵ is a *weak solution* to (1.1) if $f^\epsilon(t,x,v) \in L^1\big([0,T)\times \mathbb{R} \times (-1,1)\big)$ and

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \int_{-1}^1 f^\epsilon(\phi_t + v\phi_x)d\mu(v)dxdt + \int_{\mathbb{R}}\int_{-1}^1 f_0(x,v)\phi(0,x,v)d\mu(v)dx\\ &+\frac{1}{\epsilon}\int_0^T\int_{\mathbb{R}}\int_{-1}^1 (M_{f^\epsilon} - f^\epsilon)\phi d\mu(v)dxdt\\ =&0\end{aligned}$$

holds for any smooth function $\phi(t,x,v)$ with compact support in $\{(t,x,v) \in [0,T)\times \mathbb{R} \times (-1,1)\}$.

Definition 1.2. We say that $(\rho, J) \in L^1([0,T)\times\mathbb{R}) \times L^1([0,T)\times\mathbb{R})$ is a *weak solution* to (1.6) if

$$\int_0^T \int_{\mathbb{R}} \rho\varphi_t + J\varphi_x dxdt + \int_{\mathbb{R}} \rho_0\varphi(0,x)dx = 0$$

and

$$\int_0^T \int_{\mathbb{R}} \Big(J\varphi_t + \rho\psi(\frac{J}{\rho})\varphi_x + J\varphi\Big)dxdt + \int_{\mathbb{R}} J_0\varphi(0,x)dx = 0$$

hold for any smooth function $\varphi(t,x)$ with compact support in $\{(t,x)\in[0,T)\times\mathbb{R}\}$.

The main results of this paper are the following theorems. The first result concerns the existence of weak solution to (1.1).

Theorem 1.1. *Suppose the initial data $f_0(x,v) \geq 0$ satisfies*

$$\int_{\mathbb{R}}\int_{-1}^1 (1+|x|)f_0(x,v)d\mu(v)dx \leq C_0, \tag{1.7}$$

and

$$\int_{\mathbb{R}}\int_{-1}^1 f_0(x,v)\ln f_0(x,v)d\mu(v)dx \leq C_0. \tag{1.8}$$

Then (1.1) *has a weak solution $f^\epsilon(t,x,v) \in L^1([0,T)\times\mathbb{R}\times(-1,1))$.*

Remark 1.1. From conditions (1.7) and (1.8), we deduce later that

$$\int_{\mathbb{R}}\int_{-1}^1 f_0(x,v)|\ln f_0(x,v)|d\mu(v)dx \leq C_0.$$

Our second result verifies the validity of the entropy-based moment closure method for this BGK kinetic equation.

Theorem 1.2. *Suppose that the initial data $f_0(x,v) \geq 0$ satisfies* (1.7) *and* (1.8)*. For any fixed constant $\zeta > 0$, there exists $f(t,x,v) \in L^1([0,T) \times (-1,1) \times \mathbb{R})$ such that (ρ, J, p), defined by $\rho := \langle f \rangle$, $J := \langle vf \rangle$, $p := \langle v^2 f \rangle$, is a pair of weak solution to*

$$\begin{cases} \rho_t + \partial_x J = 0, \\ J_t + \partial_x p = 0, \\ \rho(x,0) = \rho_0(x), \quad J(x,0) = J_0(x), \end{cases}$$

and the pressure p satisfies

$$\|p - \rho\psi(\frac{J}{\rho})\|_{L^1_{loc}} \leq C\zeta,$$

where $(\rho_0, J_0) = (\langle f_0 \rangle, \langle v f_0 \rangle)$ and C is a constant independent of ζ.

It is worth mentioning that the function f in Theorem 1.2, which may not be equal to the weak solution in Theorem 1.1, is obtained as the limit of a sequence of weak solutions to the approximate system (4.2).

Remark 1.2. Given ρ_0, J_0. We can use

$$f_0(x,v) = M_{(\rho_0,J_0)} = \frac{\rho_0}{\mathbb{F}(\mathbb{G}^{-1}(J_0/\rho_0))} e^{v\mathbb{G}^{-1}(J_0/\rho_0)}$$

to rephrase the initial conditions (1.7) and (1.8) as

$$\int_{\mathbb{R}} (1+|x|)\rho_0 dx \leq C_0,$$

and

$$\int_{\mathbb{R}} \Big(\rho_0 \ln \rho_0 - \rho_0 \ln\big(\mathbb{F} \circ \mathbb{G}^{-1}(J_0/\rho_0)\big) + J_0 \mathbb{G}^{-1}(J_0/\rho_0)\Big) dx \leq C_0.$$

Our method in this paper can be used in a wide range of kinetic equations and their corresponding system of hyperbolic conservation laws, which is derived from an entropy-based moment closure of the kinetic equation.

There are some difficulties to obtain weak solutions of the conservation laws in L^1 space. First, the minimizer or equilibrium M_f is unbounded in L^∞ and we lack strong compactness for the moments of f^ϵ. The entropy inequality leads to an estimate

$$\begin{aligned} &\frac{1}{\epsilon}\int_0^T \int_{\mathbb{R}} \int_{-1}^1 (\ln f^\epsilon(t,x,v) - \ln M_{f^\epsilon(t,x,v)})(f^\epsilon(t,x,v) - M_{f^\epsilon(t,x,v)}) d\mu(v) dx dt \\ \leq &C(f_0(x,v), T), \end{aligned}$$

but what we need is a uniform bound for

$$\frac{1}{\epsilon}\int_0^T \int_{\mathbb{R}} \int_{-1}^1 |f^\epsilon(t,x,v) - M_{f^\epsilon(t,x,v)}| d\mu(v) dx dt. \tag{1.9}$$

In fact, for any $x, y > 0$, it can be proved that

$$(\ln x - \ln y)(x-y) \geq 2(\sqrt{x} - \sqrt{y})^2. \tag{1.10}$$

So the entropy estimate is not sufficient to control (1.9). Secondly, we lack regularity of the moments of f^ϵ. From the kinetic theory, we can gain some regularities by the velocity averaging lemma, but the right-hand side of (1.1) and the approximate equation (1.6) do not satisfy the conditions to apply velocity averaging lemma. Finally, for system (1.6), a natural entropy can be given by

$$\eta(\rho, J) = \rho \ln \rho - \rho \ln[\mathbb{F} \circ \mathbb{G}^{-1}(J/\rho)] + J\mathbb{G}^{-1}(J/\rho),$$

the corresponding flux is

$$q(\rho, J) = J \ln \rho - J \ln[\mathbb{F} \circ \mathbb{G}^{-1}(J/\rho)] + \rho\psi(J/\rho)\mathbb{G}^{-1}(J/\rho),$$

but this entropy-flux pair is not sufficient to gain the convergence of viscous solutions.

The rest of this paper is organized as follows. In section 2, we first review some basic lemmas that will be used in our proof. In section 3, we prove the existence of weak solutions to (1.1) by using the time splitting method with a free transport step followed by a projection partially on the equilibrium. In the last section, we establish the existence of weak solutions to the approximate equation (1.6).

2. Preliminaries

In this section, we will recall some known facts and elementary lemmas that will be used in our proof.

As a bounded subset of $L^1(\mathbb{R}^n)$ may not be weakly compact, we need the following Dunford-Pettis theorem to establish the weak compactness.

