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BLOCK-BY-BLOCK TECHNIQUE FOR A CLASS OF NONLINEAR SYSTEMS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract This work expresses an approximate approach for a class of nonlinear systems of fractional integro-differential equations. The proposed scheme uses the five-point Gauss-Lobatto quadrature method and the block-by-block technique. This procedure obtains automatically several approximate values of the problem at the same time. The analysis of convergence of the adopted approach is investigated. Moreover, it is proved that the convergence order of the method is $O(h^8)$. Some numerical examples are considered to reveal the effectiveness of the method.

Keywords System of fractional integro-differential equations, Gaussian-Lobatto quadrature formula, Block-by-block method.

MSC(2010) 65R20, 65D30, 45G10.

1. Introduction

Fractional calculus is a generalization of the integer calculus, which has attracted the attention of many researchers, due to extraordinary innovations in various scientific and technological fields, such as oscillator [15], vibration [14, 16, 29], viscous fluid [38], bioengineering [28], fractional dynamics [19], etc.

Since finding the analytical solutions of systems of fractional integro differential equations is usually difficult, researchers have introduced and extended plenty of numerical techniques to solve these systems. Some of these methods are collocation method [18, 42, 46], Adomian decomposition method [8, 20], fractional differential transform method [2, 25], homotopy analysis method [34, 45], iteration method [6, 24, 36, 44], radial basis functions method [3], wavelets method [13, 37, 41], discrete Galerkin method [22], rational Haar wavelets method [26], block-pulse functions method [7, 39, 40], Taylor expansion approach [9], Legendre tau method [23], B-Spline method [1] and least squares method [5, 21].

In this study, we suggest the block-by-block technique based upon the five-point Gauss-Lobatto quadrature formula to find the solution of a category of systems

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of fractional integro-differential equations. The proposed approach can automatically be found several values of the unknown functions simultaneously at each step. The error analysis and the convergence of the adopted approach have been proved. Moreover, it is revealed that the order of convergence for a given step size h is at least eight.

As you know, the notion of the block-by-block method was first stated in [43]. This approach has been used by many authors for the solution of various problems. In 1968, Linz [17] provided two block method for the solution of nonlinear Volterra integral equations of the second kind. In 2005, Saify [33] developed two, three, and four blocks methods and used them for the solution of a system of linear Volterra integral equations of the second kind. In 2010 and 2012, Katani and Shahmorad [10–12] have applied the block-by-block technique for the solution of nonlinear Volterra integral equations.

The setup of this work is as follows: In Section 2, some definitions regarding the fractional derivative and integral have been recalled. Section 3 is devoted to implementation of the presented scheme for the introduced problem. The analysis and convergence of the adopted approach are discussed in Section 4. In Section 5, we introduce some numerical examples to illustrate the accuracy of the introduced approach. The conclusion of the work is given in Section 6.

2. The preliminary knowledge

This section consists some definitions and results about fractional calculus that will be required for the completion of our research.

Definition 2.1. ([27]) The Mittag-Leffler functions are defined by

$$\mathbf{E}_{\sigma}(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(j\sigma + 1)}, \tag{2.1}$$

and

$$\mathbf{E}_{\sigma,\varrho}(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(j\sigma + \varrho)}, \tag{2.2}$$

where $t \in \mathbb{C}$ and $\sigma,\ \varrho \in \mathbb{R}^+$.

Definition 2.2. ([27]) The fractional integration operator of the Riemann-Liouville type of order $\alpha > 0$ is defined as

$$I^{\alpha} y(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} y(s) ds, & \alpha > 0, \\ y(t), & \alpha = 0, \end{cases}$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.3. ([27]) The fractional derivative of the Caputo type of order $\vartheta - 1 < \alpha \leq \vartheta$ of a sufficiently differentiable function y is defined as

$${}_0^C D_t^{\alpha} y(t) = \begin{cases} \dfrac{1}{\Gamma(\vartheta-\alpha)} \displaystyle\int_0^t (t-s)^{\vartheta-\alpha-1} y^{(\vartheta)}(s) ds, & \vartheta - 1 < \alpha \leq \vartheta, \\ y^{(\vartheta)}(t), & \alpha = \vartheta, \end{cases}$$

where ϑ is a positive integer.

Property 2.1. ([27]) *Based on the above definitions, we have the following significant attributes:*

$$\begin{aligned}
&{}^{C}_{0}D_t^{\alpha}\left(I^{\alpha}y(t)\right) = y(t),\\
&I^{\alpha}\left({}^{C}_{0}D_t^{\alpha}y(t)\right) = y(t) - \sum_{i=0}^{\vartheta-1} y^{(i)}(0)\tfrac{t^i}{i!}.
\end{aligned}$$

3. Implementation of the proposed method

This section provides a numerical algorithm based on the block-by-block scheme for solving system of fractional integro-differential equations:

$$ {}^{C}_{0}D_t^{\alpha}\mathbf{y}(t) = \mathbf{X}(t) + \int_0^t \mathbf{K}(t,s,\mathbf{y}(s))ds, \qquad 0 \le s \le t \le 1, \quad n-1 < \alpha \le n, \; n \in \mathbb{N} \tag{3.1}$$

with

$$\begin{aligned}
&\mathbf{y}(t) = \left[y_1(t), y_2(t), \ldots, y_p(t)\right]^T, \qquad \mathbf{X}(t) = \left[X_1(t), X_2(t), \ldots, X_p(t)\right]^T,\\
&\mathbf{K}(t,s,\mathbf{y}(s)) = \left[K_1(t,s,\mathbf{y}(s)), K_2(t,s,\mathbf{y}(s)), \ldots, K_p(t,s,\mathbf{y}(s))\right]^T,
\end{aligned}$$

and the initial conditions

$$\mathbf{y}^{(\rho)}(0) = \zeta_\rho, \qquad \rho = 0, 1, \cdots, n-1.$$

By applying the operator I^{α} on both sides of (3.1), we obtain the following equivalent problem:

$$\mathbf{y}(t) = \mathbf{x}(t) + \int_0^t \mathbf{k}(t,s,y(s))ds \tag{3.2}$$

with

$$\begin{aligned}
&\mathbf{x}(t) = \left[x_1(t), x_2(t), \ldots, x_p(t)\right]^T,\\
&\mathbf{k}(t,s,\mathbf{y}(s)) = \left[k_1(t,s,\mathbf{y}(s)), k_2(t,s,\mathbf{y}(s)), \ldots, k_p(t,s,\mathbf{y}(s))\right]^T,
\end{aligned}$$

where

$$\mathbf{x}(t) = \sum_{\rho=0}^{n-1} \zeta_\rho \frac{t^\rho}{\rho!} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathbf{X}(s)ds,$$

and

$$\mathbf{k}(t,s) = \frac{1}{\Gamma(\alpha)}\int_s^t (t-w)^{\alpha-1}\mathbf{K}(w,s)dw,$$

such that $\mathbf{x}(t)$ is the known function, $\mathbf{k}(t,s)$ is the kernel function, ${}^{C}_{0}D_t^{\alpha}$ is the fractional differential operator and $\mathbf{y}(t)$ is the unknown function.

