Journal of Applied Analysis and Computation
Volume 14, Number 1, February 2024, 182–196
Website:http://www.jaac-online.com
DOI:10.11948/20230131

A MODIFIED BLOCK PRECONDITIONER FOR COMPLEX SYMMETRIC INDEFINITE LINEAR SYSTEMS*

Wenbin Bao[1,†] and Shuxin Miao[2]

Abstract To solve the real equivalent 2×2 block linear system of complex symmetric indefinite linear systems, by introducing a preconditioning matrix in the NB preconditioner (which was proposed in [Numerical Algorithm, 74 (2017) 889-903]), a modified block preconditioner is proposed. Compared with the NB one, when choose a suitable preconditioning matrix for the new preconditioner to get faster convergence than the NB preconditioner. The unconditional convergence of the new iteration method is discussed. The eigenvalue distribution and an upper bound of the degree of the minimal polynomial of the preconditioned matrix are given. Finally, a numerical example is carried out to demonstrate the effectiveness and robustness of the proposed preconditioner.

Keywords Block two-by-two linear system, preconditioner, preconditioned matrix, eigenvalue distribution.

MSC(2010) 65F10, 65F15.

1. Introduction

We consider the following large sparse nonsingular complex symmetric linear systems

$$(W + \mathbf{i}T)(x - \mathbf{i}y) = g + \mathbf{i}f, \tag{1.1}$$

where $W \in \mathbb{R}^{n\times n}$ is symmetric indefinite and $T \in \mathbb{R}^{n\times n}$ is symmetric positive definite, the vectors $x,\ y,\ f,\ g \in R^n$ and $\mathbf{i} = \sqrt{-1}$. Such complex linear system (1.1) has varied applications in sciences and engineering, such as diffuse optical tomography [1], the frequency analysis of linear mechanical systems [19], electrical power system modeling [21] and so on; see [2–4, 11, 12, 23, 25] and references therein for other applications.

In order to avoid the complex arithmetic, one approach is to deal with one of the several real equivalent formulations of complex linear system (1.1), see for example [2–4, 9–11, 18, 20, 24–26, 30, 32]. We can transform the complex linear system

†The corresponding author.
1 College of Mathematics and Statistics, Yili Normal University, Yining 835000, China
2 College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China
*The authors were supported by Yili Normal University scientific research project (No. 2023YSYB026).
Email: baowb224@163.com(W. Bao), miaosx205@163.com(S. Miao)

(1.1) into the following real 2×2 block system [2,17,25]:

$$\mathcal{A}u = \begin{bmatrix} T & -W \\ W & T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} = b. \tag{1.2}$$

Since the structure is large and sparse of coefficient matrix $\mathcal{A}$ of the 2×2 block linear system (1.2), the direct methods may destroy the sparsity of the coefficient matrix $\mathcal{A}$, increase the storage capacity of the computer, and reduce the computational efficiency, which is not conducive to the solution of the practical problems. In order to solve the system (1.2) effectively and fast, scholars made fully utilize the sparsity of coefficient matrix $\mathcal{A}$ and proposed Krylov subspace iterative method (such as GMRES [28]), it is very competitive than other methods. The reason is that their computational processes involve only the product between the vector and the coefficient matrix.

As we known, an efficient preconditioner can improve the computational efficiency of the preconditioned Krylov subspace iteration methods. Therefore, in recent years, many research works have been developed to various efficient preconditioners. Based on the HSS iteration method [7], the HSS iteration scheme that may be suitable for solving the real equivalent 2×2 block linear system (1.2), as follows

$$\begin{cases} \begin{bmatrix} \alpha I + T & 0 \\ 0 & \alpha I + T \end{bmatrix} \begin{bmatrix} x^{k+\frac{1}{2}} \\ y^{k+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \alpha I & W \\ -W & \alpha I \end{bmatrix} \begin{bmatrix} x^{k} \\ y^{k} \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix}, \\ \begin{bmatrix} \alpha I & -W \\ W & \alpha I \end{bmatrix} \begin{bmatrix} x^{k+1} \\ y^{k+1} \end{bmatrix} = \begin{bmatrix} \alpha I - T & 0 \\ 0 & \alpha I - T \end{bmatrix} \begin{bmatrix} x^{k+\frac{1}{2}} \\ y^{k+\frac{1}{2}} \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix}, \end{cases} \tag{1.3}$$

where $k = 0, 1, 2, \cdots$, one can deduce that the HSS preconditioner for (1.2) is

$$\mathcal{P}_{HSS} = \frac{1}{2\alpha} \begin{bmatrix} \alpha I + T & 0 \\ 0 & \alpha I + T \end{bmatrix} \begin{bmatrix} \alpha I & -W \\ W & \alpha I \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \alpha I + T & -\frac{1}{\alpha}\left(\alpha I + T\right) W \\ \frac{1}{\alpha}\left(\alpha I + T\right) W & \alpha I + T \end{bmatrix}, \tag{1.4}$$

where $\alpha > 0$ is a real parameter. The difference between $\mathcal{P}_{HSS}$ and $\mathcal{A}$ is given by

$$\mathcal{Q}_{HSS} = \mathcal{P}_{HSS} - \mathcal{A} = \frac{1}{2} \begin{bmatrix} \alpha I - T & \frac{1}{\alpha}(\alpha I - T)W \\ -\frac{1}{\alpha}(\alpha I - T)W & \alpha I - T \end{bmatrix}, \tag{1.5}$$

theoretical analysis shows that all eigenvalues of the HSS preconditioned matrix $\mathcal{P}_{HSS}^{-1}\mathcal{A}$ are located in a circle centered at (1,0) with radius strictly less than one. Recently, Shen and Shi [29] by simply switching positions of some sub-matrices of the HSS preconditioner $\mathcal{P}_{HSS}$ (1.4), proposed a new variant of the HSS (VHSS) preconditioner as follows

$$\mathcal{P}_{VHSS} = \frac{1}{2\alpha} \begin{bmatrix} \alpha I + T & 0 \\ 0 & 2\alpha I \end{bmatrix} \begin{bmatrix} \alpha I & -W \\ W & T \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \alpha I + T & -\frac{1}{\alpha}\left(\alpha I + T\right) W \\ 2W & 2T \end{bmatrix}. \tag{1.6}$$

From (1.6) and (1.2), we get the difference matrices

$$\mathcal{Q}_{VHSS} = \mathcal{P}_{VHSS} - \mathcal{A} = \frac{1}{2}\begin{bmatrix} \alpha I - T & \frac{1}{\alpha}(\alpha I - T)W \\ 0 & 0 \end{bmatrix}. \tag{1.7}$$

