Journal of Applied Analysis and Computation
Volume 14, Number 1, February 2024, 283–297
Website:http://www.jaac-online.com
DOI:10.11948/20230212

DYNAMICAL BEHAVIORS OF A DISCRETE-TIME PREY-PREDATOR MODEL WITH HARVESTING EFFECT ON THE PREDATOR

Zohreh Eskandari[1], Parvaiz Ahmad Naik[2,†] and Mehmet Yavuz[3]

Abstract This study investigates the dynamics of a discrete-time prey-predator model with a harvesting effect on the predator. During the analysis of the bifurcations at the interior fixed point, we find that there are some generic bifurcations, including fold, flip, Neimark-Sacker, and strong resonance bifurcations. Using the normal form theory and the center manifold theorem, we can characterize these bifurcations. Furthermore, we determine the non-degeneracy conditions for the computed bifurcations and compute the critical normal form coefficients. Our analysis of the obtained analytical results as well as the revealing of more complex dynamical behaviors that cannot be achieved analytically is carried out using the numerical continuation method by computing several bifurcation curves emanating from the detected bifurcation points.

Keywords Prey-predator model, Harvest effect, bifurcation, numerical normal form, critical normal form coefficient.

MSC(2010) 37G35, 39A30, 39A33, 91B76, 92B05, 92D25.

1. Introduction

The dynamics of discrete-time models defined by difference equations often exhibit more complex behaviors compared to continuous-time models. In the field of biological mathematics, discrete-time models are frequently chosen, particularly when studying populations with non-overlapping generations (see [16–18, 20, 25, 26, 33, 42, 43]). One of the fundamental structures in population dynamics are the prey-predator interaction, which was extensively described by pioneers such as Lotka and Volterra [26, 42]. The size of a population can be influenced by various interactions between species, including predatory, cooperative, mutualistic, and commensalism interactions. In recent years, there has been a growing body of research focusing on the impact of harvesting on prey-predator models [1, 8, 15, 27].

[†] The corresponding author.

[1] Department of Mathematics, Faculty of Science, Fasa University, Fasa, Iran
[2] Department of Mathematics and Computer Science, Youjiang Medical University for Nationalities, Baise, Guangxi, China
[3] Department of Mathematics and Computer Sciences, Faculty of Science, Necmettin Erbakan University, Konya, Turkey
Email: z.eskandari@fasau.ac.ir(Z. Eskandari), naik.parvaiz@ymun.edu.cn(P. A. Naik), mehmetyavuz@erbakan.edu.tr(M. Yavuz)

Predicting population size requires the use of both discrete-time and continuous-time models. Discrete-time models have been found to provide more accurate simulations for non-overlapping generations [3, 4, 6, 7, 24, 29, 44]. Consequently, there has been significant attention given to the discretization of predator-prey equations governed by difference equations [14,45,46]. Bifurcation theory has been widely employed to investigate dynamical systems [5, 9–11, 13, 19, 21–23, 28, 30–32, 34–37, 41]. Specifically, bifurcation analysis for discrete-time populations has been examined in previous studies [2–4, 14, 24, 29, 45]. Sene [39], in their paper, analyzed a fractional-order chaotic system in the context of the Caputo fractional derivative via bifurcation and Lyapunov exponents. They have represented the phase portraits and observed the influence of the fractional-order on the proposed chaotic system. With the help of Lyapunov exponents, they have characterized the chaotic and hyperchaotic behaviors of the system. Recently Sene [40] presented a new chaotic system under three fractional operators namely Caputo, Atangana-Baleanu, and Caputo-Fabrizio derivatives. The calculated Lyapunov exponents for the fractional-order chaotic systems characterized the behaviors of the dynamics of their considered fractional-order system. The fractional linear multistep method and the Adams-Basford method were employed to obtain the phase portraits of the proposed fractional chaotic model.

In this paper, we consider the effects of harvesting on the predator population within the framework of the Lotka-Volterra approach, as represented by the following model:

$$\begin{cases} \frac{dp_1}{d\tau} &= ap_1(1-p_1) - cp_1p_2, \\ \frac{dp_2}{d\tau} &= bp_1p_2 - ep_2 - Hp_2. \end{cases} \tag{1.1}$$

Here, p_1 and p_2 represent the prey and predator populations, respectively. It is worth noting that the parameters a, b, c, e, and H are all positive.

Subsequently, we focus on a discrete-time version of the system (1.1) using the forward Euler scheme:

$$\begin{cases} p_1 &\mapsto p_1 + h\left(ap_1(1-p_1) - cp_1p_2\right), \\ p_2 &\mapsto p_2 + h\left(bp_1p_2 - ep_2 - Hp_2\right), \end{cases} \tag{1.2}$$

where the harvesting effect is represented by H.

The main objective of this study is to compute the critical normal form of one-parameter bifurcation as well as two-parameter bifurcation at the fixed points of the model using the inner product approach. We aim to investigate codim-1 bifurcations, such as period-doubling and Neimark-Sacker bifurcations, considering a single free parameter, as well as codim-2 bifurcations, including resonance 1:2, 1:3, and 1:4, by combining two free parameters. In this work, we verify the non-degeneracy of codim-1 and codim-2 bifurcations for (1.2) using normal form coefficients. This approach offers the advantage of avoiding direct computation of the central manifold, conversion of the linear component to the Jordan form, and significantly reduces computational complexity compared to alternative methods. Additional information can be found in [4–6, 11, 30, 31].