Now, for a given positive integer number M that is a multiple of blocksize namely 4, we divide the interval $[0,1]$ into M parts with step size $h = 1/M$ such that $t_j = jh$, $j = 0, 1, \dots, M$. For the grid point $t = t_{4m+q}$ using (3.2), we have

$$\begin{aligned}
y_1(t_{4m+q}) =& x_1(t_{4m+q}) + \int_0^{t_{4m}} k_1(t_{4m+q}, s, y_1(s), y_2(s), \dots, y_p(s))ds \\
&+ \int_{t_{4m}}^{t_{4m+q}} k_1(t_{4m+q}, s, y_1(s), y_2(s), \dots, y_p(s))ds, \\
y_2(t_{4m+q}) =& x_2(t_{4m+q}) + \int_0^{t_{4m}} k_2(t_{4m+q}, s, y_1(s), y_2(s), \dots, y_p(s))ds \\
&+ \int_{t_{4m}}^{t_{4m+q}} k_2(t_{4m+q}, s, y_1(s), y_2(s), \dots, y_p(s))ds, \\
&\vdots \\
y_p(t_{4m+q}) =& x_p(t_{4m+q}) + \int_0^{t_{4m}} k_p(t_{4m+q}, s, y_1(s), y_2(s), \dots, y_p(s))ds \\
&+ \int_{t_{4m}}^{t_{4m+q}} k_p(t_{4m+q}, s, y_1(s), y_2(s), \dots, y_p(s))ds. \qquad (3.3)
\end{aligned}$$

For $m = 0, 1, \dots, M/4 - 1$ and $q = 1, 2, 3, 4$, we assume that $Y_{i,j} \simeq y_i(t_j)$ for $i = 1, \dots, p$ is the approximate solution of $y_i(t)$ at the grid point $t = t_j$ for $j = 0, 1, \dots, M$. Also, we put $Y_{i,0} = x_i(t_0)$. Assume that the values of $Y_{i,0}, Y_{i,1}, \dots, Y_{i,4m}$ $(i = 1, 2, \dots, p)$ are given. Then the integrals defined over $[0, t_{4m}]$ are approximated by standard numerical integration rules. Moreover, the five-point Gauss-Lobatto quadrature method is used to compute the integrals appointed over $[t_{4m}, t_{4m+q}]$ via the points t_{4m}, t_{4m+1}, t_{4m+2}, t_{4m+3}, t_{4m+4}. Thus, a nonlinear algebraic system with 4n equations is solved, so that a block of unknowns is obtained at a time. As you know, the five-point Gauss-Lobatto quadrature method requires the use of weights γ_l and points $\bar{y}_l$ for $l = 0, 1, \dots, 4$. This requirement given as [4]:

$$\gamma_0 = \frac{1}{10},\ \gamma_1 = \frac{49}{90},\ \gamma_2 = \frac{32}{45},\ \gamma_3 = \frac{49}{90},\ \gamma_4 = \frac{1}{10},$$

and

$$\bar{y}_0 = -1,\ \bar{y}_1 = -\sqrt{\frac{3}{7}},\ \bar{y}_2 = 0,\ \bar{y}_3 = \sqrt{\frac{3}{7}},\ \bar{y}_4 = 1.$$

In the sequence, we set $\mathbf{Y}_j = (Y_{1,j}, Y_{2,j}, \dots, Y_{i,j}) \simeq (y_1(t_j), y_2(t_j), \dots, y_i(t_j))$. Therefore, the five-point Gauss-Lobatto quadrature method leads to the following approximations:

$$\begin{aligned}
&\int_{t_{4m}}^{t_{4m+q}} k_i\left(t_{4m+q}, s, \mathbf{y}(s)\right) ds \simeq \frac{t_{4m+q} - t_{4m}}{2}, \\
&\sum_{l=0}^{4} \gamma_l k_i \bigg(t_{4m+q}, \frac{t_{4m+q} - t_{4m}}{2} \bar{y}_l + \frac{t_{4m+q} + t_{4m}}{2}, \\
&\mathbf{y}\left(\frac{t_{4m+q} - t_{4m}}{2} \bar{y}_l + \frac{t_{4m+q} + t_{4m}}{2} \right) \bigg) \\
=&\frac{t_q}{2} \left[\frac{1}{10} \bigg(k_i\left(t_{4m+q}, t_{4m+\xi_0}, \mathbf{Y}_{4m+\xi_0}\right) + k_i\left(t_{4m+q}, t_{4m+\xi_4}, \mathbf{Y}_{4m+\xi_4}\right) \bigg) \right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{32}{45}k_i\left(t_{4m+q},t_{4m+\xi_2},\mathbf{Y}_{4m+\xi_2}\right)\\
&+\frac{49}{90}\left(k_i\left(t_{4m+q},t_{4m+\xi_1},\mathbf{Y}_{4m+\xi_1}\right)+k_i\left(t_{4m+q},t_{4m+\xi_3},\mathbf{Y}_{4m+\xi_3}\right)\right)\Bigg]\\
&\triangleq F_i(4m+q,4m),\quad i=1,2,\ldots,p,
\end{aligned}
\tag{3.4}
$$

where

$$
\xi_0=0,\,\xi_1=\left(1-\sqrt{\frac{3}{7}}\right)\frac{q}{2},\,\xi_2=\frac{q}{2},\,\xi_3=\left(1+\sqrt{\frac{3}{7}}\right)\frac{q}{2},\,\xi_4=q.
$$

If the intermediate grid points, namely ξ_j, $j=1,2,3$ are not integers. It is necessary to calculate the values of $\mathbf{Y}_{4m+\xi_j}$ in (3.4). These values are achieved with the Lagrange interpolation at the points $t_{4m+\xi_j}$, $j=1,2,3,4$, that is,

$$
\mathbf{Y}_{4m+\xi_j}\simeq\mathcal{P}\left(t_{4m}+\xi_j h\right)=\sum_{\bar{j}=0}^{4}L_{\bar{j}}(\xi_j)\mathbf{Y}_{4m+\bar{j}},
$$

where

$$
L_{\bar{j}}(\xi_j):=\prod_{\substack{r=0\\ r\neq\bar{j}}}^{4}\frac{\xi_j-r}{\bar{j}-r},\quad j=1,2,3.
$$

Since $4m$ is a multiple of 4, so we define

$$
\begin{aligned}
&\int_0^{t_{4m}}k_i(t_{4m+q},s,\mathbf{y}(s))ds\\
&=\sum_{j=1}^{m}\left(\int_{t_{4(j-1)}}^{t_{4j}}k_i(t_{4m+q},s,\mathbf{y}(s))ds\right)\\
&=\sum_{j=1}^{m}\left(\frac{t_{4j}-t_{4(j-1)}}{2}\left[\frac{1}{10}\left(k_i\left(t_{4m+q},t_{4m+\xi_0},\mathbf{Y}_{4(j-1)}\right)+k_i\left(t_{4m+q},t_{4m+\xi_4},\mathbf{Y}_{4j}\right)\right)\right.\right.\\
&\quad+\frac{32}{45}k_i\left(t_{4m+q},t_{4m+\xi_2},\mathbf{Y}_{4(j-1)+2}\right)\\
&\quad\left.\left.+\frac{49}{90}\left(k_i\left(t_{4m+q},t_{4m+\xi_1},\mathbf{Y}_{4(j-1)+\xi_1}\right)+k_i\left(t_{4m+q},t_{4m+\xi_3},\mathbf{Y}_{4(j-1)+\xi_3}\right)\right)\right]\right)\\
&\triangleq E_i(4m,0).\quad i=1,2,\ldots,p.
\end{aligned}
\tag{3.5}
$$

Consequently, by replacing (3.4) and (3.5) into (3.3), we get

$$
\begin{aligned}
Y_{1,4m+q}&=x_1(t_{4m+q})+E_1(4m,0)+F_1(4m+q,4m),\\
Y_{2,4m+q}&=x_2(t_{4m+q})+E_2(4m,0)+F_2(4m+q,4m),\\
&\vdots\\
Y_{p,4m+q}&=x_p(t_{4m+q})+E_p(4m,0)+F_p(4m+q,4m).
\end{aligned}
\tag{3.6}
$$

Eventually, at each step, from (3.6), we form a nonlinear algebraic system of 4n equations for the unknowns $\mathbf{Y}_{4m+1}$, $\mathbf{Y}_{4m+2}$, $\mathbf{Y}_{4m+3}$ and $\mathbf{Y}_{4m+4}$. By solving this system, we obtain an approximate solution for the main system at grid points.