The preconditioner $\mathcal{P}_{VHSS}$ improves the performance of the HSS preconditioner $\mathcal{P}_{HSS}$. In order to obtain a better approximation to the coefficient matrix $\mathcal{A}$, based on the relaxed techniques, Zhang and Dai [31] presented a new block (NB) preconditioner

$$\mathcal{P}_{NB} = \frac{1}{\alpha}\begin{bmatrix} \alpha I & -W \\ W & T \end{bmatrix}\begin{bmatrix} \alpha I + T & 0 \\ 0 & \alpha I \end{bmatrix} = \begin{bmatrix} \alpha I + T & -W \\ W\left(I + \frac{1}{\alpha}T\right) & T \end{bmatrix} \tag{1.8}$$

for the 2×2 block linear system (1.2), theoretical analysis shows that all eigenvalues of the new block preconditioned matrix $\mathcal{P}_{NB}^{-1}\mathcal{A}$ are located in the interval (0,1]. From (1.8), we have

$$\mathcal{Q}_{NB} = \mathcal{P}_{NB} - \mathcal{A} = \begin{bmatrix} \alpha I & 0 \\ \frac{1}{\alpha}WT & 0 \end{bmatrix}. \tag{1.9}$$

It is easy to observe that the VHSS preconditioner $\mathcal{P}_{VHSS}$ and NB preconditioner $\mathcal{P}_{NB}$ may be better approximation to the coefficient matrix $\mathcal{A}$ than the HSS preconditioner $\mathcal{P}_{HSS}$.

To further generalize the NB preconditioner and accelerate its preconditioning efficiency, in this paper, we propose a new modified NB preconditioner for the 2×2 block linear system (1.2). The proposed preconditioner is obtained by introducing a preconditioned matrix in the NB preconditioner $\mathcal{P}_{NB}$. We study the spectral properties, the eigenvalues, the eigenvector distribution and the degree of the minimal polynomial of the corresponding new preconditioned matrix. The remainder of the present paper is organized as follows. In Section 2, we propose the new preconditioner and analyze the spectral radius of the iteration matrix. In Section 3, we discuss some properties of the new preconditioned matrix and the implementation details. A numerical example is given to show the effectiveness and robustness of the proposed new preconditioner in Section 4. Finally, we end this paper with some conclusions in Section 5.

2. The proposed new preconditioner

In this section, we present a new modified NB preconditioner (MNB) for the 2×2 block linear system (1.2). It is known that the preconditioning strategy can improve the computational efficiency of a preconditioner [16], so the new preconditioner is defined as follows

$$\mathcal{P}_{MNB} = \frac{1}{\alpha}\begin{bmatrix} \alpha I & -W \\ WP^{-1} & T \end{bmatrix}\begin{bmatrix} \alpha P + T & 0 \\ 0 & \alpha I \end{bmatrix} = \begin{bmatrix} \alpha P + T & -W \\ W(I + \frac{1}{\alpha}P^{-1}T) & T \end{bmatrix}, \tag{2.1}$$

where $P \in \mathbb{R}^{n\times n}$ is symmetric positive definite preconditioning matrix. If $P = I$, then the MNB precontioner $\mathcal{P}_{MNB}$ becomes the NB preconditioner $\mathcal{P}_{NB}$ (1.8). In actual computation, the matrix P can be specially chosen so that the computation cost is as little as possible. The difference between $\mathcal{P}_{MNB}$ and $\mathcal{A}$ is given by

$$\mathcal{Q}_{MNB} = \mathcal{P}_{MNB} - \mathcal{A} = \begin{bmatrix} \alpha P & 0 \\ \frac{1}{\alpha} W P^{-1} T & 0 \end{bmatrix}. \tag{2.2}$$

Compared $\mathcal{Q}_{MNB}$ with $\mathcal{Q}_{NB}$, when choose a suitable symmetric positive definite matrix P for the $\mathcal{P}_{MNB}$ preconditioner, we can see that the MNB preconditioner $\mathcal{P}_{MNB}$ (2.1) is a better approximation to the 2×2 block real matrix $\mathcal{A}$ than the NB preconditioner $\mathcal{P}_{NB}$ (1.8).

Actually, the MNB preconditioner $\mathcal{P}_{MNB}$ is induced by the following matrix splitting:

$$\mathcal{A} = \mathcal{P}_{MNB} - \mathcal{Q}_{MNB} = \begin{bmatrix} \alpha P + T & -W \\ W(I + \frac{1}{\alpha} P^{-1} T) & T \end{bmatrix} - \begin{bmatrix} \alpha P & 0 \\ \frac{1}{\alpha} W P^{-1} T & 0 \end{bmatrix}, \tag{2.3}$$

which produces the following MNB stationary iteration method.

Method 2.1. (The MNB Iteration Method). Let α be a given positive constant and $\begin{bmatrix} x^{(0)} \\ y^{(0)} \end{bmatrix}$ be an initial guess vector. For $k = 0, 1, 2, \cdots$, until the iteration sequence $\begin{bmatrix} x^{(k)} \\ y^{(k)} \end{bmatrix}$ converges, compute

$$\begin{bmatrix} \alpha P + T & -W \\ W(I + \frac{1}{\alpha} P^{-1} T) & T \end{bmatrix} \begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = \begin{bmatrix} \alpha P & 0 \\ \frac{1}{\alpha} W P^{-1} T & 0 \end{bmatrix} \begin{bmatrix} x^{(k)} \\ y^{(k)} \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix}. \tag{2.4}$$

Hence the MNB iteration method can be written in the following fixed-point form

$$\begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = \mathcal{P}_{MNB}^{-1} \mathcal{Q}_{MNB} \begin{bmatrix} x^{(k)} \\ y^{(k)} \end{bmatrix} + \mathcal{P}_{MNB}^{-1} \begin{bmatrix} f \\ g \end{bmatrix}, \tag{2.5}$$

where

$$\mathcal{R}(\alpha) = \mathcal{P}_{MNB}^{-1} \mathcal{Q}_{MNB} = \begin{bmatrix} \alpha P + T & -W \\ W(I + \frac{1}{\alpha} P^{-1} T) & T \end{bmatrix}^{-1} \begin{bmatrix} \alpha P & 0 \\ \frac{1}{\alpha} W P^{-1} T & 0 \end{bmatrix} \tag{2.6}$$

is the iteration matrix of the MNB iteration method.

$\rho(\Gamma)$ denotes the spectral radius of the matrix Γ. It is known that the iteration method (2.4) converges if the spectral radius of the iteration matrix $\mathcal{R}(\alpha) = \mathcal{P}_{MNB}^{-1} \mathcal{Q}_{MNB}$ satisfies $\rho\left(\mathcal{R}\left(\alpha\right)\right) < 1$. Now we discuss the unconditional convergence theory of the MNB iteration method in the following Theorem 2.1.

