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VARIATION OPERATORS FOR COMMUTATORS OF ROUGH SINGULAR INTEGRALS ON WEIGHTED MORREY SPACES*

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Abstract In this paper, we establish the boundedness and compactness the variation operators of commutators of singular integrals with rough kennels $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q \in (1, \infty]$ on the weighted Lebesgue and Morrey spaces. Our main results represent significant improvements as well as natural extensions of what was known previously.

Keywords Variation operator, rough singular integrals, commutators, weighted Lebesgue and Morrey spaces, boundedness and compactness.

MSC(2010) 42B20, 42B99.

1. Introduction

Let $m \in \mathbb{N}$. The m-th iterated commutator of singular integral is defined as

$$T^m_{\Omega,b}(f)(x) = \lim_{\epsilon \to 0^+} T^m_{\Omega,b,\epsilon}(f)(x),$$

where

$$T^m_{\Omega,b,\epsilon}(f)(x) = \int_{|x-y|>\epsilon} \frac{\Omega(x-y)}{|x-y|^n}(b(x)-b(y))^m f(y)dy.$$

Here $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\Omega \in L^1(\mathrm{S}^{n-1})$ is homogeneous of zero and satisfies

$$\int_{\mathrm{S}^{n-1}} \Omega(\theta)d\sigma(\theta) = 0. \tag{1.1}$$

For convenience, we denote $T^m_{\Omega,b} = T_{\Omega,b}$ for $m = 1$ and $T^m_{\Omega,b} = T_\Omega$ for $m = 0$. Let $\mathcal{T}_\Omega = \{T_{\Omega,\epsilon}\}_{\epsilon>0}$ and $\mathcal{T}^m_{\Omega,b} = \{T^m_{\Omega,b,\epsilon}\}_{\epsilon>0}$. For $\rho > 2$, the ρ-variation operator of $\mathcal{T}_\Omega$ is defined by

$$\mathcal{V}_\rho(\mathcal{T}_\Omega)(f)(x) := \sup_{\varepsilon_i \searrow 0} \Big(\sum_{i=1}^{\infty} \Big| \int_{\varepsilon_{i+1}<|x-y|\le\varepsilon_i} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \Big|^\rho \Big)^{1/\rho},$$

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where the above sup is taken over all sequences $\{\varepsilon_i\}$ decreasing to zero. Analogously, the ρ-variation operator of $\mathcal{T}_{\Omega,b}^m$ can be given as

$$\begin{aligned}&\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f)(x)\\ &:= \sup_{\varepsilon_i\searrow 0}\Big(\sum_{i=1}^{\infty}\Big|\int_{\varepsilon_{i+1}<|x-y|\le\varepsilon_i}(b(x)-b(y))^m\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\Big|^\rho\Big)^{1/\rho},\end{aligned}$$

where the above sup is taken over all sequences $\{\varepsilon_i\}$ decreasing to zero. For convenience, we denote $\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)=\mathcal{V}_\rho(\mathcal{T}_\Omega)$ when $m=0$.

Over the last several years, a considerable amount of research has been done to study variational inequalities of various integral operators. This study was initiated by Lépingle [18] who established the variational inequality for general martingales (see [23] for a simple proof). Similar variational inequalities for the ergodic averages were obtained by Bourgain [1] via Lépingle's result. Since then, Bourgain's work has inaugurated a lot of investigations on the variational inequalities in harmonic analysis (see [3, 5, 6, 8, 14, 19, 20, 22]). The study of the variation operators for rough singular integrals began with Campbell, Jones, Reinhdd and Wierdl [3] who established the $L^p(\mathbb{R}^n)\,(1<p<\infty)$ bounds for $\mathcal{V}_\rho(\mathcal{T}_\Omega)$, provided that $\Omega\in L\log^+L(\mathrm{S}^{n-1})$. This result was essentially improved by Ding, Hong and Liu [8] to the case $\Omega\in H^1(\mathrm{S}^{n-1})$ since $L\log^+L(\mathrm{S}^{n-1})\subsetneq H^1(\mathrm{S}^{n-1})$, which is a proper inclusion. The weighted result for $\mathcal{V}_\rho(\mathcal{T}_\Omega)$ was first studied by Ma, Torrea and Xu [22] who proved that $\mathcal{V}_\rho(\mathcal{T}_\Omega)$ is bounded on $L^p(w)$ for $1<p<\infty$ and $w\in A_p(\mathbb{R}^n)$, provided that $\Omega\in \mathrm{Lip}_\alpha(\mathrm{S}^{n-1})$ for $\alpha>0$. Later on, the above result was improved by Chen, Ding, Hong and Liu [5] to the case $\Omega\in L^q(\mathrm{S}^{n-1})$ for some $q>1$. More precisely, the authors of [5] showed that if $\Omega\in L^q(\mathrm{S}^{n-1})$ for some $q>1$ satisfying (1.1), then $\mathcal{V}_\rho(\mathcal{T}_\Omega)$ is bounded on $L^p(w)$ for $q'<p<\infty$ and $w\in A_{p/q'}(\mathbb{R}^n)$. For the commutators of rough singular integrals, Chen, Ding, Hong and Liu [6] proved that if $b\in \mathrm{BMO}(\mathbb{R}^n)$ and $\Omega\in L^q(\mathrm{S}^{n-1})$ for some $q>1$ satisfying (1.1), then $\mathcal{V}_\rho(\mathcal{T}_{\Omega,b})$ is bounded on $L^p(w)$ for $q'<p<\infty$ and $w\in A_{p/q'}(\mathbb{R}^n)$. Recently, Liu and Cui [19] investigated the boundedness and compactness for the ρ-variation operators for commutators of singular integrals on the weighted Morrey spaces. For more progresses on commutators of some integrals, we refer to the papers [2, 4, 10, 11, 13, 25, 26].

Let us recall the definition of weighted Morrey spaces.

Definition 1.1. (Weighed Morrey spaces). Let $1\le p<\infty$ and $0\le\beta<1$. For a weight w defined on $\mathbb{R}^n$, the weighted Morrey space $M^{p,\beta}(w)$ is defined by

$$M^{p,\beta}(w):=\{f\in L^p_{\mathrm{loc}}(w):\|f\|_{M^{p,\beta}(w)}<\infty\},$$

where

$$\|f\|_{M^{p,\beta}(w)}:=\sup_{B\text{ balls in }\mathbb{R}^n}\Big(\frac{1}{w(B)^\beta}\int_B|f(x)|^pw(x)dx\Big)^{1/p},$$

where the supremum is taken over all balls in $\mathbb{R}^n$.

The weighted Morrey spaces $M^{p,\beta}(w)$ were originally introduced by Komori and Shirai [16] who established the bounds for the Hardy–Littlewood maximal operator, fractional integral operator and the Calderón–Zygmund singular integral operator on $M^{p,\beta}(w)$. When $\beta=0$, the space $M^{p,\beta}(w)$ is just the classical weighted Lebesgue space $L^p(w)$. When $w\equiv 1$, the space $M^{p,\beta}(w)$ reduces to the classical Morrey space $M^{p,\beta}(\mathbb{R}^n)$. More progresses on Morrey spaces were much investigated in [12, 15, 24, 27, 28].

We now introduce partial result of [19] as follows.

Theorem A ([19]). *Let $\Omega \in \mathrm{Lip}_\alpha(\mathrm{S}^{n-1})$ for some $\alpha > 0$ and Ω satisfy* (1.1). *Let $\rho > 2$, $1 < p < \infty$, $0 \le \beta < 1$ and $w \in A_p(\mathbb{R}^n)$. Then*

(i) *If $m \in \mathbb{N}$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$, then*

$$\|\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})(f)\|_{M^{p,\beta}(w)} \le C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{M^{p,\beta}(w)}, \quad \forall f \in M^{p,\beta}(w).$$

(ii) *If $m \in \mathbb{N} \setminus \{0\}$ and $b \in \mathrm{CMO}(\mathbb{R}^n)$, then $\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})$ is a compact operator on $M^{p,\beta}(w)$.*

It is well known that

$$\mathrm{Lip}_\alpha(\mathrm{S}^{n-1}) \subsetneq L^q(\mathrm{S}^{n-1}), \quad \forall \alpha > 0, \quad 1 < q \le \infty.$$

Note that the above inclusion relationship is proper. Very recently, Zhang, Liu and Zhang [30] investigated the boundedness and compactness for the ρ-variation operators for commutators of singular integrals on Morrey spaces. In order to introduce the main result of [30], let us introduce one notation. Let $1 \le q < \infty$ and set

$$F(q) := \int_0^1 \frac{w_q(\delta)}{\delta}(1 + |\log \delta|)d\delta < \infty. \tag{1.2}$$

Here $w_q(\delta)$ denotes the integral continuous modulus of Ω of degree q defined by

$$w_q(\delta) := \sup_{\|\rho\|<\delta} \Big(\int_{\mathrm{S}^{n-1}} |\Omega(\tau x') - \Omega(x')|^q d\sigma(x')\Big)^{1/q}$$

and τ is a rotation in $\mathbb{R}^n$ and $\|\tau\| := \sup_{x' \in \mathrm{S}^{n-1}} |\tau x' - x'|$. The main result of [30] can be listed as follows.