4. Convergence analysis

The aim of this section is to investigate the convergence order of the method expressed in the previous section to solve the fractional integro-differential problem introduced in this paper.

Theorem 4.1. *Let $\boldsymbol{k}$, $\boldsymbol{y}$ be functions with at least eight times differentiable in (3.2). Thus, the approximation method of (3.6) is convergent and the rate of convergence of the method with respect to the derivability of functions is at least* 8.

Proof. From (3.5) we have

$$\mathbf{E}(4m,0) := \int_0^{t_{4m}} \mathbf{k}(t_{4m+q}, s, \mathbf{y}(s))ds \simeq h\sum_{j=0}^{4m} \gamma_j \mathbf{k}(t_{4m+q}, t_j, \mathbf{Y}_j),$$

where $t_j = t_{4(m-1)+\xi_j}$, $\mathbf{Y}_j = \mathbf{Y}_{4(m-1)+\xi_j}$ and $\xi_j = \xi_{j-4(m-1)}$.

Define $\mathbf{e}_j = |\mathbf{y}(t_j) - \mathbf{Y}_j|$ where $\mathbf{e}_j = [e_{1,j}, e_{2,j}, \ldots, e_{p,j}]^T$. Then it follows from (3.3) and (3.6) that

$$\begin{aligned}
\mathbf{e}_{4m+q} &= |\mathbf{y}(t_{4m+q}) - \mathbf{Y}_{4m+q}| \\
&= \Bigg| \int_0^{t_{4m}} \mathbf{k}(t_{4m+q}, s, \mathbf{y}(s)) + \int_{t_{4m}}^{t_{4m+q}} \mathbf{k}(t_{4m+q}, s, \mathbf{y}(s)) \\
&\quad - h\sum_{j=0}^{4m} \gamma_j \mathbf{k}(t_{4m+q}, t_j, \mathbf{Y}_j) - \frac{t_q}{2}\Bigg[\frac{1}{10}\mathbf{k}(t_{4m+q}, t_{4m+\xi_0}, \mathbf{Y}_{4m+\xi_0}) \\
&\quad + \frac{49}{90}\mathbf{k}\left(t_{4m+q}, t_{4m+\xi_1}, \mathbf{Y}_{4m+\xi_1}\right) + \frac{32}{45}\mathbf{k}\left(t_{4m+q}, t_{4m+\xi_2}, \mathbf{Y}_{4m+\xi_2}\right) \\
&\quad + \frac{49}{90}\mathbf{k}\left(t_{4m+q}, t_{4m+\xi_3}, \mathbf{Y}_{4m+\xi_3}\right) + \frac{1}{10}\mathbf{k}(t_{4m+q}, t_{4m+\xi_4}, \mathbf{Y}_{4m+\xi_4})\Bigg]\Bigg|.
\end{aligned}$$

By totaling and subducting the terms

$$h\sum_{j=0}^{4m} \gamma_j \mathbf{k}(t_{4m+q}, t_j, \mathbf{y}(t_j)),\ \ \frac{t_q}{20}\mathbf{k}(t_{4m+q}, t_{4m+\xi_0}, \mathbf{y}(t_{4m+\xi_0}))\ , \cdots ,$$

$$\frac{49t_q}{180}\mathbf{k}(t_{4m+q}, t_{4m+\xi_3}, \sum_{\bar{j}=0}^{4} L_{\bar{j}}(\xi_3)\mathbf{y}(t_{4m+\bar{j}})), \frac{t_q}{20}\mathbf{k}(t_{4m+q}, t_{4m+\xi_4}, \mathbf{y}(t_{4m+\xi_4})),$$

and utilizing the Lipschitz condition for $\mathbf{k}$, we get

$$\begin{aligned}
\mathbf{e}_{4m+q} =& h\sum_{j=0}^{4m} \gamma_j \eta(t_{4m+q}, t_j)\mathbf{e}_j \\
&+ \frac{t_q}{20}\eta(t_{4m+q}, t_{4m+\xi_0})\mathbf{e}_{4m+\xi_0} + \frac{t_q}{20}\eta(t_{4m+q}, t_{4m+\xi_4})\mathbf{e}_{4m+\xi_4} \\
&+ \frac{49t_q}{180}\eta(t_{4m+q}, t_{4m+\xi_1})\left(\sum_{\bar{j}=0}^{4} L_{\bar{j}}(\xi_1)\mathbf{y}(t_{4m+\bar{j}}) - \sum_{\bar{j}=0}^{4} L_{\bar{j}}(\xi_1)\mathbf{Y}(t_{4m+\bar{j}})\right) \\
&+ \frac{32t_q}{90}\eta(t_{4m+q}, t_{4m+\xi_2})\left(\sum_{\bar{j}=0}^{4} L_{\bar{j}}(\xi_2)\mathbf{y}(t_{4m+\bar{j}}) - \sum_{\bar{j}=0}^{4} L_{\bar{j}}(\xi_2)\mathbf{Y}(t_{4m+\bar{j}})\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{49t_q}{180}\eta(t_{4m+q},t_{4m+\xi_3})\bigg(\sum_{\bar{j}=0}^{4}L_{\bar{j}}(\xi_3)\mathbf{y}(t_{4m+\bar{j}})-\sum_{\bar{j}=0}^{4}L_{\bar{j}}(\xi_3)\mathbf{Y}(t_{4m+\bar{j}})\bigg)\\
&+R_1+R_2,
\end{aligned}
$$

where $\eta(t,s)$ is continuous on $[0,1]$. Thus

$$
\begin{aligned}
&\mathbf{e}_{4m+q}\\
\leq& h\sum_{j=0}^{4m}\gamma_j l_j\mathbf{e}_j+\frac{q}{20}hl_{4m}\mathbf{e}_{4m}+\frac{q}{20}hl_{4m+q}\mathbf{e}_{4m+q}\\
&+\frac{49q}{180}hl_{4m+\xi_1}\max_{\bar{j}}\left\{L_{\bar{j}}(\xi_1)\right\}\sum_{\bar{j}=0}^{4}\mathbf{e}_{4m+\bar{j}}+\frac{32q}{90}hl_{4m+\xi_2}\max_{\bar{j}}\left\{L_{\bar{j}}(\xi_2)\right\}\sum_{\bar{j}=0}^{4}\mathbf{e}_{4m+\bar{j}}\\
&+\frac{49q}{180}hl_{4m+\xi_3}\max_{\bar{j}}\left\{L_{\bar{j}}(\xi_3)\right\}\sum_{\bar{j}=0}^{4}\mathbf{e}_{4m+\bar{j}}+R_1+R_2\\
\leq& hc\sum_{j=0}^{4m}\mathbf{e}_j+hc_1\mathbf{e}_{4m+1}+hc_2\mathbf{e}_{4m+2}+hc_3\mathbf{e}_{4m+3}+hc_4\mathbf{e}_{4m+4}+R_1+R_2,
\end{aligned}
$$

where $R\triangleq R_1+R_2$ is the error of the numerical integrations and $c,c_1,\dots,c_4$ are constants. Without loss of totality, we assume that

$$
\max_{\bar{j}=4m+1,4m+2,\dots,4m+4}\mathbf{e}_{\bar{j}}=\mathbf{e}_{4m+q},\qquad \max_{\bar{j}=1,2,\dots,4m}\mathbf{e}_{\bar{j}}=\mathbf{e}_{4m}.
$$