Lemma 2.1. *[8] Let $V_Q = (\alpha I + Q)^{-1} (\alpha I - Q)$ and $\alpha > 0$ is a positive real constant. If $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive semi-definite matrix, then $\|V_Q\|_2 \leq 1$. Furthermore, If $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, then $\|V_Q\|_2 < 1$.*

Theorem 2.1. *Assume that $W \in \mathbb{R}^{n\times n}$ is a symmetric indefinite matrix, $T \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and $\alpha > 0$ is a positive real constant. Assume that the preconditioning matrix P is symmetric positive definite, then the MNB iteration method is convergent unconditionally to the exact solution of the block two-by-two real linear system (1.2).*

Proof. From the iteration scheme (2.5), we have

$$\begin{aligned}
\mathcal{R}(\alpha) &= \mathcal{P}_{MNB}^{-1}\mathcal{Q}_{MNB}\\
&= \alpha \begin{bmatrix} \alpha P + T & -W \\ W\left(I + \frac{1}{\alpha}P^{-1}T\right) & T \end{bmatrix}^{-1} \begin{bmatrix} \alpha P & 0 \\ \frac{1}{\alpha}WP^{-1}T & 0 \end{bmatrix}\\
&= \alpha \begin{bmatrix} \alpha P + T & 0 \\ 0 & \alpha I \end{bmatrix}^{-1} \begin{bmatrix} \alpha I & -W \\ WP^{-1} & T \end{bmatrix}^{-1} \begin{bmatrix} \alpha P & 0 \\ \frac{1}{\alpha}WP^{-1}T & 0 \end{bmatrix}\\
&= \begin{bmatrix} \alpha\left(\alpha P + T\right)^{-1}P + \left(\alpha P + T\right)^{-1}W\mathbf{X} & 0 \\ \left(T + \frac{1}{\alpha}WP^{-1}W\right)^{-1}\left(\frac{1}{\alpha}WP^{-1}T - W\right) & 0 \end{bmatrix}\\
&= \begin{bmatrix} \left(\alpha P + T\right)^{-1}\left(\alpha P + W\mathbf{X}\right) & 0 \\ \mathbf{X} & 0 \end{bmatrix},
\end{aligned}$$

where $\mathbf{X} = \left(T + \frac{1}{\alpha}WP^{-1}W\right)^{-1}W\left(\frac{1}{\alpha}P^{-1}T - I\right)$. Hence, for any $\alpha > 0$, in order to prove $\rho\left(\mathcal{R}\left(\alpha\right)\right) < 1$, we only need to verify $\rho\Big(\left(\alpha P + T\right)^{-1}\left(\alpha P + W\mathbf{X}\right)\Big) < 1$. Since T is symmetric positive definite, the matrix $(\alpha P + T)^{-1}\left(\alpha P + W\mathbf{X}\right)$ is similar to $\left(\alpha I + \widehat{T}\right)^{-1}\left[\alpha I + \widehat{W}\left(\widehat{T} + \frac{1}{\alpha}\widehat{W}^2\right)^{-1}\widehat{W}\left(\frac{1}{\alpha}\widehat{T} - I\right)\right]$ with $\widehat{T} = P^{-\frac{1}{2}}TP^{-\frac{1}{2}}$ and $\widehat{W} = P^{-\frac{1}{2}}WP^{-\frac{1}{2}}$.

By making use of the Sherman-Morrison-Woodbury formula, it has

$$\left(\widehat{T} + \frac{1}{\alpha}\widehat{W}^2\right)^{-1} = \widehat{T}^{-1} - \widehat{T}^{-1}\widehat{W}\left(\alpha I + \widehat{W}\widehat{T}^{-1}\widehat{W}\right)^{-1}\widehat{T}^{-1}\widehat{W}.$$

Hence, we have

$$\begin{aligned}
\mathcal{T} &= \left(\alpha I + \widehat{T}\right)^{-1}\left[\alpha I + \widehat{W}\left(\widehat{T} + \frac{1}{\alpha}\widehat{W}^2\right)^{-1}\widehat{W}\left(\frac{1}{\alpha}\widehat{T} - I\right)\right]\\
&= \left(\alpha I + \widehat{T}\right)^{-1}\left(\alpha I + \widehat{W}\widehat{T}^{-1}\widehat{W}\right)^{-1}\left(\alpha^2 I + \widehat{W}\widehat{T}^{-1}\widehat{W}\widehat{T}\right)
\end{aligned}$$

which is similar to the matrix

$$\widehat{\mathcal{T}} = \left(\alpha I + \widehat{W}\widehat{T}^{-1}\widehat{W}\right)^{-1}\left(\alpha^2 I + \widehat{W}\widehat{T}^{-1}\widehat{W}\widehat{T}\right)\left(\alpha I + \widehat{T}\right)^{-1}.$$

By adopting the technique applied of [15], we have

$$\widehat{\mathcal{T}} = \left(\alpha I + \widehat{W}\widehat{T}^{-1}\widehat{W}\right)^{-1}\left(\alpha^2 I + \widehat{W}\widehat{T}^{-1}\widehat{W}\widehat{T}\right)\left(\alpha I + \widehat{T}\right)^{-1}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\alpha I + \widehat{W}\widehat{T}^{-1}\widehat{W}\right)^{-1}\left(\alpha I + \widehat{W}\widehat{T}^{-1}\widehat{W}\right)\left(\alpha I + \widehat{T}\right)\left(\alpha I + \widehat{T}\right)^{-1} \\
&\quad + \frac{1}{2}\left(\alpha I + \widehat{W}\widehat{T}^{-1}\widehat{W}\right)^{-1}\left(\alpha I - \widehat{W}\widehat{T}^{-1}\widehat{W}\right)\left(\alpha I - \widehat{T}\right)\left(\alpha I + \widehat{T}\right)^{-1} \\
&= \frac{1}{2}\left[I + \left(\alpha I + \widehat{W}\widehat{T}^{-1}\widehat{W}\right)^{-1}\left(\alpha I - \widehat{W}\widehat{T}^{-1}\widehat{W}\right)\left(\alpha I - \widehat{T}\right)\left(\alpha I + \widehat{T}\right)^{-1}\right].
\end{aligned}
$$