Theorem B ([30]). *Let $m \in \mathbb{N}$, $\rho > 2$, $0 \le \beta < 1$ and $1 < p < \infty$. Let $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q > 1$ satisfying* (1.1).

(i) *If $b \in \mathrm{BMO}(\mathbb{R}^n)$, then*

$$\|\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})(f)\|_{M^{p,\beta}(\mathbb{R}^n)} \lesssim_{n,p,\beta,m} \|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{M^{p,\beta}(\mathbb{R}^n)}, \quad \forall f \in M^{p,\beta}(\mathbb{R}^n).$$

(ii) *If $b \in \mathrm{CMO}(\mathbb{R}^n)$ and $F(1) < \infty$, then $\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})$ is a compact operator on $M^{p,\beta}(\mathbb{R}^n)$.*

Based on the above, it is natural to ask the following

Question 1.2. *Does Theorem A hold if $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q > 1$?*

This question can be addressed by the following result.

Theorem 1.1. *Let $m \in \mathbb{N} \setminus \{0\}$, $\rho > 2$, $0 \le \beta < 1$ and $1 < p < \infty$. Let $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q > 1$ satisfying* (1.1), *$q' < p < \infty$ and $w \in A_{p/q'}(\mathbb{R}^n)$. Then*

(i) *The operator $\mathcal{V}_\rho(\mathcal{T}_\Omega)$ is bounded on $M^{p,\beta}(w)$.*

(ii) *If* $b \in \mathrm{BMO}(\mathbb{R}^n)$, *then*

$$\|\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})(f)\|_{M^{p,\beta}(w)} \lesssim_{n,p,\beta,m} \|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{M^{p,\beta}(w)}, \quad \forall f \in M^{p,\beta}(w).$$

Theorem 1.2. *Let* $m \in \mathbb{N}\setminus\{0\}$, $\rho > 2$, $0 \le \beta < 1$ *and* $1 < p < \infty$. *Let* $\Omega \in L^q(\mathrm{S}^{n-1})$ *for some* $q > 1$ *satisfying* (1.1), $q' < p < \infty$ *and* $w \in A_{p/q'}(\mathbb{R}^n)$. *If* $b \in \mathrm{CMO}(\mathbb{R}^n)$ *and* $F(q) < \infty$, *then* $\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})$ *is a compact operator on* $M^{p,\beta}(w)$.

Remark 1.1. There are some remarks as follows:

(i) It should be pointed out that the condition $F(q) < \infty$ was firstly introduced by Chen, Ding and Wang [7] who proved that if $b \in \mathrm{CMO}(\mathbb{R}^n)$ and $F(q) < \infty$, then $T_{\Omega,b}$ is a compact operator on $M^{p,\beta}(\mathbb{R}^n)$, provided that $0 < \beta < 1$, $1 < p < \infty$ and $\Omega \in L^q(\mathrm{S}^{n-1})$ with $q > 1/(1-\beta)$ satisfying (1.1).

(ii) We remark that the conditions $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q \in (1, \infty]$ and $F(q) < \infty$ are strictly weaker than the condition $\Omega \in \mathrm{Lip}_\alpha(\mathrm{S}^{n-1})$ with some $\alpha > 0$. Thus, Theorems 1.1 and 1.2 essentially improve the conclusions of Theorem A when $q = \infty$.

(iii) Theorems 1.1 and 1.2 can be regarded as the weighted version of Theorem B.

(iv) When $\beta = 0$, Theorem 1.2 implies the compactness of $\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})$ with $m \ge 1$ on $L^p(w)$, which is new, even in the special case $m = 1$.

(v) When $0 < \beta < 1$, Theorems 1.1 and 1.2 are new, even in the special case $m = 0$.

(vi) Note that $L^q(\mathrm{S}^{n-1}) \subsetneq L(\log^+ L)^\beta(\mathrm{S}^{n-1}) \subsetneq H^1(\mathrm{S}^{n-1})$ for any $q > 1$ and $\beta \ge 1$. It is unknown whether the corresponding results in Theorems 1.1 and 1.2 also hold under the condition that $\Omega \in L(\log^+ L)^\alpha(\mathrm{S}^{n-1})$ for some $\alpha \ge 1$ or more generally $\Omega \in H^1(\mathrm{S}^{n-1})$?

(vii) The corresponding results also hold for $T^m_{\Omega,b}$ under the same conditions of Theorem 1.1.

Actually, we shall prove Theorem 1.1 by establishing a more general result. Let us give one definition.

Definition 1.3. Let $m \ge 1$ and $\vec{b} = (b_1, \ldots, b_m)$ be a suitable vector function. Let $\mathcal{T}^m_{\Omega,\vec{b}} = \{T^m_{\Omega,\vec{b},\epsilon}\}_{\epsilon>0}$ with $m \ge 1$, where $T^m_{\Omega,\vec{b},\epsilon}$ is given by

$$T^m_{\Omega,\vec{b},\epsilon}(f)(x) = \int_{|x-y|>\epsilon} \prod_{j=1}^m (b_j(x) - b_j(y))K(x,y)f(y)dy.$$

For $\rho > 2$, the ρ-variation operators of $\mathcal{T}^m_{\Omega,\vec{b}}$ is defined by

$$\mathcal{V}_\rho(\mathcal{T}^m_{K,\vec{b}})(f)(x) := \sup_{\varepsilon_i \searrow 0} \Big(\sum_{i=1}^\infty \Big|\int_{\varepsilon_{i+1}<|x-y|\le\varepsilon_i} \prod_{j=1}^m (b_j(x) - b_j(y))K(x,y)f(y)dy\Big|^\rho\Big)^{1/\rho},$$

where the above sup is taken over all sequences $\{\varepsilon_i\}$ decreasing to zero. Clearly, $\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,\vec{b}}) = \mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})$ if $b_j = b$ for all $1 \le j \le m$.

Theorem 1.1 follows from the following result.

Theorem 1.3. *Let $m \in \mathbb{N}$, $\rho > 2$, $0 \le \beta < 1$ and $1 < p < \infty$. Let $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q > 1$ satisfying (1.1). Let $\vec{b} = (b_1, \ldots, b_m)$ with each $b_j \in \mathrm{BMO}(\mathbb{R}^n)$. If $q' < p < \infty$ and $w \in A_{p/q'}(\mathbb{R}^n)$, then*

$$\|\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,\vec{b}})(f)\|_{M^{p,\beta}(w)} \lesssim_{n,p,\beta,m} \prod_{j=1}^{m} \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{M^{p,\beta}(w)}, \quad \forall f \in M^{p,\beta}(w). \tag{1.3}$$

Remark 1.2. By a slight modification of the proof of Theorem 1.2, it is not difficult to conclude that the compactness result of Theorem 1.2 also holds in the multilinear setting. That is, if $\vec{b} = (b_1, \ldots, b_m)$ with each $b_j \in \mathrm{CMO}(\mathbb{R}^n)$, then $\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,\vec{b}})$ is a compact operator on $M^{p,\beta}(w)$ for $q' < p < \infty$ and $w \in A_{p/q'}(\mathbb{R}^n)$, provided that $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q > 1$ satisfying (1.1) and $F(q) < \infty$.

Remark 1.3. We believe this is the first time that such the boundedness and compactness for variation operators for commutators of rough singular integrals on weighted Morrey spaces are studied.

The rest of this paper is organized as follows. In Section 2 we present the proof of Theorem 1.3. The proof of Theorem 1.2 will be given in Section 3. We would like to point out that Theorem 1.3 is based on a criterion on the weighted Morrey space boundedness of a class of operators (see Proposition 2.1). The proof of Theorem 1.2 is based on Theorem 1.3, some approximation arguments followed from [29], smooth truncated techniques followed from [17], some compactness characterizations of $L^p(w)$ and $M^{p,\beta}(w)$ and some known techniques following from [19]. However, some new techniques are needed in the weighted setting. The main novelty is on how to accommodate these ideas to prove the main results.

2. Proof of Theorem 1.3

Before establishing the proof of Theorem 1.3, let us present some notation and lemmas, which are the ingredients of proving Theorem 1.3.

Throughout this paper, the letter C or c, sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence, but are independent of the essential variables. If there exists a constant $c > 0$ depending only on ϑ such that $A \le cB$, we then write $A \lesssim_\vartheta B$. For any $p \in (1, \infty)$, we let p' denote the dual exponent to p defined as $1/p + 1/p' = 1$. For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x with radius r. For $t > 0$ and $B := B(x, r)$ with $x \in \mathbb{R}^n$ and $r > 0$, we denote $tB = B(x, tr)$. Let $\vec{b} = (b_1, \ldots, b_m)$ and Q be a cube of $\mathbb{R}^n$. For $1 \le j \le m$, we denote $b_{j,Q} = \frac{1}{|Q|}\int_Q b_j(x)dx$. Let $A \subset \mathbb{R}^n$, we use χ_A to denote the characteristic function on A. We now introduce an useful inequality:

$$\Big(\sum_{i=1}^{\infty} \Big| \int_{\varepsilon_{i+1} < |x-y| \le \varepsilon_i} F(x,y)dy\Big|^\rho\Big)^{1/\rho} \le \int_{\mathbb{R}^n} |F(x,y)|dy, \tag{2.1}$$

for all $x \in \mathbb{R}^n$, any arbitrary functions F defined on $\mathbb{R}^n \times \mathbb{R}^n$, where $\rho > 1$ and $\{\varepsilon_i\}$ is an increasing or decreasing sequence of positive numbers.