Then, it is easy to see that

$$
\begin{aligned}
\mathbf{e}_{4m+q}&\leq hc\sum_{j=0}^{4m+q-1}\mathbf{e}_j+hc'\ \mathbf{e}_{4m+q}+R,\\
\mathbf{e}_{4m+q}&\leq\frac{hc}{1-hc'}\sum_{j=0}^{4m+q-1}\mathbf{e}_j+\frac{R}{1-hc'},
\end{aligned}
$$

where $c'=c_1+c_2+c_3+c_4$. So, utilizing the Gronwall inequality, we obtain

$$
\mathbf{e}_{4m+q}\leq\frac{R}{1-hc'}\exp\left(\frac{c}{1-hc'}Mh\right).
$$

Thus, $\mathbf{e}_{4m+q}\to 0$ as $h\to 0$, because R tends to zero as $h\to 0$. For the functions $\mathbf{k}(t,s)$ and $\mathbf{y}(t)$ with at least eighth order derivatives, we have $R=O(h^8)$ and therefore $\mathbf{e}_{4m+q}=O(h^8)$ which complements the proof. □

5. Numerical experiments

In this part, the results of several numerical examples are considered to certify the convergence and error bound of the proposed method. All computations have been done by running programming codes in Maple 2016 software.

Example 5.1. ([35]) Consider the following system:

$$
\begin{aligned}
{}_0^C D_t^{0.5} y_1(t) &= \frac{2t^{0.5}}{\sqrt{\pi}} - \frac{t^4}{24} + \frac{1}{2}\int_0^t (t-s)^2 \sqrt{y_2(s)}ds, \\
{}_0^C D_t^{1.5} y_2(t) &= \frac{4t^{0.5}}{\sqrt{\pi}} - \frac{31t^5}{120} + \frac{1}{4}\int_0^t (t+s)^2 y_1^2(s)ds,
\end{aligned}
$$

such that $y_1(0) = 0,\ y_2(0) = y_2'(0) = 0$. The true solution for the above system is

$$
\begin{cases} y_1(t) = t, \\ y_2(t) = t^2. \end{cases}
$$

The absolute errors for $y_1(t)$ and $y_2(t)$ by the presented technique with four dimensional blocks for different values of m and their comparison with those obtained by the method expressed in [35] are shown in Tables 1 and 2. These tables confirm that our approach is more accurate than the approach expressed in [35] while m increases. Fig 1 displays the numerical and true solutions and also the absolute errors of the achieved results when $m = 14$. This figure shows the adaptability of the numerical solutions to the exact solution and the highest degree of accuracy between the exact and numerical solutions. Furthermore, the computational times of implementing the proposed method to solve this problem with $m = 4,\ 9,\ 14$ are recorded as 3.042, 8.034 and 23.244 seconds, respectively.

Table 1. A comparison between the results of our method with those obtained by other method for $y_1(t)$ in Example 5.1.

	Method of [35]			Our method		
t_i	$\widehat{m} = 4$	$\widehat{m} = 8$	$\widehat{m} = 16$	$m = 4$	$m = 9$	$m = 14$
0	1.3627E-04	2.3828E-05	4.8392E-07	0.00000	0.00000	0.00000
0.1	2.2312E-04	3.4723E-05	5.7283E-07	9.7193E-11	9.7176E-11	2.0035E-12
0.2	7.8732E-05	4.0238E-05	9.2738E-08	2.1989E-09	1.7731E-10	4.1509E-11
0.3	3.4720E-04	4.8721E-05	6.9837E-07	2.8077E-10	2.5731E-10	5.6437E-12
0.4	3.9128E-04	9.8922E-06	7.7620E-07	4.0100E-09	3.3704E-10	8.0109E-11
0.5	4.6720E-04	6.5320E-05	8.2617E-07	4.4279E-10	4.1587E-10	8.7604E-12
0.6	5.3925E-04	7.9039E-05	8.9182E-07	5.7924E-09	4.9237E-10	1.1716E-10
0.7	8.7821E-05	7.7630E-05	9.6718E-07	5.3907E-10	5.6372E-10	8.0904E-12
0.8	6.3829E-04	8.9013E-05	2.3516E-07	7.4373E-09	6.2487E-10	1.4615E-10
0.9	8.9923E-04	1.0297E-04	3.2779E-06	3.7456E-10	6.6803E-10	8.0821E-12

Example 5.2. ([30,35]) Consider the following system:

$$
\begin{aligned}
{}_0^C D_t^{\alpha} y_1(t) &= e^t - 0.0035t\sinh(2t) + 0.007t^2\left(\cosh(2t) - t\sinh(2t)\right) + 0.0047t^4 \\
&\quad + 0.0281\int_0^t ts^2 y_2^2(s)ds, \\
{}_0^C D_t^{\alpha} y_2(t) &= \cosh(t) + t^2\Big(0.012e^{2t} - 0.0958e^t - 0.0239te^{2t} + 0.0958te^t - 0.0239t^2
\end{aligned}
$$

Table 2. A comparison between the results of our method with those obtained by other method for $y_2(t)$ in Example 5.1.

	Method of [35]			Our method		
t_i	$\widehat{m}=4$	$\widehat{m}=8$	$\widehat{m}=16$	$m=4$	$m=9$	$m=14$
0	7.7282E-05	3.2819E-05	9.3829E-08	0	0	0
0.1	3.6282E-04	4.8290E-05	2.8293E-07	3.1821E-11	3.1820E-11	1.6067E-12
0.2	4.6271E-04	5.9292E-05	4.2920E-07	2.8800E-09	4.1758E-10	1.4257E-10
0.3	6.3829E-04	7.3920E-06	5.2930E-07	2.0286E-09	1.9889E-09	1.1929E-10
0.4	7.9839E-04	6.9200E-05	5.8293E-07	3.7795E-08	6.1031E-09	2.1516E-09
0.5	7.1039E-04	8.3829E-05	7.9203E-07	1.4837E-08	1.4643E-08	9.0566E-10
0.6	8.2937E-05	9.3728E-05	3.8392E-06	1.8001E-07	3.0016E-08	1.0688E-08
0.7	9.4828E-04	1.9292E-04	9.3829E-07	5.5766E-08	5.1157E-08	3.4554E-09
0.8	1.2819E-03	2.3829E-04	4.2930E-06	5.5236E-07	9.3525E-08	3.3465E-08
0.9	2.0938E-03	4.2902E-04	5.2020E-06	1.5057E-07	1.4910E-07	9.4064E-09