Then we can consider the spectral radius of the matrix $\widehat{\mathcal{T}}$, such that

$$
\begin{aligned}
\rho\left(\widehat{\mathcal{T}}\right) &\leq \frac{1}{2}\left[I + \rho\left(\left(\alpha I + \widehat{W}\widehat{T}^{-1}\widehat{W}\right)^{-1}\left(\alpha I - \widehat{W}\widehat{T}^{-1}\widehat{W}\right)\left(\alpha I - \widehat{T}\right)\left(\alpha I + \widehat{T}\right)^{-1}\right)\right] \\
&\leq \frac{1}{2}\left[I + \| \left(\alpha I + \widehat{W}\widehat{T}^{-1}\widehat{W}\right)^{-1}\left(\alpha I - \widehat{W}\widehat{T}^{-1}\widehat{W}\right)\left(\alpha I - \widehat{T}\right)\left(\alpha I + \widehat{T}\right)^{-1} \|_2\right] \\
&\leq \frac{1}{2}\left[I + \| \left(\alpha I + \widehat{W}\widehat{T}^{-1}\widehat{W}\right)^{-1}\left(\alpha I - \widehat{W}\widehat{T}^{-1}\widehat{W}\right) \|_2 \| \left(\alpha I - \widehat{T}\right)\left(\alpha I + \widehat{T}\right)^{-1} \|_2\right].
\end{aligned}
$$

Since both matrices $\widehat{T}$ and $\widehat{W}\widehat{T}^{-1}\widehat{W}$ are positive semidefinite and positive definite, respectively. According to Lemma 2.1, it has $\| \left(\alpha I - \widehat{T}\right)\left(\alpha I + \widehat{T}\right)^{-1} \|_2 = \| \left(\alpha I + \widehat{T}\right)^{-1}\left(\alpha I - \widehat{T}\right) \|_2 < 1$ and $\| \left(\alpha I + \widehat{W}\widehat{T}^{-1}\widehat{W}\right)^{-1}\left(\alpha I - \widehat{W}\widehat{T}^{-1}\widehat{W}\right) \|_2 \leq 1$ for any $\alpha > 0$, so that

$$
\| \left(\alpha I + \widehat{W}\widehat{T}^{-1}\widehat{W}\right)^{-1}\left(\alpha I - \widehat{W}\widehat{T}^{-1}\widehat{W}\right)\left(\alpha I - \widehat{T}\right)\left(\alpha I + \widehat{T}\right)^{-1} \|_2 < 1, \qquad \forall \alpha > 0.
$$

As a result, we have $\rho\left(\mathcal{R}(\alpha)\right) = \rho\left(\widehat{\mathcal{T}}\right) < 1$, which demonstrates the unconditional convergence of the MNB iteration method. □

According to the Theorem 2.1, we know that if $\alpha > 0$, the MNB iteration method converges unconditionally.

3. Theoretical analysis of the preconditioned matrix

In this section, we discuss some properties of the preconditioned matrix $\mathcal{P}_{MNB}^{-1}\mathcal{A}$, give an upper bound of the degree of the minimal polynomial and discuss upon some computational aspects of the preconditioner $\mathcal{P}_{MNB}$. Based on Theorem 2.1, the following theorem describes the eigenvalue distribution of the MNB preconditioned matrix $\mathcal{P}_{MNB}^{-1}\mathcal{A}$.

Theorem 3.1. *Assume that the conditions of Theorem 2.1 are satisfied, then for the preconditioned matrix $\mathcal{P}_{MNB}^{-1}\mathcal{A}$, the following results hold.*

(i) $\mathcal{P}_{MNB}^{-1}\mathcal{A}$ has an eigenvalue 1 with multiplicity at least n;

(ii) the remaining eigenvalues are the eigenvalues of the matrix

$$
\mathbf{Y} = (\alpha P + T)^{-1}\left[T - W\left(\alpha T + WP^{-1}W\right)^{-1} W\left(P^{-1}T - \alpha I\right)\right]
$$

and located in $(0,1)$.

Proof. It follows from (2.2) and (3.5) that

$$\begin{aligned}
\mathcal{P}_{MNB}^{-1}\mathcal{A} &= \mathcal{P}_{MNB}^{-1}\left(\mathcal{P}_{MNB}-\mathcal{Q}_{MNB}\right) \\
&= I-\mathcal{P}_{MNB}^{-1}\mathcal{Q}_{MNB} \\
&= I-\begin{bmatrix} \alpha\left(\alpha P+T\right)^{-1}P+\left(\alpha P+T\right)^{-1}W\mathbf{X} & 0 \\ \left(T+\frac{1}{\alpha}WP^{-1}W\right)^{-1}\left(\frac{1}{\alpha}WP^{-1}T-W\right) & 0 \end{bmatrix} \\
&= \begin{bmatrix} I-\alpha\left(\alpha P+T\right)^{-1}P-\left(\alpha P+T\right)^{-1}W\mathbf{X} & 0 \\ -\left(T+\frac{1}{\alpha}WP^{-1}W\right)^{-1}\left(\frac{1}{\alpha}WP^{-1}T-W\right) & I \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{Y} & 0 \\ -\mathbf{X} & I \end{bmatrix},
\end{aligned} \tag{3.1}$$

where $\mathbf{X}=\left(T+\frac{1}{\alpha}WP^{-1}W\right)^{-1}W\left(\frac{1}{\alpha}P^{-1}T-I\right)$. So, the expression of $\mathcal{P}_{MNB}^{-1}\mathcal{A}$ in (3.1) implies that it has an eigenvalue 1 with multiplicity at least n and the remaining eigenvalues are the same as those of the matrix $\mathbf{Y}$.

Moreover, the matrix $\mathbf{Y}$ is similar to

$$\widehat{\mathbf{Y}}=\left(\alpha I+\widehat{T}\right)^{-1}\left[\widehat{T}-\widehat{W}\left(\alpha\widehat{T}+\widehat{W}^2\right)^{-1}\widehat{W}\left(\widehat{T}-\alpha I\right)\right]$$

with $\widehat{T}=P^{-\frac{1}{2}}TP^{-\frac{1}{2}}$ and $\widehat{W}=P^{-\frac{1}{2}}WP^{-\frac{1}{2}}$. As $\widehat{T}$ and $\widehat{W}$ are symmetric positive definite and symmetric indefinite, respectively, it follows from [31, Theorem 1] that the eigenvalues of $\widehat{\mathbf{Y}}$ are located in $(0,1)$. Hence, the eigenvalues of $\mathbf{Y}$ are located in $(0,1)$. $\square$

Owing to the fact that the convergence of Krylov subspace methods is not only dependent on the eigenvalue distribution of the preconditioned matrix, but also on the corresponding eigenvectors of the preconditioned matrix [5, 6, 27]. Let rank$(\cdot)$ and range$(\cdot)$ denote the rank and the range space of the corresponding matrix, respectively. We discuss the eigenvector distribution of the preconditioned matrix $\mathcal{P}_{MNB}^{-1}\mathcal{A}$ in the following theorem.

