Journal of Applied Analysis and Computation
Volume 14, Number 1, February 2024, 235–262
Website:http://www.jaac-online.com
DOI:10.11948/20230194

EXECUTION OF A NOVEL DISCRETIZATION APPROACH FOR SOLVING VARIABLE-ORDER CAPUTO-RIESZ TIME-SPACE FRACTIONAL SCHRÖDINGER EQUATIONS

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Abstract This work deals with the variable-order Caputo-Riesz (VO-CR) time-space fractional Schrödinger equations with the help of the Pell discretization method. For the first step, we separate the proposed problem into real and imaginary parts. Then, expanding the functions with respect to Pell polynomials and utilizing the required operational matrices. The operational matrices, together with the Pell discretization method, reduce the problem into a system of algebraic equations. It should be noted that the technique of obtaining the operational matrices strongly affects the precision of the numerical method process. Finally, we implement the proposed approach in several numerical experiments to confirm the theoretical scheme. And also, the comparison of obtained results with some existing methods is displayed in tables.

Keywords Fractional Schrödinger equations, Pell polynomials, modified operational matrix, Caputo-Riesz fractional derivative, variable-order Caputo derivative.

MSC(2010) 35R11, 65M70.

1. Introduction

In recent years, fractional derivatives with various definitions and features have been introduced. So that the mathematical models involving fractional derivatives are powerful instruments to describe the behavior of real-life phenomena [5, 27, 28, 38].

The nonlinear Schrödinger equation has appeared in various fields in quantum (quantum mechanics, quantum gravity, open quantum systems), photonics, plasma physics, and other branches of physics [1, 11, 23, 24]. On the other hand, fractional differential equations are used more frequently to describe the behaviour of physical phenomena, because fractional derivatives enable the description of the memory and hereditary features of diverse substances [29]. Two decades ago, the fractional Schrödinger equation was generalized by Laskin [20–22], which plays an important

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role in fractional quantum mechanics. The Schrödinger equation has been studied by many researchers applying various numerical techniques. For instance, Bhrawy et al. [3] applied the Jacobi spectral method for approximating the solution of multi-dimensional time-fractional Schrödinger equations. Zhang et al. [39] provided the residual power series method for time-fractional Schrödinger equations. Bhrawy and Zaky [4] constructed the Jacobi-Gauss-Lobatto collocation approach with exponential accuracy to solve VO-fractional Schrödinger equations. Moreover, many other methods for solving the fractional Schrödinger equations have been existed, which can be found in Refs. [13, 15, 36, 37, 40]. Therefore, the Schrödinger equation including the VO-fractional derivative has been studied in a limited number of papers. This issue motivated us to introduce an efficient numerical method to solve this equation.

In this paper, we introduce a precision algorithm to calculate the approximate solution of VO-CR-time-space fractional Schrödinger equations. To reach the desired aim, we use the Pell polynomials together with the spectral approach and appropriate operational matrices. Due to the numerical process, the proposed problem reduces to a system of algebraic equations and the approximate solution is determined. This work includes a new technique for obtaining the operational matrices. In the procedure of presenting the components of these matrices, we apply the properties of Pell polynomials. So that, the structure of these matrices plays an important role in the accuracy of the proposed method.

It is worth noting that spectral methods are recognized as powerful and efficient methods for solving diverse differential equations. Many researchers have illustrated their interest in using these methods. For example, Babaei et al. [2] used the Sinc collocation method for finding the approximate solution of VO-fractional integro-partial differential equations. Dehestani and Ordokhani [7] provided a new method based on Legendre–Gauss–Lobatto quadrature and discrete shifted Hahn polynomials for solving Caputo–Fabrizio fractional Volterra partial integro-differential equations. Ghimire et al. [12] applied hybrid Chebyshev polynomial for elliptic partial differential equations. Khader and Saad [18] solved the fractional Fisher equation with the help of the Chebyshev spectral collocation method. Liu et al. [26] considered the fully discrete spectral methods for solving time fractional nonlinear Sine–Gordon equation. Interested readers to this subject can observe more papers in [8, 9, 14, 34].

Pell polynomials have been derived from the Pell numbers, which were initially examined by the Greek mathematician Archimedes around 200 BC [19]. Then, John Pell's work involved solving what are now known as Pell equations and discovering a recursive relation that gave rise to the concept of Pell numbers. In recent years, Pell polynomials have been recognized as important contributors to the field of number theory, and they have found applications in various areas including cryptography, coding theory, Astrophysics, and combinatorics [19, 31, 35]. The Pell polynomials [16] have been used in a few research papers containing numerical analysis [32, 33]. An important advantage of these polynomials is that the coefficients of individual terms are integers. Therefore, the coefficients of these polynomials are not creating a computational error.

We consider the variable-order Caputo-Riesz time-space fractional Schrödinger equation as follows:

$$i\mathbf{D}_t^{\alpha(x,t)}\psi(x,t) = -\eta\frac{\partial^\beta}{\partial|x|^\beta}\psi(x,t) + \vartheta|\psi|^2\psi(x,t) + \varphi(x)\psi(x,t) + R(x,t), \tag{1.1}$$

$$0 < \alpha(x,t) \leq 1, \quad 1 < \beta \leq 2, \quad (x,t) \in [0,1] \times [0,1],$$

subject to the initial condition and boundary conditions

$$\psi(x,0) = k(x), \quad \psi(0,t) = s(t), \quad \psi(1,t) = r(t). \tag{1.2}$$

Here, η and ϑ are constants, $\varphi(x)$ denotes trapping potential and, $\psi(x,t)$, $R(x,t)$, $k(x)$, $s(t)$ and $r(t)$ are complex functions. So that, we define the imaginary and real parts of these functions as follows:

$$\begin{aligned}
&\psi(x,t) = \mathcal{U}(x,t) + i\mathcal{V}(x,t),\\
&R(x,t) = f(x,t) + ig(x,t),\\
&\varphi(x) = \varphi_0(x) + i\varphi_1(x),\\
&k(x) = k_0(x) + ik_1(x),\\
&s(t) = s_0(t) + is_1(t),\\
&r(t) = r_0(t) + ir_1(t).
\end{aligned}$$

Therefore, due to the above relations, Eq. (1.1) can be rewritten as follows:

$$\begin{aligned}
&\left(-\mathbf{D}_t^{\alpha(x,t)}\mathcal{V}(x,t) + \eta\frac{\partial^\beta}{\partial|x|^\beta}\mathcal{U}(x,t) - \vartheta(\mathcal{U}^2 - \mathcal{V}^2)\mathcal{U}(x,t) - \varphi_0(x)\mathcal{U}(x,t) - f(x,t)\right)\\
&+ i\left(\mathbf{D}_t^{\alpha(x,t)}\mathcal{U}(x,t) + \eta\frac{\partial^\beta}{\partial|x|^\beta}\mathcal{V}(x,t) - \vartheta(\mathcal{U}^2 - \mathcal{V}^2)\mathcal{V}(x,t)\right.\\
&\left.-\varphi_1(x)\mathcal{V}(x,t) - g(x,t)\right)\\
&=0.
\end{aligned}\tag{1.3}$$

Next, by separating the real and imaginary parts of the equation mentioned above a couple system of variable-order Caputo-Riesz time-space fractional partial differential equations is obtained:

$$\begin{aligned}
&-\mathbf{D}_t^{\alpha(x,t)}\mathcal{V}(x,t) = -\eta\frac{\partial^\beta}{\partial|x|^\beta}\mathcal{U}(x,t) + \vartheta(\mathcal{U}^2 + \mathcal{V}^2)\mathcal{U}(x,t) + \varphi_0(x)\mathcal{U}(x,t) + f(x,t),\\
&\mathbf{D}_t^{\alpha(x,t)}\mathcal{U}(x,t) = -\eta\frac{\partial^\beta}{\partial|x|^\beta}\mathcal{V}(x,t) + \vartheta(\mathcal{U}^2 + \mathcal{V}^2)\mathcal{V}(x,t) + \varphi_1(x)\mathcal{V}(x,t) + g(x,t),
\end{aligned}\tag{1.4}$$

with the initial and boundary conditions

$$\begin{aligned}
&\mathcal{U}(x,0) = k_0(x), \quad \mathcal{V}(x,0) = k_1(x),\\
&\mathcal{U}(0,t) = s_0(t), \quad \mathcal{V}(0,t) = s_1(t),\\
&\mathcal{U}(1,t) = r_0(t), \quad \mathcal{V}(1,t) = r_1(t).
\end{aligned}\tag{1.5}$$

The fractional derivatives in the main problem are in variable-order Caputo fractional (VO-CF) [6] and Caputo-Riesz (CR) [30] sense, respectively. So that, these derivatives are defined as follows:

$$\mathbf{D}_t^{\alpha(x,t)}f(t) = \frac{1}{\Gamma(1-\alpha(x,t))}\int_0^t \frac{f^{(q)}(\eta)}{(t-\eta)^{\alpha(x,t)+1-q}}d\eta, \quad q-1 < \alpha(x,t) \leq q \in \mathbb{N}, \tag{1.6}$$

and

$$\frac{\partial^{\beta}}{\partial|x|^{\beta}}h(x)=\frac{1}{2\cos\frac{\pi\beta}{2}}\left(D^{\beta}_{*+}h(x)+D^{\beta}_{*-}h(x)\right),\quad \beta\neq 1. \tag{1.7}$$

Here, D^{β}_{*+} and D^{β}_{*-} denote the left- and right-sided Riemann-Liouville fractional derivatives of order β, which are defined by left- and right-sided Caputo fractional derivatives:

$$\begin{aligned}D^{\beta}_{*+}h(x)&=D^{\beta}_{+}h(x)+\sum_{i=0}^{\lceil\beta\rceil-1}\frac{h^{(i)}(0)}{\Gamma(i+1-\beta)}x^{i-\beta},\\ D^{\beta}_{*-}h(x)&=D^{\beta}_{-}h(x)+\sum_{i=0}^{\lceil\beta\rceil-1}\frac{(-1)^{i}h^{(i)}(1)}{\Gamma(i+1-\beta)}(1-x)^{i-\beta}.\end{aligned} \tag{1.8}$$

The outline of this paper is structured as follows. The next section is devoted to the Pell polynomials and their attributes, which contain the relation between Pell polynomials and Taylor polynomials. Section 3 presents the algorithm for obtaining the pseudo operational matrix of VO-CF-derivative. The idea of presenting the modified operational matrices of RF-derivative is discussed in Section 4. In Section 5, the modified operational matrix of integration with the help of the Pell polynomials feature is presented. The numerical procedure for getting the approximation solution of VO-CR-time-space fractional Schrödinger equations is provided in Section 6. In Section 7, the theoretical approach is implemented in several examples. Finally, conclusions and remarks are given in Section 8.