We start with the definition of A_p weight class.

Definition 2.1. **(A_p weight).** A weight is a nonnegative, locally integrable function on $\mathbb{R}^n$ that takes values in $(0,\infty)$ almost everywhere. For $1<p<\infty$, a weight w is said to be in the Muckenhoupt weight class $A_p(\mathbb{R}^n)$ if there exists a positive constant C such that

$$\sup_{Q \text{ cubes in } \mathbb{R}^n} \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\Big)^{p-1} \le C. \tag{2.2}$$

The smallest constant C in inequality (2.2) is the corresponding A_p constant of w, which is denoted by $[w]_{A_p}$.

We now recall the definition of $\mathrm{BMO}(\mathbb{R}^n)$.

Definition 2.2. **($\mathrm{BMO}(\mathbb{R}^n)$ space).** The $\mathrm{BMO}(\mathbb{R}^n)$ space is given by

$$\mathrm{BMO}(\mathbb{R}^n) := \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \quad \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \|M^\sharp f\|_{L^\infty(\mathbb{R}^n)} < \infty\},$$

where $M^\sharp$ is the sharp maximal function, i.e.

$$M^\sharp f(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)-f_Q|dy,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ that contain the given point x.

The following result presents some properties for $A_p(\mathbb{R}^n)$ weights and $\mathrm{BMO}(\mathbb{R}^n)$ functions, which are very useful in the proofs of main results.

Lemma 2.1. ([19]). *Let $1<p<\infty$ and $w\in A_p(\mathbb{R}^n)$. Then*

(i) *There exists a constant $\theta\in(0,1)$ such that $w^{1+\theta}\in A_p(\mathbb{R}^n)$. Both θ and $[w^{1+\theta}]_{A_p}$ depend only on n, p and the A_p constant of w.*

(ii) *There exists a constant $\epsilon\in(0,1)$ such that $w\in A_{p-\epsilon}(\mathbb{R}^n)$.*

(iii) *The measure $w(x)dx$ is doubling, i.e. for all $\lambda>1$ we have*

$$\sup_{\mathrm{Q} \text{ cubes in } \mathbb{R}^n}\frac{w(\lambda Q)}{w(Q)} \le [w]_{A_p}\lambda^{np}.$$

(iv) *There exists a constant $\gamma_w>1$ such that*

$$\inf_{\mathrm{Q} \text{ cubes in } \mathbb{R}^n}\frac{w(2Q)}{w(Q)} \ge \gamma_w.$$

(v) *If $b\in\mathrm{BMO}(\mathbb{R}^n)$, then*

$$\sup_{Q \text{ cubes in } \mathbb{R}^n}\Big(\frac{1}{w(Q)}\int_Q |b(x)-b_Q|^p w(x)dx\Big)^{1/p} \simeq_{p,[w]_{A_p}} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

For convenience, we always use the weighted Morrey spaces associated to cubes. Let $1\le p<\infty$ and $0\le\beta<1$. For a weight w defined on $\mathbb{R}^n$, the weighted Morrey space associated to cubes is defined by

$$\widetilde{M}^{p,\beta}(w) := \{f\in L^p_{\mathrm{loc}}(w) : \|f\|_{\widetilde{M}^{p,\beta}(w)} < \infty\},$$

where

$$\|f\|_{\widetilde{M}^{p,\beta}(w)} := \sup_{\text{Q cubes in } \mathbb{R}^n} \Big(\frac{1}{w(Q)^\beta}\int_Q |f(x)|^p w(x)dx\Big)^{1/p},$$

where the supremum is taken over all cubes in $\mathbb{R}^n$. In [19], the authors pointed out that if the weight w is doubling, then $\widetilde{M}^{p,\beta}(w) = M^{p,\beta}(w)$, i.e.

$$\|f\|_{\widetilde{M}^{p,\beta}(w)} \simeq \|f\|_{M^{p,\beta}(w)}. \tag{2.3}$$

For $\Omega \in L^1(\mathrm{S}^{n-1})$, the maximal operator with rough kernel Ω is defined by

$$M_\Omega(f)(x) = \sup_{r>0} \frac{1}{r^n}\int_{|y|\le r} |\Omega(y')f(x-y)|dy.$$

The following lemmas were proved by Lu, Ding and Yan [21].

Lemma 2.2. ([21]). *Let $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q > 1$ satisfying* (1.1). *If $q' < p < \infty$ and $w \in A_{p/q'}(\mathbb{R}^n)$, then M_Ω is bounded on $L^p(w)$.*

Lemma 2.3. ([21]). *Let $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q \in [1,\infty)$ satisfying* (1.1). *Then for $R > 0$, there exists a constant $C > 0$ independent of R such that for $x \in \mathbb{R}^n$ with $|x| < R/2$,*

$$\Big(\int_{R<|y|<2R}\Big|\frac{\Omega(y-x)}{|y-x|^n} - \frac{\Omega(y)}{|y|^n}\Big|^q dy\Big)^{1/q} \le CR^{-\frac{n}{q'}}\Big(\frac{|x|}{R} + \int_{|x|/(2R)}^{|x|/R}\frac{w_q(\delta)}{\delta}d\delta\Big).$$

Here $w_q(\delta)$ is given as in Theorem 1.1.

Applying Lemma 2.3, we have

Lemma 2.4. *Let $\eta > 0$ and $|h| < \frac{\eta}{4}e^{-1/\eta}$. Suppose that Ω satisfies* (1.1) *and $F(q) < \infty$ with $q \in (1,\infty)$. Then there exists a constant $C > 0$ independent of η and h such that*

$$\int_{|x-y|>\eta}\Big|\frac{\Omega(x-y+h)}{|x-y+h|^n} - \frac{\Omega(x-y)}{|x-y|^n}\Big||f(y)|dy \le C\eta(1+F(q))(M(|f|^{q'})(x))^{1/q'}. \tag{2.4}$$

Here M is the Hardy-Littlewood maximal operator defined on $\mathbb{R}^n$.

Proof. By a change of variable and Hölder's inequality, we have

$$\begin{aligned}
&\int_{|x-y|>\eta}\Big|\frac{\Omega(x-y+h)}{|x-y+h|^n} - \frac{\Omega(x-y)}{|x-y|^n}\Big||f(y)|dy \\
&\le \sum_{k=0}^\infty \int_{2^k\eta<|x-y|\le 2^{k+1}\eta}\Big|\frac{\Omega(x-y+h)}{|x-y+h|^n} - \frac{\Omega(x-y)}{|x-y|^n}\Big||f(y)|dy \\
&\le \sum_{k=0}^\infty \int_{2^k\eta<|z|\le 2^{k+1}\eta}\Big|\frac{\Omega(z+h)}{|z+h|^n} - \frac{\Omega(z)}{|z|^n}\Big||f(x-z)|dz \\
&\le \sum_{k=0}^\infty \Big(\int_{2^k\eta<|z|\le 2^{k+1}\eta}\Big|\frac{\Omega(z+h)}{|z+h|^n} - \frac{\Omega(z)}{|z|^n}\Big|^q dz\Big)^{1/q} \\
&\quad \times\Big(\int_{2^k\eta<|z|\le 2^{k+1}\eta}|f(x-z)|^{q'}dz\Big)^{1/q'}.
\end{aligned} \tag{2.5}$$

Invoking Lemma 2.3, we have

$$\begin{aligned}
&\Big(\int_{2^k\eta<|z|\le 2^{k+1}\eta}\Big|\frac{\Omega(z+h)}{|z+h|^n}-\frac{\Omega(z)}{|z|^n}\Big|^q dz\Big)^{1/q}\\
&\le C(2^k\eta)^{-n/q'}\Big(\frac{|h|}{2^k\eta}+\int_{|h|/(2^{k+1}\eta)}^{|h|/(2^k\eta)}\frac{w_q(\delta)}{\delta}d\delta\Big)\\
&\le C(2^k\eta)^{-n/q'}\Big(2^{-k}e^{-1/\eta}\\
&\quad+\frac{1}{1+k+\eta^{-1}}\int_{|h|/(2^{k+1}\eta)}^{|h|/(2^k\eta)}\frac{w_q(\delta)}{\delta}(1+|\log\delta|)d\delta\Big).
\end{aligned}\tag{2.6}$$

Combining (2.5) with (2.6) implies that

$$\begin{aligned}
&\int_{|x-y|>\eta}\Big|\frac{\Omega(x-y+h)}{|x-y+h|^n}-\frac{\Omega(x-y)}{|x-y|^n}\Big||f(y)|dy\\
&\le C\sum_{k=0}^{\infty}\Big(2^{-k}e^{-1/\eta}+\frac{1}{1+k+\eta^{-1}}\int_{|h|/(2^{k+1}\eta)}^{|h|/(2^k\eta)}\frac{w_q(\delta)}{\delta}(1+|\log\delta|)d\delta\Big)\\
&\quad\times\Big(\frac{1}{(2^k\eta)^n}\int_{2^k\eta<|z|\le 2^{k+1}\eta}|f(x-z)|^{q'}dz\Big)^{1/q'}\\
&\le C(M(|f|^{q'})(x))^{1/q'}\Big(\sum_{k=0}^{\infty}2^{-k}e^{-1/\eta}+\eta\sum_{k=0}^{\infty}\int_{|h|/(2^{k+1}\eta)}^{|h|/(2^k\eta)}\frac{w_q(\delta)}{\delta}(1+|\log\delta|)d\delta\Big)\\
&\le C(1+F(q))\eta(M(|f|^{q'})(x))^{1/q'}.
\end{aligned}$$

This proves (2.4) and completes the proof. □

In order to prove Theorem 1.3, we shall establish the following result.