Lemma 2.1 ([9])**.** *A bounded subset $\mathcal{F} \subset L^1(\mathbb{R}^n)$ is precompact with respect to the weak topology of L^1 if and only if the following hold:*

(i) *$\mathcal{F}$ is uniformly integrable:*

$$\int_A |f(x)|dx \to 0, \ \text{as } |A| \to 0, \quad \text{uniformly in } f \in \mathcal{F};$$

(ii) *$\mathcal{F}$ is tight:*

$$\int_{|x|>R} |f(x)|dx \to 0, \ \text{as } R \to \infty, \quad \text{uniformly in } f \in \mathcal{F}.$$

Remark 2.1. Note that $\mathcal{F}$ is uniformly integrable is equivalently to

$$\int_{|f(z)|>C} |f(x)|dx \to 0, \ \text{as } C \to \infty, \quad \text{uniformly in } f \in \mathcal{F}.$$

A metric space is compact if and only if it is complete and totally bounded, and since we are interested in compactness results for subsets of L^1, we need the following Kolmogorov-Riesz theorem in [11].

Lemma 2.2 ([11])**.** *Let $1 \le p < \infty$, a subset $\mathcal{F} \subset L^p(\mathbb{R}^n)$ is totally bounded if and only if the following hold:*

(i) *$\mathcal{F}$ is bounded;*

(ii) *For any* $\varepsilon > 0$*, there exists some* R *such that for every* $f \in \mathcal{F}$

$$\int_{|x|>R} |f(x)|^p dx < \varepsilon^p.$$

(iii) *For any* $\varepsilon > 0$*, there exists some* $\delta > 0$ *such that for every* $f \in \mathcal{F}$ *and* $y \in \mathbb{R}^n$ *with* $|y| < \delta$*,*

$$\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p dx < \varepsilon^p.$$

Corollary 2.1 ([11])**.** *Let* $1 \leq p < \infty$*,* $\Omega \subset \mathbb{R}^n$ *be an open set. Define*

$$f_K(x) := \begin{cases} f(x), & x \in K, \\ 0, & \textit{otherwise}. \end{cases}$$

A subset $\mathcal{F} \subset L^p_{loc}(\Omega)$ *is totally bounded if and only if*

(i) *For every compact set* $K \subset \Omega$*, there exists some* M *such that*

$$\int |f_K(x)|^p dx < M, \quad f \in \mathcal{F}.$$

(ii) *For every* $\varepsilon > 0$ *and every compact set* $K \subset \Omega$ *, there exists some* $\delta > 0$*, such that*

$$\int |f_K(x+y) - f_K(x)|^p dx < \varepsilon^p, \quad |y| < \delta, f \in \mathcal{F}.$$

Lemma 2.3 ([1])**.** *Let* $1 \leq p < \infty$ *and* $\Omega \subset \mathbb{R}^n$ *be an open set. A bounded subset* $\mathcal{F} \subset L^p(\Omega)$ *is precompact if and only if for every* $\varepsilon > 0$*, there exists some* $\delta > 0$ *and a subset* $G \Subset \Omega$*, such that the following inequalities hold:*

$$\int_{\Omega \setminus \bar{G}} |f(x)|^p dx < \varepsilon^p, \quad f \in \mathcal{F},$$
$$\int_{\Omega} |\tilde{f}(x+y) - \tilde{f}(x)|^p dx < \varepsilon^p, \quad |y| < \delta, \ f \in \mathcal{F},$$

where

$$\tilde{f}(x) := \begin{cases} f(x), & x \in G, \\ 0, & \textit{otherwise}. \end{cases}$$

We use the following lemma given in [4] to prove strong compactness for the moments of f_n.

Lemma 2.4 ([4])**.** *Let* Ω *be an open set of* $\mathbb{R}^{d+1}_{x,t}$*, and* f_n *be a bounded sequence in* $L^2_{loc}(\Omega \times \mathbb{R}^d_v)$ *such that*

$$\Big\{ \int (\partial_t f_n + div_x(v f_n))\sigma(v) dv \Big\}$$

is precompact in $H^{-1}_{loc}(\Omega)$ *for any* $\sigma(v) \in C^\infty(\mathbb{R}^d_v)$*. Then* $\{\int f_n \sigma(v) dv\}$ *is precompact in* $L^2_{loc}(\Omega)$ *for any* $\sigma(v) \in C^\infty(\mathbb{R}^d_v)$*.*

In order to apply the above lemma to our system we introduce the following corollary.

Corollary 2.2. *Let Ω be an open set of $\mathbb{R}^{d+1}_{x,t}$, $D \subset \mathbb{R}^d_v$ be a bounded open set, and f_n be a bounded sequence in $L^2_{loc}(\Omega \times D)$ such that*

$$\Big\{ \int_D (\partial_t f_n + div_x(v f_n))\sigma(v)dv \Big\}$$

is precompact in $H^{-1}_{loc}(\Omega)$ for any $\sigma(v) \in C^\infty(\bar{D})$. Then $\{\int_D f_n \sigma(v) dv\}$ is precompact in $L^2_{loc}(\Omega)$ for any $\sigma(v) \in C^\infty(\bar{D})$.

To prove the corollary, we first extend f_n to $\tilde{f}_n$ by letting $\tilde{f}_n(t,x,v) = f_n(t,x,v)$ when $v \in D$ and $\tilde{f}_n(t,x,v) = 0$ otherwise, then apply Lemma 2.4 to $\tilde{f}_n$.

3. Existence of the weak solutions to the kinetic equation

Proof of Theorem 1.1. We use a few steps to prove the theorem.

Step 1. By time splitting method, we first construct a sequence of approximate functions $f^\epsilon_{\Delta t}$ which will be proved converging to a weak solution of (1.1) in the following steps. Then we derive some estimates for $f^\epsilon_{\Delta t}$ and $M_{f^\epsilon_{\Delta t}}$.

Choose a positive integer N and let $\Delta t := \frac{T}{N}$, we iteratively define $f^\epsilon_{\Delta t}(t,x,v)$ in the time interval $(k\Delta t, (k+1)\Delta t]$, $k = 0, 1, \cdots, N-1$, through

$$t = 0: \qquad f^\epsilon_{\Delta t}(t,x,v) = f_0(x,v), \tag{3.1}$$

$$t \in (k\Delta t, (k+1)\Delta t): \quad f^\epsilon_{\Delta t}(t,x,v) = f^\epsilon_{\Delta t}(k\Delta t, x - (t - k\Delta t)v, v), \tag{3.2}$$

$$\begin{aligned} t = (k+1)\Delta t: \qquad f^\epsilon_{\Delta t}((k+1)\Delta t, x, v) &= e^{-\frac{\Delta t}{\epsilon}} f^\epsilon_{\Delta t}((k+1)\Delta t-, x, v) \\ &\quad + (1 - e^{-\frac{\Delta t}{\epsilon}}) M_{f^\epsilon_{\Delta t}((k+1)\Delta t-, x, v)}. \end{aligned} \tag{3.3}$$

By construction, the above approximate functions satisfies

$$\begin{cases} \partial_t f^\epsilon_{\Delta t} + v\partial_x f^\epsilon_{\Delta t} = (1 - e^{-\frac{\Delta t}{\epsilon}}) \sum\limits_k \big(M_{f^\epsilon_{\Delta t}(t,x,v)} - f^\epsilon_{\Delta t}(t,x,v) \big) \delta(t - k\Delta t), \\ f^\epsilon_{\Delta t}(0,x,v) = f_0(x,v). \end{cases} \tag{3.4}$$