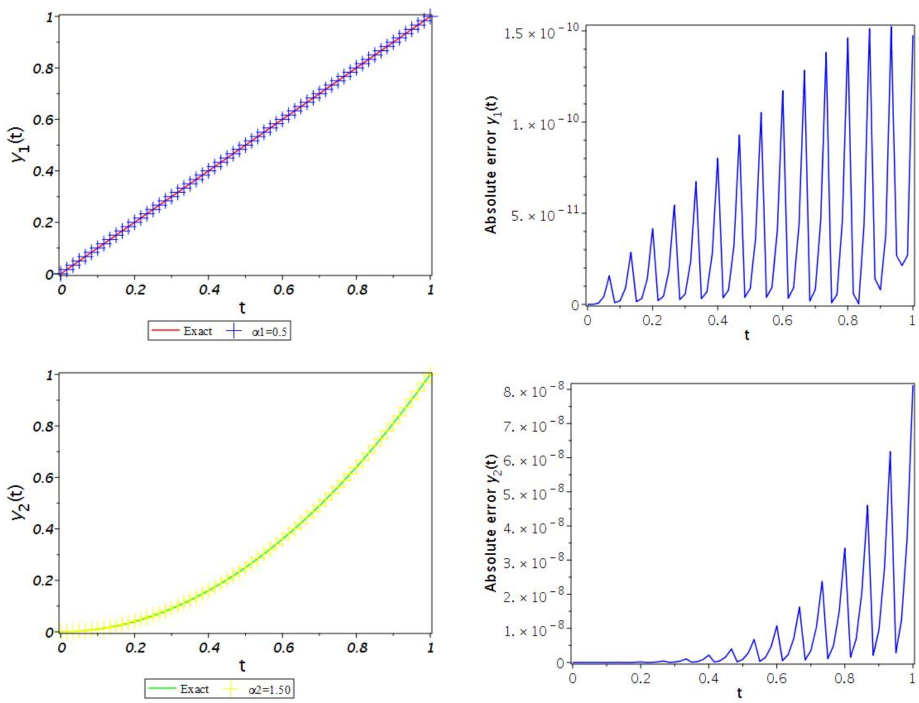


Figure 1. Graphs of the true and approximate solutions (left), and associated absolute error functions (right) with $m = 14$ in Example 5.1.

$$+ 0.0838\Big) + 0.0479 \int_0^t t^2 s y_1^2(s) ds,$$

where $0 < \alpha \le 1$ and $y_1(0) = y_2(0) = 0$. The analytic solution for this above system, whenever $\alpha = 1$ is

$$\begin{cases} y_1(t) = e^t - 1, \\ y_2(t) = \sinh(t). \end{cases}$$

The absolute errors generated for $y_1(t)$ and $y_2(t)$ by the proposed approach with four-dimensional blocks whenever $m = 4$ together with their comparison with those obtained by the techniques expressed in [30, 35] are reported in Tables 3 and 4. Also, the numerical results whenever $\alpha = 0.25$, 0.50 and 0.75 are given in Table 5. These tables confirm the high accuracy of our method. Fig 2 shows the numerical results for $\alpha = 0.25$, 0.50 and 0.75, and also the absolute errors of the derived results whenever $m = 4$. This figure shows the adaptability of the numerical solutions to the exact solution so that whenever α approaches to 1, the obtained solutions approach to the exact solution with $\alpha = 1$, and also shows the highest degree of accuracy between the exact and numerical solutions. Furthermore, the computational times of implementing the proposed method for this problem whenever $\alpha = 0.25$, 0.50, 0.75 and 1 are recorded as 3.744, 3.572, 3.510 and 2.652 seconds, respectively.

Table 3. A comparison between the results of our method with those obtained by other methods for $y_1(t)$ in Example 5.2.

t_i	Exact solution	Method of [35]	Method of [30]	Our method
0	0	0	0	0
0.1	0.105170918	0.105170920	0.105171462	0.105170918
0.2	0.221402758	0.221402767	0.221402843	0.221402758
0.3	0.349858807	0.349858815	0.349858925	0.349858807
0.4	0.491824697	0.491824712	0.491824876	0.491824697
0.5	0.648721270	0.648721281	0.648721538	0.648721270
0.6	0.822118800	0.822118817	0.822118921	0.822118800
0.7	1.013752707	1.013752712	1.013752950	1.013752707
0.8	1.225540928	1.225540935	1.225541126	1.225540928
0.9	1.459603111	1.459603123	1.459603498	1.459603111

Example 5.3. ([31, 32]) Consider the following system

$$\begin{aligned} {}_0^C D_t^\alpha y_1(t) &= \cosh(t) - \frac{1}{2}\sinh^2(t) - \frac{1}{6}t^4 - \frac{1}{2}t^2 + \int_0^t (t-s)\left(y_1^2(s) + y_2^2(s)\right) ds, \\ {}_0^C D_t^\alpha y_2(t) &= -(1+4t)\cosh(t) + 8\sinh(t) - 4t + \int_0^t (t-s)\left(y_1^2(s) - y_2^2(s)\right) ds \end{aligned}$$

where $1 < \alpha \leq 2$ and $y_1(0) = y_1'(0) = y_2'(0) = 1$ and $y_2(0) = -1$. The true solution for the above system whenever $\alpha = 2$ is

$$\begin{cases} y_1(t) = t + \cosh(t), \\ y_2(t) = t - \cosh(t). \end{cases}$$

The absolute errors produced for $y_1(t)$ and $y_2(t)$ via the proposed technique with four-dimensional blocks whenever $m = 24$, as well as the results of the methods

Table 4. A comparison between the results of our method with those obtained by other methods for $y_2(t)$ in Example 5.2.

t_i	Exact solution	Method of [35]	Method of [30]	Our method
0	0	0	0	0
0.1	0.100166750	0.100166759	0.100166876	0.100166750
0.2	0.201336002	0.201336013	0.201336208	0.201336002
0.3	0.304520293	0.304520310	0.304520421	0.304520293
0.4	0.410752325	0.410752320	0.410752509	0.410752325
0.5	0.521095305	0.521095318	0.521095690	0.521095305
0.6	0.636653582	0.636653597	0.636653633	0.636653582
0.7	0.758583701	0.758583721	0.758583924	0.758583701
0.8	0.888105982	0.888106004	0.888106427	0.888105982
0.9	1.026516725	1.026516743	1.026516942	1.026516725

Table 5. The outcomes yielded by our approach in Example 5.2 with $\alpha = 0.25$, 0.50 and 0.75

	$y_1(t)$			$y_2(t)$		
t_i	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.6723185	0.3815924	0.2049578	0.6226195	0.3577771	0.1938906
0.2	0.8670167	0.5776149	0.3654521	0.7483512	0.5100275	0.3281187
0.3	1.0412073	0.7578445	0.5256209	0.8429800	0.6329907	0.4493537
0.4	1.2149144	0.9382375	0.6926722	0.9284892	0.7445513	0.5656538
0.5	1.3958359	1.1256280	0.8704338	1.0133356	0.8523691	0.6811461
0.6	1.5884466	1.3242581	1.0617419	1.1022723	0.9610076	0.7986561
0.7	1.7961530	1.5374217	1.2691004	1.1989685	1.0738420	0.9204716
0.8	2.0218625	1.7680363	1.4949407	1.3063616	1.1937052	1.0486528
0.9	2.2685741	2.0189663	1.7417598	1.4281423	1.3234236	1.1852193

utilized in [31, 32] where $\alpha = 2$ are shown in Tables 6 and 7. Also, the numerical results whenever $\alpha = 1.25$, 1.50 and 1.75 are given in Table 8. These tables show that our approach for solving this example is more accurate than the techniques given in [31,32]. Also, the last row of this table illustrates the norm-2 of the absolute errors of our technique and those obtained by the method given in [31,32] for $\alpha = 2$. It can be observed that the results yielded by the current method are more accurate than the ones extracted by the method given in [31,32]. The absolute error functions and the numerical results for $\alpha = 1.25$, 1.50 and 1.75 whenever $m = 24$ are plotted in Fig 3, which representative the superiority of the raised technique. Also, the computational time of our method is 96.798 seconds, the method presented in [31] is 1512.60 seconds and the method presented in [32] is 40879.60 seconds. It can be seen that the computational time of our technique are much less than the methods given in [31,32].