2. Fixed points and their stability

In model (1.2), there are three possible fixed points as follows:

1. The origin fixed point $\mathcal{F}_{pp}^{0} = (0,0)$.
2. The boundary fixed point $\mathcal{F}_{pp}^{1} = (1,0)$.
3. The positive fixed point $\mathcal{F}_{pp}^{*} = \left(\frac{H+e}{b}, -\frac{a(H-b+e)}{bc}\right)$.

As described in reference [38], each fixed point in system (1.2) is analyzed for its local stability.

3. Bifurcation analysis

This section explores the dynamics of model (1.2) around the positive fixed point due to its biological significance. Here, we study the conditions for the existence of bifurcations as well as the non-degeneracy of those bifurcations. It is necessary to calculate critical coefficients for the normal forms for the model reduced to the corresponding center manifold in order to determine the non-degeneracy conditions for each bifurcation.

3.1. Period-doubling bifurcation

If $b = b_{pd}$ where

$$b_{pd} = \frac{ha\left(H^2h + 2\,Heh + e^2h + 2\,H + 2\,e\right)}{Hah^2 + aeh^2 + 4},$$

the linear part of model (1.2) at $\mathcal{F}_{pp}^{*}$ is as follows:

$$\mathcal{L}_1 = \begin{pmatrix} -\frac{Hah^2+aeh^2-hH-he+2}{hH+he+2} & -\frac{c\left(Hah^2+aeh^2+4\right)}{a(hH+he+2)} \\ 2\,\frac{(ha-2)(H+e)ha}{c(Hah^2+aeh^2+4)} & 1 \end{pmatrix}.$$

Following are the multipliers of $\mathcal{L}_1$:

$$\mu_1 = -1, \quad \mu_2 = -\frac{Hah^2 + aeh^2 - 3\,hH - 3\,he - 2}{hH + he + 2}.$$

Considering that $\mu_2 \neq \pm 1$, the central manifold corresponding to model (1.2) is a one-dimensional manifold, and model (1.2) is restricted to the central manifold as follows:

$$u \mapsto -u + \alpha_{pd}u^3 + \mathcal{O}(u^4),$$

where

$$\alpha_{pd} = \frac{(2 + (H+e)\,h)^2 \left(-8 + a\,(H+e)^3\,h^4 - (H+e)^3\,h^3 + 6\,(H+e)^2\,h^2 + (4\,H + 4\,e)\,h\right) c^2}{(ha-2)^2\,(-4 + a\,(H+e)\,h^2 + (-4\,H - 4\,e)\,h)\,(H+e)^2}.$$

The non-degeneracy condition of the period-doubling bifurcation is $\alpha_{pd} \neq 0$.

Therefore the following theorem is obtained.

Theorem 3.1. *If $b = b_{pd}$ where*

$$b_{pd} = \frac{ha\left(H^2h + 2\,Heh + e^2h + 2\,H + 2\,e\right)}{Hah^2 + aeh^2 + 4},$$

the fixed point $\mathcal{F}_{pp}^{*}$ of model (1.2) tolerates a generic period-doubling bifurcation.

3.2. Neimark-Sacker bifurcation

If $b = b_{ns}$ where

$$b_{ns} = \frac{H\,h + e\,h + 1}{h},$$

the linear part of model (1.2) at $\mathcal{F}^*_{pp}$ is as follows:

$$\mathcal{L}_1 = \begin{pmatrix} -\frac{H\,a\,h^2 + a\,e\,h^2 - H\,h - e\,h - 1}{H\,h + e\,h + 1} & -\frac{h^2 c\,(H+e)}{H\,h + e\,h + 1} \\ \frac{a}{c} & 1 \end{pmatrix}.$$

Following are the multipliers of $\mathcal{L}_1$:

$$\mu_{1,2} = -1/2\,\frac{Hah^2 + aeh^2 - 2\,Hh - 2\,eh - 2}{Hh + eh + 1} + \frac{i/2\sqrt{\Delta}}{Hh + eh + 1},$$

where

$$\Delta = -h^2 a\left(H^2ah^2 + 2\,Haeh^2 + ae^2h^2 - 4\,H^2h - 8\,Heh - 4\,e^2h - 4\,H - 4\,e\right).$$

Considering that $\Delta > 0$, the central manifold corresponding to model (1.2) is a two-dimensional manifold, and model (1.2) is restricted to the central manifold as follows:

$$\omega \mapsto \mu_1\omega + \delta_{ns}\omega^2\overline{\omega} + \mathcal{O}(|\omega|^4), \quad \omega \in \mathbb{C}.$$

Any time $\sigma_{ns} = \Re\left(\mu_2\delta_{ns}\right) \neq 0$, we have the generic Neimark-Sacker bifurcation. Thus we get the following theorem.

Theorem 3.2. *If $b = b_{ns}$ where*

$$b_{ns} = \frac{H\,h + e\,h + 1}{h},$$

the fixed point $\mathcal{F}^*_{pp}$ of model (1.2) tolerates a generic Neimark-Sacker bifurcation.