Proposition 2.1. *Let $0<\beta<1$, $m\in\mathbb{N}$ and $\Omega\in L^q(\mathrm{S}^{n-1})$ for some $q>1$ satisfying (1.1). Let $\vec{b}=(b_1,\dots,b_m)$ with each $b_j\in\mathrm{BMO}(\mathbb{R}^n)$ for $1\le j\le m$ and T be a linear or sublinear operator satisfying*

$$|T(f)(x)|\le C_1\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^n}\prod_{j=1}^m|b_j(x)-b_j(y)||f(y)|dy \tag{2.7}$$

and

$$\|T(f)\|_{L^p(w)}\le C_2\prod_{j=1}^m\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{L^p(w)},\quad \forall f\in L^p(w) \tag{2.8}$$

for $q'<p<\infty$ and $w\in A_{p/q'}(\mathbb{R}^n)$. Then we have

$$\|T(f)\|_{M^{p,\beta}(w)}\lesssim_{m,n,p,\beta,\Omega,C_1,C_2}\prod_{j=1}^m\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{M^{p,\beta}(w)},\quad \forall f\in M^{p,\beta}(w). \tag{2.9}$$

Proof. Let $f\in\widetilde{M}^{p,\beta}(w)$ and $\beta\in(0,1)$. Fix a cube $Q=Q(x_0,r)$. To prove (2.9), by (2.3), it suffices to show that

$$\Big(\frac{1}{w(Q)^\beta}\int_Q|T(f)(x)|^p w(x)dx\Big)^{1/p}\le C\prod_{j=1}^m\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w)}, \tag{2.10}$$

where $C > 0$ is independent of x_0, r and $b_1, \dots, b_m$.

Let us decompose f as $f = f\chi_{2Q} + f\chi_{(2Q)^c}$. Then we have

$$\begin{aligned}
&\Big(\frac{1}{w(Q)^\beta}\int_Q |T(f)(x)|^p w(x)dx\Big)^{1/p}\\
\le&\Big(\frac{1}{w(Q)^\beta}\int_Q |T(f\chi_{2Q})(x)|^p w(x)dx\Big)^{1/p}\\
&+\Big(\frac{1}{w(Q)^\beta}\int_Q |T(f\chi_{(2Q)^c})(x)|^p w(x)dx\Big)^{1/p}\\
=:& I_1 + I_2.
\end{aligned}\tag{2.11}$$

By (2.8) and Lemma 2.1 (iii), we have

$$\begin{aligned}
I_1 &\lesssim_{C_2,p} \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\Big(\frac{1}{w(Q)^\beta}\int_{2Q} |f(x)|^p w(x)dx\Big)^{1/p}\\
&\lesssim_{C_2,p} \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\Big(\Big(\frac{w(2Q)}{w(Q)}\Big)^\beta \frac{1}{w(2Q)^\beta}\int_{2Q} |f(x)|^p w(x)dx\Big)^{1/p}\\
&\lesssim_{C_2,n,\beta,p} \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w)}.
\end{aligned}\tag{2.12}$$

In view of (2.11) and (2.12), for (2.10) it suffices to show that

$$I_2 \lesssim_{n,\beta,p} \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w)}.\tag{2.13}$$

Fix $x \in Q$. Note that for any $k \ge 1$ and $z \in 2^{k+1}Q \setminus 2^kQ$. It holds that

$$\begin{aligned}
2^{k+2}r &\ge (2^{k+1}+1)r \ge |z-x_0|_\infty + |x-x_0|_\infty\\
&\ge |x-z|_\infty \ge |z-x_0|_\infty - |x-x_0|_\infty \ge (2^k-1)r \ge 2^{k-1}r.
\end{aligned}\tag{2.14}$$

It follows from (2.7) and (2.14) that

$$\begin{aligned}
&|T(f\chi_{(2B)^c})(x)|\\
\le& C_1\int_{(2Q)^c} \frac{|\Omega(x-z)|}{|x-z|^n}\prod_{j=1}^m |b_j(x)-b_j(z)||f(z)|dz\\
=& C_1\sum_{k=1}^\infty \frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q\setminus 2^kQ} |\Omega(x-z)|\prod_{j=1}^m |b_j(x)-b_j(z)||f(z)|dz.
\end{aligned}\tag{2.15}$$

Fix $k \ge 1$ and let $E = \{1, \dots, m\}$. Observe that

$$\begin{aligned}
\prod_{j=1}^m |b_j(x)-b_j(z)| &= \prod_{j=1}^m (|b_j(x)-b_{j,2^{k+1}Q}| + |b_j(z)-b_{j,2^{k+1}Q}|)\\
&= \sum_{\tau\subset E}\Big(\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{k+1}Q}|\Big)\Big(\prod_{\nu\in E\setminus\tau}|b_\nu(z)-b_{\nu,2^{k+1}Q}|\Big).
\end{aligned}$$

This together with (2.15) implies that

$$
\begin{aligned}
&|T(f\chi_{(2B)^c})(x)| \\
&\leq C_1 \sum_{\tau\subset E}\Big(\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{k+1}Q}|\Big) \\
&\quad\times\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q\setminus 2^kQ}|\Omega(x-z)|\Big(\prod_{\nu\in E\setminus\tau}|b_\nu(z)-b_{\nu,2^{k+1}Q}|\Big)|f(z)|dz.
\end{aligned}
\tag{2.16}
$$

Fix $\tau\subset E$. By Lemma 2.1 (i), there exists $\epsilon\in(0,1)$ such that $w^{1+\epsilon}\in A_{p/q'}(\mathbb{R}^n)$. Let $t=\frac{p((p/q')'-1)(1+\epsilon)}{(p/q')'(1+\epsilon)-\epsilon}$. One can easily check that $t\in(q',p)$ and there exists $\delta\in(1,\infty)$ such that $1/t+1/q+1/\delta=1$. By Hölder's inequality,

$$
\begin{aligned}
&\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q\setminus 2^kQ}|\Omega(x-z)|\Big(\prod_{\nu\in E\setminus\tau}|b_\nu(z)-b_{\nu,2^{k+1}Q}|\Big)|f(z)|dz \\
&\leq\frac{1}{|2^{k+1}Q|}\Big(\int_{2^{k+1}Q\setminus 2^kQ}|\Omega(x-z)|^qdz\Big)^{1/q}\Big(\int_{2^{k+1}Q\setminus 2^kQ}|f(z)|^tdz\Big)^{1/t} \\
&\quad\times\Big(\int_{2^{k+1}Q\setminus 2^kQ}\Big(\prod_{\nu\in E\setminus\tau}|b_\nu(z)-b_{\nu,2^{k+1}Q}|\Big)^\delta dz\Big)^{1/\delta}.
\end{aligned}
\tag{2.17}
$$

By (2.14) and some changes of variables, one has

$$
\begin{aligned}
\int_{2^{k+1}Q\setminus 2^kQ}|\Omega(x-z)|^qdz&\leq\int_{2^{k-1}r\leq|x-z|_\infty\leq 2^{k+2}r}|\Omega(x-z)|^qdz \\
&\leq\int_{2^{k-1}r\leq|z|_\infty\leq 2^{k+2}r}|\Omega(z)|^qdz \\
&\leq\int_{2^{k-1}r\leq|z|\leq 2^{k+2}\sqrt{n}r}|\Omega(z)|^qdz \\
&\leq\int_{\mathrm{S}^{n-1}}|\Omega(\theta)|^qd\sigma(\theta)\int_{2^{k-1}r}^{2^{k+2}\sqrt{n}r}u^{n-1}du \\
&\lesssim_n (2^kr)^n\|\Omega\|^q_{L^q(\mathrm{S}^{n-1})}.
\end{aligned}
\tag{2.18}
$$