2. Pell polynomials

The Pell polynomials are defined by the following recurrence relation [16]:

$$P_{m+1}(x)=2xP_m(x)+P_{m-1}(x),\quad P_0(x)=1,\quad P_1(x)=x,\quad m\geq 1. \tag{2.1}$$

Also, the explicit formula of Pell polynomials is defined as follows [16]:

$$P_m(x)=\sum_{n=0}^{\left[\frac{m-1}{2}\right]}\binom{m-n-1}{n}(2x)^{m-2n-1},\quad m\neq 0. \tag{2.2}$$

The Binet formula to get Pell polynomials is defined as follows [19]:

$$P_m(x)=\frac{W^m(x)-V^m(x)}{W(x)-V(x)},$$

where

$$W(x)=\frac{U(x)+\sqrt{U^2(x)+4Q(x)}}{2},\quad V(x)=\frac{U(x)-\sqrt{U^2(x)+4Q(x)}}{2}.$$

By considering the above relation and $W(x)=2x, Q(x)=1$, the Pell polynomials can be obtained

$$P_m(x)=\frac{\left(x+\sqrt{x^2+1}\right)^m-\left(x-\sqrt{x^2+1}\right)^m}{2^m\sqrt{x^2+1}},\quad m\geq 0.$$

Due to the Pell polynomials statement, these polynomials can be defined with respect to Taylor polynomials [10]. So that, the elements of the coefficient matrix are obtained as follows [32]:

$$P(x) = QT(x), \tag{2.3}$$

where

$$P(x) = [P_0(x), P_1(x), \ldots, P_M(x)]^T, \quad T(x) = [1, x, \ldots, x^M]^T,$$

and

$$Q = [q_{nm}], \quad q_{n1} = \begin{cases} 1, \text{ if } n \text{ even}, \\ 0, \text{ otherwise}, \end{cases}$$

$$q_{nm} = 2q_{n-1,m-1} + 2q_{n-2,m}, \quad q_{nn} = 2^{n-1}, \quad n > m.$$

Moreover, the components of the Taylor vector can be expanded with Pell polynomials [32]:

$$x^n = \begin{cases} 2^{1-n} \displaystyle\sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{k} P_{n-2k}(x), & n \text{ is odd}, \\ 2^{1-n} \displaystyle\sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{k} P_{n-2k}(x) + (-1)^{[\frac{n}{2}]+1} 2^{-n} \binom{n}{\frac{n}{2}} P_0(x), & n \text{ is even}. \end{cases} \tag{2.4}$$

A function $f(x)$ defined over $[0,1]$ may be approximated by Pell polynomials as:

$$f(x) = \sum_{i=0}^{\infty} c_i P_i(x),$$

If the infinite expansion in the above equation is truncated, we get

$$f(x) \simeq \sum_{i=0}^{M} c_i P_i(x) = C^T P(x),$$

where the elements of the coefficient vector are calculated by applying the regular grid points. According to Taylor expansion and Eq. (2.4), we get the expansion in terms of Pell polynomials as follows:

$$\begin{aligned} f(x) = &\sum_{j=0}^{\infty} \sum_{k=0}^{[\frac{2j+1}{2}]} (-1)^k \binom{2j+1}{k} \frac{2^{-2j} f^{(2j+1)}(0)}{(2j+1)!} P_{2j+1-2k}(x) \\ &+ \sum_{j=0}^{\infty} \sum_{k=0}^{[j]} (-1)^k \binom{2j}{k} \frac{2^{1-2j} f^{(2j)}(0)}{(2j)!} P_{2j-2k}(x) \\ &+ \sum_{j=0}^{\infty} (-1)^{[j]+1} \binom{2j}{j} \frac{2^{-2j} f^{(2j)}(0)}{(2j)!} P_0(x). \end{aligned} \tag{2.5}$$

Also, the truncated form of the above series with $M+1$ sentences is defined as follows:

$$f_M(x) = \sum_{j=0}^{M} \sum_{k=0}^{[\frac{2j+1}{2}]} (-1)^k \binom{2j+1}{k} \frac{2^{-2j} f^{(2j+1)}(0)}{(2j+1)!} P_{2j+1-2k}(x) \tag{2.6}$$

$$+\sum_{j=0}^{M}\sum_{k=0}^{[j]}(-1)^k\binom{2j}{k}\frac{2^{1-2j}f^{(2j)}(0)}{(2j)!}P_{2j-2k}(x)$$
$$+\sum_{j=0}^{M}(-1)^{[j]+1}\binom{2j}{j}\frac{2^{-2j}f^{(2j)}(0)}{(2j)!}P_0(x).$$

Hence, we have

$$\begin{aligned}|f(x)-f_M(x)| \leq & \sum_{k=0}^{[\frac{2M+3}{2}]}\binom{2M+3}{k}\frac{C_1}{2^{2M+2}(2M+3)!}|P_{2M+3-2k}(x)| \\ & + \sum_{k=0}^{[M+1]}\binom{2M+2}{k}\frac{C_2}{2^{2M+1}(2M+2)!}|P_{2M+2-2k}(x)| \\ & + \binom{2M+2}{M+1}\frac{C_2}{2^{2M+2}(2M+2)!}|P_0(x)|, \end{aligned} \tag{2.7}$$

where

$$C_1 \geq \sup_{x\in[0,1]}|f^{(2M+3)}(x)|, \quad C_2 \geq \sup_{x\in[0,1]}|f^{(2M+2)}(x)|.$$

Therefore, the following inequality can be obtained:

$$\begin{aligned}\|f(x)-f_M(x)\|_\infty \leq & \sum_{k=0}^{[\frac{2M+3}{2}]}\binom{2M+3}{k}\frac{C_1Q_1}{2^{2M+2}(2M+3)!} \\ & + \sum_{k=0}^{[M+1]}\binom{2M+2}{k}\frac{C_2Q_2}{2^{2M+1}(2M+2)!} \\ & + \binom{2M+2}{M+1}\frac{C_2}{2^{2M+2}(2M+2)!}, \end{aligned} \tag{2.8}$$

where $Q_1 \geq \sup_{x\in[0,1]}|P_{2M+3-2k}(x)|$, and $Q_2 \geq \sup_{x\in[0,1]}|P_{2M+2-2k}(x)|$.

3. Pseudo operational matrix of VO-CF-derivative

This section aims to present details of the method to obtain the pseudo operational matrix of VO-CF-derivative. So that we obtain

$$\mathbf{D}_t^{\alpha(x,t)}P(t) \simeq t^{q-\alpha(x,t)}\Lambda(x,t)P(t), \quad q-1<\alpha(x,t)\leq q, \tag{3.1}$$

where $\Lambda(x,t)$ denotes the pseudo operational matrix of VO-CF-derivative. Now, to calculate the elements of the proposed matrix, we use Eq. (2.2) and the feature of VO-CF-derivative:

$$\begin{aligned}&\mathbf{D}_t^{\alpha(x,t)}P_n(t) \\ &= \sum_{m=0}^{[\frac{n-1}{2}]}\binom{n-m-1}{m}2^{n-2m-1}\mathbf{D}_t^{\alpha(x,t)}\left(t^{n-2m-1}\right)\end{aligned} \tag{3.2}$$

$$= \sum_{m=\lceil\frac{n-1-q}{2}\rceil}^{\left[\frac{n-1}{2}\right]} 2^{n-2m-1}\binom{n-m-1}{m}\frac{\Gamma(n-2m)}{\Gamma(n-2m-\alpha(x,t))}t^{n-2m-1-\alpha(x,t)}$$

$$= t^{q-\alpha(x,t)} \sum_{m=\lceil\frac{n-1-q}{2}\rceil}^{\left[\frac{n-1}{2}\right]} \mu_{n,m}(x,t)t^{n-2m-1-q},$$

where

$$\mu_{n,m}(x,t) = 2^{n-2m-1}\binom{n-m-1}{m}\frac{\Gamma(n-2m)}{\Gamma(n-2m-\alpha(x,t))}.$$

Next, to continue the process of obtaining the component of the considered matrix, we evaluate $t^{n-2m-1-q}$ in view of Pell polynomials:

$$t^{n-2m-1-q} \simeq \sum_{i=0}^{N} a_i P_i(t).$$

As a result, by substituting the above approximation in Eq. (3.2), we get

$$\begin{aligned}
&\mathbf{D}_t^{\alpha(x,t)}P_n(t) \qquad (3.3)\\
&\simeq t^{q-\alpha(x,t)}\sum_{i=0}^{N}\left(\sum_{m=\lceil\frac{n-1-q}{2}\rceil}^{\left[\frac{n-1}{2}\right]}\mu_{n,m}(x,t)a_i\right)P_i(t)\\
&= t^{q-\alpha(x,t)}\left[\sum_{m=\lceil\frac{n-1-q}{2}\rceil}^{\left[\frac{n-1}{2}\right]}\lambda_{0,n,m}(x,t)\ \ldots\ \sum_{m=\lceil\frac{n-1-q}{2}\rceil}^{\left[\frac{n-1}{2}\right]}\lambda_{N,n,m}(x,t)\right]P(t),
\end{aligned}$$

where $\lambda_{i,n,m}(x,t) = \mu_{n,m}(x,t)a_i$. To clarify the above process, we consider $N = 2$ and $\alpha(x,t) = 0.5$. Then, we obtain

$$\mathbf{D}_t^{0.5}P(t) \simeq t^{1-0.5}\Lambda(x,t)P(t),$$

where

$$\Lambda(x,t) = \begin{bmatrix} 0 & 0 & 0\\ 2.256758334191025 & 0 & 0\\ 0 & 3.0090111122547 & 0\end{bmatrix}.$$

4. Modified operational matrices of RF-derivative

In this section, our goal is to present the technique of obtaining the modified operational matrices of RF-derivative for basis function. Thus, we achieve

$$\frac{\partial^\beta}{\partial|x|^\beta}P(x) \simeq x^{-\beta}\mathbf{\Theta}P(x) + (1-x)^{-\beta}\mathbf{\Delta}P(x), \quad 1<\beta<2. \qquad (4.1)$$

Herein, $\mathbf{\Theta}$ and $\mathbf{\Delta}$ are the modified operational matrices of RF-derivative. To reach the desired goal, we apply the Eqs. (2.1), (2.2) and definition of RF-derivative on Pell polynomials:

$$\frac{\partial^\beta}{\partial|x|^\beta}P_0(x) = \frac{1}{2\cos\frac{\pi\beta}{2}}\left(\frac{1}{\Gamma(1-\beta)}x^{-\beta} + \frac{1}{\Gamma(1-\beta)}(1-x)^{-\beta}\right) \qquad (4.2)$$

$$
\begin{aligned}
&= x^{-\beta}\frac{1}{2\cos\frac{\pi\beta}{2}\Gamma(1-\beta)}P_0(x) + (1-x)^{-\beta}\frac{1}{2\cos\frac{\pi\beta}{2}\Gamma(1-\beta)}P_0(x)\\
&= x^{-\beta}\left[\begin{array}{cccc}\frac{1}{2\cos\frac{\pi\beta}{2}\Gamma(1-\beta)} & 0\ 0 & \ldots & 0\end{array}\right]P(x)\\
&\quad +(1-x)^{-\beta}\left[\begin{array}{cccc}\frac{1}{2\cos\frac{\pi\beta}{2}\Gamma(1-\beta)} & 0\ 0 & \ldots & 0\end{array}\right]P(x).
\end{aligned}
$$