3.3. 1:2 resonance bifurcation

If $b = b_{r_2}$ and $H = H_{r_2}$ where

$$b_{r_2} = \frac{a}{a\,h - 4}, \quad H_{r_2} = -\frac{a\,eh^2 - 4\,eh - 4}{h\,(a\,h - 4)},$$

the linear part of model (1.2) at $\mathcal{F}^*_{pp}$ is as follows:

$$\mathcal{L}_1 = \begin{pmatrix} -3 & -4\,\frac{c}{a} \\ \frac{a}{c} & 1 \end{pmatrix}.$$

Following are the multipliers of $\mathcal{L}_1$:

$$\mu_{1,2} = -1.$$

Considering that $\mu_{1,2} = -1$, the central manifold corresponding to model (1.2) is a two-dimensional manifold, and model (1.2) is restricted to the central manifold as follows:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} -u_1 + u_2 \\ -u_2 + \sigma_{r_2} u_1^3 + \delta_{r_2} u_1^2 u_2 \end{pmatrix} + \mathcal{O}\left(\|u\|^4\right), \quad u = (u_1, u_2)^T,$$

where

$$\sigma_{r_2} = -1/8\, \frac{\left(a^2 + 16\, c^2\right) h^2 \left(a\, h - 2\right) \left(a\, h - 10\right)}{\left(a\, h - 4\right)^2},$$
$$\delta_{r_2} = 1/8\, \frac{h^2 \left(a^4 h^2 + 32\, a^2 c^2 h^2 + 4\, a^3 h - 96\, a\, c^2 h - 12\, a^2 + 256\, c^2\right)}{\left(a\, h - 4\right)^2}.$$

The non-degeneracy conditions of the 1:2 resonance bifurcation, in this case, are as follows:

1. $\sigma_{r_2} \neq 0$,
2. $\delta_{r_2} \neq -3\, \sigma_{r_2}$.

Hence, the next theorem can be stated.

Theorem 3.3. *If $b = b_{r_2}$ and $H = H_{r_2}$ where*

$$b_{r_2} = \frac{a}{a\, h - 4}, \quad H_{r_2} = -\frac{a\, e h^2 - 4\, e h - 4}{h \left(a\, h - 4\right)},$$

the fixed point $\mathcal{F}_{pp}^*$ of model (1.2) *tolerates a 1:2 resonance bifurcation.*

3.4. 1:3 resonance bifurcation

If $b = b_{r_3}$ and $H = H_{r_3}$ where

$$b_{r_3} = \frac{a}{a\, h - 3}, \quad H_{r_3} = -\frac{a\, e h^2 - 3\, e h - 3}{h\, \left(a\, h - 3\right)},$$

the linear part of model (1.2) at $\mathcal{F}_{pp}^*$ is as follows:

$$\mathcal{L}_1 = \begin{pmatrix} -2 & -3\, \frac{c}{a} \\ \frac{a}{c} & 1 \end{pmatrix}.$$

Following are the multipliers of $\mathcal{L}_1$:

$$\mu_{1,2} = -\frac{1}{2} \pm i\, \frac{\sqrt{3}}{2}.$$

Considering that $\mu_{1,2} = -\frac{1}{2} \pm i\, \frac{\sqrt{3}}{2}$, the central manifold corresponding to model (1.2) is a two-dimensional manifold, and model (1.2) is restricted to the central manifold as follows:

$$u \mapsto \mu_1 u + \beta_{r_3} \overline{u}^2 + \sigma_{r_3} u^2 \overline{u} + \mathcal{O}(|u|^4), \quad u \in \mathbb{C},$$

where

$$\beta_{r_3} = \frac{h\,c\,\left(i\sqrt{3} - 2\,a\,h + 3\right)}{2\,a\,h - 6},$$
$$\sigma_{r_3} = 1/2\,\frac{c^2h^2\left(i\sqrt{3}a^2h^2 - 12\,i\sqrt{3}a\,h - a^2h^2 + 15\,i\sqrt{3} + 15\right)}{\left(a\,h - 3\right)^2}.$$

The non-degeneracy conditions of the 1:3 resonance bifurcation, in this case, are as follows:

1. $\beta_{r_3} \neq 0$,
2. $\Re\left(\frac{\mu_2\sigma_{r_3}}{|\beta_{r_3}|^2} - 1\right) \neq 0$.

Therefore, we have proved the next theorem.