On the other hand, there exists $\{r_i\}_{i\in E\setminus\tau}\subset(1,\infty)$ such that $\sum_{i\in E\setminus\tau}1/r_i=1$. By Hölder's inequality and a well-known property for $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$, we obtain

$$
\begin{aligned}
&\Big(\int_{2^{k+1}Q\setminus 2^kQ}\Big(\prod_{\nu\in E\setminus\tau}|b_\nu(z)-b_{\nu,2^{k+1}Q}|\Big)^\delta dz\Big)^{1/\delta} \\
&\leq\prod_{\nu\in E\setminus\tau}\Big(\int_{2^{k+1}Q}|b_\nu(z)-b_{\nu,2^{k+1}Q}|^{\delta r_\nu}dz\Big)^{1/(\delta r_\nu)} \\
&\lesssim_{m,n}\prod_{\nu\in E\setminus\tau}|2^{k+1}Q|^{1/(\delta r_\mu)}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\lesssim_{m,n}|2^{k+1}Q|^{1/\delta}\prod_{\nu\in E\setminus\tau}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}.
\end{aligned}
\tag{2.19}
$$

Let $s=p/t$. One can easily check that $(1+\epsilon)\Big(1-\Big(\frac{p}{q'}\Big)'\Big)=1-s'$ and $\frac{1}{s't}=\frac{1}{t}-\frac{1}{p}=$

$\Big(\frac{1}{q'}-\frac{1}{p}\Big)\frac{1}{1+\epsilon}$. By Hölder's inequality,

$$\begin{aligned}
&\Big(\int_{2^{k+1}Q}|f(z)|^t dz\Big)^{1/t}\\
&\leq \Big(\int_{2^{k+1}Q}|f(z)|^p w(x)dz\Big)^{1/p}\Big(\int_{2^{k+1}Q}w(x)^{1-s'}dz\Big)^{1/(s't)}\\
&\leq w(2^{k+1}Q)^{\beta/p}\|f\|_{\widetilde{M}^{p,\beta}(w)}\Big(\int_{2^{k+1}Q}w(x)^{1-s'}dz\Big)^{1/(s't)}.
\end{aligned}\tag{2.20}$$

Since $w^{1+\epsilon}\in A_{p/q'}(\mathbb{R}^n)$ and $(1+\epsilon)(1-(p/q')')=1-s'$, then

$$\begin{aligned}
&\int_{2^{k+1}Q}w(z)^{1-s'}dz\\
&=\int_{2^{k+1}Q}w(z)^{(1+\epsilon)(1-(p/q')')}dz\\
&\leq [w^{1+\epsilon}]_{A_{p/q'}}^{\frac{1}{p/q'-1}}|2^{l+1}Q|^{(p/q')'}\Big(\int_{2^{l+1}Q}w(x)^{1+\epsilon}dx\Big)^{-1/(p/q'-1)}.
\end{aligned}\tag{2.21}$$

By Hölder's inequality, one has

$$w(2^{k+1}Q)=\int_{2^{k+1}Q}w(x)dx\leq\Big(\int_{2^{k+1}Q}w(x)^{1+\epsilon}dx\Big)^{1/(1+\epsilon)}|2^{k+1}Q|^{\epsilon/(1+\epsilon)},$$

which together with (2.21) implies that

$$\begin{aligned}
&\int_{2^{k+1}Q}w(z)^{1-s'}dz\\
&\leq [w^{1+\epsilon}]_{A_{p/q'}}^{\frac{1}{p/q'-1}}|2^{k+1}Q|^{(p/q')'}\Big(|2^{k+1}Q|^{-\epsilon}w(2^{k+1}Q)^{1+\epsilon}\Big)^{-1/(p/q'-1)}\\
&\leq [w^{1+\epsilon}]_{A_{p/q'}}^{\frac{1}{p/q'-1}}|2^{k+1}Q|^{(p/q')'+\frac{q'\epsilon}{p-q'}}w(2^{k+1}Q)^{-\frac{q'}{p-q'}(1+\epsilon)}.
\end{aligned}\tag{2.22}$$

Note that $1/(s't)=\big(1/q'-1/p\big)/(1+\epsilon)$. Then

$$\frac{q'(1+\epsilon)}{(p-q')s't}=\frac{1}{p},\qquad \frac{(p/q')'}{s't}+\frac{q'\epsilon}{(p-q')s't}=\frac{p+q'\epsilon}{pq'(1+\epsilon)}.$$

This together with (2.22) and (2.20) implies that

$$\begin{aligned}
&\Big(\int_{2^{k+1}Q}|f(z)|^t dz\Big)^{1/t}\\
&\lesssim_{p,q} w(2^{k+1}Q)^{\frac{\beta}{p}-\frac{q'(1+\epsilon)}{(p-q')s't}}|2^{k+1}Q|^{\frac{(p/q')'}{s't}+\frac{q'\epsilon}{(p-q')s't}}\|f\|_{\widetilde{M}^{p,\beta}(w)}\\
&\lesssim_{p,q} w(2^{k+1}Q)^{\frac{\beta-1}{p}}|2^{k+1}Q|^{\frac{p+q'\epsilon}{pq'(1+\epsilon)}}\|f\|_{\widetilde{M}^{p,\beta}(w)}.
\end{aligned}\tag{2.23}$$

It follows from (2.17)-(2.19) and (2.23) that

$$\begin{aligned}
&\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q\backslash 2^kQ}|\Omega(x-z)|\Big(\prod_{\nu\in E\backslash\tau}|b_\nu(z)-b_{\nu,2^{k+1}Q}|\Big)|f(z)|dz\\
&\leq\frac{1}{|2^{k+1}Q|}(2^kr)^{n/q}\|\Omega\|_{L^q(\mathrm{S}^{n-1})}w(2^{k+1}Q)^{\frac{\beta-1}{p}}|2^{k+1}Q|^{\frac{p+q'\epsilon}{pq'(1+\epsilon)}}\|f\|_{\widetilde{M}^{p,\beta}(w)}\\
&\quad\times|2^{k+1}Q|^{1/\delta}\prod_{\nu\in E\backslash\tau}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\lesssim_{m,n,p,q,\Omega}\prod_{\nu\in E\backslash\tau}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}w(2^{k+1}Q)^{\frac{\beta-1}{p}}\|f\|_{\widetilde{M}^{p,\beta}(w)},
\end{aligned}\tag{2.24}$$

where in the last inequality of (2.24) we have used $-\frac{1}{t}+\frac{p+q'\epsilon}{pq'(1+\epsilon)}=0$ and $\frac{1}{q}+\frac{1}{t}+\frac{1}{\delta}=1$. Combining (2.24) with (2.16) implies that

$$I_2 \lesssim_{C_1,m,n,p,q,\Omega} \sum_{\tau\subset E}\prod_{\nu\in E\setminus\tau}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w)} I_3, \tag{2.25}$$

where

$$I_3 := \Big(\frac{1}{w(Q)^\beta}\int_Q\Big(\sum_{k=1}^\infty w(2^{k+1}Q)^{\frac{\beta-1}{p}}|\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{k+1}Q}|\Big)^p w(x)dx\Big)^{1/p}.$$

By Lemma 2.1 (iv) and Minkowski's inequality, we have

$$\begin{aligned} I_3 &\le \Big(\frac{1}{w(Q)}\int_Q\Big(\sum_{k=1}^\infty\Big(\frac{w(2^{k+1}Q)}{w(Q)}\Big)^{\frac{\beta-1}{p}}\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{k+1}Q}|\Big)^p w(x)dx\Big)^{1/p}\\ &\le \sum_{k=1}^\infty \gamma_w^{-\frac{(1-\beta)(k+1)}{p}}\Big(\frac{1}{w(Q)}\int_Q\Big(\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{k+1}Q}|\Big)^p w(x)dx\Big)^{1/p}. \end{aligned}$$

One can choose $\{t_i\}_{i\in\tau}$ such that $t_i\in(1,\infty)$ and $\sum_{i\in\tau}1/t_i=1$. By Hölder's inequality, one has

$$\begin{aligned} &\Big(\frac{1}{w(Q)}\int_Q\Big(\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{k+1}Q}|\Big)^p w(x)dx\Big)^{1/p}\\ &\le w(Q)^{-1/p}\prod_{\mu\in\tau}\Big(\int_Q\Big(|b_\mu(x)-b_{\mu,2^{k+1}Q}|\Big)^{pt_\mu}w(x)dx\Big)^{1/(pt_\mu)}. \end{aligned}$$

Fix $\mu\in\tau$. By Minkowski's inequality, Lemma 2.1(v) and the fact that $|b_{\mu,Q}-b_{\mu,2^{k+1}Q}|\lesssim_n (k+1)\|b_\mu\|_{\mathrm{BMO}(\mathbb{R}^n)}$, we have

$$\begin{aligned} &\Big(\int_Q|b_\mu(x)-b_{\mu,2^{k+1}Q}|^{pt_\mu}w(x)dx\Big)^{1/(pt_\mu)}\\ &\le w(Q)^{1/(pt_\mu)}|b_{\mu,Q}-b_{\mu,2^{k+1}Q}|+\Big(\int_Q|b_\mu(x)-b_{\mu,Q}|^{pt_\mu}w(x)dx\Big)^{1/(pt_\mu)}\\ &\lesssim_{n,m}(k+1)w(Q)^{1/(pt_\mu)}\|b_\mu\|_{\mathrm{BMO}(\mathbb{R}^n)}. \end{aligned}$$