And also, for $m = 1, 2, \ldots, M$, we have

$$
\begin{aligned}
\frac{\partial^\beta}{\partial|x|^\beta}P_m(x) &= \frac{1}{2\cos\frac{\pi\beta}{2}}\left\{D^\beta_{*+}P_m(x) + D^\beta_{*-}P_m(x)\right\} \qquad (4.3)\\
&= \frac{1}{2\cos\frac{\pi\beta}{2}}\left\{\sum_{i=0}^{\left[\frac{m-1}{2}\right]}\eta_i\left(D^\beta_{*+}\left[x^{m-2i-1}\right] + D^\beta_{*-}\left[x^{m-2i-1}\right]\right)\right\},
\end{aligned}
$$

where $\eta_i = \binom{m-i-1}{i}2^{m-2i-1}$. Besides, the right- and left-sided Riemann-Liouville fractional derivatives of x^{m-2i-1} are expanded in view of Pell polynomials as follows:

$$
\begin{aligned}
D^\beta_{*+}\left[x^{m-2i-1}\right] &= D^\beta_{+}x^{m-2i-1} + \sum_{j=0}^{\lceil\beta\rceil-1}\frac{(x^{m-2i-1})^{(j)}\big|_{x=0}}{\Gamma(j+1-\beta)}x^{j-\beta} \qquad (4.4)\\
&= \frac{1}{\Gamma(1-\beta)}\int_0^x\frac{(\xi^{m-2i-1})''}{(x-\xi)^{\beta-1}}d\xi + \sum_{j=0}^{\lceil\beta\rceil-1}\frac{(x^{m-2i-1})^{(j)}\big|_{x=0}}{\Gamma(j+1-\beta)}x^{j-\beta}\\
&\simeq x^{-\beta}\sum_{j=0}^{M}a_jP_j(x),
\end{aligned}
$$

and

$$
\begin{aligned}
&D^\beta_{*-}\left[x^{m-2i-1}\right]\\
&= D^\beta_{-}x^{m-2i-1} + \sum_{j=0}^{\lceil\beta\rceil-1}\frac{(-1)^i(x^{m-2i-1})^{(j)}\big|_{x=1}}{\Gamma(j+1-\beta)}(1-x)^{j-\beta}\\
&= \frac{-1}{\Gamma(1-\beta)}\int_x^1\frac{(\xi^{m-2i-1})''}{(\xi-x)^{\beta-1}}d\xi \qquad (4.5)\\
&\quad + \sum_{j=0}^{\lceil\beta\rceil-1}\frac{(-1)^i(x^{m-2i-1})^{(j)}\big|_{x=1}}{\Gamma(j+1-\beta)}(1-x)^{j-\beta}\\
&\simeq (1-x)^{-\beta}\sum_{j=0}^{M}b_jP_j(x).
\end{aligned}
$$

Therefore, by replacing Eqs. (4.4) and (4.5) in Eq. (4.3), we get

$$
\frac{\partial^\beta}{\partial|x|^\beta}P_m(x)
$$

$$\simeq \frac{1}{2\cos\frac{\pi\beta}{2}}\left\{\sum_{i=0}^{\left[\frac{m-1}{2}\right]}\eta_i\left(x^{-\beta}\sum_{j=0}^{M}a_jP_j(x)+(1-x)^{-\beta}\sum_{j=0}^{M}b_jP_j(x)\right)\right\}$$
$$=x^{-\beta}\sum_{j=0}^{M}\left(\sum_{i=0}^{\left[\frac{m-1}{2}\right]}\frac{\eta_i a_j}{2\cos\frac{\pi\beta}{2}}\right)P_j(x)+(1-x)^{-\beta}\sum_{j=0}^{M}\left(\sum_{i=0}^{\left[\frac{m-1}{2}\right]}\frac{\eta_i b_j}{2\cos\frac{\pi\beta}{2}}\right)P_j(x)$$
$$=x^{-\beta}\sum_{j=0}^{M}\theta^{\beta}_{ijm}P_j(x)+(1-x)^{-\beta}\sum_{j=0}^{M}\delta^{\beta}_{ijm}P_j(x). \tag{4.6}$$

Ultimately, we obtain the general form of proposed matrices as follows:

$$\mathbf{\Theta}=\begin{bmatrix}\frac{1}{2\cos\frac{\pi\beta}{2}\Gamma(1-\beta)} & 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \Theta^{\beta}_{i0M} & \Theta^{\beta}_{i1M} & \Theta^{\beta}_{i2M} & \dots & \Theta^{\beta}_{iMM}\end{bmatrix},$$

and

$$\mathbf{\Delta}=\begin{bmatrix}\frac{1}{2\cos\frac{\pi\beta}{2}\Gamma(1-\beta)} & 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \delta^{\beta}_{i0M} & \delta^{\beta}_{i1M} & \delta^{\beta}_{i2M} & \dots & \delta^{\beta}_{iMM}\end{bmatrix}.$$

In this method, two elements $x^{-\beta}$ and $(1-x)^{-\beta}$ are excluded from the approximation process. This technique greatly enhances the precision of finding the operational matrices of the RF-derivative.

5. Modified operational matrix of integration

The objective of this section is to introduce an accurate integral operational matrix for Pell polynomials. To achieve this purpose, we divide the integral of Pell polynomials into two parts. The first part consists of an operational matrix of integration, while the second part involves a complement vector. We consider

$$\int_0^x P(\xi)d\xi=\Upsilon P(x)+Y(x), \tag{5.1}$$

where Υ denotes the modified operational matrix of integration and $Y(x)$ is called complement vector. By applying classical integration to these polynomials, the desired matrix and vector are obtained. Thus, the components of Υ are computed as follows:

$$\Upsilon=[\gamma_{i,j}],\quad i,j=1,2,\dots,M+1, \tag{5.2}$$

where

$$\gamma_{i,j}=\begin{cases}\frac{-1}{4}, & i=2,\quad j=1,\\ \frac{-1}{i}, & j=1,\quad i \text{ is even},\\ \frac{1}{2i}, & j=i+1,\ \text{ or } j=i-1,\\ 0, & \text{otherwise},\end{cases}\qquad i,j=1,2,\dots,M+1.$$

And also, we have

$$Y(x) = [y_i(x)], \quad y_i(x) = \begin{cases} 0, & i = 1, 2, \dots, M, \\ \frac{1}{2i} P_i(x), & i = M+1. \end{cases} \tag{5.3}$$

Similarly, the above process is established for variable t. Hence, we have

$$\int_0^t P(\eta) d\eta = \tilde{\Upsilon} P(t) + \tilde{Y}(t). \tag{5.4}$$

Therefore, according to the above formula, for $M = 5$, we get

$$\Upsilon = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{-1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 \\ \frac{-1}{4} & 0 & \frac{1}{8} & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & \frac{1}{10} & 0 & \frac{1}{10} \\ \frac{-1}{6} & 0 & 0 & 0 & \frac{1}{12} & 0 \end{bmatrix}, \quad Y(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{12} P_6(x) \end{bmatrix}.$$

6. Numerical procedure

In this section, we aim is to introduce the numerical algorithm for solving VO-CR-time-space fractional Schrödinger equations defined in Eq. (8.4). First, we take approximations of $\frac{\partial^3 \mathcal{U}}{\partial x^2 \partial t}(x,t)$ and $\frac{\partial^3 \mathcal{V}}{\partial x^2 \partial t}(x,t)$ in view of Pell polynomials as follows:

$$\frac{\partial^3 \mathcal{U}}{\partial x^2 \partial t}(x,t) \simeq P^T(x) A P(t), \quad \frac{\partial^3 \mathcal{V}}{\partial x^2 \partial t}(x,t) \simeq P^T(x) B P(t). \tag{6.1}$$

Then, to find the approximation of other functions that exist in the proposed problem, we use the introduced operational matrices. Hence, by integrating Eq. (6.1) with respect to variable t and using the operational matrix of integration, we obtain

$$\begin{aligned} \frac{\partial^2 \mathcal{U}}{\partial x^2}(x,t) &\simeq P^T(x) A \left(\tilde{\Upsilon} P(t) + \tilde{Y}(t) \right) + k_0''(x), \\ \frac{\partial^2 \mathcal{V}}{\partial x^2}(x,t) &\simeq P^T(x) B \left(\tilde{\Upsilon} P(t) + \tilde{Y}(t) \right) + k_1''(x). \end{aligned} \tag{6.2}$$

Next, by integrating Eq. (6.2) with respect to the variable x and utilizing the modified operational matrix of integration, we can deduce

$$\begin{aligned} \frac{\partial \mathcal{U}}{\partial x}(x,t) &\simeq \left(P^T(x) \Upsilon^T + Y^T(x) \right) A \left(\tilde{\Upsilon} P(t) + \tilde{Y}(t) \right) + k_0'(x) - k_0'(0) + \frac{\partial \mathcal{U}}{\partial x}(0,t), \\ \frac{\partial \mathcal{V}}{\partial x}(x,t) &\simeq \left(P^T(x) \Upsilon^T + Y^T(x) \right) B \left(\tilde{\Upsilon} P(t) + \tilde{Y}(t) \right) + k_1'(x) - k_1'(0) + \frac{\partial \mathcal{V}}{\partial x}(0,t). \end{aligned} \tag{6.3}$$

In the above relations, the values of $\frac{\partial \mathcal{U}}{\partial x}(0,t)$ and $\frac{\partial \mathcal{V}}{\partial x}(0,t)$ are unknown. So, we take integral from these equations according to variable x on the interval 0 to 1 as follows:

$$\begin{aligned}\frac{\partial \mathcal{U}}{\partial x}(0,t) &\simeq r_0(t) - s_0(t) - \left(\int_0^1 P^T(x)dx\Upsilon^T + \int_0^1 Y^T(x)dx\right) A\left(\tilde{\Upsilon}P(t) + \tilde{Y}(t)\right)\\ &\quad -k_0(1) + k_0(0) + k_0'(0),\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}\frac{\partial \mathcal{V}}{\partial x}(0,t) &\simeq r_1(t) - s_1(t) - \left(\int_0^1 P^T(x)dx\Upsilon^T + \int_0^1 Y^T(x)dx\right) B\left(\tilde{\Upsilon}P(t) + \tilde{Y}(t)\right)\\ &\quad -k_1(1) + k_1(0) + k_1'(0).\end{aligned} \tag{6.5}$$

By substituting these two aforementioned relations into Eq. (6.3), we can fully calculate $\frac{\partial \mathcal{U}}{\partial x}(x,t)$ and $\frac{\partial \mathcal{V}}{\partial x}(x,t)$. At last, by integrating from Eq. (6.3) concerning to x, we get

$$\begin{aligned}\mathcal{U}(x,t) &\simeq \left(\left[P^T(x)\Upsilon^T + Y^T(x)\right]\Upsilon^T + Z^T(x)\right) A\left(\tilde{\Upsilon}P(t) + \tilde{Y}(t)\right) + k_0(x) - k_0(0)\\ &\quad -xk_0'(0) + x\left[r_0(t) - s_0(t) - \left(\int_0^1 P^T(x)dx\Upsilon^T + \int_0^1 Y^T(x)dx\right)\right.\\ &\quad \left.\times A\left(\tilde{\Upsilon}P(t) + \tilde{Y}(t)\right) - k_0(1) + k_0(0) + k_0'(0)\right] + s_0(t),\end{aligned} \tag{6.6}$$

and

$$\begin{aligned}\mathcal{V}(x,t) &\simeq \left(\left[P^T(x)\Upsilon^T + Y^T(x)\right]\Upsilon^T + Z^T(x)\right) B\left(\tilde{\Upsilon}P(t) + \tilde{Y}(t)\right) + k_1(x) - k_1(0)\\ &\quad -xk_1'(0) + x\left[r_1(t) - s_1(t) - \left(\int_0^1 P^T(x)dx\Upsilon^T + \int_0^1 Y^T(x)dx\right)\right.\\ &\quad \left.\times B\left(\tilde{\Upsilon}P(t) + \tilde{Y}(t)\right) - k_1(1) + k_1(0) + k_1'(0)\right] + s_1(t),\end{aligned} \tag{6.7}$$

where

$$Z(x) = [z_i(x)], \quad z_i(x) = \begin{cases} 0, & i = 1,2,\ldots,M,\\ \frac{1}{2i}\int_0^x P_i(\xi)d\xi, & i = M+1.\end{cases}$$