From (1.2), (1.5), (1.7) and the construction of $f^\epsilon_{\Delta t}(t,x,v)$, we know

$$\int_{\mathbb{R}} \int_{-1}^1 f^\epsilon_{\Delta t}(t,x,v) d\mu(v) dx = \int_{\mathbb{R}} \int_{-1}^1 f_0(x,v) d\mu(v) dx \le C_0. \tag{3.5}$$

Direct calculation yields that

$$\begin{aligned} &\int_{\mathbb{R}} \int_{-1}^1 |x| f^\epsilon_{\Delta t}(k\Delta t-, x, v) d\mu(v) dx \\ \le &\int_{\mathbb{R}} \int_{-1}^1 |x| f^\epsilon_{\Delta t}((k-1)\Delta t, x, v) d\mu(v) dx + \Delta t \int_{\mathbb{R}} \int_{-1}^1 f_0(x,v) d\mu(v) dx, \end{aligned}$$

then we can deduce iteratively that

$$\int_{\mathbb{R}} \int_{-1}^1 |x| f^\epsilon_{\Delta t}(t,x,v) d\mu(v) dx$$

$$\leq \int_{\mathbb{R}} \int_{-1}^{1} |x| f_0(x,v) d\mu(v) dx + N\Delta t \int_{\mathbb{R}} \int_{-1}^{1} f_0(x,v) d\mu(v) dx \leq C_0(1+T). \tag{3.6}$$

Condition (1.7) is used to derive the last inequality.

On the other hand, multiply $(\ln f^\epsilon_{\Delta t}(t,x,v)+1)$ to the first line of (3.4), integrate with respect to v and x, we obtain that

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}} \int_{-1}^{1} f^\epsilon_{\Delta t}(t,x,v) \ln f^\epsilon_{\Delta t}(t,x,v) d\mu(v) dx + \sum_k (1 - e^{-\frac{\Delta t}{\epsilon}}) \delta(t - k\Delta t) \\
&\times \int_{\mathbb{R}} \int_{-1}^{1} \Big(f^\epsilon_{\Delta t}(t,x,v) - M_{f^\epsilon_{\Delta t}(t,x,v)} \Big) \Big(\ln f^\epsilon_{\Delta t}(t,x,v) - \ln M_{f^\epsilon_{\Delta t}(t,x,v)} \Big) d\mu(v) dx \\
=&0,
\end{aligned}$$

since $\int (f - M_f) \ln M_f dv = 0$ from the definition of equilibrium state. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{-1}^{1} f^\epsilon_{\Delta t} \ln f^\epsilon_{\Delta t} d\mu(v) dx + (1 - e^{-\frac{\Delta t}{\epsilon}}) \sum_k \int_{\mathbb{R}} \int_{-1}^{1} \big(f^\epsilon_{\Delta t}(k\Delta t, x, v) - M_{f^\epsilon_{\Delta t}(k\Delta t,x,v)} \big) \\
&\times \big(\ln f^\epsilon_{\Delta t}(k\Delta t, x, v) - \ln M_{f^\epsilon_{\Delta t}(k\Delta t,x,v)} \big) d\mu(v) dx \\
=&\int_{\mathbb{R}} \int_{-1}^{1} f_0(x,v) \ln f_0(x,v) d\mu(v) dx.
\end{aligned} \tag{3.7}$$

Since the second term in the above equality is non-negative, we obtain from (1.8) that

$$\int_{\mathbb{R}} \int_{-1}^{1} f^\epsilon_{\Delta t}(t,x,v) \ln f^\epsilon_{\Delta t}(t,x,v) d\mu(v) dx \leq \int_{\mathbb{R}} \int_{-1}^{1} f_0(x,v) \ln f_0(x,v) d\mu(v) dx \leq C_0. \tag{3.8}$$

Moreover, we have from (3.6) and (3.8) that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \int_{-1}^{1} f^\epsilon_{\Delta t}(t,x,v) \big| \ln f^\epsilon_{\Delta t}(t,x,v) \big| d\mu(v) dx dt \\
=&\int_0^T \int_{\mathbb{R}} \int_{-1}^{1} f^\epsilon_{\Delta t}(t,x,v) \ln f^\epsilon_{\Delta t}(t,x,v) d\mu(v) dx dt \\
&- 2 \int_{\{(t,x,v) | f^\epsilon_{\Delta t}(t,x,v) < 1\}} f^\epsilon_{\Delta t}(t,x,v) \ln f^\epsilon_{\Delta t}(t,x,v) d\mu(v) dx dt \\
\leq&\int_0^T \int_{\mathbb{R}} \int_{-1}^{1} f^\epsilon_{\Delta t}(t,x,v) \ln f^\epsilon_{\Delta t}(t,x,v) d\mu(v) dx dt \\
&+ 2 \int_{\{e^{-|x|} \leq f^\epsilon_{\Delta t}(t,x,v) < 1\}} f^\epsilon_{\Delta t}(t,x,v) \big| \ln f^\epsilon_{\Delta t}(t,x,v) \big| d\mu(v) dx dt \\
&+ 2 \int_{\{f^\epsilon_{\Delta t}(t,x,v) < e^{-|x|}\}} f^\epsilon_{\Delta t}(t,x,v) \big| \ln f^\epsilon_{\Delta t}(t,x,v) \big| d\mu(v) dx dt \\
\leq&C_0 T + 2 \int_0^T \int_{\mathbb{R}} \int_{-1}^{1} f^\epsilon_{\Delta t}(t,x,v) |x| d\mu(v) dx dt + C \int_0^T \int_{\mathbb{R}} \int_{-1}^{1} e^{-\frac{|x|}{2}} d\mu(v) dx dt \\
\leq&C(C_0, T).
\end{aligned} \tag{3.9}$$

Combining (3.5), (3.6) and (3.9), we have proved for any $T>0$ that

$$\int_0^T\int_{\mathbb{R}}\int_{-1}^1 \big(1+|x|+|\ln f_{\Delta t}^\epsilon(t,x,v)|\big) f_{\Delta t}^\epsilon(t,x,v)d\mu(v)dxdt \le C(C_0,T). \tag{3.10}$$

Using similar method we can also obtain

$$\int_0^T\int_{\mathbb{R}}\int_{-1}^1 \big(1+|x|+|\ln M_{f_{\Delta t}^\epsilon(t,x,v)}|\big) M_{f_{\Delta t}^\epsilon(t,x,v)}d\mu(v)dxdt \le C(C_0,T). \tag{3.11}$$

Step 2. Now we prove weak compactness of $f_{\Delta t}^\epsilon$ and $M_{f_{\Delta t}^\epsilon}$ in $L^1([0,T)\times\mathbb{R}\times(-1,1))$.