Theorem 3.2. *Let the MNB preconditioner $\mathcal{P}_{MNB}$ be defined as in (2.1). Assume that the conditions of Theorem 2.1 are satisfied, then the preconditioned matrix $\mathcal{P}_{MNB}^{-1}\mathcal{A}$ has $n+j$ $(0\le j\le n)$ linearly independent eigenvectors. There are*

(1) n linearly independent eigenvectors $\begin{bmatrix} 0 \\ v_l \end{bmatrix}$ $(l=1,\cdots,n)$ corresponding to the eigenvalue 1, where v_l $(l=1,\cdots,n)$ are arbitrary linearly independent vectors.

(2) j $(0\le j\le n)$ linearly independent eigenvectors $\begin{bmatrix} u_l^\zeta \\ v_l^\zeta \end{bmatrix}$ $(1\le l\le j_\zeta)$ corresponding to the nonunit eigenvalues $\zeta\neq 1$, where u_l^ζ $(1\le l\le j_\zeta)$ satisfies $\left[T+W\left(\alpha T+WP^{-1}W\right)^{-1}W\left(P^{-1}T-\alpha I\right)\right]u_l^\zeta=\zeta\left(\alpha P+T\right)u_l^\zeta$ and v_l^ζ $(1\le l\le j_\zeta)$ satisfies $v_l^\zeta=\frac{1}{\zeta-1}\left(\alpha T+WP^{-1}W\right)^{-1}W\left(\alpha I-P^{-1}T\right)u_l^\zeta$.

Proof. Let ζ be an eigenvalue of the preconditioned matrix $\mathcal{P}_{MNB}^{-1}\mathcal{A}$ and $\begin{bmatrix} u \\ v \end{bmatrix}$ be the corresponding eigenvector. From (3.1), this is equivalent to $\begin{bmatrix} u \\ v \end{bmatrix} \neq 0$, and we have

$$\begin{bmatrix} \mathbf{Y} & 0 \\ -\mathbf{X} & I \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \zeta \begin{bmatrix} u \\ v \end{bmatrix},$$

or, equivalently,

$$\begin{cases} \left[T - W\left(\alpha T + WP^{-1}W\right)^{-1} W\left(P^{-1}T - \alpha I\right)\right] u = \zeta\left(\alpha P + T\right) u, \\ \left(\alpha T + WP^{-1}W\right)^{-1} W\left(\alpha I - P^{-1}T\right) u = \left(\zeta - 1\right) v. \end{cases} \tag{3.2}$$

If $\zeta = 1$, then the second equation of (3.2) becomes $W\left(\alpha I - P^{-1}T\right) u = 0$. Substituting $W\left(\alpha I - P^{-1}T\right) u = 0$ into the first equation of (3.2), we have $\alpha P u = 0$. As $\alpha > 0$ and P is nonsingular, $\alpha P u = 0$ leads to $u = 0$. Hence, there are n linearly independent eigenvectors $\begin{bmatrix} 0 \\ v_l \end{bmatrix}$ $(l = 1, \cdots, n)$ corresponding to the eigenvalue 1, where v_l $(l = 1, \cdots, n)$ are arbitrary linearly independent vectors.

If $\zeta \neq 1$, it follows from the second equation of (3.2) that

$$v = \frac{1}{\zeta - 1}\left(\alpha T + WP^{-1}W\right)^{-1} W\left(\alpha I - P^{-1}T\right) u. \tag{3.3}$$

In this case $u \neq 0$, for otherwise, we have $v = 0$ from (3.3), which contradicts with $\begin{bmatrix} u \\ v \end{bmatrix}$ being an eigenvector. If there exists $u \neq 0$ such that the first equation in (3.2) is satisfied, then there will be j $(0 \leq j \leq n)$ linearly independent eigenvectors $\begin{bmatrix} u_l^\zeta \\ v_l^\zeta \end{bmatrix}$ $(1 \leq l \leq j_\zeta)$ corresponding to the eigenvalues $\zeta \neq 1$, where u_l^ζ $(1 \leq l \leq j_\zeta)$ satisfies $\left[T - W\left(\alpha T + WP^{-1}W\right)^{-1} W\left(P^{-1}T - \alpha I\right)\right] u_l^\zeta = \zeta\left(\alpha P + T\right) u_l^\zeta$ and v_l^ζ $(1 \leq l \leq j_\zeta)$ satisfies (3.3).

Finally, we prove that the $n + j$ eigenvectors are linearly independent.

The proof of this theorem is completed. □

In the following, we obtain an upper bound of the degree of the minimal polynomial of the preconditioned matrix $\mathcal{P}_{MNB}^{-1}\mathcal{A}$. It is well known that Krylov subspace theory states that iteration with any method of optimality property [27] in exact arithmetic will terminate as soon as the degree of the minimal polynomial is attained.

Theorem 3.3. *Let the MNB preconditioner $\mathcal{P}_{MNB}$ be defined as in (2.1). Assume that the conditions of Theorem 2.1 are satisfied, then the degree of the minimal polynomial of the preconditioned matrix $\mathcal{P}_{MNB}^{-1}\mathcal{A}$ is at most $n + 1$.*

Proof. It follows from (3.1) that the preconditioned matrix $\mathcal{P}_{MNB}^{-1}\mathcal{A}$ has the form

$$\mathcal{P}_{MNB}^{-1}\mathcal{A} = \begin{bmatrix} \mathbf{Y} & 0 \\ -\mathbf{X} & I \end{bmatrix}.$$

Let $\mu_i (i = 1, \cdots, n)$ be the eigenvalues of the matrix $\mathbf{Y}$. Then the characteristic polynomial of the preconditioned matrix $\mathcal{P}_{MNB}^{-1}\mathcal{A}$ is

$$\Phi_{\mathcal{P}_{MNB}^{-1}\mathcal{A}}(\mu) = (\mu - 1)^n \prod_{i=1}^{n} (\mu - \mu_i).$$

Let $\Psi(\mu) = (\mu - 1) \prod_{i=1}^{n} (\mu - \mu_i)$, then

$$\begin{aligned} \Psi\left(\mathcal{P}_{MNB}^{-1}\mathcal{A}\right) &= \left(\mathcal{P}_{MNB}^{-1}\mathcal{A} - I\right) \prod_{i=1}^{n} \left(\mathcal{P}_{MNB}^{-1}\mathcal{A} - \mu_i I\right) \\ &= \begin{bmatrix} (\mathbf{Y} - I) \prod_{i=1}^{n} (\mathbf{Y} - \mu_i I) & 0 \\ -\mathbf{X} \prod_{i=1}^{n} (\mathbf{Y} - \mu_i I) & 0 \end{bmatrix}. \end{aligned}$$