Next, from Eqs. (6.6), (6.7) and pseudo operational matrix of VO-CF-derivative, we will have

$$\begin{aligned}&\mathbf{D}_t^{\alpha(x,t)}\mathcal{U}(x,t)\\ &\simeq \left(\left[P^T(x)\Upsilon^T + Y^T(x)\right]\Upsilon^T + Z^T(x)\right) A\left(t^{1-\alpha(x,t)}\tilde{\Upsilon}\Lambda(x,t)P(t) + W(t)\right)\\ &\quad +x\left[\mathbf{D}_t^{\alpha(x,t)}r_0(t) - \mathbf{D}_t^{\alpha(x,t)}s_0(t) - \left(\int_0^1 P^T(x)dx\Upsilon^T + \int_0^1 Y^T(x)dx\right)A\right.\\ &\quad \left.\times\left(t^{1-\alpha(x,t)}\tilde{\Upsilon}\Lambda(x,t)P(t) + W(t)\right)\right] + \mathbf{D}_t^{\alpha(x,t)}s_0(t),\end{aligned} \tag{6.8}$$

and

$$\mathbf{D}_t^{\alpha(x,t)}\mathcal{V}(x,t)$$

$$
\begin{aligned}
&\simeq \left(\left[P^T(x)\Upsilon^T + Y^T(x)\right]\Upsilon^T + Z^T(x)\right) B \left(t^{1-\alpha(x,t)}\tilde{\Upsilon}\Lambda(x,t)P(t) + W(t)\right) \\
&\quad + x\left[\mathbf{D}_t^{\alpha(x,t)} r_1(t) - \mathbf{D}_t^{\alpha(x,t)} s_1(t) - \left(\int_0^1 P^T(x)dx\Upsilon^T + \int_0^1 Y^T(x)dx\right) B \right. \\
&\quad \left. \times \left(t^{1-\alpha(x,t)}\tilde{\Upsilon}\Lambda(x,t)P(t) + W(t)\right)\right] + \mathbf{D}_t^{\alpha(x,t)} s_1(t),
\end{aligned}
\tag{6.9}
$$

where

$$
W(t) = [w_i(t)], \quad w_i(x) = \begin{cases} 0, & i = 1, 2, \ldots, N, \\ \frac{1}{2i}\mathbf{D}_t^{\alpha(x,t)} P_i(t), & i = N+1. \end{cases}
$$

Also, from Eqs. (6.6), (6.7), Riesz fractional derivative and modified operational matrices of RF-derivative, we obtain the following relations:

$$
\begin{aligned}
&\frac{\partial^\beta}{\partial|x|^\beta}\mathcal{U}(x,t) \\
&\simeq \left(\left[\left\{x^{-\beta}P^T(x)\mathbf{\Theta}^T + (1-x)^{-\beta}P^T(x)\mathbf{\Delta}^T\right\}\Upsilon^T + \frac{\partial^\beta}{\partial|x|^\beta}Y^T(x)\right]\Upsilon^T + \frac{\partial^\beta}{\partial|x|^\beta}Z^T(x)\right) \\
&\quad \times A\left(\tilde{\Upsilon}P(t) + \tilde{Y}(t)\right) + \frac{\partial^\beta}{\partial|x|^\beta}\left\{k_0(x) - k_0(0) - xk_0'(0)\right\} \\
&\quad + \left\{\frac{\partial^\beta}{\partial|x|^\beta}x\right\}\left[r_0(t) - s_0(t) - \left(\int_0^1 P^T(x)dx\Upsilon^T + \int_0^1 Y^T(x)dx\right) A\left(\tilde{\Upsilon}P(t) + \tilde{Y}(t)\right)\right. \\
&\quad \left. - k_0(1) + k_0(0) + k_0'(0)\right] + s_0(t)\left\{\frac{1}{2\cos\frac{\pi\beta}{2}}\left(\frac{1}{\Gamma(1-\beta)}x^{-\beta} + \frac{1}{\Gamma(1-\beta)}(1-x)^{-\beta}\right)\right\},
\end{aligned}
\tag{6.10}
$$

and

$$
\begin{aligned}
&\frac{\partial^\beta}{\partial|x|^\beta}\mathcal{V}(x,t) \\
&\simeq \left(\left[\left\{x^{-\beta}P^T(x)\mathbf{\Theta}^T + (1-x)^{-\beta}P^T(x)\mathbf{\Delta}^T\right\}\Upsilon^T + \frac{\partial^\beta}{\partial|x|^\beta}Y^T(x)\right]\Upsilon^T + \frac{\partial^\beta}{\partial|x|^\beta}Z^T(x)\right) \\
&\quad \times B\left(\tilde{\Upsilon}P(t) + \tilde{Y}(t)\right) + \frac{\partial^\beta}{\partial|x|^\beta}\left\{k_1(x) - k_1(0) - xk_1'(0)\right\} \\
&\quad + \left\{\frac{\partial^\beta}{\partial|x|^\beta}x\right\}\left[r_1(t) - s_1(t) - \left(\int_0^1 P^T(x)dx\Upsilon^T + \int_0^1 Y^T(x)dx\right) B\left(\tilde{\Upsilon}P(t) + \tilde{Y}(t)\right)\right. \\
&\quad \left. - k_1(1) + k_1(0) + k_1'(0)\right] + s_1(t)\left\{\frac{1}{2\cos\frac{\pi\beta}{2}}\left(\frac{1}{\Gamma(1-\beta)}x^{-\beta} + \frac{1}{\Gamma(1-\beta)}(1-x)^{-\beta}\right)\right\}.
\end{aligned}
\tag{6.11}
$$

Now, to find the approximate solutions, we substitute Eqs. (6.1)-(6.11) in Eq. (1.4) and collocate the obtained system at nodal points of Newton–Cotes [10]. This process provides a system of algebraic equations with $2(M+1)(N+1)$ unknown coefficients. Finally, by using Newton's iteration method and Eqs. (6.6) and (6.7), the approximate solutions are gained.

7. Error analysis and convergence

This section demonstrates the convergence of Pell polynomials expansion of the function $u(x,t)$. Hence, to reach the goal, we apply the multi-variable Taylor formula [17]:

$$u(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^i t^j}{\Gamma(i+1)\Gamma(j+1)}\frac{\partial^{i+j}u(x,t)}{\partial x^i\partial t^j}\bigg|_{(0,0)}.$$

Next, by replacing the expansion of x^i and t^j defined in Eq. (2.4) in the above formula, we deduce

$$\begin{aligned}
&u(x,t) \qquad (7.1)\\
&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{[\frac{2i+1}{2}]}\sum_{l=0}^{[\frac{2j+1}{2}]}\frac{2^{-2i-2j}(-1)^{k+l}}{\Gamma(2i+2)\Gamma(2j+2)}\binom{2i+1}{k}\binom{2j+1}{l}\\
&\quad\times P_{2i-2k+1}(x)P_{2j-2l+1}(t)\frac{\partial^{2i+2j+2}u(x,t)}{\partial x^{2i+1}\partial t^{2j+1}}\bigg|_{(0,0)}\\
&\quad+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{[\frac{2i+1}{2}]}\sum_{l=0}^{[j]}\frac{2^{1-2j}(-1)^{k+l}}{\Gamma(2i+2)\Gamma(2j+1)}\binom{2i+1}{k}\binom{2j}{l}\\
&\quad\times P_{2i-2k+1}(x)P_{2j-2l}(t)\frac{\partial^{2i+2j+1}u(x,t)}{\partial x^{2i+1}\partial t^{2j}}\bigg|_{(0,0)}\\
&\quad+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{[\frac{2i+1}{2}]}\frac{2^{-2j}(-1)^{k+[j]+1}}{\Gamma(2i+2)\Gamma(2j+1)}\binom{2i+1}{k}\binom{2j}{j}\\
&\quad\times P_{2i-2k+1}(x)P_0(t)\frac{\partial^{2i+2j+1}u(x,t)}{\partial x^{2i+1}\partial t^{2j}}\bigg|_{(0,0)}\\
&\quad+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{[i]}\sum_{l=0}^{[\frac{2j+1}{2}]}\frac{2^{1-2i}(-1)^{k+l}}{\Gamma(2i+1)\Gamma(2j+2)}\binom{2i}{k}\binom{2j+1}{l}\\
&\quad\times P_{2i-2k}(x)P_{2j-2l+1}(t)\frac{\partial^{2i+2j+1}u(x,t)}{\partial x^{2i}\partial t^{2j+1}}\bigg|_{(0,0)}\\
&\quad+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{[j]}\frac{2^{-2i}(-1)^{[i]+l+1}}{\Gamma(2i+1)\Gamma(2j+2)}\binom{2i}{i}\binom{2j+1}{l}\\
&\quad\times P_0(x)P_{2j-2l+1}(t)\frac{\partial^{2i+2j+1}u(x,t)}{\partial x^{2i}\partial t^{2j+1}}\bigg|_{(0,0)}\\
&\quad+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{[i]}\sum_{l=0}^{[j]}\frac{2^{2-2i-2j}(-1)^{k+l}}{\Gamma(2i+1)\Gamma(2j+1)}\binom{2i}{k}\binom{2j}{l}\\
&\quad\times P_{2i-2k}(x)P_{2j-2l}(t)\frac{\partial^{2i+2j}u(x,t)}{\partial x^{2i}\partial t^{2j}}\bigg|_{(0,0)}\\
&\quad+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{[i]}\frac{2^{1-2i-2j}(-1)^{k+[j]+1}}{\Gamma(2i+1)\Gamma(2j+1)}\binom{2i}{k}
\end{aligned}$$

$$\times \binom{2j}{j} P_{2i-2k}(x) P_0(t) \frac{\partial^{2i+2j} u(x,t)}{\partial x^{2i} \partial t^{2j}}\bigg|_{(0,0)}$$

$$+ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{[j]} \frac{2^{1-2i-2j}(-1)^{[i]+1+l}}{\Gamma(2i+1)\Gamma(2j+1)} \binom{2i}{i}\binom{2j}{l} P_0(x) P_{2j-2l}(t) \frac{\partial^{2i+2j} u(x,t)}{\partial x^{2i}\partial t^{2j}}\bigg|_{(0,0)}$$

$$+ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{2^{-2i-2j}(-1)^{[i]+[j]+2}}{\Gamma(2i+1)\Gamma(2j+1)} \binom{2i}{i}\binom{2j}{j} P_0(x) P_0(t) \frac{\partial^{2i+2j} u(x,t)}{\partial x^{2i}\partial t^{2j}}\bigg|_{(0,0)}.$$