Note that for any $R,M>1$ we have from (3.10) that

$$\begin{aligned}
&\int_0^T\int_{|x|>R}\int_{-1}^1 |f_{\Delta t}^\epsilon(t,x,v)|d\mu(v)dxdt\\
=&\int_0^T\int_{|x|>R}\int_{-1}^1 f_{\Delta t}^\epsilon(t,x,v)d\mu(v)dxdt\\
\le&\frac{1}{R}\int_0^T\int_{|x|>R}\int_{-1}^1 |x|f_{\Delta t}^\epsilon(t,x,v)d\mu(v)dxdt\\
\le&\frac{C(C_0,T)}{R}\to 0,\quad \text{as } R\to+\infty,\\
&\int_{|f_{\Delta t}^\epsilon|>M}|f_{\Delta t}^\epsilon(t,x,v)|d\mu(v)dxdt\\
=&\int_{f_{\Delta t}^\epsilon>M} f_{\Delta t}^\epsilon(t,x,v)d\mu(v)dxdt\\
\le&\frac{1}{|\ln M|}\int_0^T\int_{\mathbb{R}_x}\int_{-1}^1 |\ln f_{\Delta t}^\epsilon| f_{\Delta t}^\epsilon d\mu(v)dxdt\\
\le&\frac{C(C_0,T)}{|\ln M|}\to 0,\quad \text{as } M\to+\infty.
\end{aligned}$$

From Lemma 2.1 and Remark 2.1 (Dunford-Pettis theorem), we know that $f_{\Delta t}^\epsilon$ is weakly compact in $L^1([0,T)\times\mathbb{R}\times(-1,1))$. Similarly, by using (3.11), we can deduce $M_{f_{\Delta t}^\epsilon(t,x,v)}$ is weakly compact in $L^1([0,T)\times\mathbb{R}\times(-1,1))$. Therefore, there exist subsequences of $\{f_{\Delta t}^\epsilon\}$, $\{M_{f_{\Delta t}^\epsilon}\}$, still denoted using the same subscript, and functions f^ϵ, g^ϵ such that

$$f_{\Delta t}^\epsilon \xrightharpoonup{\text{w}} f^\epsilon \quad \text{in } L^1([0,T)\times\mathbb{R}\times(-1,1)), \qquad \text{as } \Delta t\to 0, \tag{3.12}$$

$$M_{f_{\Delta t}^\epsilon} \xrightharpoonup{\text{w}} g^\epsilon \quad \text{in } L^1([0,T)\times\mathbb{R}\times(-1,1)), \qquad \text{as } \Delta t\to 0. \tag{3.13}$$

Step 3. Set $\Omega:=(0,T)\times\mathbb{R}$. Now we prove precompactness of

$$\int_{-1}^1 \sigma(v) f_{\Delta t}^\epsilon(t,x,v)d\mu(v)$$

in $L^2_{loc}(\Omega)$ for any $\sigma(v)\in C^\infty([-1,1])$.

We introduce the cut-off function $\chi_\alpha \in C^\infty(\mathbb{R}^+)$ such that

$$\chi_\alpha(s) := \begin{cases} s, & \text{if } 0 \le s \le \alpha, \\ \in [\alpha, \alpha+2], & \text{if } s \in [\alpha, \alpha+4], \\ \alpha+2, & \text{if } s \ge \alpha+4, \end{cases} \tag{3.14}$$

and $\chi_\alpha'(s) \le 1$.

Due to the construction of $f^\epsilon_{\Delta t}$ in (3.1)-(3.3), function $\chi_\alpha(f^\epsilon_{\Delta t}(t,x,v))$ satisfies the following for $t \in [k\Delta t, (k+1)\Delta t)$:

$$\begin{cases} \partial_t \chi_\alpha(f^\epsilon_{\Delta t}) + v\partial_x \chi_\alpha(f^\epsilon_{\Delta t}) = 0, \quad t \in (k\Delta t, (k+1)\Delta t), \\ \chi_\alpha(f^\epsilon_{\Delta t})|_{t=k\Delta t} = \chi_\alpha(f^\epsilon_{\Delta t}(k\Delta t, x, v)). \end{cases}$$

For $t \in (0,T)$, function $\chi_\alpha(f^\epsilon_{\Delta t})$ satisfies

$$\begin{cases} \partial_t \chi_\alpha(f^\epsilon_{\Delta t}) + v\partial_x \chi_\alpha(f^\epsilon_{\Delta t}) = \sum_k \Big(\chi_\alpha\big(f^\epsilon_{\Delta t}(t)\big) - \chi_\alpha\big(f^\epsilon_{\Delta t}(t-)\big)\Big)\delta(t-k\Delta t), \\ \chi_\alpha(f^\epsilon_{\Delta t})|_{t=0} = \chi_\alpha(f_0). \end{cases}$$

For any $\sigma(v) \in C^\infty([-1,1])$, we have

$$\begin{aligned} &\int_{-1}^1 \sigma(v)\big(\partial_t \chi_\alpha(f^\epsilon_{\Delta t}) + v\partial_x \chi_\alpha(f^\epsilon_{\Delta t})\big) d\mu(v) \\ =& \int_{-1}^1 \sigma(v) \sum_k \Big(\chi_\alpha\big(f^\epsilon_{\Delta t}(t,x,v)\big) - \chi_\alpha\big(f^\epsilon_{\Delta t}(t-,x,v)\big)\Big)\delta(t-k\Delta t) d\mu(v). \end{aligned} \tag{3.15}$$

Now we claim that the right hand side of (3.15) is bounded in $\mathcal{M}_{loc}(\Omega)$, the space of Radon measures.

In fact, for any $\phi(t,x) \in C_c(\Omega)$, we have

$$\begin{aligned} &\sum_k \int_\Omega \int_{-1}^1 \sigma(v) \Big(\chi_\alpha\big(f^\epsilon_{\Delta t}(t)\big) - \chi_\alpha\big(f^\epsilon_{\Delta t}(t-)\big)\Big)\delta(t-k\Delta t) d\mu(v)\phi(t,x) dx dt \\ =& \sum_k \int_{\mathbb{R}} \int_{-1}^1 \sigma(v) \Big(\chi_\alpha\big(f^\epsilon_{\Delta t}(k\Delta t)\big) - \chi_\alpha\big(f^\epsilon_{\Delta t}(k\Delta t-)\big)\Big) d\mu(v)\phi(k\Delta t, x) dx \\ \le& C\|\phi\|_{C_c(\Omega)} \sum_k \int_{\mathbb{R}} \int_{-1}^1 \Big| \chi_\alpha\big(f^\epsilon_{\Delta t}(k\Delta t)\big) - \chi_\alpha\big(f^\epsilon_{\Delta t}(k\Delta t-)\big)\Big| d\mu(v) dx \\ \le& C\|\phi\|_{C_c(\Omega)} \sum_k \int_{\mathbb{R}} \int_{-1}^1 \Big| f^\epsilon_{\Delta t}(k\Delta t) - f^\epsilon_{\Delta t}(k\Delta t-)\Big| d\mu(v) dx \\ \le& C\|\phi\|_{C_c(\Omega)} (1-e^{-\frac{\Delta t}{\epsilon}}) \sum_k \int_{\mathbb{R}} \int_{-1}^1 \Big| M_{f^\epsilon_{\Delta t}(k\Delta t-)} - f^\epsilon_{\Delta t}(k\Delta t-)\Big| d\mu(v) dx \\ \le& C\|\phi\|_{C_c(\Omega)} \sum_k (1-e^{-\frac{\Delta t}{\epsilon}}) \int_{\mathbb{R}} \int_{-1}^1 \Big| \sqrt{M_{f^\epsilon_{\Delta t}(k\Delta t-)}} - \sqrt{f^\epsilon_{\Delta t}(k\Delta t-)}\Big|^2 d\mu(v) dx \\ &+ C\|\phi\|_{C_c(\Omega)} \sum_k (1-e^{-\frac{\Delta t}{\epsilon}}) \int_{\mathbb{R}} \int_{-1}^1 \Big| \sqrt{M_{f^\epsilon_{\Delta t}(k\Delta t-)}} + \sqrt{f^\epsilon_{\Delta t}(k\Delta t-)}\Big|^2 d\mu(v) dx. \end{aligned} \tag{3.16}$$