It follows from Hamilton-Caylay theorem that $\prod_{i=1}^{n} (\mathbf{Y} - \mu_i I) = 0$. Therefore, the degree of the minimal polynomial of the preconditioned matrix $\mathcal{P}_{MNB}^{-1}\mathcal{A}$ is at most $n + 1$. $\square$

Next, we discuss upon some computational aspects of the preconditioner $\mathcal{P}_{MNB}$. At each step of the MNB iteration method 2.1 or applying $\mathcal{P}_{MNB}$ within a Krylov subspace method, we need to solve the following linear system:

$$\frac{1}{\alpha} \begin{bmatrix} \alpha I & -W \\ WP^{-1} & T \end{bmatrix} \begin{bmatrix} \alpha P + T & 0 \\ 0 & \alpha I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \tag{3.4}$$

where $[z_1^T,\ z_2^T]^T$ and $[r_1^T,\ r_2^T]^T$ are the current and the generalized residual vectors, respectively. It is easy to verify that

$$\begin{bmatrix} \alpha I & -W \\ WP^{-1} & T \end{bmatrix}^{-1} = \begin{bmatrix} I & \frac{1}{\alpha}W \\ 0 & I \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha}I & 0 \\ 0 & \left(T + \frac{1}{\alpha}WP^{-1}W\right)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\frac{1}{\alpha}WP^{-1} & I \end{bmatrix}. \tag{3.5}$$

Therefore, we can obtain the following algorithm to solve the generalized residual equation (3.4).

Algorithm 3.1. For a given generalized residual vector $[r_1^T,\ r_2^T]^T$, the current vector $[z_1^T,\ z_2^T]^T$ is computed by the following steps:

(i) compute $u_1 = r_2 - \frac{1}{\alpha}WP^{-1}r_1$;

(ii) solve $\left(T + \frac{1}{\alpha}WP^{-1}W\right) z_2 = u_1$;

(iii) compute $u_2 = r_1 + Wz_2$;

(iv) solve $(\alpha P + T)\, z_1 = u_2$.

From Algorithm 3.1, it is known that at each iteration step, it is required to solve two linear subsystems with coefficient matrices $T + \frac{1}{\alpha} W P^{-1} W$ and $\alpha P + T$. Since the matrix $T + \frac{1}{\alpha} W P^{-1} W$ and $\alpha P + T$ are symmetric positive definite for all $\alpha > 0$. Therefore, the conjugate gradient (CG) or the preconditioned conjugate gradient (PCG) method can be employed to solve the sub-system of linear equations with the coefficient matrix $T + \frac{1}{\alpha} W P^{-1} W$ by a prescribed accuracy in practice. In addition, the sub-linear equations with the coefficient matrix $T + \frac{1}{\alpha} W P^{-1} W$ can also be solved by some iterative methods such as the Cholesky or LU factorization in combination with AMD or column AMD reordering [13].

4. Numerical experiments

In this section, a numerical example is presented to illustrate the feasibility and effectiveness of the proposed MNB iteration method and the MNB preconditioner for the equivalent real block two-by-two linear system (1.2). To demonstrate the advantages of the MNB preconditioner over the HSS preconditioner [7], the VHSS preconditioner [29] and the NB preconditioner [31], we compare these iteration methods from aspects of the number of iteration steps (denoted by "IT") and the elapsed CPU times in seconds (denoted by "CPU"). In actual implementations, the initial guess is chosen to be the zero vector and the iteration is terminated once the current residuals satisfies RES $= \frac{\|b-\mathcal{A}u\|_2}{\|b\|_2} \leq 10^{-6}$ or the number of iteration steps exceeds $k_{\max} = 1500$. In additional, all codes are run in MATLAB (version R2016b) in double precision and all experiments are performed on an Intel Core (i3-2310M CPU, 6G RAM) Windows 7 system.

We know that the convergent rate of Krylov subspace methods with an ideal preconditioner should be dependent of the suitable selection of the parameter α. Choosing an optimal parameter α to implement the preconditioner $\mathcal{P}_{MNB}$ is definitely important in actual computation. Enlightened by the idea of [22], we investigate the choice for the parameter α of $\mathcal{P}_{MNB}$ by minimizing the Frobenius norm of the difference between $\mathcal{P}_{MNB}$ and $\mathcal{A}$.

The Frobenius norm of $\mathcal{Q}_{MNB}$ is

$$\begin{aligned} \|\mathcal{Q}_{MNB}\|_F &= \left\| \begin{bmatrix} \alpha P & 0 \\ \frac{1}{\alpha} W P^{-1} T & 0 \end{bmatrix} \right\|_F \\ &= \alpha^2 \operatorname{tr}\left(P^2\right) + \frac{1}{\alpha^2} \operatorname{tr}\left(T P^{-1} W^2 P^{-1} T\right) \\ &= \min, \end{aligned} \tag{4.1}$$

where $\operatorname{tr}(\cdot)$ denotes the trace of a given matrix. Then, from (4.1) we obtain the parameter

$$\alpha_{MNB} = \left(\frac{\operatorname{tr}(T P^{-1} W^2 P^{-1} T)}{\operatorname{tr}(P^2)} \right)^{\frac{1}{4}} \tag{4.2}$$

for the MNB preconditioner $\mathcal{P}_{MNB}$.

Similarly, the parameters α in different preconditioners are selected by the ways

- for the HSS preconditioner [7, Corollary 2.3]:

$$\alpha_{HSS} = \sqrt{\mu_{\min}\mu_{\max}};$$

Table 1. Parameters α for Example 4.1

Grids	8×8	16×16	32×32	48×48
α_{HSS}	1.5303	0.8194	0.4235	0.2854
α_{NB}	2.7734	4.2550	4.8735	4.9920
α_{VHSS}	1.5303	0.8194	0.4235	0.2854
α_{MNB}	8.1041	8.1544	9.5218	9.7848

- for the VHSS preconditioner [29, Theorem 3.3]:

$$\alpha_{VHSS} = \sqrt{\mu_{\min}\mu_{\max}};$$

- for the NB preconditioner [31]:

$$\alpha_{NB} = \left(\frac{\mathrm{tr}(TW^2T)}{n}\right)^{\frac{1}{4}}.$$

Here $\mu_{\min}$ and $\mu_{\max}$ are the smallest and the largest eigenvalues of the symmetric positive definite matrix T.

Example 4.1. Let the submatrices in $\mathcal{A}$ be given by $W = -\frac{(3-\sqrt{3})\omega^2}{(m+1)^2}I + I\otimes V_m + V_m\otimes I$ and $T = \frac{(3+\sqrt{3})\tau^2}{(m+1)^2}I + I\otimes V_m + V_m\otimes I$, where ω and τ are two positive parameters, and $V_m = \frac{1}{h^2}\mathrm{tridiag}\,(-1,\ 2,\ -1) \in \mathbb{R}^{m\times m}$.