Therefore, the error bound of the approximate function $u_{MN}(x,t)$, which is obtained by truncating the function $u(x,t)$ to $M+1$ sentences with respect to x and $N+1$ sentences with respect to t, is calculated as follows:

$$\begin{aligned}
&|u(x,t) - u_{MN}(x,t)| \qquad (7.2)\\
&\leq \sum_{k=0}^{[\frac{2M+3}{2}]}\sum_{l=0}^{[\frac{2N+3}{2}]} \frac{2^{-2M-2N-4}C_1}{\Gamma(2M+4)\Gamma(2N+4)}\binom{2M+3}{k}\binom{2N+3}{l}\\
&\times |P_{2M+3-2k}(x)|\,|P_{2N+3-2l}(t)|\\
&+ \sum_{k=0}^{[\frac{2M+3}{2}]}\sum_{l=0}^{[N+1]} \frac{2^{-2N-1}C_2}{\Gamma(2M+4)\Gamma(2N+3)}\binom{2M+3}{k}\\
&\times \binom{2N+2}{l} |P_{2M+3-2k}(x)|\,|P_{2N+2-2l}(t)|\\
&+ \sum_{k=0}^{[\frac{2M+3}{2}]} \frac{2^{-2N-2}C_2}{\Gamma(2M+4)\Gamma(2N+3)}\binom{2M+3}{k}\\
&\times \binom{2N+2}{N+1} |P_{2M+3-2k}(x)|\\
&+ \sum_{k=0}^{[M+1]}\sum_{l=0}^{[\frac{2N+3}{2}]} \frac{2^{-2M-1}C_3}{\Gamma(2M+3)\Gamma(2N+4)}\binom{2M+2}{k}\binom{2N+3}{l}\\
&\times |P_{2M+2-2k}(x)|\,|P_{2N+3-2l}(t)|\\
&+ \sum_{l=0}^{[N+1]} \frac{2^{-2M-2}C_3}{\Gamma(2M+3)\Gamma(2N+4)}\binom{2M+2}{M+1}\binom{2N+3}{l} |P_{2N+3-2l}(t)|\\
&+ \sum_{k=0}^{[M+1]}\sum_{l=0}^{[N+1]} \frac{2^{-2M-2N-2}C_4}{\Gamma(2M+3)\Gamma(2N+3)}\binom{2M+2}{k}\binom{2N+2}{l}\\
&\times |P_{2M+2-2k}(x)|\,|P_{2N+2-2l}(t)|\\
&+ \sum_{k=0}^{[M+1]} \frac{2^{-2M-2N-3}C_4}{\Gamma(2M+3)\Gamma(2N+3)}\binom{2M+2}{k}\binom{2N+2}{N+1} |P_{2M+2-2k}(x)|\\
&+ \sum_{l=0}^{[N+1]} \frac{2^{-2M-2N-3}C_4}{\Gamma(2M+3)\Gamma(2N+3)}\binom{2M+2}{M+1}\binom{2N+2}{l} |P_{2N+2-2l}(t)|
\end{aligned}$$

$$+\frac{2^{-2M-2N-4}C_4}{\Gamma(2M+3)\Gamma(2N+3)}\binom{2M+2}{M+1}\binom{2N+2}{N+1},$$

where $C_1 \geq \sup_{x\in[0,1]\times[0,1]}\left|\frac{\partial^{2i+2j+2}u(x,t)}{\partial x^{2i+1}\partial t^{2j+1}}\right|$, $C_2 \geq \sup_{x\in[0,1]\times[0,1]}\left|\frac{\partial^{2i+2j+1}u(x,t)}{\partial x^{2i+1}\partial t^{2j}}\right|$, $C_3 \geq \sup_{x\in[0,1]\times[0,1]}\left|\frac{\partial^{2i+2j+1}u(x,t)}{\partial x^{2i}\partial t^{2j+1}}\right|$, $C_4 \geq \sup_{x\in[0,1]\times[0,1]}\left|\frac{\partial^{2i+2j}u(x,t)}{\partial x^{2i}\partial t^{2j}}\right|$. Consequently, according to the above result, we get

$$\begin{aligned}
&\|u(x,t)-u_{MN}(x,t)\|_\infty \qquad (7.3)\\
&\leq \sum_{k=0}^{[\frac{2M+3}{2}]}\sum_{l=0}^{[\frac{2N+3}{2}]}\frac{2^{-2M-2N-4}C_1V_1\bar{V}_1}{\Gamma(2M+4)\Gamma(2N+4)}\binom{2M+3}{k}\binom{2N+3}{l}\\
&\quad+\sum_{k=0}^{[\frac{2M+3}{2}]}\sum_{l=0}^{[N+1]}\frac{2^{-2N-1}C_2V_1\bar{V}_2}{\Gamma(2M+4)\Gamma(2N+3)}\binom{2M+3}{k}\binom{2N+2}{l}\\
&\quad+\sum_{k=0}^{[\frac{2M+3}{2}]}\frac{2^{-2N-2}C_2V_2}{\Gamma(2M+4)\Gamma(2N+3)}\binom{2M+3}{k}\binom{2N+2}{N+1}\\
&\quad+\sum_{k=0}^{[M+1]}\sum_{l=0}^{[\frac{2N+3}{2}]}\frac{2^{-2N-1}C_3\bar{V}_1V_2}{\Gamma(2M+3)\Gamma(2N+4)}\binom{2M+2}{k}\binom{2N+3}{l}\\
&\quad+\sum_{l=0}^{[N+1]}\frac{2^{-2M-2}C_3\bar{V}_1}{\Gamma(2M+3)\Gamma(2N+4)}\binom{2M+2}{M+1}\binom{2N+3}{l}\\
&\quad+\sum_{k=0}^{[M+1]}\sum_{l=0}^{[N+1]}\frac{2^{-2M-2N-2}C_4V_2\bar{V}_2}{\Gamma(2M+3)\Gamma(2N+3)}\binom{2M+2}{k}\binom{2N+2}{l}\\
&\quad+\sum_{k=0}^{[M+1]}\frac{2^{-2M-2N-3}C_4V_2}{\Gamma(2M+3)\Gamma(2N+3)}\binom{2M+2}{k}\binom{2N+2}{N+1}\\
&\quad+\sum_{l=0}^{[N+1]}\frac{2^{-2M-2N-3}C_4\bar{V}_2}{\Gamma(2M+3)\Gamma(2N+3)}\binom{2M+2}{M+1}\binom{2N+2}{l}\\
&\quad+\frac{2^{-2M-2N-4}C_4}{\Gamma(2M+3)\Gamma(2N+3)}\binom{2M+2}{M+1}\binom{2N+2}{N+1}\\
&=\Pi_{MN},
\end{aligned}$$

where

$$V_1 \geq \sup_{x\in[0,1]}|P_{2M+3-2k}(x)|, \quad \bar{V}_1 \geq \sup_{t\in[0,1]}|P_{2N+3-2l}(t)|,$$

$$V_2 \geq \sup_{x\in[0,1]}|P_{2M+2-2k}(x)|, \quad \bar{V}_2 \geq \sup_{t\in[0,1]}|P_{2N+2-2l}(t)|.$$

From the above inequality, it can be inferred that the approximate solution converges to the exact solution by increasing the number of Pell polynomials (M, N).

8. Numerical results

In this section, we implement the proposed approach, which was explained in the previous section, in several examples. To verify the accuracy and applicability of the proposed numerical algorithm, we obtain the maximum absolute error and least square error in some examples:

$$
\begin{aligned}
L_\infty - \text{error} &= \max_{0\leq x,t\leq S} \left|u(x,t) - u_{MN}(x,t)\right|,\\
L_2 - \text{error} &= \left(\sum_{i=0}^{S} \left|u(x_i,t_i) - u_{MN}(x_i,t_i)\right|^2\right)^{\frac{1}{2}},\\
RMS &= \left(\frac{1}{S}\sum_{i=1}^{S} \left|u(x_i,t_i) - u_{MN}(x_i,t_i)\right|^2\right)^{\frac{1}{2}},
\end{aligned}
$$

where $u(x,t)$ and $u_{MN}(x,t)$ represent the exact solution and approximate solution, respectively. Additionally, to demonstrate the substantial accuracy improvement achieved by increasing the number of basis functions, we determine the convergence order by using the following formula:

$$
CO = \frac{\left|\log(\frac{E_2}{E_1})\right|}{\log\left(\left[\frac{M+1}{M+1}\right]^2\right)},
$$

where E_1 and E_2 represent the first and second absolute errors obtained by the proposed method, and M (or N) and $\bar{M}$ (or $\bar{N}$) are the number of basis functions in each implementation of the numerical algorithm. All numerical computations were performed on a personal computer and codes were written in MATLAB software.

Example 8.1. For the first example, we consider following linear VO-CR-time-space fractional Schrödinger equation:

$$
\begin{aligned}
&i\mathbf{D}_t^{\alpha(x,t)}\psi(x,t) - \frac{\partial^\beta}{\partial|x|^\beta}\psi(x,t)\\
&= 2(i-1)\sin(\pi x)\frac{t^{2-\alpha(x,t)}}{\Gamma(3-\alpha(x,t))}\\
&\quad + \frac{\partial^\beta}{\partial|x|^\beta}(1+i)t^2\sin(\pi x), \quad 0<\alpha(x,t)\leq 1, \quad 1<\beta<2,
\end{aligned}
$$

subject to the initial condition $\psi(x,0) = 0$ and boundary conditions $\psi(0,t) = \psi(1,t) = 0$. The corresponding analytical solution of the problem $\psi(x,t) = (1+i)t^2\sin(\pi x)$. The results of this example are demonstrated in Tables 1 and 2 and, Figures 1 and 2. The errors of the real part and imaginary part of the approximate solution for different choices of $\alpha(x,t)$ are listed in Tables 1 and 2. Also, the graphical representations of the absolute error of the real part and imaginary part of the approximate solution for various values of $\alpha(x,t)$ and M, N are shown in Figures 1 and 2. As you see in the results, the error tends to zero as the basis functions term increases.

Table 1. Errors of the real part and imaginary part of the approximate solution with $\alpha(x,t) = 0.5$, $\beta = 1.3$ and $N = 3$ of Example 8.1.

	Re-part			Im-part		
M	L_2-error	L_∞-error	RMS	L_2-error	L_∞-error	RMS
3	4.4717×10^{-3}	2.7221×10^{-3}	1.3482×10^{-3}	3.3273×10^{-3}	2.1918×10^{-3}	1.0032×10^{-3}
5	1.0074×10^{-4}	5.7579×10^{-5}	3.0377×10^{-5}	7.2466×10^{-5}	4.3899×10^{-5}	2.1849×10^{-5}
7	1.4873×10^{-6}	7.7813×10^{-7}	4.4845×10^{-7}	1.0555×10^{-6}	6.0691×10^{-7}	3.1824×10^{-7}

Table 2. Errors for the real part and imaginary part of the approximate solution with $\alpha(x,t) = 1 - 0.5\exp(-xt)$, $\beta = 1.6$ and $N = 3$ of Example 8.1.