For the first term in the last line of the above inequality, we obtain from (1.10), (3.7) and (3.9) that

$$
\begin{aligned}
&2(1-e^{-\frac{\Delta t}{\epsilon}})\sum_k \int_{\mathbb{R}}\int_{-1}^{1}\Big(\sqrt{f^\epsilon_{\Delta t}(k\Delta t)}-\sqrt{M_{f^\epsilon_{\Delta t}(k\Delta t)}}\Big)^2 d\mu(v)dxdt\\
\leq&(1-e^{-\frac{\Delta t}{\epsilon}})\sum_k \int_{\mathbb{R}}\int_{-1}^{1}\big(f^\epsilon_{\Delta t}(k\Delta t)-M_{f^\epsilon_{\Delta t}(k\Delta t)}\big)\\
&\times\big(\ln f^\epsilon_{\Delta t}(k\Delta t)-\ln M_{f^\epsilon_{\Delta t}(k\Delta t)}\big)d\mu(v)dx\\
\leq&\int_{\mathbb{R}}\int_{-1}^{1} f^\epsilon_{\Delta t}(t)|\ln f^\epsilon_{\Delta t}(t)|d\mu(v)dx+\int_{\mathbb{R}_x}\int_{-1}^{1} f_0(x,v)|\ln f_0|d\mu(v)dx\leq C(C_0,T).
\end{aligned}
\tag{3.17}
$$

For the second term in the last line of (3.16), it is easy to see from (3.5) that

$$
\begin{aligned}
&\sum_k(1-e^{-\frac{\Delta t}{\epsilon}})\int_{\mathbb{R}}\int_{-1}^{1}\Big|\sqrt{M_{f^\epsilon_{\Delta t}(k\Delta t-)}}+\sqrt{f^\epsilon_{\Delta t}(k\Delta t-)}\Big|^2 d\mu(v)dx\\
\leq&2\sum_k(1-e^{-\frac{\Delta t}{\epsilon}})\int_{\mathbb{R}}\int_{-1}^{1}\Big(M_{f^\epsilon_{\Delta t}(k\Delta t-)}+f^\epsilon_{\Delta t}(k\Delta t-)\Big)d\mu(v)dx\\
\leq&\frac{C(C_0,T)}{\epsilon},
\end{aligned}
\tag{3.18}
$$

since $1-e^{-\frac{\Delta t}{\epsilon}}\leq\frac{\Delta t}{\epsilon}$ and $T=N\Delta t$.

Substitute estimates (3.17) and (3.18) into (3.16), we know that

$$
\begin{aligned}
&\sum_k\int_{\Omega}\int_{-1}^{1}\sigma(v)\Big(\chi_\alpha\big(f^\epsilon_{\Delta t}(t)\big)-\chi_\alpha\big(f^\epsilon_{\Delta t}(t-)\big)\Big)\delta(t-k\Delta t)d\mu(v)\phi(t,x)dxdt\\
\leq&C(f_0,T)(1+\frac{1}{\epsilon})
\end{aligned}
$$

for any $\sigma(v)\in C^\infty([-1,1])$ and $\phi(t,x)\in C_c(\Omega)$. Therefore, the right hand side of (26) is bounded in $\mathcal{M}_{loc}(\Omega)$ (the space of Radon measures), by using the embedding $\mathcal{M}_{loc}(\Omega)\hookrightarrow W^{-1,p}(\Omega)$, $p\in[1,2)$, we could deduce $\int_{-1}^1\sigma(v)[\partial_t\chi_\alpha(f^\epsilon_{\Delta t})+v\partial_x\chi_\alpha(f^\epsilon_{\Delta t})]d\mu(v)$ is compact in $W^{-1,p}(\Omega)$.

On the other hand, since $\chi_\alpha(f^\epsilon_{\Delta t}(t,x,v))\in L^\infty([0,T)\times(-1,1)\times\mathbb{R}_x)$, so

$$
\int_{-1}^{1}\sigma(v)[\partial_t\chi_\alpha(f^\epsilon_{\Delta t})+v\partial_x\chi_\alpha(f^\epsilon_{\Delta t})]d\mu(v)
$$

is bounded in $W^{-1,\infty}_{loc}(\Omega)$.

By using the embedding theorem Theorem 2.3.2 in [18], we could deduce $\int_{-1}^1\sigma(v)\big(\partial_t\chi_\alpha(f^\epsilon_{\Delta t})+v\partial_x\chi_\alpha(f^\epsilon_{\Delta t})\big)d\mu(v)$ is precompact in $H^{-1}_{loc}(\Omega)$. Finally, by Corollary 2.2, we obtain $\int_{-1}^1\sigma(v)\chi_\alpha(f^\epsilon_{\Delta t})d\mu(v)$ is precompact in $L^2_{loc}(\Omega)$.

Step 4. Now, we prove the precompactness of $\int_{-1}^1\sigma(v)f^\epsilon_{\Delta t}d\mu(v)$ in $L^1_{loc}(\Omega)$. It is sufficient to prove, for any compact set $K\subset\Omega$, that

$$
\lim_{(\tau,h)\to(0,0)}\int_K\Big|\int_{-1}^{1}\sigma(v)\big(f^\epsilon_{\Delta t}(t+\tau,x+h,v)-f^\epsilon_{\Delta t}(t,x,v)\big)d\mu(v)\Big|dxdt=0
$$

uniformly.

We omit the superscript ϵ, and denote $f_{\Delta t}^{\alpha} := \chi_\alpha(f_{\Delta t}^{\epsilon})$, $f_{\Delta t}^{1-\alpha} := f_{\Delta t}^{\epsilon} - \chi_\alpha(f_{\Delta t}^{\epsilon})$. Then for any $K \subset \Omega$,

$$
\begin{aligned}
&\int_K \Big| \int_{-1}^{1} \sigma(v)\big(f_{\Delta t}^{\epsilon}(t+\tau, x+h, v) - f_{\Delta t}^{\epsilon}(t,x,v)\big) d\mu(v) \Big| dxdt \\
\leq &\int_K \Big| \int_{-1}^{1} \sigma(v)\big(f_{\Delta t}^{\alpha}(t+\tau, x+h, v) - f_{\Delta t}^{\alpha}(t,x,v)\big) d\mu(v) \Big| dxdt \\
&+ \int_K \Big| \int_{-1}^{1} \sigma(v)\big(f_{\Delta t}^{1-\alpha}(t+\tau, x+h, v) - f_{\Delta t}^{1-\alpha}(t,x,v)\big) d\mu(v) \Big| dxdt \\
= &I_1 + I_2.
\end{aligned}
$$