It follows that

$$\begin{aligned} I_3 &\lesssim_{n,m} w(Q)^{-1/p}\sum_{k=1}^\infty\gamma_w^{-\frac{(1-\beta)(k+1)}{p}}\prod_{\mu\in\tau}(k+1)w(Q)^{1/(pt_\mu)}\|b_\mu\|_{\mathrm{BMO}(\mathbb{R}^n)}\\ &\lesssim_{n,m}\prod_{\mu\in\tau}\|b_\mu\|_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{k=1}^\infty\gamma_w^{-\frac{(1-\beta)(k+1)}{p}}(k+1)^m\\ &\lesssim_{n,m,p,\beta}\prod_{\mu\in\tau}\|b_\mu\|_{\mathrm{BMO}(\mathbb{R}^n)}. \end{aligned}$$

This together with (2.25) leads to (2.13). This completes the proof of Proposition 2.1. □

Remark 2.1. (i) One can easily check that M_Ω satisfies (2.7) with $m=0$. By Lemma 2.2 and Proposition 2.1, we see that M_Ω is bounded on $M^{p,\beta}(w)$ if $\Omega\in L^q(\mathrm{S}^{n-1})$ for some $q>1$ satisfying (1.1), $q'<p<\infty$, $0<\beta<1$ and $w\in A_{p/q'}(\mathbb{R}^n)$.

(ii) It is clear that T_Ω satisfies (2.7) with $m = 0$. Applying [21, Theorem 2.2.3] and Proposition 2.1, we see that T_Ω is bounded on $M^{p,\beta}(w)$ if $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q > 1$ satisfying (1.1), $q' < p < \infty$, $0 \le \beta < 1$ and $w \in A_{p/q'}(\mathbb{R}^n)$.

(iii) It was shown in [9, Theorem 3] that the maximal singular integral operator

$$T_\Omega^*(f)(x) = \sup_{\epsilon>0} \Big| \int_{|x-y|\ge\epsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \Big|$$

is bounded on $L^p(w)$ for $q' < p < \infty$ and $w \in A_{p/q'}(\mathbb{R}^n)$, provided that $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q > 1$ satisfying (1.1). This together with Proposition 2.1 implies that T_Ω^* is bounded on $M^{p,\beta}(w)$, provided that $b \in \mathrm{BMO}(\mathbb{R}^n)$, $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q > 1$ satisfying (1.1), $q' < p < \infty$, $0 \le \beta < 1$ and $w \in A_{p/q'}(\mathbb{R}^n)$.

(iv) It was shown in [21, Theorem 2.4.4] that $T_{\Omega,b}$ is bounded on $L^p(w)$ for $q' < p < \infty$ and $w \in A_{p/q'}(\mathbb{R}^n)$, provided that $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q > 1$ satisfying (1.1). One can easily check that $T_{\Omega,b}$ satisfies (2.7) with $m = 1$. These above facts together with Proposition 2.1 imply that $T_{\Omega,b}$ is bounded on $M^{p,\beta}(w)$, provided that $b \in \mathrm{BMO}(\mathbb{R}^n)$, $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q > 1$ satisfying (1.1), $q' < p < \infty$, $0 \le \beta < 1$ and $w \in A_{p/q'}(\mathbb{R}^n)$.

We now prove Theorem 1.3.

Proof. [Proof of Theorem 1.3] It was shown in [5] that $\mathcal{V}_\rho(\mathcal{T}_\Omega)$ is bounded on $L^p(w)$ for $q' < p < \infty$ and $w \in A_{p/q'}(\mathbb{R}^n)$. It was also shown in [6, Corollary 1.4] that $\mathcal{V}_\rho(\mathcal{T}^1_{\Omega,b})$ is bounded on $L^p(w)$ for $q' < p < \infty$ and $w \in A_{p/q'}(\mathbb{R}^n)$ if $b \in \mathrm{BMO}(\mathbb{R}^n)$. These together with [6, Theorem 1.1] imply

$$\|\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,\vec{b}})(f)\|_{L^p(w)} \lesssim_{n,p,\beta,m} \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^p(w)}, \quad \forall f \in L^p(w).$$

This proves Theorem 1.3 for the case $\beta = 0$. On the other hand, one can check that $\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,\vec{b}})$ satisfies (2.7). These together with Proposition 2.1 leads to (1.3). □

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2. In order to prove Theorem 1.2, we need the following proposition, which gives some characterizations that a subset in $M^{p,\beta}(w)$ is a strongly pre-compact set.

Proposition 3.1. ([19]). *Let $1 < p < \infty$, $0 \le \beta < 1$ and $w \in A_p(\mathbb{R}^n)$. A subset $\mathfrak{F}$ of $M^{p,\beta}(w)$ is strongly pre-compact set in $M^{p,\beta}(w)$ if $\mathfrak{F}$ satisfies the following conditions:*

(i) *$\mathfrak{F}$ is bounded, i.e.*

$$\sup_{f\in\mathfrak{F}} \|f\|_{M^{p,\beta}(w)} < \infty;$$

(ii) *$\mathfrak{F}$ uniformly vanishes as infinity, i.e.*

$$\lim_{N\to+\infty} \|f\chi_{E_N}\|_{M^{p,\beta}(w)} = 0, \quad \text{uniformly for all } f \in \mathfrak{F},$$

where $E_N = \{x \in \mathbb{R}^n; |x| > N\}$.

(iii) $\mathfrak{F}$ *is uniformly translation continuous, i.e.*

$$\lim_{r\to 0}\sup_{h\in B(0,r)}\|f(\cdot+h)-f(\cdot)\|_{M^{p,\beta}(w)}=0,\quad \text{uniformly for all } f\in\mathfrak{F}.$$

Now we prove Theorem 1.2.

Proof. [Proof of Theorem 1.2] Let ρ, β, p, Ω, w be given as in Theorem 1.2. At first, we shall prove that if $\mathcal{V}_\rho(\mathcal{T}^m_{K,b})$ is compact for any $b\in\mathcal{C}^\infty_c(\mathbb{R}^n)$, then $\mathcal{V}_\rho(\mathcal{T}^m_{K,b})$ is compact for any $b\in \mathrm{CMO}(\mathbb{R}^n)$. Actually, for a fixed $b\in\mathrm{CMO}(\mathbb{R}^n)$ and $\varepsilon\in(0,1)$, there exists $b_\varepsilon\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $\|b_\varepsilon-b\|_{\mathrm{BMO}(\mathbb{R}^n)}<\varepsilon$. Clearly,

$$b_\varepsilon^m-b^m=(b_\varepsilon-b)(b_\varepsilon^{m-1}+b_\varepsilon^{m-2}b+\cdots+b^{m-1}).$$

For convenience, we set

$$\vec{b}_1=(b_\varepsilon-b,\overbrace{b_\varepsilon,\cdots,b_\varepsilon}^{m-1}),\vec{b}_2=((b_\varepsilon-b,\overbrace{b_\varepsilon,\cdots,b_\varepsilon}^{m-2},b),\cdots,\vec{b}_m=((b_\varepsilon-b,\overbrace{b,\cdots,b}^{m-1}).$$

It is not difficult to see that

$$\begin{aligned}&|\mathcal{V}_\rho(\mathcal{T}^m_{K,b_\varepsilon})(f)(x)-\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f)(x)|\\ \le{}&\sup_{\epsilon_i\searrow 0}\Big(\sum_{i=1}^\infty\Big|\int_{\epsilon_{i+1}<|x-z|\le\epsilon_i}((b_\varepsilon(x)-b_\varepsilon(z))^m-(b(x)-b(z))^m)\\ &\times K(x,z)f(z)dz\Big|^\rho\Big)^{1/\rho}\\ \le{}&\sum_{j=1}^m\mathcal{V}_\rho(\mathcal{T}^m_{K,\vec{b}_j})(f)(x).\end{aligned}$$

Combining this with Theorem 1.3 and Minkowski's inequality implies that

$$\begin{aligned}&\|\mathcal{V}_\rho(\mathcal{T}^m_{K,b_\varepsilon})(f)-\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f)\|_{M^{p,\beta}(w)}\\ \le{}&\sum_{j=1}^m\|\mathcal{V}_\rho(\mathcal{T}^m_{K,\vec{b}_j})(f)\|_{M^{p,\beta}(w)}\lesssim_{n,p,\beta}\varepsilon\|f\|_{M^{p,\beta}(w)},\end{aligned}$$

which combining with [29, p. 278, Theorem (iii)] implies that to obtain the compactness for $\mathcal{V}_\rho(\mathcal{T}^m_{K,b})$ with $b\in\mathrm{CMO}(\mathbb{R}^n)$, it suffices to prove the compactness for $\mathcal{V}_\rho(\mathcal{T}^m_{K,b})$ with $b\in\mathcal{C}^\infty_c(\mathbb{R}^n)$.