	Re-part			Im-part		
M	L_2-error	L_∞-error	RMS	L_2-error	L_∞-error	RMS
3	3.8396×10^{-3}	2.3418×10^{-3}	1.1576×10^{-3}	2.5550×10^{-3}	1.7257×10^{-3}	7.7036×10^{-4}
5	9.0530×10^{-5}	5.2245×10^{-5}	2.7295×10^{-5}	5.8825×10^{-5}	3.6780×10^{-5}	1.7736×10^{-5}
7	1.4124×10^{-6}	7.4366×10^{-7}	4.2587×10^{-7}	9.1203×10^{-7}	5.0859×10^{-7}	2.7498×10^{-7}

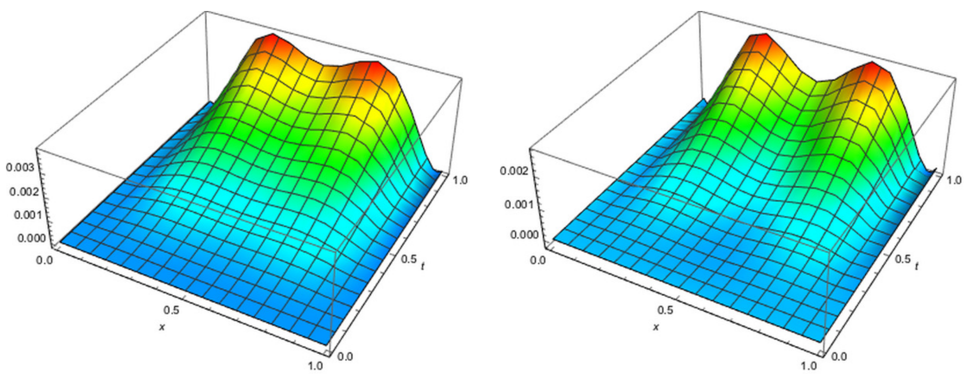

Figure 1. The absolute error of the real part (left) and imaginary part of the approximate solution (right) obtained with $\alpha(x,t) = 0.2, \beta = 1.8$ and $M = N = 3$ of Example 8.1.

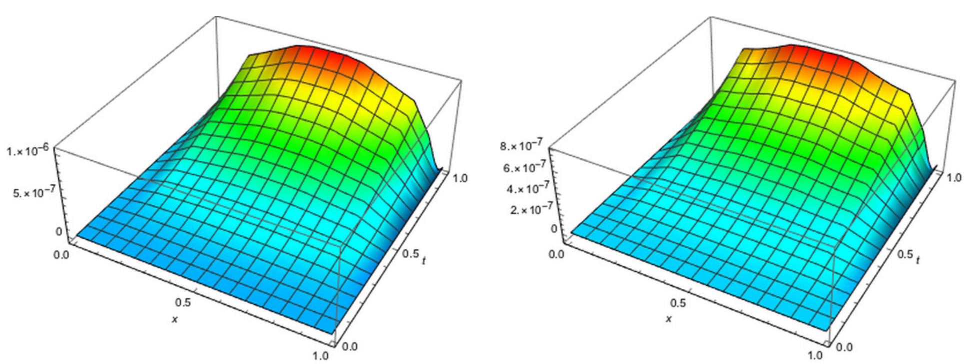

Figure 2. The absolute error of the real part (left) and imaginary part of the approximate solution (right) obtained with $\alpha(x,t) = 0.2, \beta = 1.8$ and $M = 3, N = 7$ of Example 8.1.

Example 8.2. We consider the following linear VO-CR-time-space fractional

Table 3. Errors for the real part and imaginary part of the approximate solution with $\alpha(x,t)=1$ and $\beta=1.9$ of Example 8.2.

	Present method			
	Re-part		Im-part	
$M=N$	L_2-error	L_∞-error	L_2-error	L_∞-error
3	7.8761×10^{-4}	5.4589×10^{-3}	2.9942×10^{-3}	1.6666×10^{-3}
5	2.32139×10^{-5}	1.1740×10^{-5}	6.1491×10^{-5}	3.1698×10^{-5}
7	2.6602×10^{-7}	1.3291×10^{-7}	8.8129×10^{-7}	4.2117×10^{-7}
	Jacobi-Gauss-Lobatto collocation method [4]			
	Re-part		Im-part	
M	L_∞-error		L_∞-error	
5	1.987×10^{-3}		1.423×10^{-3}	
10	1.725×10^{-7}		1.355×10^{-7}	

Schrödinger equation:

$$i\mathbf{D}_t^{\alpha(x,t)}\psi(x,t)=\frac{\partial^\beta}{\partial|x|^\beta}\psi(x,t)+R(x,t),\quad 0<\alpha(x,t)\le1,\quad 1<\beta<2,$$

subject to the initial condition $\psi(x,0)=\sin(\pi x)$ and boundary conditions $\psi(0,t)=\psi(1,t)=0$. The corresponding $R(x,t)$ are computed according to the analytical solution of the problem $\psi(x,t)=\sin(\pi x)\exp(it)$. Tables 3 and 4 show the numerical results of this example. In Table 3, the least square error and maximum absolute error for the real and imaginary parts of the approximate solution are presented. Also, the comparison of the maximum absolute error obtained with the present method and the Jacobi-Gauss-Lobatto collocation method [4] are listed in Table 3. The results from this table demonstrate that the proposed method, with fewer basis functions, achieves higher accuracy compared to the Jacobi-Gauss-Lobatto collocation method. In addition, the absolute errors of the real and imaginary parts of the approximate solution for different choices of $\alpha(x,t)$ are presented in Table 4.

Example 8.3. We consider the following nonlinear VO-C-time-fractional Schrödinger equation:

$$i\mathbf{D}_t^{\alpha(x,t)}\psi(x,t)+\frac{\partial^2}{\partial x^2}\psi(x,t)+|\psi|^2\psi(x,t)=R(x,t),\quad 0<\alpha(x,t)\le1,$$

subject to the initial condition $\psi(x,0)=x^2(1-x)^2$ and boundary conditions $\psi(0,t)=\psi(1,t)=0$. The $R(x,t)$ are computed according to the analytical solution of the problem $\psi(x,t)=x^2(1-x)^2\exp(it)$. To demonstrate the role of $\alpha(x,t)$ and the numbers of basis functions M and N, we present the least square error, maximum absolute error and root mean square error in Table 5. From Table 5, it is evident that, as N is increased, the error decreases. Figures 3 and 4 depict the approximation of the real and imaginary parts at various times and the absolute error of the real and imaginary parts at $t=1$. These graphical representations

Table 4. The absolute errors for the real part and imaginary part of the approximate solution with $\beta = 1.5$ and $M = N = 5$ of Example 8.2.

	$\alpha(x,t) = 0.2$		$\alpha(x,t) = 0.8$		$\alpha(x,t) = 0.8 + 0.005\cos(xt)\sin(x)$	
$x = t$	Re-part	Im-part	Re-part	Im-part	Re-part	Im-part
0	0	0	0	0	0	0
0.1	3.1776×10^{-7}	4.6641×10^{-6}	1.8043×10^{-6}	3.5307×10^{-6}	1.8075×10^{-6}	3.5184×10^{-6}
0.2	3.9903×10^{-7}	1.5134×10^{-5}	4.7357×10^{-6}	1.4212×10^{-5}	4.7683×10^{-6}	1.4199×10^{-5}
0.3	2.0039×10^{-7}	2.2976×10^{-5}	5.9877×10^{-6}	2.4002×10^{-5}	6.0383×10^{-6}	2.4021×10^{-5}
0.4	1.3342×10^{-6}	3.3407×10^{-5}	3.7771×10^{-6}	3.6019×10^{-5}	3.8048×10^{-6}	3.6068×10^{-5}
0.5	4.4978×10^{-6}	4.4553×10^{-5}	7.9589×10^{-7}	4.7561×10^{-5}	8.0365×10^{-7}	4.7606×10^{-5}
0.6	7.5601×10^{-6}	4.9310×10^{-5}	4.8053×10^{-6}	5.2204×10^{-5}	4.8211×10^{-6}	5.2223×10^{-5}
0.7	1.1339×10^{-5}	5.1813×10^{-5}	9.4364×10^{-6}	5.4661×10^{-5}	9.4379×10^{-6}	5.4668×10^{-5}
0.8	1.7803×10^{-5}	5.6819×10^{-5}	1.7176×10^{-5}	5.9252×10^{-5}	1.7175×10^{-5}	5.9267×10^{-5}
0.9	1.5003×10^{-5}	3.8325×10^{-5}	1.5203×10^{-5}	3.9571×10^{-5}	1.5210×10^{-5}	3.9583×10^{-5}
1.0	6.8921×10^{-40}	2.5029×10^{-40}	4.9177×10^{-40}	1.3076×10^{-40}	3.8988×10^{-40}	1.3752×10^{-40}

indicate that the approximate solutions have excellent agreement with the exact solution.

Table 5. Errors for the real part and imaginary part of the approximate solution with $M = 4$ of Example 8.3.

$\alpha(x,t) = 0.2$				
	$N = 2$		$N = 4$	
	Re-part	Im-part	Re-part	Im-part
L_2-error	2.5022×10^{-5}	1.1653×10^{-5}	3.5619×10^{-8}	1.8414×10^{-8}
L_∞-error	1.6434×10^{-5}	7.8161×10^{-6}	2.5915×10^{-8}	1.4345×10^{-8}
RMS	7.5446×10^{-6}	3.5137×10^{-6}	1.0739×10^{-8}	5.5523×10^{-9}
$\alpha(x,t) = 0.85$				
	$N = 2$		$N = 4$	
	Re-part	Im-part	Re-part	Im-part
L_2-error	4.2045×10^{-5}	3.6184×10^{-5}	1.8133×10^{-7}	1.8523×10^{-7}
L_∞-error	2.1635×10^{-5}	2.2711×10^{-5}	1.0943×10^{-7}	1.0162×10^{-7}
RMS	1.2677×10^{-5}	1.0910×10^{-5}	5.4673×10^{-8}	5.5851×10^{-8}
$\alpha(x,t) = \frac{3+\sin(x)\cos(t)}{4}$				
	$N = 2$		$N = 4$	
	Re-part	Im-part	Re-part	Im-part
L_2-error	3.0901×10^{-5}	2.3907×10^{-5}	5.3283×10^{-8}	5.5854×10^{-8}
L_∞-error	1.7922×10^{-5}	1.4965×10^{-5}	2.6443×10^{-8}	3.2744×10^{-8}
RMS	9.3172×10^{-6}	7.2082×10^{-6}	1.6065×10^{-8}	1.6841×10^{-8}

Example 8.4. We consider the following nonlinear VO-CR-time-space fractional

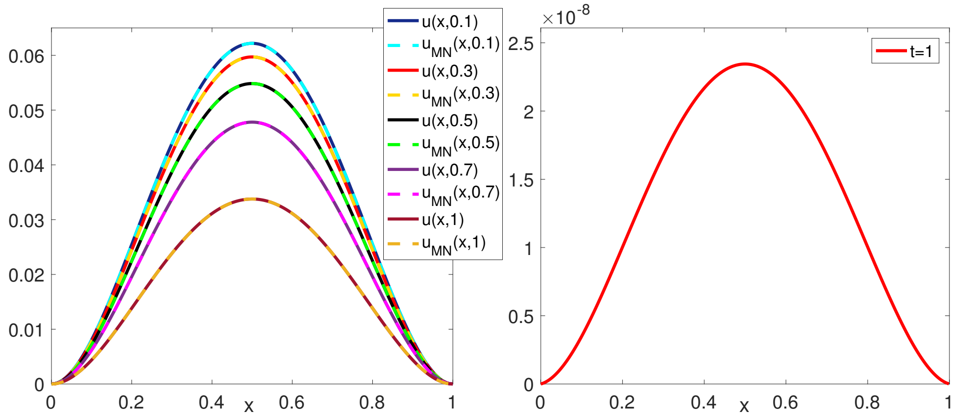

Figure 3. The approximation of the real part at different time (left) and absolute error of real part at $t = 1$ (right) for $\alpha(x, t) = 0.5$ and $M = 4, N = 5$ of Example 8.3.