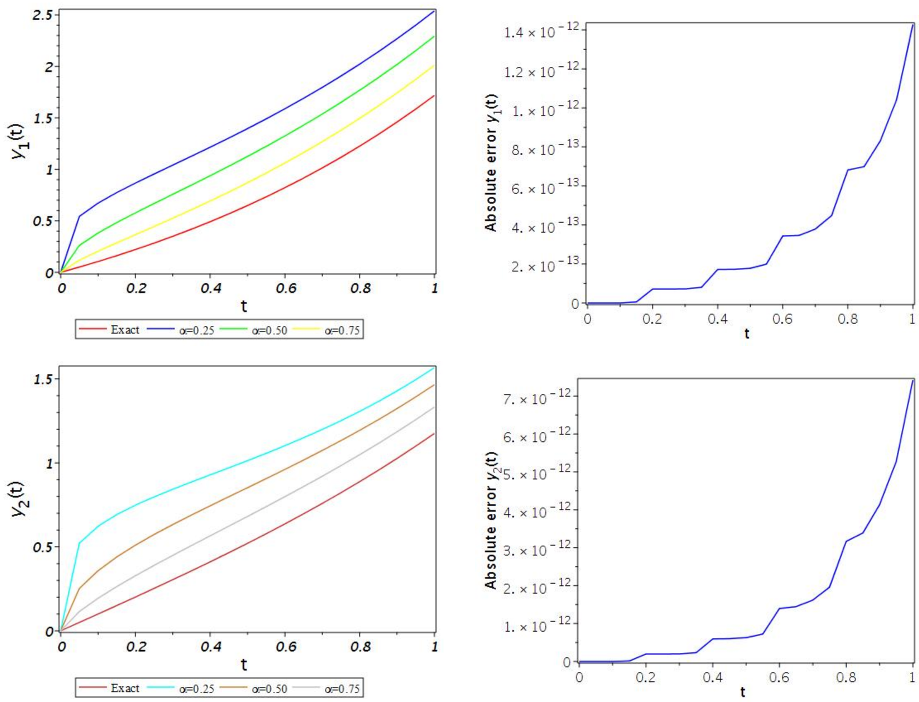


Figure 2. Graphs of the true solution and approximate solutions with some values of α (left), and associated absolute error functions whenever $\alpha = 1$ (right) with $m = 4$ in Example 5.2.

Table 6. A comparison between the results of our method with those obtained by other methods in Example 5.3 with $\alpha = 3$.

	$y_1(t)$		
t_i	Method of [31]	Method of [32]	Our method
0	1.2218E-04	4.6916E-11	0.00000000
0.1	3.9268E-05	3.6593E-13	2.1449E-19
0.2	1.4908E-05	2.4092E-13	3.0908E-18
0.3	4.0600E-05	2.3670E-13	1.4875E-17
0.4	3.6491E-05	3.8591E-13	4.6719E-17
0.5	3.4295E-07	5.8278E-11	1.1366E-16
0.6	4.0454E-05	5.1581E-13	2.3681E-16
0.7	4.8619E-05	2.0517E-13	4.4129E-16
0.8	1.8376E-05	1.4788E-13	7.6013E-16
0.9	5.8046E-05	7.1054E-13	1.2309E-15
$\|.\|\|_2$	1.6546E-04	7.4824E-11	6.4053E-16

Example 5.4. ([13]) Consider the system

$$ {}_0^C D_t^{\alpha} y_1(t) = -2t - 2t^3 - \frac{2}{5}t^5 + \int_0^t \left(y_1^2(s) + y_2^2(s) \right) ds, $$

Table 7. A comparison between the results of our method with those obtained by other methods in Example 5.3 with $\alpha = 3$.

	$y_2(t)$		
t_i	Method of [31]	Method of [32]	Our method
0	1.2218E-04	4.6916E-11	0.00000000
0.1	3.9267E-05	3.6515E-13	4.1633E-19
0.2	1.4919E-05	2.4336E-13	1.1453E-18
0.3	4.0655E-05	2.4403E-13	2.3039E-18
0.4	3.6655E-05	3.6804E-13	5.7206E-18
0.5	2.4369E-08	5.8232E-11	1.3701E-17
0.6	4.1125E-05	4.5519E-13	3.1298E-17
0.7	4.9651E-05	3.0687E-13	6.5077E-17
0.8	1.9757E-05	3.1641E-13	1.2570E-16
0.9	5.6400E-05	4.4642E-13	2.2672E-16
$\Vert.\Vert_2$	1.6558E-04	7.4787E-11	1.1948E-16

Table 8. The outcomes yielded by our approach in Example 5.3 with $\alpha = 2.25$, 2.50 and 2.75.

	$y_1(t)$			$y_2(t)$		
t_i	$\alpha = 2.25$	$\alpha = 2.50$	$\alpha = 2.75$	$\alpha = 2.25$	$\alpha = 2.50$	$\alpha = 2.75$
0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	1.1497030	1.1238158	1.11106723	-0.9497006	-0.9238155	-0.9110672
0.2	1.3187454	1.2676004	1.2373350	-0.9186902	-0.8675909	-0.8373337
0.3	1.4986950	1.4249477	1.3762819	-0.8983467	-0.8248761	-0.7762708
0.4	1.6880772	1.5940700	1.5270876	-0.8867829	-0.7937685	-0.7270348
0.5	1.8869603	1.7744130	1.6895187	-0.8833592	-0.7734898	-0.6893415
0.6	2.0962495	1.9660982	1.8637056	-0.8879067	-0.7637877	-0.6632278
0.7	2.3174811	2.1697386	2.0500629	-0.9004454	-0.7647059	-0.6489556
0.8	2.5527615	2.3863711	2.2492598	-0.9210395	-0.7764681	-0.6469618
0.9	2.8047682	2.6174399	2.4622128	-0.9497007	-0.7994052	-0.6578284

$$
{}_{0}^{C}D_t^{\alpha} y_2(t) = -\frac{2}{3}t^3 - \frac{1}{5}t^5 + \int_0^t (t-s)\left(y_1^2(s) - y_2^2(s)\right) ds,
$$

where $2 < \alpha \leq 3$ and $y_1(0) = y_1'(0) = y_2(0) = 1$, $y_1''(0) = y_2''(0) = 2$ and $y_2'(0) = -1$. The true solution of this system is

$$
\begin{cases} y_1(t) = 1 + t + t^2, \\ y_2(t) = 1 - t + t^2. \end{cases}
$$

A comparison between the results extracted using the method of [13] with those obtained by our method in the case of $\alpha = 3$ is listed in Table 9 and one can see that our approach is more capable than the technique represented in [13]. The absolute error functions and the numerical results for $\alpha = 2.25, 2.50, 2.75$ whenever