Theorem 3.4. *If $b = b_{r_3}$ and $H = H_{r_3}$ where*

$$b_{r_3} = \frac{a}{a\,h - 3}, \quad H_{r_3} = -\frac{a\,eh^2 - 3\,eh - 3}{h\,\left(a\,h - 3\right)},$$

the fixed point $\mathcal{F}^*_{pp}$ of model (1.2) *tolerates a 1:3 resonance bifurcation.*

3.5. 1:4 resonance bifurcation

If $b = b_{r_4}$ and $H = H_{r_4}$ where

$$b_{r_4} = \frac{a}{ah - 2}, \quad H_{r_4} = -\frac{aeh^2 - 2\,eh - 2}{h\,(ah - 2)},$$

the linear part of model (1.2) at $\mathcal{F}^*_{pp}$ is as follows:

$$\mathcal{L}_1 = \begin{pmatrix} -1 & -2\,\frac{c}{a} \\ \frac{a}{c} & 1 \end{pmatrix}.$$

Following are the multipliers of $\mathcal{L}_1$:

$$\mu_{1,2} = \pm i.$$

Considering that $\mu_{1,2} = \pm i$, the central manifold corresponding to model (1.2) is a two-dimensional manifold, and model (1.2) is restricted to the central manifold as follows:

$$u \mapsto \mu_1 u + \sigma_{r_4} u^2\overline{u} + \delta_{r_4}\overline{u}^3 + \mathcal{O}(|u|^4), \quad u \in \mathbb{C},$$

where

$$\sigma_{r_4} = -3/2\,\frac{h^2c^2\left(ia^2h^2 + 2\,a^2h^2 - 6\,ah + 4 - 2\,i\right)}{\left(ah - 2\right)^2},$$
$$\delta_{r_4} = 1/2\,\frac{h^2c^2\left(2\,ia^2h^2 - a^2h^2 - 4\,iah + 6\,ah - 6\right)}{\left(ah - 2\right)^2}.$$

As long as the following conditions are met, the 1:4 resonance bifurcation is generic:

1. $\delta_{r_4} \neq 0$,
2. $A_{r_4} = -\frac{i\,\sigma_{r_4}}{|\delta_{r_4}|} \neq 0$.

So the next theorem is presented.

Theorem 3.5. *If $b = b_{r_4}$ and $H = H_{r_4}$ where*

$$b_{r_4} = \frac{a}{ah-2}, \quad H_{r_4} = -\frac{aeh^2 - 2\,eh - 2}{h\,(ah-2)},$$

the fixed point $\mathcal{F}^*_{pp}$ of model (1.2) *tolerates a 1:4 resonance bifurcation.*

4. Numerical continuation

To confirm the analytical results, we use MatcontM, which is a toolbox of Matlab and works based on the numerical continuation method, for more detals see [12, 21, 23]. Here b is considered as free parameter and fixed parameters are as follows:

$$a = 5, \quad c = 1, \quad H = 1, \quad h = 0.9, \quad e = 0.2.$$

A Neimark-Sacker bifurcation occurs for $b = b_{ns} = 2.311111$ at $\mathcal{F}^*_{pp} = (0.519231, 2.403846)$ with the frist Lyapunov coefficient $\sigma_{ns} = -2.344082$. Since $\sigma_{ns} < 0$, we conclude that the Neimark-Sacker bifurcation is supercritical. This phenomenon is shown in Fig 1. A period-doubling bifurcation occurs for $b = b_{pd} = 1.8772011$ at $\mathcal{F}^*_{pp} = (0.639250, 1.8033752)$ with $\alpha_{pd} = -1.645355$. The period doubles when a curve emanates from a period-doubling bifurcation is shown in Fig 3.

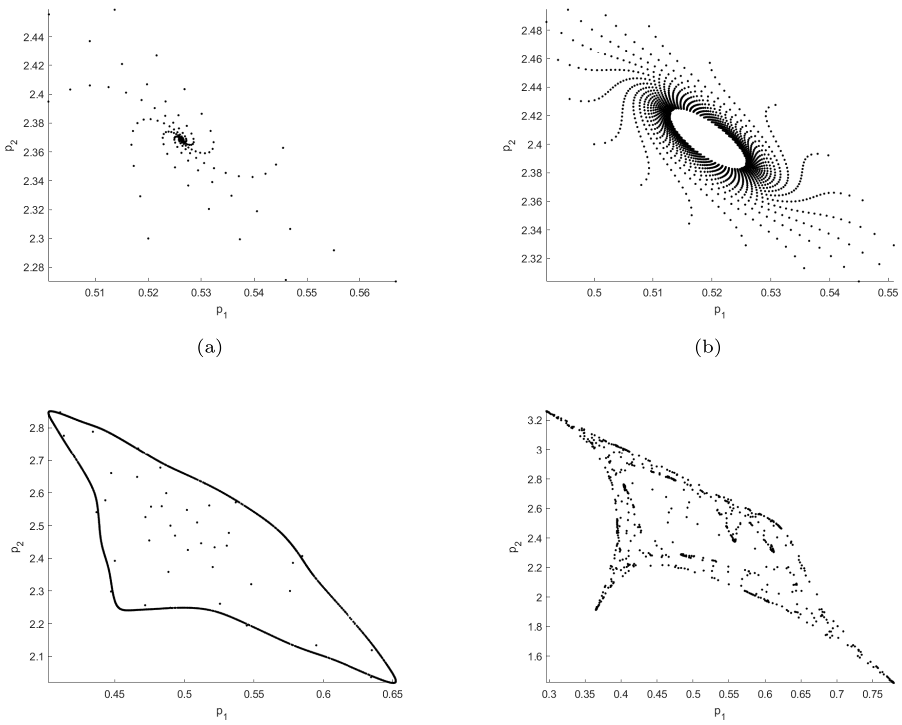
Remark 4.1. Fig. 2 illustrates the maximum Lyapunov exponent for $b \succeq 2.4$, indicating that chaos exists. Positive Lyapunov exponents are generally considered to be a sign of chaos.