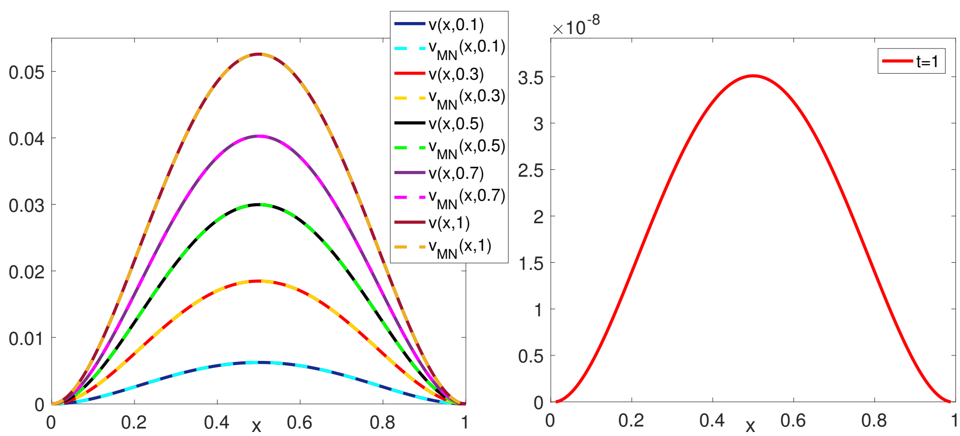

Figure 4. The approximation of the imaginary part at different time (left) and absolute error of imaginary part at $t = 1$ (right) for $\alpha(x, t) = 0.5$ and $M = 4, N = 5$ of Example 8.3.

Schrödinger equation [25]:

$$
\begin{aligned}
&i\mathbf{D}_t^{\alpha(x,t)}\psi(x,t)\\
&= -\frac{\partial^\beta}{\partial|x|^\beta}\psi(x,t) - 2|\psi|^2\psi(x,t) + R(x,t), \quad 0 < \alpha(x,t) \leq 1, \quad 1 < \beta < 2,
\end{aligned}
$$

subject to the initial condition $\psi(x, 0) = 0$ and boundary conditions $\psi(0, t) = \psi(1, t) = 0$. The corresponding $R(x, t)$ are computed according to the analytical solution of the problem $\psi(x, t) = t^3 x^2 (1 - x)^2$. The comparison of least square error obtained by the present scheme and finite difference method [25] are represented in Table 6. Based on the results from the table, it can be concluded that the proposed method exhibits greater accuracy compared to the finite difference method. Moreover, the approximation of the real and imaginary parts at various times and the absolute error of the real and imaginary parts at $t = 1$ are depicted in Figures 5 and 6. In Figures 5 and 6 (right), we demonstrated the behavior of the approximate solution in different values of time t with various choices of $\alpha(x,t)$. In addition, to illustrate the accuracy of the method, we plotted the absolute error in Figures 5 and 6 (left).

Table 6. L_2-error of the approximate solution obtained with $M = 4, N = 3$ and $\alpha(x,t) = 0.2$ of Example 8.4.

	$\beta = 1.3$	$\beta = 1.5$	$\beta = 1.7$	$\beta = 1.9$
Present method	6.7511×10^{-15}	4.1838×10^{-13}	3.7891×10^{-13}	9.0541×10^{-13}
Finite difference method [25] with $h = \frac{1}{128}$	8.4784×10^{-6}	8.6193×10^{-6}	9.1175×10^{-6}	1.0282×10^{-5}

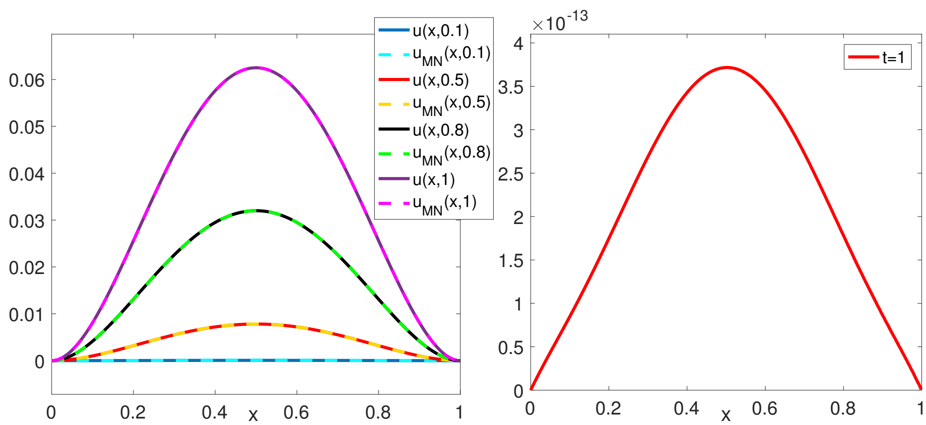

Figure 5. The approximation at different time (left) and absolute error at $t = 1$ (right) for $\alpha(x,t) = 0.5$, $\beta = 1.5$ and $M = 4, N = 5$ of Example 8.4.

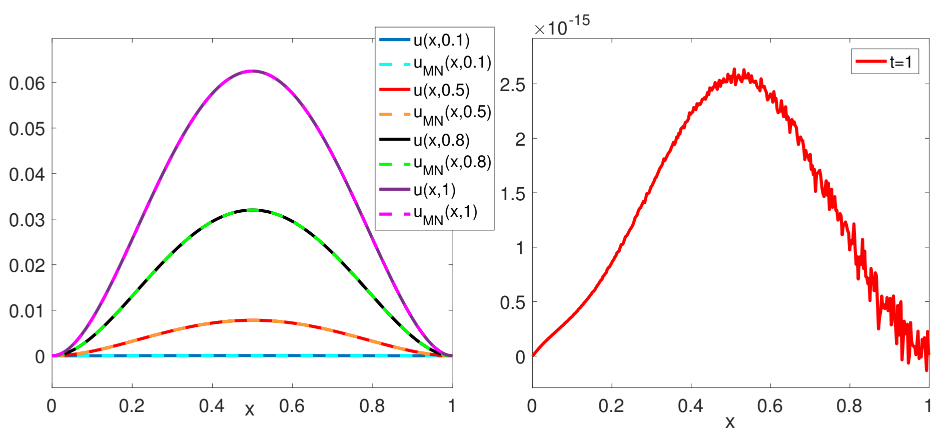

Figure 6. The approximation at different time (left) and absolute error at $t = 1$ (right) for $\alpha(x,t) = \frac{2+\sin(xt)}{400}$, $\beta = 1.5$ and $M = 4, N = 5$ of Example 8.4.

Example 8.5. We consider the following nonlinear VO-C-time fractional

Schrödinger equation [25]:

$$
\begin{aligned}
&i\mathbf{D}_t^{\alpha(x,t)}\psi(x,t)+\frac{\partial^2}{\partial x^2}\psi(x,t)+|\psi|^2\psi(x,t)+\sin(x)\psi(x,t)\\
=&R(x,t),\quad 0<\alpha(x,t)\leq 1,
\end{aligned}
$$

subject to the initial condition $\psi(x,0)=0$ and boundary conditions $\psi(0,t)=t^3,\psi(1,t)=t^3\exp(i)$. The corresponding $R(x,t)$ are computed according to the analytical solution of the problem $\psi(x,t)=t^3\exp(ix)$. According to the proposed method, we assume

$$
\frac{\partial^3\mathcal{U}}{\partial x^2\partial t}(x,t)\simeq P^T(x)AP(t),\quad \frac{\partial^3\mathcal{V}}{\partial x^2\partial t}(x,t)\simeq P^T(x)BP(t).
$$

Then, by integrating from the above relation with respect to t and x, respectively. We infer

$$
\frac{\partial^2\mathcal{U}}{\partial x^2}(x,t)\simeq P^T(x)A\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right),\quad \frac{\partial^2\mathcal{V}}{\partial x^2}(x,t)\simeq P^T(x)B\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right),
$$

and

$$
\begin{aligned}
\frac{\partial\mathcal{U}}{\partial x}(x,t)&\simeq\left(P^T(x)\Upsilon^T+Y^T(x)\right)A\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right)+\frac{\partial\mathcal{U}}{\partial x}(0,t),\\
\frac{\partial\mathcal{V}}{\partial x}(x,t)&\simeq\left(P^T(x)\Upsilon^T+Y^T(x)\right)B\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right)+\frac{\partial\mathcal{V}}{\partial x}(0,t).
\end{aligned}\tag{8.1}
$$

In view of the method, we obtain

$$
\begin{aligned}
\frac{\partial\mathcal{U}}{\partial x}(0,t)&\simeq t^3\cos(1)-t^3-\left(\int_0^1P^T(x)dx\Upsilon^T+\int_0^1Y^T(x)dx\right)A\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right),\\
\frac{\partial\mathcal{V}}{\partial x}(0,t)&\simeq t^3\sin(1)-\left(\int_0^1P^T(x)dx\Upsilon^T+\int_0^1Y^T(x)dx\right)B\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right).
\end{aligned}\tag{8.2}
$$

By substituting Eq. (8.2) in Eq. (8.1), we get

$$
\begin{aligned}
\frac{\partial\mathcal{U}}{\partial x}(x,t)\simeq&\left(P^T(x)\Upsilon^T+Y^T(x)\right)A\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right)+t^3\cos(1)-t^3\\
&-\left(\int_0^1P^T(x)dx\Upsilon^T+\int_0^1Y^T(x)dx\right)A\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right),\\
\frac{\partial\mathcal{V}}{\partial x}(x,t)\simeq&\left(P^T(x)\Upsilon^T+Y^T(x)\right)B\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right)+t^3\sin(1)\\
&-\left(\int_0^1P^T(x)dx\Upsilon^T+\int_0^1Y^T(x)dx\right)B\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right).
\end{aligned}
$$

Therefore, by integrating the above formula with respect to x, we obtain the real and imaginary parts of the approximate solution as follows:

$$
\begin{aligned}
&\mathcal{U}(x,t)\\
\simeq&\left(\left[P^T(x)\Upsilon^T+Y^T(x)\right]\Upsilon^T+Z^T(x)\right)A\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right)\\
&+x\left(t^3\cos(1)-t^3-\left(\int_0^1P^T(x)dx\Upsilon^T+\int_0^1Y^T(x)dx\right)\right.
\end{aligned}\tag{8.3}
$$

$$
\begin{aligned}
&\times A\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right)\Big)+t^3,\\
&\mathcal{V}(x,t)\\
\simeq&\left(\left[P^T(x)\Upsilon^T+Y^T(x)\right]\Upsilon^T+Z^T(x)\right)B\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right)\\
&+x\left(t^3\sin(1)-\left(\int_0^1 P^T(x)dx\Upsilon^T+\int_0^1 Y^T(x)dx\right)B\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right)\right).
\end{aligned}
$$