The estimate of I_2 comes from the equintegrability (3.10). For any $\kappa > 0$, there exists a suitably large constant α, such that

$$
\begin{aligned}
I_2 = &\int_K \Big| \int_{-1}^{1} \sigma(v)\big(f_{\Delta t}^{1-\alpha}(t+\tau, x+h, v) - f_{\Delta t}^{1-\alpha}(t,x,v)\big) d\mu(v) \Big| dxdt \\
\leq &C \int_{f_{\Delta t} \geq \alpha} |f_{\Delta t}| d\mu(v) dxdt \\
< &\frac{\kappa}{2}.
\end{aligned}
$$

For the above fixed α, from precompactness of $\int_{-1}^{1} \sigma(v)\chi_\alpha(f_{\Delta t}^{\epsilon}) d\mu(v)$ in $L_{loc}^2(\Omega)$, there exists $\delta > 0$, such that for any $h^2 + \tau^2 < \delta^2$, we have

$$
\begin{aligned}
I_1 = &\int_\Omega \Big| \int_{-1}^{1} \sigma(v)\Big(f_{\Delta t}^{\alpha}|_K(t+\tau, x+h, v) - f_{\Delta t}^{\alpha}|_K(t,x,v)\Big) d\mu(v) \Big| dxdt \\
\leq &C(K)\Big(\int_\Omega \Big| \int_{-1}^{1} \sigma(v)\Big(f_{\Delta t}^{\alpha}|_K(t+\tau, x+h, v) - f_{\Delta t}^{\alpha}|_K(t,x,v)\Big) d\mu(v) \Big|^2 dxdt\Big)^{\frac{1}{2}} \\
< &\frac{\kappa}{2}.
\end{aligned}
$$

Combining the above argument, we know that for any $\kappa > 0$, there exists $\delta > 0$ such that for any $h^2 + \tau^2 < \delta^2$

$$
\int_K \Big| \int_{-1}^{1} \sigma(v)\big(f_{\Delta t}^{\epsilon}(t+\tau, x+h, v) - f_{\Delta t}^{\epsilon}(t,x,v)\big) d\mu(v) \Big| dxdt < \kappa.
$$

Then from Corollary 2.1, the Kolmogorov-Riesz theorem, we obtain the precompactness of

$$
\Big\{ \int_{-1}^{1} \sigma(v) f_{\Delta t}^{\epsilon}(t,x,v) d\mu(v) \Big\}
$$

in $L_{loc}^1(\Omega)$. There exists a subsequence, still denoted by $\{\int_{-1}^{1} \sigma(v) f_{\Delta t}^{\epsilon}(t,x,v) d\mu(v)\}$, satisfying

$$
\lim_{\Delta t \to 0} \Big\| \int_{-1}^{1} \sigma(v) f_{\Delta t}^{\epsilon} d\mu(v) - \int_{-1}^{1} \sigma(v) f^{\epsilon} d\mu(v) \Big\|_{L_{loc}^1(\Omega)} = 0,
$$

where f^ϵ is the weak limit of $\{f_{\Delta t}^{\epsilon}\}$ obtained in (3.12).

By tightness and uniform integrability of $f^\epsilon_{\Delta t}(t,x,v)$, we know from Lemma 2.1 that

$$\lim_{\Delta t\to 0}\Big\|\int_{-1}^1 \sigma(v) f^\epsilon_{\Delta t} d\mu(v) - \int_{-1}^1 \sigma(v) f^\epsilon d\mu(v)\Big\|_{L^1(\Omega)} = 0.$$

Hence there exists a further subsequence, still denoted by $\int_{-1}^1 \sigma(v) f^\epsilon_{\Delta t}(t,x,v)d\mu(v)$, that converges almost everywhere to $\int_{-1}^1 \sigma(v) f^\epsilon(t,x,v)d\mu(v)$.

Step 5. Now we prove that the macroscopic quantities of $f^\epsilon_{\Delta t}$ converge to those of f^ϵ, and $M_{f^\epsilon_{\Delta t}} \to M_{f^\epsilon}$.

Let $\rho^\epsilon_{\Delta t}(t,x) := \int_{-1}^1 f^\epsilon_{\Delta t} d\mu(v)$, and $J^\epsilon_{\Delta t}(t,x) := \int_{-1}^1 v f^\epsilon_{\Delta t} d\mu(v)$, then we have from the results of Step 4 that

$$\rho^\epsilon_{\Delta t}(t,x) \to \rho^\epsilon(t,x), \qquad J^\epsilon_{\Delta t}(t,x) \to J^\epsilon(t,x), \quad \text{a.e. in } \Omega,$$

where $\rho^\epsilon, J^\epsilon$ are the macroscopic quantities of f^ϵ.

Set $E := \{(t,x) | \rho^\epsilon(t,x) > 0\}$. For any fixed $(t,x) \in E$, let

$$u^\epsilon_{\Delta t}(t,x) := \frac{J^\epsilon_{\Delta t}(t,x)}{\rho^\epsilon_{\Delta t}(t,x)} \to \frac{J^\epsilon(t,x)}{\rho^\epsilon(t,x)} = u^\epsilon(t,x), \quad \text{a.e. in } \Omega\cap E,$$

and obviously $u^\epsilon_{\Delta t}$, $u^\epsilon \in (-1,1)$ by mean value theorem. Then we have the equilibrium state of $f^\epsilon_{\Delta t}$

$$M_{f^\epsilon_{\Delta t}(t,x,v)} \to M_{f^\epsilon(t,x,v)} \quad \text{as } \Delta t \to 0, \quad \text{a.e. in } (\Omega\cap E)\times(-1,1),$$

where M_{f^ϵ} are the equilibrium state of f^ϵ.

For any $(t,x) \in E^c$, $\rho^\epsilon = 0$ and for any compact set $K \subset \Omega$,

$$\int_K \int_{-1}^1 |M_{f^\epsilon_{\Delta t}(t,x,v)} - 0| d\mu(v) dx dt \to 0 \quad \text{as } \Delta t \to 0,$$

so we have

$$M_{f^\epsilon_{\Delta t}(t,x,v)} \to 0, \quad \text{a.e. in } (\Omega\cap E^c)\times(-1,1), \quad \text{as } \Delta t \to 0.$$

Finally, we arrive at

$$M_{f^\epsilon_{\Delta t}(t,x,v)} \to M_{f^\epsilon(t,x,v)} \quad \text{a.e. in } \Omega\times(-1,1), \quad \text{as } \Delta t \to 0.$$

Step 6. Now we prove that the approximate solution of (3.4) tends to weak solution of (1.1).