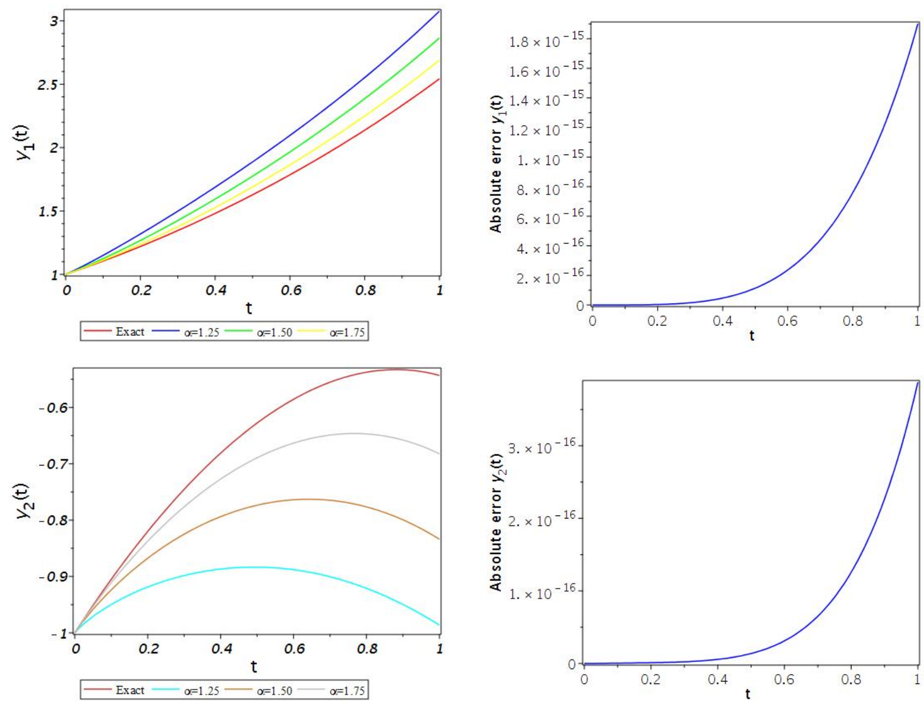


Figure 3. Graphs of the true solution and approximate solutions with some values of α (left), and associated absolute error functions whenever $\alpha = 2$ (right) with $m = 24$ in Example 5.3.

Table 9. A comparison between the results of our method with those obtained by other method in Example 5.4.

	$y_1(t)$		$y_2(t)$	
	Method of [13]	Our method	Method of [13]	Our method
0	4.0000E-20	0.0000	3.0000E-20	0.0000
0.1	0.0000000	0.0000	1.0000E-20	0.0000
0.2	0.0000000	0.0000	2.3000E-19	0.0000
0.3	1.0000E-19	0.0000	6.3000E-19	0.0000
0.4	2.0000E-19	0.0000	1.2000E-18	0.0000
0.5	3.0000E-19	0.0000	2.1000E-18	0.0000
0.6	5.0000E-19	0.0000	3.1000E-18	0.0000
0.7	7.0000E-19	0.0000	4.4000E-18	0.0000
0.8	1.0000E-18	0.0000	5.9000E-18	0.0000
0.9	1.4000E-18	0.0000	7.6000E-18	0.0000
1	1.7000E-18	0.0000	9.6000E-18	0.0000

$m = 4$ are plotted in Fig 3, which representative the superiority of the raised technique. Also, the computational times of implementing the presented method f with $\alpha = 2.25$, 2.50, 2.75 , 3 are recorded as 3.495, 3.448, 3.354 and 3.167 seconds, respectively.

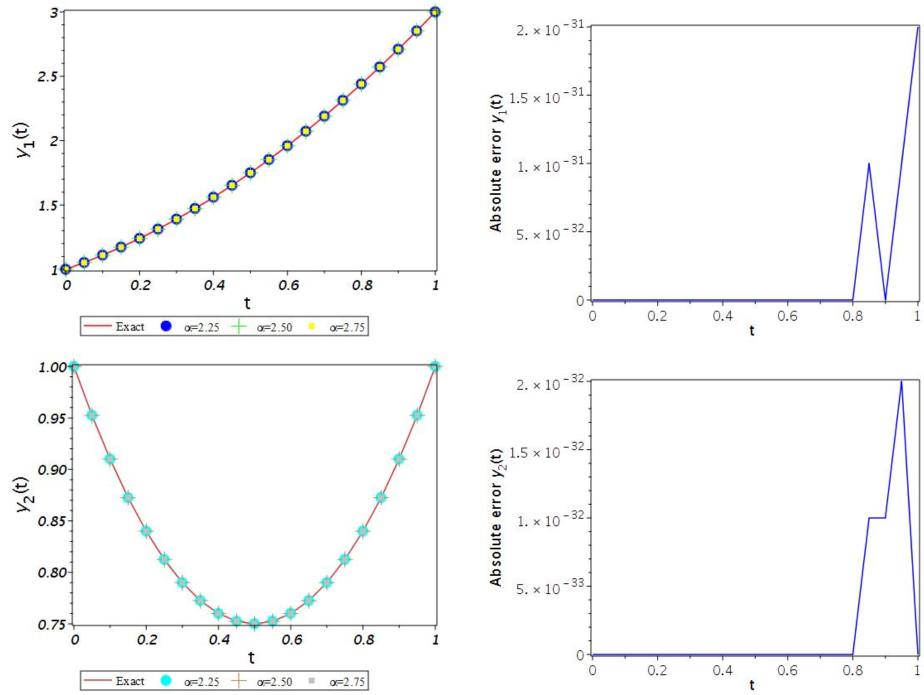


Figure 4. Graphs of the true solution and approximate solutions with some values of α (left), and associated absolute error functions whenever $\alpha = 3$ (right) with $m = 4$ in Example 5.4.

Example 5.5. Consider the following system

$$\begin{aligned}
{}^C_0D_t^\alpha y_1(t) =& t^{2-\alpha}\mathbf{E}_{1,3-\alpha}(t) + \frac{e^{2t}}{2}(t-1) + \frac{e^{4t}}{4}(3t-1) + \frac{3}{4}(t+1) \\
&+ \int_0^t \big((t-2s)\, y_1^2(s) + (t-4s)\, y_2^2(s)\big)\, ds, \\
{}^C_0D_t^\alpha y_2(t) =& 4t^{2-\alpha}\mathbf{E}_{1,3-\alpha}(2t) + \frac{e^{4t}}{4}(3t-s) + \frac{e^{6t}}{6}(5t-1) + \frac{5}{12}(t+1) \\
&+ \int_0^t \big((t-4s)\, y_2^2(s) + (t-6s)\, y_3^2(s)\big)\, ds, \\
{}^C_0D_t^\alpha y_3(t) =& 9t^{2-\alpha}\mathbf{E}_{1,3-\alpha}(3t) + \frac{e^{2t}}{2}(t-1) + \frac{e^{6t}}{6}(5t-1) + \frac{2}{3}(t+1) \\
&+ \int_0^t \big((t-6s)\, y_3^2(s) + (t-2s)\, y_1^2(s)\big)\, ds,
\end{aligned}$$

where $1 < \alpha \leq 2$ and $y_1(0) = y_2(0) = y_3(0) = y_1'(0) = 1$, $y_2'(0) = 2$ and $y_3'(0) = 3$. The true solution of this system is

$$\begin{cases} y_1(t) = e^t, \\ y_2(t) = e^{2t}, \\ y_3(t) = e^{3t}. \end{cases}$$