On the Neimark-Sacker bifurcation curve presented in Fig 4(a) we detected the following bifurcations:

1. Resonance 1 : 4 bifurcation for $a = a_{r_4} = 2.833333$ and $m = m_{r_4} = 0.9$ at $\mathcal{F}^*_{pp} = (1.5, 2.133333)$ with $A_{r_4} = -2.117336 + 2.58679 \times 10^{-1}\, i$. Since $|A_{r_4}| > 1$ two-fold curves of the fourth iterate emanate from the R4 point of model (1.2), see Fig 5.
2. Resonance 1 : 3 bifurcation for $a = a_{r_3} = 3.5$ and $m = m_{r_3} = 4.5$ at $\mathcal{F}^*_{pp} = (1.5, 8.0)$ with $\Re\left(\frac{\mu_2 \sigma_{r_3}}{|\beta_{r_3}|^2} - 1\right) = -1.047619$.
3. Resonance 1 : 2 bifurcation for $a = a_{r_2} = 4.1666667$ and $m = m_{r_2} = 22.5$ at $\mathcal{F}^*_{pp} = (1.542.66667,)$ with $\sigma_{r_2} = 1.684908 \times 10^{-3}$ and $\delta_{r_2} = -6.908183 \times 10^{-3}$.

On the period-doubling bifurcation curve presented in Fig 4(b) we detected the following bifurcations:

1. Resonance 1 : 2 bifurcation for $a = a_{r_2} = 4.1666667$ and $m = m_{r_2} = 22.5$ at $\mathcal{F}^*_{pp} = (1.542.66667,)$ with $\sigma_{r_2} = 1.684908 \times 10^{-3}$ and $\delta_{r_2} = -6.908183 \times 10^{-3}$.



(a) (b)

(c) (d)

Figure 1. Phase portraits of model (1.2). (a) A stable fixed point for $b = 2.28$. (b) A closed invariant curve for $b = 2.3111111$. (c) The broken invariant closed curve for $b = 2.4$. (d) A chaotic attractor for $b = 2.5$.

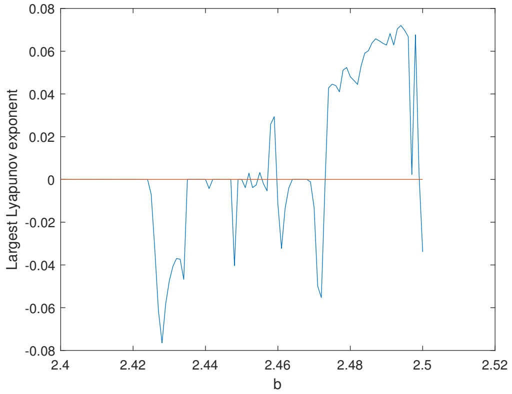


Figure 2. The maximum Lyapunov exponent corresponding to Fig. 1.

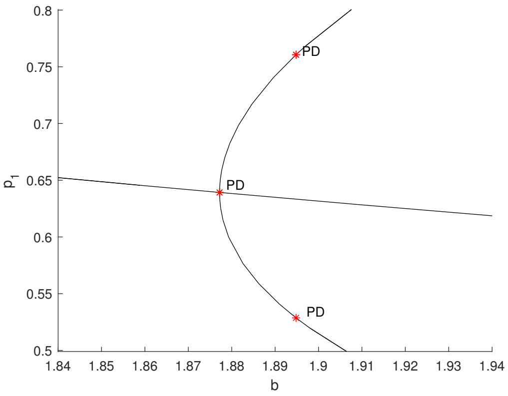
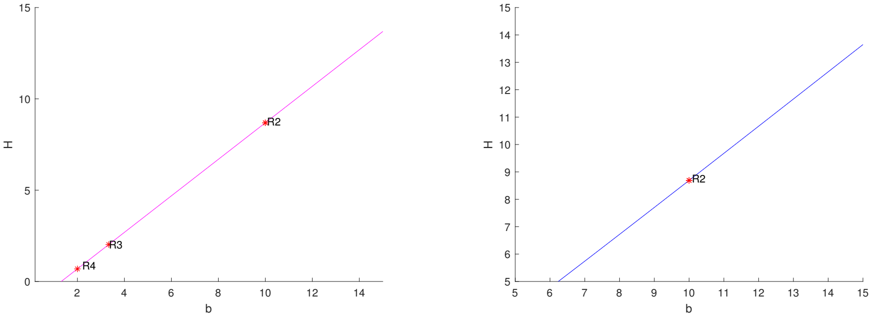


Figure 3. The period-doubling cascad.