In fact, from (3.4), for any smooth function $\phi(t,x,v)$ with compact support in $[0,T)\times\mathbb{R}\times(-1,1)$, we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \int_{-1}^1 f^\epsilon_{\Delta t}(\phi_t + v\phi_x) d\mu(v) dx dt + \int_{\mathbb{R}} \int_{-1}^1 f_0(x,v)\phi(0,x,v) d\mu(v) dx \\
&+ \frac{1-e^{-\frac{\Delta t}{\epsilon}}}{\Delta t} \sum_k \Delta t \int_{\mathbb{R}} \int_{-1}^1 \big(M_{f^\epsilon_{\Delta t}}(k\Delta t) - f^\epsilon_{\Delta t}(k\Delta t)\big)\phi(k\Delta t) d\mu(v) dx \\
=&0.
\end{aligned} \tag{3.19}$$

By Lemma 2.3 and the fact $0 \le t - [\frac{t}{\Delta t}]\Delta t \le \Delta t$, we have that

$$
\begin{aligned}
&\Big|\sum_k \Delta t \int_{\mathbb{R}}\int_{-1}^{1} \big(M_{f^\epsilon_{\Delta t}} - f^\epsilon_{\Delta t}\big)\phi d\mu(v)dx(k\Delta t) \\
&\quad - \int_0^T \int_{\mathbb{R}}\int_{-1}^{1} \big(M_{f^\epsilon} - f^\epsilon\big)\phi d\mu(v)dxdt\Big| \\
=&\Big|\int_0^T \int_{\mathbb{R}}\int_{-1}^{1} \big(M_{f^\epsilon_{\Delta t}} - f^\epsilon_{\Delta t}\big)([\frac{t}{\Delta t}]\Delta t, x, v)\phi([\frac{t}{\Delta t}]\Delta t, x, v) - \big(M_{f^\epsilon_{\Delta t}} - f^\epsilon_{\Delta t}\big)(t,x,v) \\
&\quad \times \phi(t,x,v)d\mu(v)dxdt\Big| + \Big|\int_0^T \int_{\mathbb{R}}\int_{-1}^{1} \big(M_{f^\epsilon_{\Delta t}} - f^\epsilon_{\Delta t} - M_{f^\epsilon} + f^\epsilon\big)\phi d\mu(v)dxdt\Big| \\
\to& 0.
\end{aligned}
$$

Therefore, we have by passing Δt to zero in (3.19) that

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}}\int_{-1}^{1} f^\epsilon\big(\phi_t + v\phi_x\big)d\mu(v)dxdt + \int_{\mathbb{R}}\int_{-1}^{1} f_0(x,v)\phi(0,x,v)d\mu(v)dx \\
&+ \frac{1}{\epsilon}\int_0^T \int_{\mathbb{R}}\int_{-1}^{1} \big(M_{f^\epsilon} - f^\epsilon\big)\phi d\mu(v)dxdt \\
=&0.
\end{aligned}
$$

The limiting function f^ϵ is a weak solution of (1.1). The proof of Theorem 1.1 is complete. □

4. Approximate equation

We expect that f^ϵ tends to M_{f^ϵ} as ϵ tends to 0, and the conservation laws

$$
\int_{-1}^{1} \begin{pmatrix} 1 \\ v \end{pmatrix} \big(\partial_t f^\epsilon(t,x,v) + v\partial_x f^\epsilon(t,x,v)\big)d\mu(v) = 0 \tag{4.1}
$$

decay to the closed system of the compressible Euler equations (1.6) of gas-dynamics. However, as stated in the introduction, it is difficult to obtain strong compactness.

For any fixed constant $\zeta > 0$, define

$$
\begin{aligned}
D(t,x) :=\Big\{(t,x)\Big| &\int_{-1}^{1} \Big(\sqrt{M_{f^\epsilon(t,x,v)}} - \sqrt{f^\epsilon(t,x,v)}\Big)^2 d\mu(v) \\
&\ge \zeta \int_{-1}^{1} \Big(\sqrt{M_{f^\epsilon(t,x,v)}} + \sqrt{f^\epsilon(t,x,v)}\Big)^2 d\mu(v)\Big\},
\end{aligned}
$$

and consider the following modified equation:

$$
\begin{cases}
f^\epsilon_t + v\partial_x f^\epsilon = \dfrac{1}{\epsilon}\big(M_{f^\epsilon} - f^\epsilon\big)\mathbf{1}_{D(t,x)}, \\
f^\epsilon(0,x,v) = f_0(x,v),
\end{cases} \tag{4.2}
$$

where $\mathbf{1}_D$ represents the characteristic function of D. We still use f^ϵ to denote the solution of (4.2).

Proof of Theorem 1.2. We use three steps to prove the theorem.

Step 1. Weak compactness of $f^\epsilon(t,x,v)$ and $M_{f^\epsilon(t,x,v)}$ in $L^1([0,T) \times \mathbb{R} \times (-1,1))$.

Multiply $(\ln f^\epsilon + 1)$ to (4.2), integrate with respect to v and x, then we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\int_{-1}^{1} f^\epsilon(t,x,v)\ln f^\epsilon(t,x,v)d\mu(v)dx\\
&+\frac{1}{\epsilon}\int_{(t,x)\in D(t,x)}\int_{-1}^{1}\big(\ln f^\epsilon(t,x,v)-\ln M_{f^\epsilon(t,x,v)}\big)\\
&\times\big(f^\epsilon(t,x,v)-M_{f^\epsilon(t,x,v)}\big)d\mu(v)dxdt\\
=&0.
\end{aligned}$$

Using a similar method as in the proof of Theorem 1.1, we obtain estimates

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\int_{-1}^{1}(1+|x|+|\ln f^\epsilon(t,x,v)|)f^\epsilon(t,x,v)d\mu(v)dxdt \le C(f_0(x,v),T),\\
&\int_0^T\int_{\mathbb{R}}\int_{-1}^{1}(1+|x|+|\ln M_{f^\epsilon(t,x,v)}|)M_{f^\epsilon(t,x,v)}d\mu(v)dxdt \le C(f_0(x,v),T),
\end{aligned}$$

and the entropy inequality

$$\begin{aligned}
&\frac{1}{\epsilon}\int_{(t,x)\in D(t,x)}\int_{-1}^{1}(\ln f^\epsilon(t,x,v)-\ln M_{f^\epsilon(t,x,v)})\\
&\times(f^\epsilon(t,x,v)-M_{f^\epsilon(t,x,v)})d\mu(v)dxdt\\
\le& C(f_0(x,v),T).
\end{aligned}$$

From the entropy inequality we have

$$\frac{1}{\epsilon}\int_{(t,x)\in D(t,x)}\int_{-1}^{1}\Big(\sqrt{f^\epsilon(t,x,v)}-\sqrt{M_{f^\epsilon(t,x,v)}}\Big)^2 d\mu(v)dxdt \le C(f_0(x,v),T).$$

From Dunford-Pettis theorem, we obtain $f^\epsilon(t,x,v)$ and $M_{f^\epsilon(t,x,v)}$ are weakly compact in $L^1([0,T)\times\mathbb{R}\times(-1,1))$. Therefore, there exist subsequences of $\{f^\epsilon\}$, $\{M_{f^\epsilon}\}$, still denoted using the same subscript, and functions f, g such that

$$f^\epsilon \xrightarrow{\text{w}} f \quad \text{in } L^1([0,T)\times\mathbb{R}\times(-1,1)), \qquad \text{as } \epsilon\to 0, \tag{4.3}$$

$$M_{f^\epsilon} \xrightarrow{\text{w}} g \quad \text{in } L^1([0,T)\times\mathbb{R}\times(-1,1)), \qquad \text{as } \epsilon\to 0. \tag{4.4}$$

Step 2. Precompactness of $\int_{-1}^1 \sigma_i(v)f^\epsilon(t,x,v)d\mu(v)$ in $L^2_{loc}(\Omega)$ for $i=0,1$.