With the assistance of the pseudo operational matrix of VO-CF-derivative and Eq. (8.3), we obtain:

$$
\begin{aligned}
&\mathbf{D}_t^{\alpha(x,t)}\mathcal{U}(x,t)\\
\simeq&\left(\left[P^T(x)\Upsilon^T+Y^T(x)\right]\Upsilon^T+Z^T(x)\right)A\left(t^{1-\alpha(x,t)}\tilde{\Upsilon}\Lambda(x,t)P(t)+W(t)\right)\\
&+x\left(\frac{\Gamma(4)}{\Gamma(4-\alpha(x,t))}t^{3-\alpha(x,t)}\cos(1)-\frac{\Gamma(4)}{\Gamma(4-\alpha(x,t))}t^{3-\alpha(x,t)}\right.\\
&\left.-\left(\int_0^1 P^T(x)dx\Upsilon^T+\int_0^1 Y^T(x)dx\right)A\left(t^{1-\alpha(x,t)}\tilde{\Upsilon}\Lambda(x,t)P(t)+W(t)\right)\right)\\
&+\frac{\Gamma(4)}{\Gamma(4-\alpha(x,t))}t^{3-\alpha(x,t)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{D}_t^{\alpha(x,t)}\mathcal{V}(x,t)\\
\simeq&\left(\left[P^T(x)\Upsilon^T+Y^T(x)\right]\Upsilon^T+Z^T(x)\right)B\left(t^{1-\alpha(x,t)}\tilde{\Upsilon}\Lambda(x,t)P(t)+W(t)\right)\\
&+x\left(\frac{\Gamma(4)}{\Gamma(4-\alpha(x,t))}t^{3-\alpha(x,t)}\sin(1)-\left(\int_0^1 P^T(x)dx\Upsilon^T+\int_0^1 Y^T(x)dx\right)B\right.\\
&\left.\times\left(t^{1-\alpha(x,t)}\tilde{\Upsilon}\Lambda(x,t)P(t)+W(t)\right)\right).
\end{aligned}
$$

Also, by utilizing the modified operational matrices of RF-derivative and Eq. (8.3), we get

$$
\begin{aligned}
&\frac{\partial^\beta}{\partial|x|^\beta}\mathcal{U}(x,t)\\
\simeq&\left(\left[\left\{x^{-\beta}P^T(x)\boldsymbol{\Theta}^T+(1-x)^{-\beta}P^T(x)\boldsymbol{\Delta}^T\right\}\Upsilon^T+\frac{\partial^\beta}{\partial|x|^\beta}Y^T(x)\right]\Upsilon^T+\frac{\partial^\beta}{\partial|x|^\beta}Z^T(x)\right)\\
&\times A\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right)\\
&+\left\{\frac{\partial^\beta}{\partial|x|^\beta}x\right\}\left(t^3\cos(1)-t^3-\left(\int_0^1 P^T(x)dx\Upsilon^T+\int_0^1 Y^T(x)dx\right)A\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right)\right)\\
&+t^3\left\{\frac{1}{2\cos\frac{\pi\beta}{2}}\left(\frac{1}{\Gamma(1-\beta)}x^{-\beta}+\frac{1}{\Gamma(1-\beta)}(1-x)^{-\beta}\right)\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial^\beta}{\partial|x|^\beta}\mathcal{V}(x,t)\\
\simeq&\left(\left[\left\{x^{-\beta}P^T(x)\boldsymbol{\Theta}^T+(1-x)^{-\beta}P^T(x)\boldsymbol{\Delta}^T\right\}\Upsilon^T+\frac{\partial^\beta}{\partial|x|^\beta}Y^T(x)\right]\Upsilon^T+\frac{\partial^\beta}{\partial|x|^\beta}Z^T(x)\right)\\
&\times B\left(\tilde{\Upsilon}P(t)+\tilde{Y}(t)\right)
\end{aligned}
$$

Table 7. Errors for the real part and imaginary part of the approximate solution with $\alpha(x,t) = \frac{2+\sin(xt)}{400}$ and $N = 3$ of Example 8.5.

	$M=3$		$M=6$	
	Re-part	Im-part	Re-part	Im-part
L_2-error	9.1663×10^{-7}	6.1757×10^{-7}	1.0753×10^{-11}	2.1291×10^{-11}
L_∞-error	6.9541×10^{-7}	4.6015×10^{-7}	9.3471×10^{-12}	1.7410×10^{-11}
RMS	2.7637×10^{-7}	1.8621×10^{-7}	3.2422×10^{-12}	6.4197×10^{-12}

$$+\left\{\frac{\partial^\beta}{\partial|x|^\beta}x\right\}\left(t^3\sin(1) - \left(\int_0^1 P^T(x)dx\Upsilon^T + \int_0^1 Y^T(x)dx\right)B\left(\tilde{\Upsilon}P(t) + \tilde{Y}(t)\right)\right).$$

Finally, by substituting the aforementioned approximation relations into the problem, we obtain a system of equations. By considering the proposed method for $\alpha(x,t) = 0.75 + 0.2\exp(-xt)$ and $M = N = 3$, we obtain

$$\begin{aligned}
&\mathcal{U}(x,t)\\
&= 4.48254715 \times 10^{-7}tx^3 - 9.32423071 \times 10^{-8}tx^2 - 2.16448669 \times 10^{-8}tx^4\\
&\quad -2.97548156 \times 10^{-9}tx^5 - 1.35588257 \times 10^{-7}tx + 7.93509567 \times 10^{-7}t^2x^2\\
&\quad -2.275562 \times 10^{-6}t^2x^3 - 0.49951534t^3x^2 + 1.65741381 \times 10^{-6}t^2x^4\\
&\quad -2.17917575 \times 10^{-3}t^3x^3 + 2.86803747 \times 10^{-6}t^4x^2 - 3.77099928 \times 10^{-7}t^2x^5\\
&\quad +0.045994907t^3x^4 - 1.08899919 \times 10^{-5}t^4x^3 - 0.00396327t^3x^5\\
&\quad +6.32634849 \times 10^{-6}t^4x^4 - 6.24147275 \times 10^{-7}t^4x^5 + 2.01739089 \times 10^{-7}t^2x\\
&\quad -3.48113691 \times 10^{-5}t^3x + 2.31975325 \times 10^{-6}t^4x + t^3,
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{V}(x,t)\\
&= 3.25696787 \times 10^{-7}tx^3 - 2.17428054 \times 10^{-8}tx^2 - 5.385105192 \times 10^{-9}tx^4\\
&\quad -1.01640368 \times 10^{-7}tx^5 - 1.96928508 \times 10^{-7}tx + 7.41767571 \times 10^{-7}t^2x^2\\
&\quad -3.48209618 \times 10^{-6}t^2x^3 + 2.06107326 \times 10^{-4}t^3x^2 + 1.42413001 \times 10^{-7}t^2x^4\\
&\quad -0.167553541t^3x^3 + 1.98394158 \times 10^{-7}t^4x^2 + 1.01522004 \times 10^{-6}t^2x^5\\
&\quad +0.00158145t^3x^4 - 2.45692854 \times 10^{-6}t^4x^3 + 0.00725267t^3x^5\\
&\quad -4.08437306 \times 10^{-6}t^4x^4 + 3.43564331 \times 10^{-6}t^4x^5\\
&\quad +1.58269557 \times 10^{-6}t^2x + 0.99998429t^3x + 2.90726414 \times 10^{-6}t^4x.
\end{aligned}$$

Table 7 displays the least square error, maximum absolute error and root mean square error for a different choice of M with $\alpha(x,t) = \frac{2+\sin(xt)}{400}$ and $N = 3$. Also, the absolute error of the real part and imaginary part of the approximate solution are illustrated in Figure 7. From this table, it can be concluded that by increasing the value of basis functions M, the error tends to zero. Furthermore, we listed the convergence order for different values of M in Table 8. Table 8 indicates more Pell polynomials improve the accuracy rapidly.

Table 8. Convergence order for the real part and imaginary part of the approximate solution with $\alpha(x,t) = \frac{2+\sin(xt)}{400}$ and $N = 3$ of Example 8.5.

$x = t$	$M = 3, \bar{M} = 6$		$M = 5, \bar{M} = 6$	
	Re-part	Im-part	Re-part	Im-part
0.2	8.815267	9.305897	14.482894	11.929048
0.4	9.658216	10.297097	16.332050	14.076232
0.6	10.047746	10.395906	16.186109	14.549069
0.8	9.736902	9.277343	14.541508	11.814301

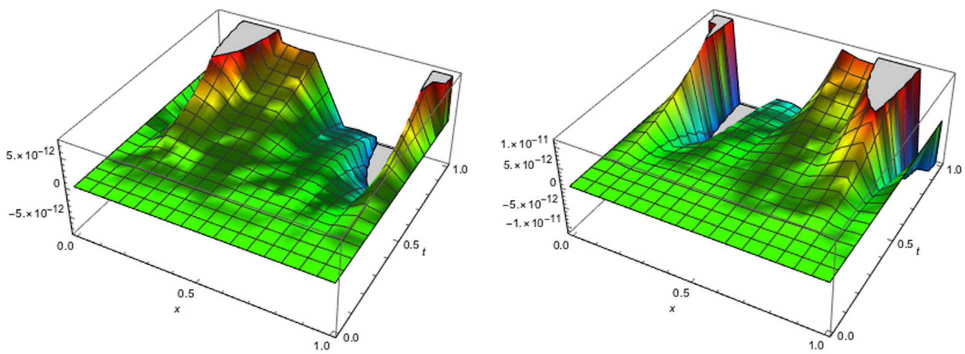

Figure 7. The absolute error of the real part (left) and imaginary part (right) of approximate solution obtained with $\alpha(x,t) = 0.75 + 0.2\exp(-xt)$ and $M = 6, N = 3$ of Example 8.5.

9. Conclusion

This paper deals with the discretization method combined with Pell polynomials for solving variable-order Caputo-Riesz time-space fractional Schrödinger equations. In order to achieve an accurate and convenient technique for solving the proposed problem, we applied a novel method to obtain the operational matrix. It is also worth mentioning that the algorithm considered in this study leads to the conversion of the problem into a system of equations. Finally, several test problems have been included to demonstrate the validity and accuracy of the numerical technique. Furthermore, the obtained results have been compared with those from the Jacobi-Gauss-Lobatto collocation method [4] and the finite difference method [25]. The proposed technique offers several advantages, including the accurate relationship between Pell polynomials and Taylor polynomials, as well as accurate operational matrices. These properties contribute to the superiority of this method when compared to the Jacobi-Gauss-Lobatto collocation method and the finite difference method. The comparison tables clearly show that the Pell spectral method yields more accurate results compared to the two aforementioned methods.

Acknowledgements

We express our sincere thanks to the anonymous referees for their valuable suggestions that improved the final manuscript.

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