(a) (b)

Figure 4. (a) , (b) The NS and PD curves, respectively

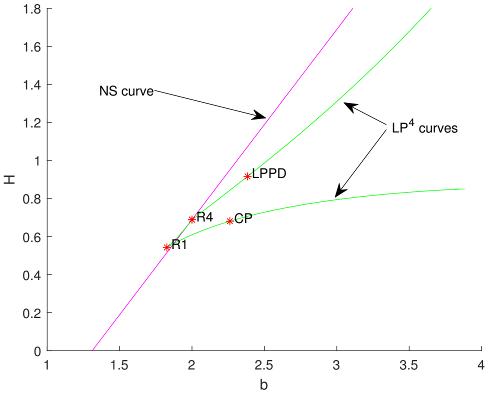


Figure 5. Two-fold curves of the fourth iterate emanate from the R4 point of model (1.2).

5. Discussion and ecological implications

Bifurcations in discrete prey-predator models can have important ecological implications. These models describe the interactions between a population of prey and a population of predators, where the prey population is consumed by the predator population.

The period-doubling bifurcation is a type of bifurcation that can occur in discrete prey-predator models. It is characterized by the doubling of the period of oscillations in the population dynamics, which can have important ecological implications.

In ecological terms, the period-doubling bifurcation can lead to an increase in the amplitude of population cycles, which can result in larger population booms and busts. This can have significant effects on the stability of the ecosystem, as the larger fluctuations in population size can lead to increased competition for resources, predation pressure, and other ecological interactions. In addition, the period-doubling bifurcation can lead to the emergence of chaotic dynamics in the population cycles, which can be difficult to predict and manage. Chaotic dynamics can lead to unpredictable changes in the ecosystem, which can make it difficult to develop effective conservation and management strategies. Overall, the period-doubling bifurcation in discrete prey-predator models highlights the importance of understanding the dynamics of populations and their interactions in ecological systems. By studying these models, we can gain insight into the underlying mechanisms that drive population cycles and other ecological processes, and develop more effective strategies to promote ecosystem stability and resilience.

The Neimark-Sacker bifurcation is a type of bifurcation that can occur in discrete prey-predator models. It is characterized by the emergence of a stable limit cycle in the population dynamics, which can have important ecological implications.

In ecological terms, the Neimark-Sacker bifurcation can lead to the emergence of regular oscillations in the population cycles. These regular oscillations can have significant effects on the stability of the ecosystem, as they can lead to predictable changes in population dynamics. This can make it easier to develop effective conservation and management strategies, as managers can anticipate when population sizes will rise and fall. The Neimark-Sacker bifurcation can also lead to the emergence of complex patterns in population dynamics. This can have significant effects on the stability of the ecosystem, as the complex patterns can make it more difficult to predict changes in population sizes. This can make it more challenging to develop effective conservation and management strategies. Overall, the Neimark-Sacker bifurcation in discrete prey-predator models highlights the importance of understanding the dynamics of populations and their interactions in ecological systems. By studying these models, we can gain insight into the underlying mechanisms that drive population cycles and other ecological processes, and develop more effective strategies to promote ecosystem stability and resilience.

The Strong resonance bifurcation is a type of bifurcation that can occur in discrete prey-predator models. It is characterized by the emergence of periodic or chaotic population dynamics, which can have important ecological implications.

In ecological terms, the Strong resonance bifurcation can lead to the emergence of large-amplitude population cycles, which can result in more severe boom-and-bust cycles in the ecosystem. This can have significant effects on the stability of the ecosystem, as the larger fluctuations in population size can lead to increased competition for resources, predation pressure, and other ecological interactions. The

Strong resonances bifurcation can also lead to the emergence of chaotic dynamics in the population cycles. Chaotic dynamics can lead to unpredictable changes in the ecosystem, which can make it difficult to develop effective conservation and management strategies. In addition, the presence of chaotic dynamics can lead to the emergence of unstable equilibria or the disappearance of equilibria, which can result in the extinction of one or both populations. Overall, the Strong resonance bifurcation in discrete prey-predator models highlights the importance of understanding the dynamics of populations and their interactions in ecological systems. By studying these models, we can gain insight into the underlying mechanisms that drive population cycles and other ecological processes, and develop more effective strategies to promote ecosystem stability and resilience.

6. Concluding remarks

In this article, we have extensively investigated a predator-prey model that incorporates the harvesting effect on the predator population. Our focus has been on studying the dynamics of the model around the positive fixed point, which holds considerable biological significance. Through our analysis, we have identified the conditions for the existence of various bifurcations and established the non-degeneracy of these bifurcations.

To determine the non-degeneracy conditions for each bifurcation, we calculated critical coefficients of the normal forms, specifically reducing the model to the corresponding center manifold. Our examination has enabled us to explore the occurrence of flip, Neimark-Sacker, and strong resonance 1:2, 1:3, and 1:4 bifurcations.

Additionally, by illustrating the curves associated with each bifurcation, we have observed a consistent agreement between our numerical simulations and the analytical results. This congruence provides further validation and confidence in the accuracy of our findings.

In conclusion, our study has contributed to a deeper understanding of the dynamics of predator-prey models with harvesting effects. The significant outcome of our research lies in the identification and analysis of various bifurcation phenomena exhibited by the model. These findings shed light on the complex dynamics that can arise in real-world ecological systems.