Using the cutoff function χ_α defined in (3.14), we obtain

$$\begin{cases}
\partial_t\chi_\alpha(f^\epsilon)+v\partial_x\chi_\alpha(f^\epsilon)=\dfrac{1}{\epsilon}\chi_\alpha'(f^\epsilon)\big(M_{f^\epsilon}-f^\epsilon\big)\mathbf{1}_{D(t,x)},\\
\chi_\alpha(f^\epsilon(0,x,v))=\chi_\alpha(f_0(x,v)).
\end{cases} \tag{4.5}$$

Due to the definition of $D(t,x)$, the term on right hand side of (4.5) satisfies

$$\begin{aligned}
&\frac{1}{\epsilon}\int_{(t,x)\in D(t,x)}\int_{-1}^{1}\Big|\chi_\alpha'(f^\epsilon)(M_{f^\epsilon}-f^\epsilon)\Big|d\mu(v)dxdt\\
\le&\frac{1}{\epsilon}\int_{(t,x)\in D(t,x)}\int_{-1}^{1}\big(\sqrt{M_{f^\epsilon}}-\sqrt{f^\epsilon}\big)\big(\sqrt{M_{f^\epsilon}}+\sqrt{f^\epsilon}\big)d\mu(v)dxdt
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\zeta}\frac{1}{\epsilon}\int_{(t,x)\in D(t,x)}\int_{-1}^{1}\left(\sqrt{f^\epsilon}-\sqrt{M_{f^\epsilon}}\right)^2 d\mu(v)dxdt\\
&\leq \frac{1}{\zeta}C(f_0(x,v),T).
\end{aligned}
$$

So it is bounded in $L^1_{loc}(\Omega)$, hence precompact in $W^{-1.p}(\Omega)$ for any $1 \leq p < 2$. On the other hand, the left hand side of (4.5) is bounded in $W^{-1,\infty}_{loc}(\Omega)$ because $\chi_\alpha(f^\epsilon(t,x,v)) \in L^\infty(\Omega\times(-1,1))$. Therefore,

$$
\int_{-1}^{1}\sigma(v)\big(\partial_t\chi_\alpha(f^\epsilon)+v\partial_x\chi_\alpha(f^\epsilon)\big)d\mu(v)
$$

is precompact in $H^{-1}_{loc}(\Omega)$.

By similar argument as in the proof of Theorem 1.1, it yields that

$$
\begin{aligned}
\rho^\epsilon &= \int_{-1}^{1} f^\epsilon d\mu(v) \to \int_{-1}^{1} f d\mu(v) = \rho \quad \text{in } L^1_{loc}(\Omega), && \text{as } \epsilon\to 0,\\
J^\epsilon &= \int_{-1}^{1} v f^\epsilon d\mu(v) \to \int_{-1}^{1} v f d\mu(v) = J \quad \text{in } L^1_{loc}(\Omega), && \text{as } \epsilon\to 0.
\end{aligned}
$$

It is obvious that

$$
\begin{aligned}
\rho^\epsilon\psi(u^\epsilon)-\rho\psi(u) =&(\rho^\epsilon-\rho)\psi(u^\epsilon)+\rho\psi'(\eta)(u^\epsilon-u)\\
=&(\rho^\epsilon-\rho)\psi(u^\epsilon)+(J^\epsilon-J)\psi'(\eta)+(\rho^\epsilon-\rho)u^\epsilon\psi'(\eta),
\end{aligned}
$$

where $\psi(u^\epsilon)$, $\psi'(\eta)$ and u^ϵ are uniformly bounded, and η lies between u and u^ϵ. Therefore

$$
\int_{-1}^{1} v^2 M_{f^\epsilon(t,x,v)} d\mu(v) = \rho^\epsilon\psi(u^\epsilon) \to \rho\psi(u) = \int_{-1}^{1} v^2 M_{f(t,x,v)} d\mu(v), \quad \text{in } L^1_{loc}(\Omega).
$$

Step 3. To prove that $\| \int_{-1}^{1} v^2\big(M_{f(t,x,v)}-f(t,x,v)\big)d\mu(v)\|_{L^1_{loc}(\Omega)} \leq C\zeta$.

In fact,

$$
\begin{aligned}
&\Big\|\int_{-1}^{1} v^2\Big(M_f-f\Big)d\mu(v)\Big\|_{L^1_{loc}(\Omega)}\\
\leq&\Big\|\int_{-1}^{1} v^2\Big(M_{f(t,x,v)}-M_{f^\epsilon(t,x,v)}\Big)d\mu(v)\Big\|_{L^1_{loc}(\Omega)}\\
&+\Big\|\int_{-1}^{1} v^2\Big(M_{f^\epsilon(t,x,v)}-f^\epsilon(t,x,v)\Big)d\mu(v)\Big\|_{L^1_{loc}(\Omega)}\\
&+\Big\|\int_{-1}^{1} v^2\Big(f^\epsilon(t,x,v)-f(t,x,v)\Big)d\mu(v)\Big\|_{L^1_{loc}(\Omega)}.
\end{aligned}
$$

From the precompactness of

$$
\int_{-1}^{1}\sigma(v)f^\epsilon(t,x,v)d\mu(v)
$$

in $L^1_{loc}(\Omega)$, we have

$$\Big\| \int_{-1}^{1} v^2 \Big(M_{f(t,x,v)} - M_{f^\epsilon(t,x,v)} \Big) d\mu(v) \Big\|_{L^1_{loc}(\Omega)} \to 0, \quad \text{as } \epsilon \to 0,$$

and

$$\Big\| \int_{-1}^{1} v^2 \Big(f^\epsilon(t,x,v) - f(t,x,v) \Big) d\mu(v) \Big\|_{L^1_{loc}(\Omega)} \to 0, \quad \text{as } \epsilon \to 0.$$

For any compact set $K \subset \Omega$,

$$\begin{aligned}
&\int_K \Big| \int_{-1}^{1} v^2 \Big(M_{f^\epsilon(t,x,v)} - f^\epsilon(t,x,v) \Big) d\mu(v) \Big| dxdt \\
\leq &\int_{K \cap D(t,x)} \Big| \int_{-1}^{1} v^2 \Big(M_{f^\epsilon(t,x,v)} - f^\epsilon(t,x,v) \Big) d\mu(v) \Big| dxdt \\
&+ \int_{K \cap D^c(t,x)} \Big| \int_{-1}^{1} v^2 \Big(M_{f^\epsilon(t,x,v)} - f^\epsilon(t,x,v) \Big) d\mu(v) \Big| dxdt \\
= &\frac{1}{\zeta} \int_{K \cap D(t,x)} \Big| \int_{-1}^{1} \Big(\sqrt{M_{f^\epsilon(t,x,v)}} - \sqrt{f^\epsilon(t,x,v)} \Big)^2 d\mu(v) \Big| dxdt \\
&+ \zeta \int_{K \cap D^c(t,x)} \Big| \int_{-1}^{1} \Big(\sqrt{M_{f^\epsilon(t,x,v)}} + \sqrt{f^\epsilon(t,x,v)} \Big)^2 d\mu(v) \Big| dxdt \\
\leq &C\zeta \qquad \text{as } \epsilon \to 0.
\end{aligned}$$

This completes the proof of Theorem 1.2. □

Remark 4.1. In the last step of the above proof we can also obtain that

$$\Big\| \int_{-1}^{1} v^2 \big(M_{f(t,x,v)} - f(t,x,v) \big) d\mu(v) \Big\|_{L^1(\Omega)} \leq C\zeta,$$

but what we consider is weak solutions of conservation laws, so it is enough to prove

$$\| \int_{-1}^{1} v^2 \big(M_{f(t,x,v)} - f(t,x,v) \big) d\mu(v) \|_{L^1_{loc}(\Omega)} \leq C\zeta.$$

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