In fact, the submatrices W and T arise from the complex symmetric linear system

$$\left[\left(K - \left(3-\sqrt{3}\right)\omega^2 I\right) + \mathbf{i}\left(K + (3+\sqrt{3})\tau^2 I\right)\right] z = c,$$

where K is the five-point center difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0,\ 1]\times[0,\ 1]$ with the mesh-size $h = \frac{1}{m+1}$, and has the form $K = I\otimes V_m + V_m\otimes I$, see [14,25,31] for more details.

It was shown [29] that the matrix T is symmetric positive definite, and the matrix W is symmetric indefinite when

$$2\,(m+1)\sqrt{\frac{1-\cos\frac{\pi}{m+1}}{3-\sqrt{3}}} < \omega < 2\,(m+1)\sqrt{\frac{1-\cos\frac{m\pi}{m+1}}{3-\sqrt{3}}}.$$

In actual computations, we take grids $m = 8,\ 16,\ 32,\ 48$, and set $\omega = 20$ and $\tau = 1$, which leads to symmetric indefinite matrices W. We set the preconditioning matrix P in the MNB preconditioner as $P = \frac{1}{10}T$. The parameters α_{HSS}, α_{VHSS}, α_{NB} and α_{MNB} for Example 4.1 are listed in Table 1.

In Table 2, we report the numerical results of the preconditioned GMRES methods for Example 4.1, in which, "$\mathcal{I}$"denotes the GMRES method without preconditioning.

In Figure 1, the residual curves of these preconditioned GMRES iteration methods for 32×32 grids and 48×48 grids are plotted.

Table 2. Numerical results for Example 4.1

Preconditioners		8	16	32	48
	IT	19	48	139	243
$\mathcal{I}$	CPU	0.5430	0.0466	0.3964	2.1481
	RES	6.0975e-7	5.9261e-7	9.8885e-7	9.8990e-7
	IT	14	30	46	58
$\mathcal{P}_{HSS}$	CPU	0.1112	0.0343	0.1742	0.4973
	RES	5.0855	9.1730e-7	8.5704e-7	9.6066e-7
	IT	7	11	19	26
$\mathcal{P}_{NB}$	CPU	0.0580	0.0171	0.0504	0.2002
	RES	5.9350e-7	2.3300e-7	8.0364e-7	9.2139e-7
	IT	7	12	18	21
$\mathcal{P}_{VHSS}$	CPU	0.0067	0.0211	0.0539	0.1531
	RES	7.7071e-7	6.5798e-7	6.3403e-7	9.9609e-7
	IT	6	7	5	5
$\mathcal{P}_{MNB}$	CPU	0.0056	0.0090	0.0357	0.1077
	RES	9.0508e-7	6.7195e-7	6.8945e-7	4.0270e-7

From Table 2 and Figure 1, we can see that all the HSS, the VHSS, the NB and the MNB preconditioners improve computing efficiency of the original GMRES method greatly. The iteration steps indicate that the MNB preconditioned GMRES method returns better results than the VHSS, the NB and the HSS preconditioned GMRES methods. In addition, the numbers of iteration steps of the MNB preconditioned GMRES method grow slower with the grid size than those of the HSS, the VHSS and the NB preconditioned GMRES methods.

To further show the efficiency of the proposed MNB preconditioner, a typical eigenvalues distribution (32×32 grids) of the HSS preconditioned matrix, the VHSS preconditioned matrix, the NB preconditioned matrix and the MNB preconditioned matrix are plotted in Figure 2.

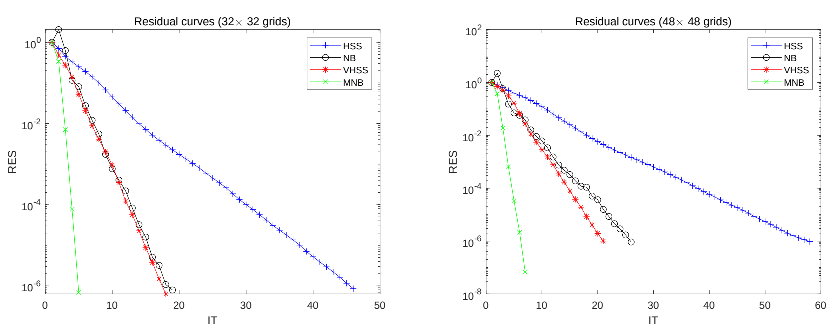

Figure 1. Residual curves of different preconditioned iteration methods for Example 4.1: 32×32 grids (left) and 48×48 grids (right)

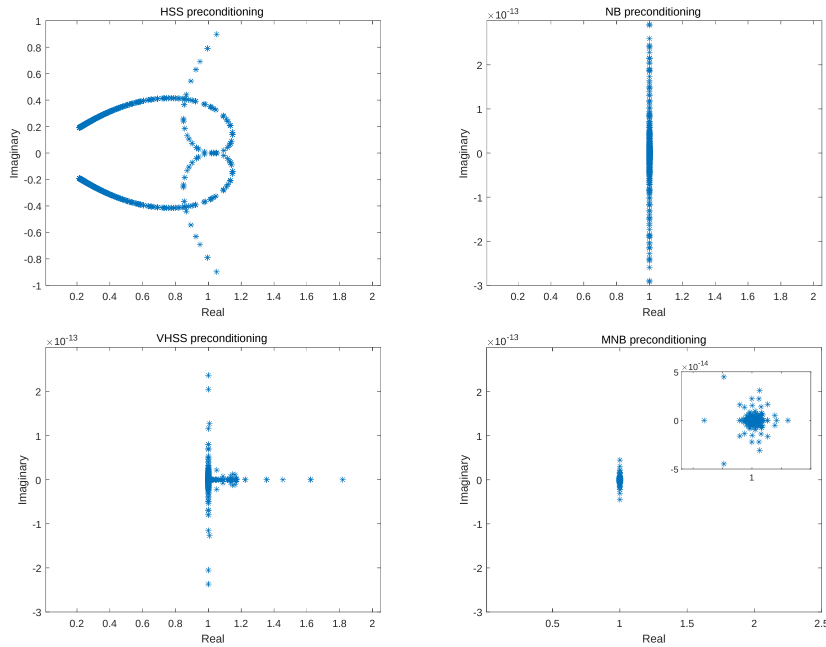

Figure 2. The eigenvalue distributions of preconditioned matrices for Example 4.1

From Figure 2, we can see that both the eigenvalues of the HSS preconditioned matrix, the VHSS preconditioned matrix, the NB preconditioned matrix and the MNB preconditioned matrix are located in a circle centered at (1, 0) with radius strictly less than 1. Moreover, the eigenvalues of the MNB preconditioned matrix are more clustered than that of the HSS preconditioned matrix, the VHSS preconditioned matrix and the NB preconditioned matrix.