Looking ahead, there are several promising directions for future research in this domain. Firstly, it would be valuable to extend our analysis to more complex ecological models that incorporate additional factors such as spatial dynamics or multiple predator-prey interactions. Furthermore, exploring the impact of alternative harvesting strategies and management approaches on the stability and sustainability of predator-prey systems could yield important insights for ecological conservation and resource management practices. Lastly, investigate the implications of our findings in practical applications, such as designing effective strategies for controlling invasive species or optimizing fisheries management, holds great potential for further advancements in the field.

By addressing the significant outcomes of our study and providing detailed future research directions, our conclusion serves to summarize the key findings of our research while also highlighting the potential for further exploration and application in the field of predator-prey dynamics.

Data availability statement

The data that support the findings of this study are available within the article.

Acknowledgments

The authors would like to thank the reviewers and editors of this paper for their careful attention to detail and constructive feedback that improved the presentation of the paper greatly.

References

[1] S. Chakraborty, S. Pal and N. Bairagi, *Predator-prey interaction with harvesting: Mathematical study with biological ramifications,* Applied Mathematical Modelling, 2012, 36(9), 4044–4059.

[2] Z. Eskandari, J. Alidousti, Z. Avazzadeh and J. T. Machado, *Dynamics and bifurcations of a discrete-time prey-predator model with Allee effect on the prey population,* Ecological Complexity, 2021, 48, 100962.

[3] Z. Eskandari, Z. Avazzadeh and R. K. Ghaziani, *Theoretical and numerical bifurcation analysis of a predator-prey system with ratio-dependence,* Mathematical Sciences, 2023, 18, 1–12.

[4] Z. Eskandari, Z. Avazzadeh, R. K. Ghaziani and B. Li, *Dynamics and bifurcations of a discrete-time Lotka-Volterra model using nonstandard finite difference discretization method,* Mathematical Methods in the Applied Sciences, 2022, 46(7), 7045–7059.

[5] Z. Eskandari, R. K. Ghaziani and Z. Avazzadeh, *Bifurcations of a discrete-time SIR epidemic model with logistic growth of the susceptible individuals,* International Journal of Biomathematics, 2023, 16(6), 2250120.

[6] M. Farman, M. F. Tabassum, P. A. Naik and S. Akram, *Numerical treatment of a nonlinear dynamical hepatitis B model: An evolutionary approach,* The European Physical Journal Plus, 2020, 135 (12), 941.

[7] H. I. Freedman, *Deterministic Mathematical Models in Population Ecology,* Marcel Dekker Incorporated, 57, 1980.

[8] X. Gao, S. Ishag, S. Fu, W. Li and W. Wang, *Bifurcation and turing pattern formation in a diffusive ratio-dependent predator-prey model with predator harvesting,* Nonlinear Analysis: Real World Applications, 2020, 51, 102962.

[9] M. Gholami, R. K. Ghaziani and Z. Eskandari, *Three-dimensional fractional system with the stability condition and chaos control,* Mathematical Modeling and Numerical Simulation with Applications, 2022, 2(1), 41–47.

[10] M. B. Ghori, Y. Kang Y. Chen, *Emergence of stochastic resonance in a two-compartment hippocampal pyramidal neuron model,* Journal of Computational Neuroscience, 2022, 50(2), 217–240.

[11] M. B. Ghori, P. A. Naik, J. Zu, Z. Eskandari and M. Naik, *Global dynamics and bifurcation analysis of a fractional-order SEIR epidemic model with saturation incidence rate,* Mathematical Methods in the Applied Sciences, 2022, 45(7), 3665–3688.

[12] W. Govaerts, R. K. Ghaziani, Y. A. Kuznetsov and H. G. Meijer, *Numerical methods for two-parameter local bifurcation analysis of maps,* SIAM Journal on Scientific Computing, 2007, 29(6), 2644–2667.

[13] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields,* Springer Science & Business Media, 42, 2013.

[14] Z. He and X. Lai, *Bifurcation and chaotic behavior of a discrete-time predator-prey system,* Nonlinear Analysis: Real World Applications, 2011, 12(1), 403–417.

[15] C. M. Heggerud and K. Lan, *Local stability analysis of ratio-dependent predator-prey models with predator harvesting rates,* Applied Mathematics and Computation, 2015, 270, 349–357.

[16] C. S. Holling, *The functional response of predators to prey density and its role in mimicry and population regulation,* The Memoirs of the Entomological Society of Canada, 1965, 97(S45), 5–60.

[17] S. R. J. Jang, *Allee effects in a discrete-time host-parasitoid model,* Journal of Difference Equations and Applications, 2006, 12(2), 165–181.

[18] S. R. J. Jang, *Discrete-time host-parasitoid models with Allee effects: Density dependence versus parasitism,* Journal of Difference Equations and Applications, 2011, 17(04), 525–539.

[19] H. Joshi, M. Yavuz and I. Stamova, *Analysis of the disturbance effect in intracellular calcium dynamic on fibroblast cells with an exponential kernel law,* Bulletin of Biomathematics, 2023, 1(1), 24–39.

[20] L. E. Keshet, *Mathematical models in biology,* SIAM, 2005, 1–586.

[21] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory,* Springer Science & Business Media, 112, 2013.

[22] Y. A. Kuznetsov and H. G. Meijer, *Numerical normal forms for codim 2 bifurcations of fixed points with at most two critical eigenvalues,* SIAM Journal on Scientific Computing, 2005, 26(6), 1932–1954.