In what follows, we let $b\in\mathcal{C}^\infty_c(\mathbb{R}^n)$. We want to show that $\mathcal{V}_\rho(\mathcal{T}^m_{K,b})$ is compact on $M^{p,\beta}(w)$. For any $\eta>0$, we define the function Ω_η by

$$\Omega_\eta(z)=\Omega(z)\Big(1-\varphi\Big(\frac{2}{\eta}|z|\Big)\Big).$$

It is clear that $\Omega_\eta\in L^q(\mathrm{S}^{n-1})$ and Ω_η satisfies (1.1). By [30, (3.4)], one has

$$|\mathcal{V}_\rho(\mathcal{T}^m_{\Omega_\eta,b})(f)(x)-\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})(f)(x)|\lesssim_n(|b(x)|+\|b\|_{L^\infty(\mathbb{R}^n)})^{m-1}\eta M_\Omega f(x).$$

This together with Remark 2.1 (i) implies that

$$\|\mathcal{V}_\rho(\mathcal{T}^m_{\Omega_\eta,b})(f)-\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})(f)\|_{M^{p,\beta}(w)}\le C\eta\|f\|_{M^{p,\beta}(w)},\quad\forall f\in M^{p,\beta}(w).\tag{3.1}$$

By (3.1) and [29, p. 278, Theorem (iii)], to prove the compactness for $\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})$, it suffices to prove the compactness for $\mathcal{V}_\rho(\mathcal{T}^m_{\Omega_\eta,b})$ when $\eta > 0$ is small enough. For $\beta > 0$, let

$$\mathcal{F} := \{\mathcal{V}_\rho(\mathcal{T}^m_{\Omega_\eta,b})(f) : \|f\|_{M^{p,\beta}(w)} \le 1\}.$$

To prove the compactness for $\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})$, it is enough to show that $\mathcal{F}$ satisfies the conditions (i)-(iii) of Proposition 3.1 when $\eta > 0$ is small enough. Let $\eta \in (0,1)$. By Theorem 1.1 and (3.1), we have

$$\begin{aligned}
&\|\mathcal{V}_\rho(\mathcal{T}^m_{\Omega_\eta,b})(f)\|_{M^{p,\beta}(w)} \\
&\le \|\mathcal{V}_\rho(\mathcal{T}^m_{\Omega_\eta,b})(f) - \mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})(f)\|_{M^{p,\beta}(w)} + \|\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})(f)\|_{M^{p,\beta}(w)} \\
&\le C\|f\|_{M^{p,\beta}(w)} \le C,
\end{aligned}$$

when $\|f\|_{M^{p,\beta}(w)} \le 1$. This yields that $\mathcal{F}$ satisfies Proposition 3.1 (i).

Let b be supported in a ball $B = B(0,r)$. Fix $f \in M^{p,\beta}(\mathbb{R}^n)$ with $\|f\|_{M^{p,\beta}(\mathbb{R}^n)} \le 1$ and $E_N := \{x \in \mathbb{R}^n : |x| > N\}$ with $N \ge \max\{nr, 1\}$. By (2.1) and a change of variable, we have

$$\begin{aligned}
&\mathcal{V}_\rho(\mathcal{T}^m_{\Omega_\eta,b})(f)(x) \\
&\le \int_{\mathbb{R}^n} |(b(x)-b(z))^m f(z)| \frac{|\Omega_\eta(x-z)|}{|x-z|^n} dz \\
&\lesssim_n (|b(x)| + \|b\|_{L^\infty(\mathbb{R}^n)})^m \int_{|z|\le r, |x-z|\le \frac{\eta}{2}} |f(z)| \frac{|\Omega(x-z)|}{|x-z|^n} dz \\
&\lesssim_n ((N-1)r)^{-n} (|b(x)| + \|b\|_{L^\infty(\mathbb{R}^n)})^m \int_{|x-z|\le\frac{\eta}{2}} |f(z)||\Omega(x-z)| dz \\
&\lesssim_n ((N-1)r)^{-n} \Big(\frac{\eta}{2}\Big)^n (|b(x)| + \|b\|_{L^\infty(\mathbb{R}^n)})^m M_\Omega f(x),
\end{aligned} \tag{3.2}$$

when $x \in E_N$. By (3.2) and Remark 2.1 (i), one has

$$\|T^m_{\Omega_\eta,b}(f)\chi_{E_N}\|_{M^{p,\beta}(w)} \lesssim_{n,m} ((N-1)r)^{-n} \Big(\frac{\eta}{2}\Big)^n \|b\|^m_{L^\infty(\mathbb{R}^n)} \|f\|_{M^{p,\beta}(w)}, \tag{3.3}$$

which implies that $\mathcal{F}$ satisfies Proposition 3.1 (ii).

Finally, we verify the condition (iii) of Proposition 3.1. It suffices to show that for any $\epsilon \in (0, 1/4)$, there exist a small number $\eta > 0$, $\delta > 0$ and $\gamma > 0$ such that

$$\|\mathcal{V}_\rho(\mathcal{T}^m_{\Omega_\eta,b})(f)(\cdot + h) - \mathcal{V}_\rho(\mathcal{T}^m_{\Omega_\eta,b})(f)(\cdot)\|_{M^{p,\beta}(w)} \le C(\epsilon + \epsilon^\gamma) \tag{3.4}$$

when $|h| < \delta$. We set $\eta = \epsilon$, $\delta = \frac{\epsilon}{16} e^{-1/\epsilon}$ and let $|h| < \delta$. It is clear that $|h| < \frac{\epsilon}{16}$ and $\frac{|h|}{\eta} < \epsilon$. By the definition of $\mathcal{V}_\rho(\mathcal{T}^m_{\Omega,b})$, we have

$$\begin{aligned}
&|\mathcal{V}_\rho(\mathcal{T}^m_{\Omega_\eta,b})(f)(x+h) - \mathcal{V}_\rho(\mathcal{T}^m_{\Omega_\eta,b})(f)(x)| \\
&\le \sup_{\varepsilon_i \searrow 0} \Big(\sum_{i=1}^\infty \Big| \int_{\varepsilon_{i+1} < |x+h-y| \le \varepsilon_i} (b(x+h) - b(y))^m \frac{\Omega_\eta(x+h-y)}{|x+h-y|^n} f(y) dy \\
&\quad - \int_{\varepsilon_{i+1} < |x-y| \le \varepsilon_i} (b(x) - b(y))^m \frac{\Omega_\eta(x-y)}{|x-y|^n} f(y) dy \Big|^\rho \Big)^{1/\rho} \\
&\le J_1 f(x) + J_2 f(x) + J_3 f(x).
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}J_1f(x) :=& \sup_{\varepsilon_i\searrow 0}\Big(\sum_{i=1}^{\infty}\Big|\int_{\varepsilon_{i+1}<|x-y|\le\varepsilon_i}((b(x+h)-b(y))^m-(b(x)-b(y))^m)\\&\times\frac{\Omega_\eta(x-y)}{|x-y|^n}f(y)dy\Big|^\rho\Big)^{1/\rho},\\ J_2f(x) :=& \sup_{\varepsilon_i\searrow 0}\Big(\sum_{i=1}^{\infty}\Big|\int_{\varepsilon_{i+1}<|x-y|\le\varepsilon_i}(b(x+h)-b(y))^m\\&\times\Big(\frac{\Omega_\eta(x+h-y)}{|x+h-y|^n}-\frac{\Omega_\eta(x-y)}{|x-y|^n}\Big)f(y)dy\Big|^\rho\Big)^{1/\rho},\\ J_3f(x) :=& \sup_{\varepsilon_i\searrow 0}\Big(\sum_{i=1}^{\infty}\Big|\int_{\mathbb{R}^n}(b(x+h)-b(y))^m\frac{\Omega_\eta(x+h-y)}{|x+h-y|^n}f(y)\\&\times(\chi_{\varepsilon_{i+1}<|x+h-y|\le\varepsilon_i}(y)-\chi_{\varepsilon_{i+1}<|x-y|\le\varepsilon_i}(y))dy\Big|^\rho\Big)^{1/\rho}.\end{aligned}$$

For J_1f. It was shown in [30] (see [30, (3.12)]) that

$$J_1f(x)\lesssim_{m,n,b}|h|\sum_{j=0}^{m}|b(x)|^j\Big(\sum_{\mu=0}^{m-1}\mathcal{V}_\rho(\mathcal{T}_\Omega)(b^\mu f)(x)+M_\Omega f(x)\Big). \tag{3.6}$$

This together with the boundedness part of Theorem 1.1, Remark 2.1 (i) and Minkowski's inequality implies that

$$\begin{aligned}\|J_1f\|_{M^{p,\beta}(w)}&\lesssim_{m,n,b}|h|\Big(\sum_{\mu=0}^{m-1}\|\mathcal{V}_\rho(\mathcal{T}_\Omega)(b^\mu f)\|_{M^{p,\beta}(w)}+\|M_\Omega f\|_{M^{p,\beta}(w)}\Big)\\&\lesssim_{m,n,b,p,\beta}|h|.\end{aligned} \tag{3.7}$$