5. Concluding remarks

By applying matrix preconditioning and relaxation techniques, based on the NB preconditioner, we proposed a new modified block preconditioner for a class of 2×2 block linear systems. When choose a suitable symmetric positive definite matrix P for the new preconditioner, we observe that the new preconditioner may be better approximation to the coefficient matrix $\mathcal{A}$ than the NB preconditioner. The convergence theories for the new iteration method and some properties of the proposed new preconditioned matrix were studied. Finally, a numerical example with a particular choice of the preconditioning matrix P showed that our proposed the new preconditioner outperforms the existing HSS preconditioner, the VHSS preconditioner and the NB preconditioner. However, in order to improve the computational efficiency of the new preconditioner, searching a more efficient preconditioning matrix P and the optimal parameter α is a further work for us.

Acknowledgments

The author would like to express our sincere thanks to the anonymous reviewers for their valuable suggestions which greatly improved the presentation of this paper.

References

[1] S. R. Arridge, *Optical tomography in medical imaging*, Inverse Probl., 1999, 15, 41–93.

[2] O. Axelsson and A. Kucherov, *Real valued iterative methods for solving complex symmetric linear systems*, Numer. Linear Algebra Appl., 2000, 7, 197–218.

[3] O. Axelsson, M. Neytcheva and B. Ahmad, *A comparison of iterative methods to solve complex valued linear algebraic systems*, Numer. Algor., 2014, 66, 811–841.

[4] O. Axelsson, M. Pourbagher and D. K. Salkuyeh, *Efficient iteration methods for complex systems with an indefinite matrix term*, Calcolo, 2022, 59, 15.

[5] Z. Bai, *Sharp error bounds of some Krylov subspace methods for non-Hermitian linear systems*, Appl. Math. Comput., 2000, 109, 273–285.

[6] Z. Bai, *Motivations and realizations of Krylov subspace methods for large sparse linear systems*, J. Comput. Appl. Math., 2015, 283, 71–78.

[7] Z. Bai, G. H. Golub and M. K. Ng, *Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems*, SIAM J. Matrix Anal. Appl., 2003, 24, 603–626.

[8] Z. Bai and A. Hadjidimos, *Optimization of extrapolated Cayley transform with non-Hermitian positive definite matrix*, Linear Algebra Appl., 2014, 463, 322–339.

[9] F. B. Balani and M. Hajarian, *On the generalized AOR and CG iteration methods for a class of block two-by-two linear systems*, Numer. Algor., 2022, 90, 669–685.

[10] W. Bao, *An improved preconditioner for 2×2 block linear system arising from complex linear system*, Japan J. Indust. Appl. Math., 2021, 38, 859–875.

[11] M. Benzi and D. Bertaccini, *Block preconditioning of real-valued iterative algorithms for complex linear systems*, IMA J. Numer. Anal., 2008, 28, 598–618.

[12] D. Bertaccini, *Efficient solvers for sequences of complex symmetric linear systems*, Electron. Trans. Numer. Anal., 2004, 18, 49–64.

[13] Y. Cao, J. Du and Q. Niu, *Shift-splitting preconditioners for saddle point problems*, J. Comput. Appl. Math., 2014, 272, 239–250.

[14] Y. Cao and Z. Ren, *Two variants of the PMHSS iteration method for a class of complex symmetric indefinite linear systems*, Appl. Math. Comput., 2015, 264, 61–71.

[15] F. Chen, *On convergence of EVHSS iteration method for solving generalized saddle-point linear systems*, Appl. Math. Lett., 2018, 86, 30–35.

[16] K. Chen, *Matrix Preconditioning Techniques and Applications*, Cambridge University Press, Cambridge, 2005.

[17] D. Day and M. A. Heroux, *Solving complex-valued linear systems via equivalent real formulations*, SIAM J. Sci. Comput., 2001, 23(2), 480–498.

[18] B. B. Fariba and H. Masoud, *Modified block product preconditioner for a class of complex symmetric linear systems*, Linear Multilinear Algebra., 2023, 71, 1521–1535.

[19] A. Feriani, F. Perotti and V. Simoncini, *Iterative system solvers for the frequency analysis of linear mechanical systems*, Comput. Methods Appl. Mech. Engrg., 2000, 190, 1719–1739.

[20] D. Hezari, V. Edalatpour and D. K Salkuyeh, *Preconditioned GSOR iterative method for a class of complex symmetric system of linear equations*, Numer. Linear Algebra Appl., 2015, 22, 338–356.

[21] V. E. Howle and S. A. Vavasis, *An iterative method for solving complex-symmetric systems arising in electrical power modeling*, SIAM J. Matrix Anal. Appl., 2005, 26, 1150–1178.

[22] Y. Huang, *A practical formula for computing optimal parameters in the HSS iteration methods*, J. Comput. Appl. Math., 2014, 225, 142–149.

[23] Z. Liang, O. Axelsson and G. Zhang, *Efficient iterative solvers for a complex valued two-by-two block linear system with application to parabolic optimal control problems*, Appl. Numer. Math., 2020, 152, 422–445.

[24] Z. Liang and G. Zhang, *On SSOR iteration method for a class of block two-by-two linear systems*, Numer. Algor., 2016, 71, 655–671.

[25] S. Miao, *A new preconditioner for a class of* 2×2 *block linear systems*, Japan J. Indust. Appl. Math., 2020, 37, 913–928.

[26] M. Pourbagher and D. K. Salkuyeh, *A new two-parameter iteration method for indefinite complex symmetric linear systems*, Japan J. Indust. Appl. Math., 2022, 39, 145–163.

[27] Y. Saad, *Iterative Methods for Sparse Linear Systems*, SIAM, Philadephia, 2003.

[28] Y. Saad and M. H. Schultz, *GMRES: A generalised minimal residual algorithm for solving nonsymmetric linear systems*, SIAM J. Sci. Stat. Comput., 1986, 7(3), 856–869.

[29] Q. Shen and Q. Shi, *A variant of the HSS preconditioner for complex symmetric indefinite linear systems*, Comput. Math. Appl., 2018, 75, 850–863.

[30] T. Wang and L. Lu, *Alternating-directional PMHSS iteration method for a class of two-by-two block linear systems*, Appl. Math. Lett., 2016, 58, 159–164.

[31] J. Zhang and H. Dai, *A new block preconditioner for complex symmetric indefinite linear systems*, Numer. Algor., 2017, 74, 889–903.

[32] Q. Zheng and L. Lu, *A shift-splitting preconditioner for a class of block two-by-two linear systems*, Appl. Math. Lett., 2017, 66, 54–60.