[23] Y. A. Kuznetsov and H. G. Meijer, *Numerical Bifurcation Analysis of Maps: From Theory to Software,* Cambridge University Press, 2019.

[24] X. Liu and D. Xiao, *Bifurcations in a discrete time Lotka-Volterra predator-prey system,* Discrete and Continuous Dynamical Systems-B, 2006, 6(3), 559–572.

[25] G. Livadiotis, L. Assas, B. Dennis, S. Elaydi and E. Kwessi, *A discrete-time host-parasitoid model with an Allee effect,* Journal of Biological Dynamics, 2015, 9(1), 34–51.

[26] A. J. Lotka, *Elements of physical biology,* Nature, 1925, 116, 461.

[27] V. Madhusudanan, K. Anitha, S. Vijaya and M. Gunasekaran, *Complex effects in discrete time prey-predator model with harvesting on prey,* The International Journal of Engineering and Science, 2014, 3(4), 1–5.

[28] P. A. Naik and Z. Eskandari, *Nonlinear dynamics of a three-dimensional discrete-time delay neural network,* International Journal of Biomathematics, 2024, 17(4), 2350057.

[29] P. A. Naik, Z. Eskandari, Z. Avazzadeh and J. Zu, *Multiple bifurcations of a discrete-time prey-predator model with mixed functional response,* International Journal of Bifurcation and Chaos, 2022, 32(04), 2250050.

[30] P. A. Naik, Z. Eskandari, A. Madzvamuse, Z. Avazzadeh and J. Zu, *Complex dynamics of a discrete-time seasonally forced SIR epidemic model,* Mathematical Methods in the Applied Sciences, 2023, 46(6), 7045–7059.

[31] P. A. Naik, Z. Eskandari and H. E. Shahraki, *Flip and generalized flip bifurcations of a two-dimensional discrete-time chemical model,* Mathematical Modelling and Numerical Simulation with Applications, 2021, 1(2), 95–101.

[32] P. A. Naik, Z. Eskandari, H. E. Shahraki and K. M. Owolabi, *Bifurcation analysis of a discrete-time prey-predator model,* Bulletin of Biomathematics, 2023, 1(2), 111–123.

[33] P. A. Naik, Z. Eskandari, M. Yavuz and J. Zu, *Complex dynamics of a discrete-time Bazykin-Berezovskaya prey-predator model with a strong Allee effect,* Journal of Computational and Applied Mathematics, 2022, 413, 114401.

[34] P. A. Naik and K. R. Pardasani, *Three dimensional finite element model to study calcium distribution in oocytes,* Network Modeling Analysis in Health Informatics and Bioinformatics, 2017, 6, 16.

[35] S. Noeiaghdam, M. Suleman and H. Budak, *Solving a modified nonlinear epidemiological model of computer viruses by homotopy analysis method,* Mathematical Sciences, 2018, 12, 211–222.

[36] K. M. Owolabi, B. Karaagac and D. Baleanu, *Pattern formation in superdiffusion predator-prey-like problems with integer- and noninteger-order derivatives,* Mathematical Methods in the Applied Sciences, 2021, 44(5), 4018–4036.

[37] K. M. Owolabi, E. Pindza and A. Atangana, *Analysis and pattern formation scenarios in the superdiffusive system of predation described with Caputo operator,* Chaos Solitons & Fractals, 2021, 152(399), 111468.

[38] A. K. G. Ozlem and M. Feckan, *Stability, neimark-sacker bifurcation and chaos control for a prey-predator system with harvesting effect on predator,* Miskolc Mathematical Notes, 2021, 22(2), 663–679.

[39] N. Sene, *Analysis of a fractional-order chaotic system in the context of the Caputo fractional derivative via bifurcation and Lyapunov exponents,* Journal of King Saud University-Science, 2021, 33(1), 101275.

[40] N. Sene, *On the modeling and numerical discretizations of a chaotic system via fractional operators with and without singular kernels,* Mathematical Sciences, 2023, 17, 517–537.

[41] N. Sene, *Second-grade fluid with Newtonian heating under Caputo fractional derivative: analytical investigations via Laplace transforms,* Mathematical Modeling and Numerical Simulation with Applications, 2022, 2(1), 13–25.

[42] V. Volterra, *Fluctuations in the abundance of a species considered mathematically,* Nature, 1926, 118, 558–560.

[43] W. X. Wang, Y. B. Zhang and C. Z. Liu, *Analysis of a discrete-time predator-prey system with Allee effect,* Ecological Complexity, 2011, 8(1), 81–85.

[44] M. Yavuz and N. Sene, *Stability analysis and numerical computation of the fractional predator-prey model with the harvesting rate,* Fractal and Fractional, 2020, 4, 35.

[45] L. G. Yuan and Q. G. Yang, *Bifurcation, invariant curve and hybrid control in a discrete-time predator-prey system,* Applied Mathematical Modelling, 2015, 39(8), 2345–2362.

[46] M. Zhao, C. Li and J. Wang, *Complex dynamic behaviors of a discrete-time predator-prey system,* Journal of Applied Analysis and Computation, 2017, 7(2), 478–500.