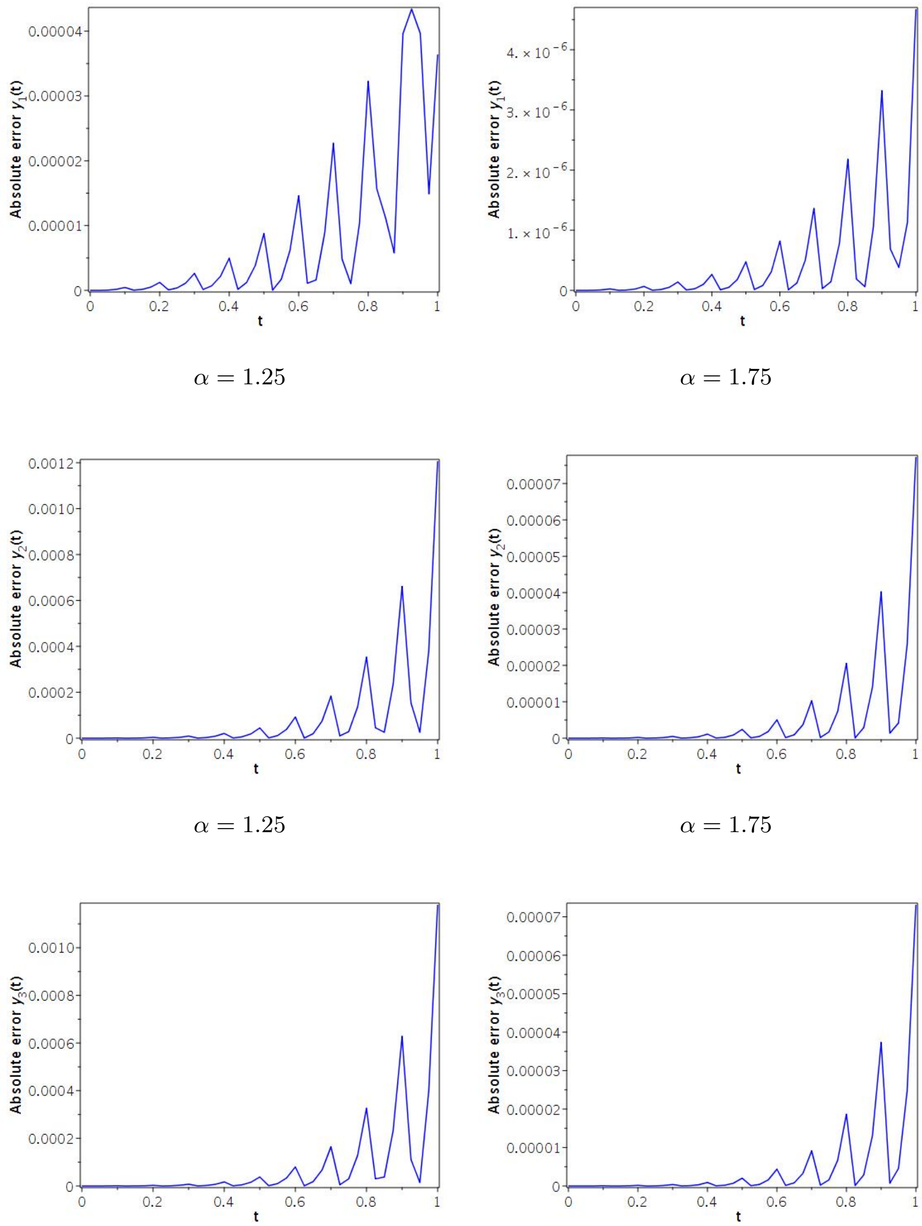
Table 10. The outcomes yielded by our approach in Example 5.5 with $\alpha = 2$.

	$y_1(t)$		$y_2(t)$		$y_3(t)$	
t_i	$m = 4$	$m = 9$	$m = 4$	$m = 9$	$m = 4$	$m = 9$
0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.1	9.0195E-13	4.6539E-13	2.4826E-11	6.3752E-12	2.3868E-11	5.9216E-12
0.2	2.8171E-10	1.2802E-12	4.2114E-09	1.7678E-11	3.9362E-09	1.6436E-11
0.3	2.7781E-10	2.5210E-12	3.2685E-09	3.2660E-11	2.9994E-09	3.0230E-11
0.4	8.8588E-10	3.9447E-12	1.5929E-08	3.9454E-11	1.5064E-08	3.5689E-11
0.5	6.9728E-10	4.7091E-12	6.1407E-09	7.7371E-14	5.4708E-09	4.4727E-12
0.6	1.6747E-09	3.1174E-12	3.7634E-08	1.8369E-10	3.6006E-08	1.8631E-10
0.7	3.9418E-10	2.8702E-12	2.4859E-08	7.3597E-10	2.5201E-08	7.3237E-10
0.8	1.3993E-09	1.1566E-11	3.7399E-08	2.1351E-09	3.6073E-08	2.1225E-09
0.9	2.5071E-09	1.0587E-12	2.7219E-07	5.3306E-09	2.6961E-07	5.3280E-09
1	2.8243E-09	1.2501E-10	1.8383E-07	1.2055E-08	1.8654E-07	1.2178E-08

Table 11. The outcomes yielded by our approach in Example 5.5 with $\alpha = 1.25$ and 1.75

	$y_1(t)$		$y_2(t)$		$y_3(t)$	
t_i	$\alpha = 1.25$	$\alpha = 1.75$	$\alpha = 1.25$	$\alpha = 1.75$	$\alpha = 1.25$	$\alpha = 1.75$
0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.1	4.4785E-07	2.4380E-08	1.0764E-06	5.9134E-08	4.2108E-07	4.5184E-08
0.2	1.2312E-06	6.5548E-08	3.5472E-06	1.9058E-07	2.7642E-06	1.4872E-07
0.3	2.6147E-06	1.3858E-07	9.0447E-06	4.8330E-07	7.2362E-06	3.8700E-07
0.4	4.9595E-06	2.6424E-07	2.0705E-05	1.1081E-06	1.7041E-05	9.1180E-07
0.5	8.7622E-06	4.7486E-07	4.4572E-05	2.4042E-06	3.7757E-05	2.0319E-06
0.6	1.4579E-05	8.1863E-07	9.1938E-05	5.0348E-06	8.0150E-05	4.3648E-06
0.7	2.2699E-05	1.3617E-06	1.8312E-04	1.0276E-05	1.6427E-04	9.1230E-06
0.8	3.2266E-05	2.1800E-06	3.5319E-04	2.0535E-05	3.2609E-04	1.8640E-05
0.9	3.9595E-05	3.3193E-06	6.6091E-04	4.0240E-05	6.2822E-04	3.7300E-05
1	3.6412E-05	4.6761E-06	1.2071E-03	7.7299E-05	1.1804E-03	7.3119E-05

The outcomes derived by our method for $y_1(t)$, $y_2(t)$ and $y_3(t)$ with four-dimensional blocks for $m = 4$ and 9 and some values of α are listed in Table 10. Table 11 displays the absolute errors obtained with different fractional orders. The outcome shows that as α tends to 2, the obtained solutions approach to the exact solution with $\alpha = 2$. It can be observed that the present technique for solving this system has a high capability. The absolute error functions and the numerical results for $\alpha = 1.25$, 1.50, 1.75 whenever $m = 9$ are displayed in Fig 5. This figure representative the superiority of the established technique. Furthermore, the computational times for this problem whenever $\alpha = 1.25$, 1.50, 1.75, 2 are recorded as 33.462, 30.218, 28.470 and 26.317 seconds, respectively.



$\alpha = 1.25$ $\alpha = 1.75$

$\alpha = 1.25$ $\alpha = 1.75$

$\alpha = 1.25$ $\alpha = 1.75$

Figure 5. Graphs of the absolute error functions for some values of α with $m = 9$ in Example 5.5.

6. Conclusion

This work suggested a numerical scheme to solve a class of nonlinear systems of fractional integro-differential equations. The presented technique was written via five-point Gauss-Lobatto quadrature method and the block-by-block technique. The order of convergence of the adopted method was proved to be $O(h^8)$. Also, a short computing time was needed to implement this method. Numerical experiments were considered to reveal the effectiveness of the method. The derived outcomes confirmed the high accuracy of the approach.

Data availability

Data sharing is not applicable to this study.

Conflict of interest

The authors declare that there is not any conflict of interest regarding this paper.

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