For J_2f. By (2.1), Lemma 2.4 and a change of variables, one has

$$\begin{aligned}J_2f(x)\le&\sup_{\varepsilon_i\searrow 0}\Big(\sum_{i=1}^{\infty}\Big|\int_{\epsilon_{i+1}<|x-y|\le\epsilon_i}(b(x+h)-b(y))\\&\times\Big(\frac{\Omega_\eta(x+h-y)}{|x+h-y|^n}-\frac{\Omega_\eta(x-y)}{|x-y|^n}\Big)f(y)\chi_{|x-y|>\frac{\eta}{4}}(y)dy\Big|^\rho\Big)^{1/\rho}\\ \le&(|b(x+h)|+\|b\|_{L^\infty(\mathbb{R}^n)})\\&\times\int_{|x-y|>\frac{\eta}{4}}\Big|\frac{\Omega_\eta(x-y+h)}{|x-y+h|^n}-\frac{\Omega_\eta(x-y)}{|x-y|^n}\Big||f(y)|dy\\ \le&C(|b(x+h)|+\|b\|_{L^\infty(\mathbb{R}^n)})(1+F(q))\eta(M(|f|^{q'})(x))^{1/q'}.\end{aligned} \tag{3.8}$$

Note that $p>q'$ and $w\in A_{p/q'}(\mathbb{R}^n)$. By the $M^{p/q',\beta}(w)$ boundedness for M, we get

$$\|(M(|f|^{q'}))^{1/q'}\|_{M^{p,\beta}(w)}=\|M(|f|^{q'})\|^{1/q'}_{M^{p/q',\beta}(w)}=\||f|^{q'}\|^{1/q'}_{M^{p/q',\beta}(w)}=\|f\|_{M^{p,\beta}(w)}.$$

This together with (3.8) leads to

$$\|J_2f\|_{M^{p,\beta}(w)}\le C\|b\|^m_{L^\infty(\mathbb{R}^n)}\eta\|f\|_{M^{p,\beta}(w)}\lesssim_{m,b,f}\epsilon. \tag{3.9}$$

It remains to estimate $J_3 f$. By Lemma 2.1 (ii), there exists $\epsilon \in (0,1)$ such that $w \in A_{p/q'-\epsilon}(\mathbb{R}^n)$. Let $s = \frac{p}{p-\epsilon}$. It is clear that $s < q$ because of $p > q'$. Moreover, $\epsilon = p/s'$. It was proved in [30] (see [30, (3.22)]) that

$$\begin{aligned} J_3 f(x) &\lesssim_{n,s} (|b(x+h)| + \|b\|_{L^\infty(\mathbb{R}^n)})^m |h|^{1/s'} \\ &\quad \times \Big(\int_{|y| \geq \frac{\eta}{4}} \frac{|\Omega_\eta(y+h) f(x-y)|^s}{|y|^{n+s-1}} dy \Big)^{1/s}. \end{aligned} \tag{3.10}$$

By Hölder's inequality, one finds

$$\begin{aligned} &\int_{|y|\geq \frac{\eta}{4}} \frac{|\Omega_\eta(y+h) f(x-y)|^s}{|y|^{n+s-1}} dy \\ &\leq \sum_{j=0}^{\infty} \int_{2^{j-2}\eta \leq |y| \leq 2^{j-1}\eta} \frac{|\Omega_\eta(y+h) f(x-y)|^s}{|y|^{n+s-1}} dy \\ &\leq \sum_{j=0}^{\infty} \Big(\int_{2^{j-2}\eta \leq |y| \leq 2^{j-1}\eta} |f(x-y)|^{s(q/s)'} dy \Big)^{1/(q/s)'} \\ &\quad \times \Big(\int_{2^{j-2}\eta \leq |y| \leq 2^{j-1}\eta} \frac{|\Omega_\eta(y+h)|^q}{|y|^{(n+s-1)q/s}} dy \Big)^{s/q}. \end{aligned} \tag{3.11}$$

Observe that

$$\begin{aligned} &\Big(\int_{2^{j-2}\eta \leq |y| \leq 2^{j-1}\eta} |f(x-y)|^{s(q/s)'} dy \Big)^{1/(q/s)'} \\ &\leq (2^{j-1}\eta)^{n(1-s/q)} (M(|f|^{sq/(q-s)})(x))^{1-s/q}. \end{aligned} \tag{3.12}$$

On the other hand, note that $|y| \geq |y+h| - |h| \geq \frac{3}{8}|y+h|$ when $|h| \leq \frac{\eta}{8}$ and $|y+h| \geq \frac{\eta}{2}$. Then we have

$$\begin{aligned} &\int_{2^{j-2}\eta \leq |y| \leq 2^{j-1}\eta} \frac{|\Omega_\eta(y+h)|^q}{|y|^{(n+s-1)q/s}} dy \\ &\leq \int_{2^{j-2}\eta \leq |y| \leq 2^{j-1}\eta, |y+h| \geq \frac{\eta}{2}} \frac{|\Omega(y+h)|^{q'}}{|y|^{(n+s-1)q/s}} dy \\ &\leq C \int_{2^{j-2}\eta \leq |y| \leq 2^{j-1}\eta, |y+h| \geq \frac{\eta}{2}} \frac{|\Omega(y+h)|^q}{|y+h|^{(n+s-1)q/s}} dy \\ &\leq C \int_{2^{j-3}\eta \leq |y+h| \leq 2^j\eta} \frac{|\Omega(y+h)|^q}{|y+h|^{(n+s-1)q/s}} dy \\ &\leq C \int_{2^{j-3}\eta \leq |z| \leq 2^j\eta} \frac{|\Omega(z)|^q}{|z|^{(n+s-1)q/s}} dz \\ &\leq C \|\Omega\|^q_{L^q(\mathrm{S}^{n-1})} \int_{2^{j-3}\eta}^{2^j\eta} r^{n-(n+s-1)q/s-1} dr \\ &\leq C \|\Omega\|^q_{L^q(\mathrm{S}^{n-1})} (2^j\eta)^{n-(n+s-1)q/s}. \end{aligned} \tag{3.13}$$

It follows from (3.11)-(3.13) that

$$\begin{aligned} &\int_{|y|\geq \frac{\eta}{4}} \frac{|\Omega_\eta(y+h) f(x-y)|^s}{|y|^{n+s-1}} dy \\ &\leq C \|\Omega\|^s_{L^q(\mathrm{S}^{n-1})} (M(|f|^{sq/(q-s)})(x))^{1-s/q} \end{aligned}$$

$$\begin{aligned}
&\times \sum_{j=0}^{\infty}(2^{j-1}\eta)^{n(1-s/q)}(2^j\eta)^{ns/q-(n+s-1)}\\
&\leq C\|\Omega\|_{L^q(\mathrm{S}^{n-1})}^s(M(|f|^{sq/(q-s)})(x))^{1-s/q}\sum_{j=0}^{\infty}(2^{j-1}\eta)^{-s+1}\\
&\leq C\|\Omega\|_{L^q(\mathrm{S}^{n-1})}^s\eta^{-s+1}(M(|f|^{sq/(q-s)})(x))^{1-s/q}. \qquad (3.14)
\end{aligned}$$

In light of (3.10) and (3.14) we would have

$$J_3f(x)\lesssim_{n,s,\Omega}(|b(x+h)|+\|b\|_{L^\infty(\mathbb{R}^n)})^m\Big(\frac{|h|}{\eta}\Big)^{1/s'}(M(|f|^{sq/(q-s)})(x))^{1/s-1/q}. \quad (3.15)$$

Note that $w\in A_{p/q'-p/s'}(\mathbb{R}^n)$. By the $M^{p/q'-p/s',\beta}(w)$ bounds for M, we see that

$$\begin{aligned}
\|(M(|f|^{sq/(q-s)}))^{1/s-1/q}\|_{M^{p,\beta}(w)}&=\|M(|f|^{sq/(q-s)})\|_{M^{p/q'-p/s',\beta}(w)}^{1/s-1/q}\\
&=\||f|^{sq/(q-s)}\|_{M^{p/q'-p/s',\beta}(w)}^{1/s-1/q}\\
&=\|f\|_{M^{p,\beta}(w)},
\end{aligned}$$

which together with (3.15) implies that

$$\|J_3f\|_{M^{p,\beta}(w)}\lesssim_{m,n,s,\Omega,b}\Big(\frac{|h|}{\eta}\Big)^{1/s'}\|f\|_{M^{p,\beta}(w)}\lesssim_{m,n,s,\Omega,q,b,f}\epsilon^{1/s'}. \quad (3.16)$$

It follows from (3.5), (3.7), (3.9) and (3.16) that

$$\|\mathcal{V}_\rho(\mathcal{T}_{\Omega_\eta,b}^m)(f)(\cdot+h)-\mathcal{V}_\rho(\mathcal{T}_{\Omega_\eta,b}^m)(f)(\cdot)\|_{M^{p,\beta}(w)}\lesssim_{m,n,b,s,p,q,\beta}(\epsilon+\epsilon^{1/s'}).$$

This leads to (3.4) and finishes the proof of Theorem 1.2. □

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